Lecture 2

Last week: typical behaviour when $c(x,x+1) \in (\kappa, \frac{1}{\kappa})$ for $\kappa \in (0,1)$. $\rightsquigarrow$ Convergence to $(B_t)_t$ via the environment seen from the walker.

If $\mathbb{E}\left[c(x,x+1)\right] = +\infty$ then this fails.

$(c(x,x+1))_{x\in\mathbb{Z}}$ i.i.d.

$\hookrightarrow$ Choose $c(0,1)$ to be heavy-tailed

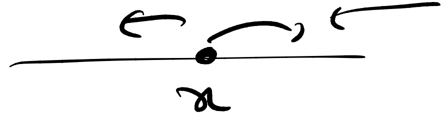
think log.

$$\mathbf{P}\left(c(0,1) > t\right) = L(t) \cdot t^{-\alpha}, \quad \alpha \in (0,1).$$

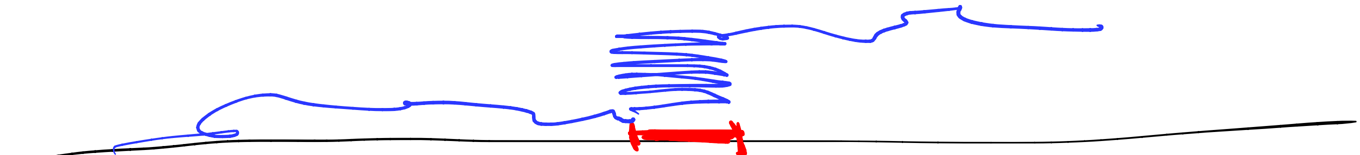
$\longrightarrow$ L: slowly varying: $\frac{L(at)}{L(t)} \xrightarrow{t\to+\infty} 1, \ \forall a \in \mathbb{R}.$

(X_n) random walk on this environment.

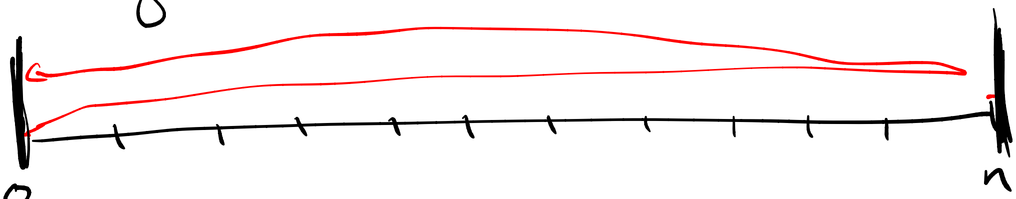
X_n :



$$\frac{c(x,x+1)}{c(x,x+1)+c(x-1,x)}.$$



<u>Q°</u>: How much time for X to reach distance n from 0?



$$r(x,x+1) = \frac{1}{c(x,x+1)}.$$

Book Lyons & Peres : Commute-time identity

$$E_0^\omega[T_n] + E_n^\omega[T_0] = 2\left(\sum_{x=0}^{n-1} c(x,x+1)\right)\left(\sum_{x=0}^{n-1} r(x,x+1)\right)$$

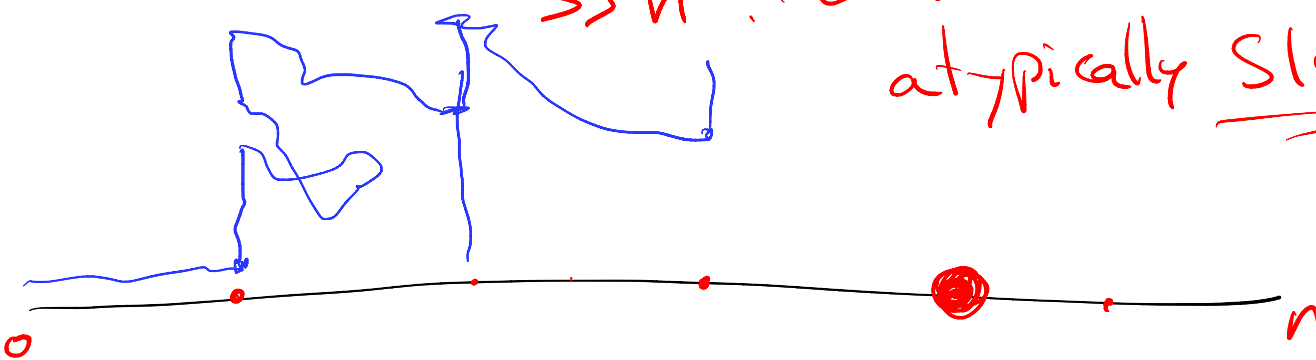
$T_k = \inf\{\ell \geq 0 : X_\ell = k\}$

This time is of order: $n^{\frac{1}{\alpha}} \times n$

$\sum C(u, u+1)$ = sum of i.i.d. stable r.v.'s.

For X, it should take time $n^{1+\frac{1}{\alpha}}$ to reach distance.

$\gg n^2$. $\rightsquigarrow$ this walk is atypically slow.



Bouchard trap model:

On $\mathbb{Z}$, assign a random average holding time τ_x to each vertex $x \in \mathbb{Z}$, (τ_x) i.i.d. values in $(0,\infty)$.

Define a continuous random walk $(X_t)_{t\geq 0}$, $X_0 = 0$, Markov process with generator:

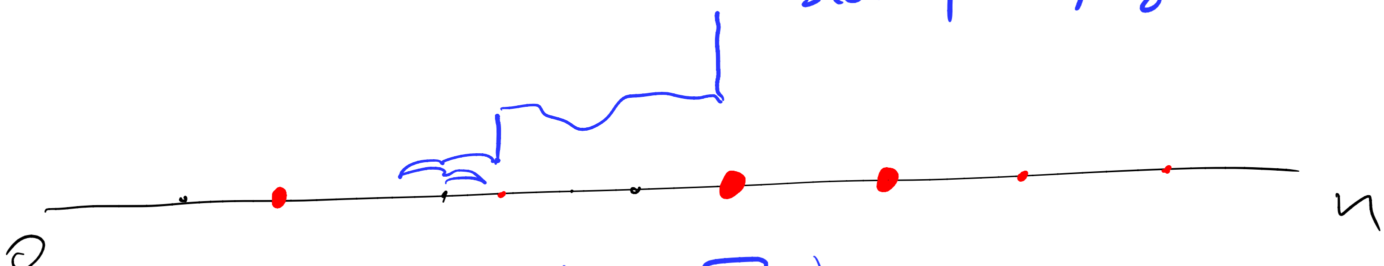
$$\mathcal{L}f(x) = \sum_{y \in \{x-1,x+1\}} \frac{1}{2\tau_x}\left(f(y) - f(x)\right).$$

$\Rightarrow$ when at x, the RW wait for an exponential time of average τ_x, and then jump ± 1 w.p. $\frac{1}{2}$.

* If $\mathbb{E}[\tau_0] < +\infty$, then typical behaviour $\rightsquigarrow$ Brownian motion.

* Assume that τ_0 is such that

$$\mathbb{P}(\tau_0 \geq u) = \underbrace{L(u)}_{\text{slowly varying}} \cdot u^{-\alpha}, \quad \alpha \in (0,1)$$



limit given by Fontes, Isopi & Newman (2002)

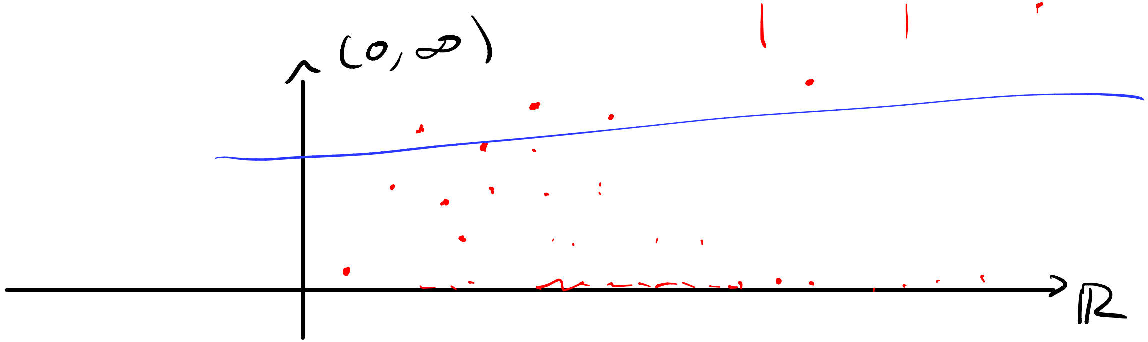
Look at the limit of the environment as a Poisson point process on $\mathbb{R} \times (0,\infty)$

position of a trap

depth of a trap ($\sim n^{\frac{1}{\alpha}}$)

Consider $\mathcal{V} = \sum_{x \in \mathbb{Z}} \tau_x \delta_x$ and take the limit.

$\rightsquigarrow$ PPP with intensity $dx \cdot \alpha \cdot v^{-1-\alpha} \, dv$.



This PPP is dense in $\mathbb{R}$

Limit Process !

Conditionally on the PPP above, say ϱ, the F.I.N. diffusion (Z_t) is expressed as a time-change of Brownian motion $(B_t)_t$

* $\ell(t,y)$ = local time of $(B_s)_s$ at time t, at y.
* $\phi_\varrho(t) = \int_{\mathbb{R}} \ell(t,y)\,\varrho(dy)$.
* $\psi_\varrho(s) = \inf\{t>0 : \phi_\varrho(t) > s\}$.

$$Z(s) = B_{\psi_\varrho(s)}.$$ ← FIN diffusion.

F.I.N. proved that, for the Bouchaud trap model.

$$X^\varepsilon(t) = \varepsilon X\left(\frac{t}{\varepsilon^{1+\frac{1}{\alpha}} L(\varepsilon)}\right) \xrightarrow[(d)]{\varepsilon\to 0} Z(t)$$

The FIN diffusion is anomalous

$$Z(t) \overset{(d)}{=} \lambda^{-1} Z\left(t \cdot \lambda^{\frac{1+\alpha}{\alpha}}\right)$$

$$\rho([0,x]) = \lambda^{-\frac{1}{\alpha}} \rho([0,\lambda x]).$$

The measure associated to ρ: $\sum_i w_i \delta_{x_i}$, where (x_i, w_i) are the atoms of ρ is discrete.

$w_x(t) = \mathbb{P}(Z_t = x)$ is supported on the atoms of ρ.

Aging results

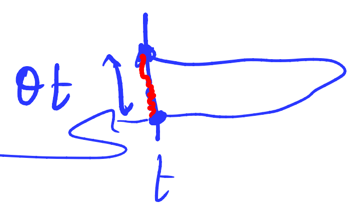
$$\sum_{y \in \mathbb{R}} \left[\mathbb{P}(X_t^{\varepsilon} = y)\right]^2 \xrightarrow[\varepsilon \to 0]{} \sum_{y \in \mathbb{R}} P(Z_t = y)^2$$

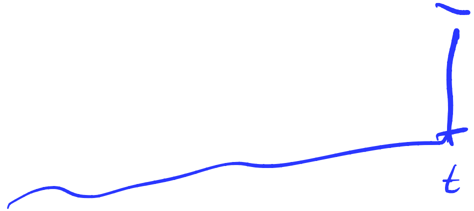
for the quenched measure.

Ben Arous & Černý

$$\lim_{t \to \infty} \mathbb{E}\left[P\left(X(t+\theta \cdot t) = X(t) \mid \rho\right)\right] = R(\theta) > 0$$

$$R(\theta) = \mathbb{E}\left[P\left(Z(1+\theta) = Z(1) \mid \rho\right)\right] > 0$$





$$\lim_{t\to+\infty} E\left[P\left(X(t') = X(t),\ \forall t' \in \left[t, t + \theta t^{\frac{1}{1+\alpha}} L(t)\right] \,\middle|\, \rho \right)\right]$$

$$= \Pi(\theta) = \int_0^\infty e^{-\frac{\theta}{u}}\, dF(u)$$

related to the law of $\rho(z_1)$

Let's come back to RW on RC

Let's allow $C(x, x+1) \in (0, \infty)$ a.s.

Assume:

$$P(c(0,1)>t) = L_\infty(t)\, t^{-\alpha_\infty}$$

L_∞, L_0 slowly varying.

$$P(\underbrace{r(0,1)}_{\frac{1}{c(0,1)}} > t) = L_0(t) \cdot t^{-\alpha_0}, \quad c(0,1) < \frac{1}{t}.$$

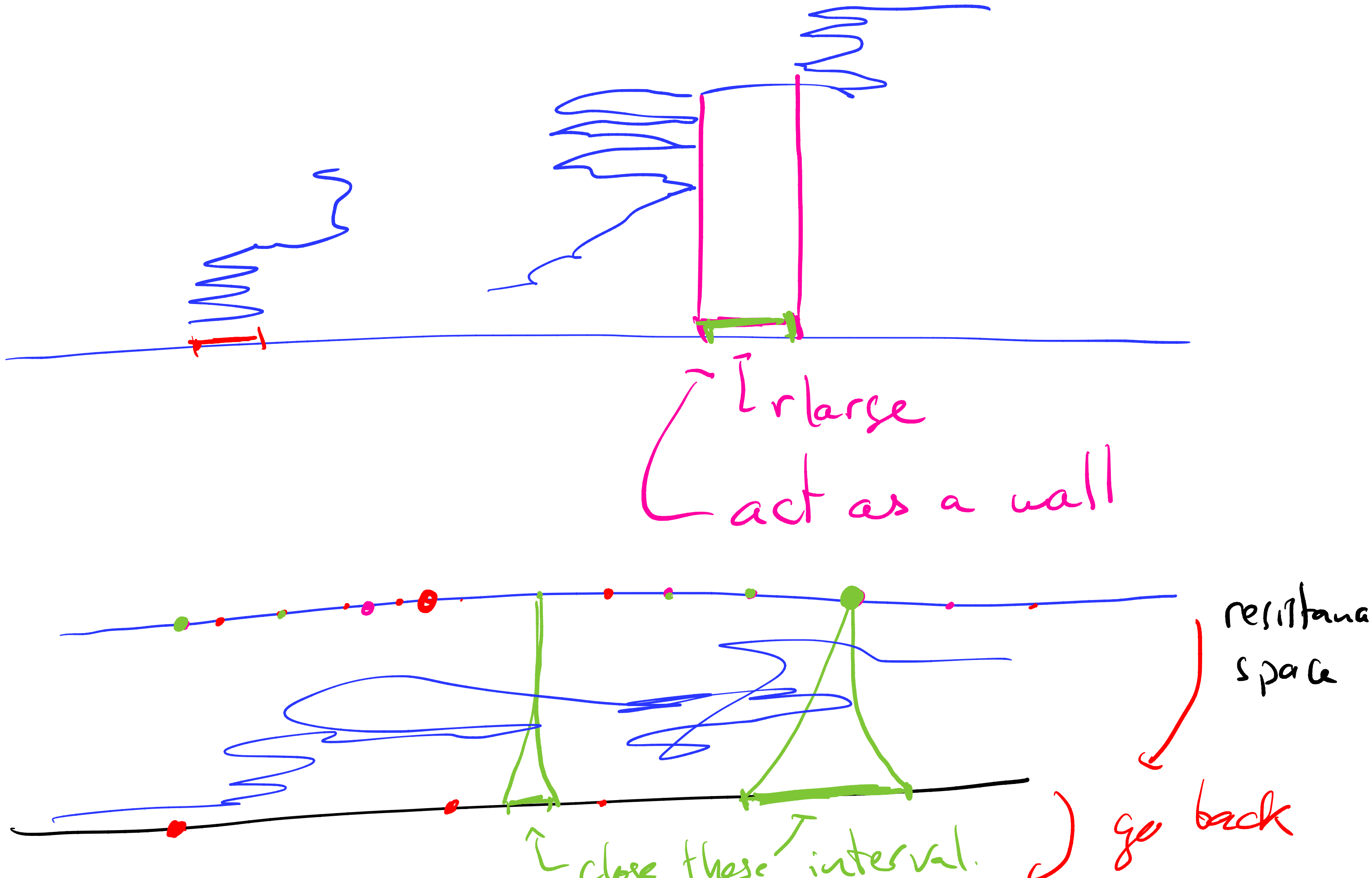
$\alpha_\infty, \alpha_0 \in (0,1)$

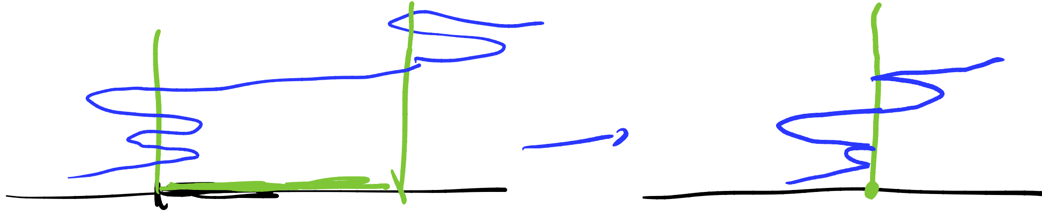
Consider the continuous-time RW $(X_t)_{t\geq 0}$, $X_0 = 0$, with generator

$$(Lf)(x) = \sum_{y\sim x} \frac{c(x,y)}{\pi(x)} (f(y) - f(x))$$

$\pi(x) = c(x-1,x) + c(x,x+1)$.

Holding time $\sim \mathrm{Exp}(1)$.


large
act as a wall
resistance
space
go back
close these interval.



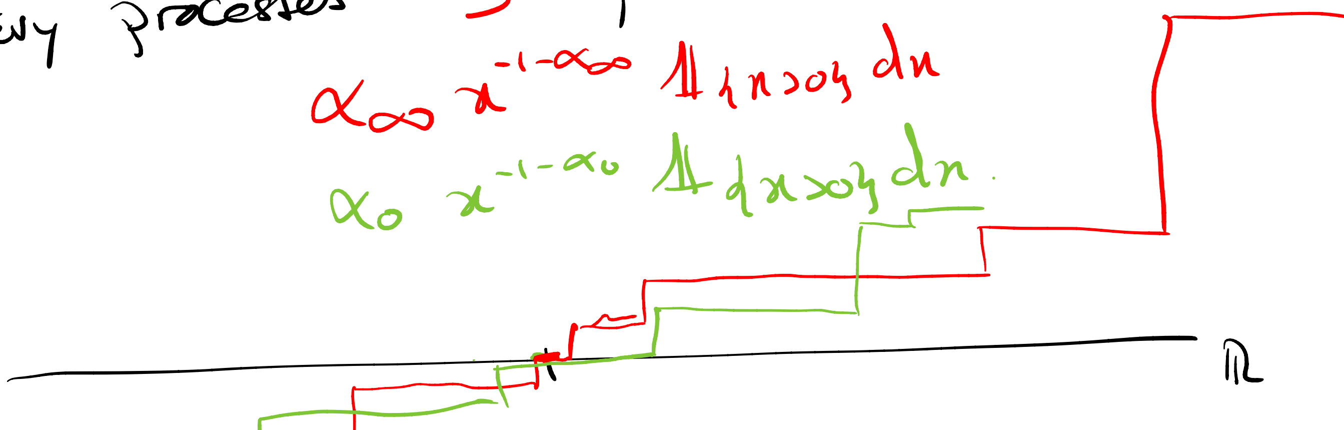
* What is the limit process?

Consider a standard Brownian motion (B_t).

Independently, consider two independent two-sided Lévy processes S^{α_∞}, S^{α_0} with Lévy measures

$$\alpha_\infty x^{-1-\alpha_\infty} \mathbb{1}_{\{x>0\}}\, dx,$$

$$\alpha_0 x^{-1-\alpha_0} \mathbb{1}_{\{x>0\}}\, dx.$$



$$\mu([a,b]) = \int_{(S^{\alpha_0})^{-1}(a)}^{(S^{\alpha_0})^{-1}(b)} d S^{\alpha_\infty}(dv)$$

generalized inverse of S^{α_0}.

$$H_t = \inf\left\{ s \geq 0 : \int_{\mathbb{R}} L_s^B(x)\,\mu(dx) > t \right\}$$

local time of B.

Set $Z_t = (S^{\alpha_0})^{-1}(B_{H_t})$

$$d_{n,0} = \inf\{t > 0 : P(r(0,1) > t) \leq \tfrac{1}{n}\} \approx n^{\frac{1}{\alpha_0}}$$

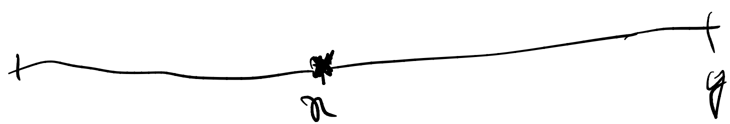
$$d_{n,\infty} = \inf\{t > 0 : P(c(0,1) > t) \leq \tfrac{1}{n}\} \approx n^{\frac{1}{\alpha_\infty}}$$

Th: The law, under $\mathbb{P}$, of $\left(\frac{X_{\lfloor t d_{n,0} d_{n,\infty} \rfloor}}{n}\right)_{t\in[0,T]}$

converges weakly to the law of $(Z_t)_{t\in[0,1]}$, as probability measures on $D([0,\infty),\mathbb{R})$

* How to prove such a limit?

The law of the occupation measure of (X_n) is



given by the resistances.

Knowing the behavior of the environment

$\Longrightarrow$ the behavior of the random walk

Goal: Prove the convergence of the environment - in which sense?

1) Vague convergence: $(\nu^{(n)})$ a sequence of measure, ν a measure. $\nu^{(n)} \xrightarrow{v} \nu$ if: $\forall f$ cont. on $\mathbb{R}$ with bounded support, real valued,

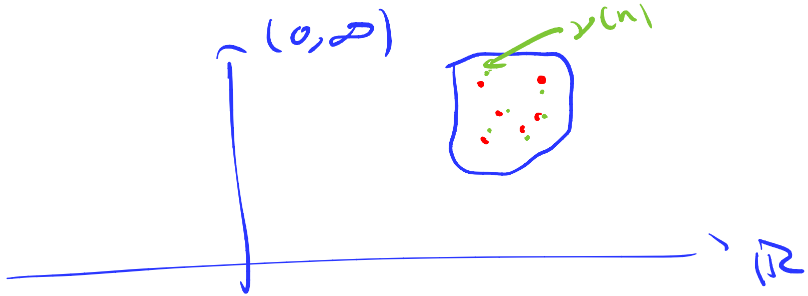
$$\int_{\mathbb{R}} f(y)\,\nu^{(n)}(dy) \longrightarrow \int_{\mathbb{R}} f(y)\,\nu(dy)$$

2) Point Process convergence

$V = \bigcup (y_i, \omega_i)$ atoms of ν

$V_n = \bigcup (y_i^{(n)}, \omega_i^{(n)})$ — $\nu^{(n)}$

$\nu^{(n)} \xrightarrow{P.P.} \nu$ if: $\forall U \subset \mathbb{R} \times (0,\infty]$, open set whose closure is compact, & such that its boundary contains no point of V, the number of point $|U \cap V^{(n)}|$ is finite and equal to $|U \cap V|$ for n large enough



2) is implied by 1) together with

$\forall \ell$, $\exists\, j_\ell(n)$ subsequence s.t.

$$\left(x^{(n)}_{j_\ell(n)}, w^{(n)}_{j_\ell(n)}\right) \longrightarrow (x_\ell, w_\ell).$$

For our problem, RW in RC, you can prove convergence of the following measures:

$$\nu^{\alpha_\infty,(n)} = \frac{1}{d_{n,\infty}} \sum_{\substack{x \in \mathbb{Z} \\ |x| \le K\cdot n}} \delta_{\frac{x}{n}} \cdot c(x,x+1)$$

$$\nu^{\alpha_0,(n)} = \frac{1}{d_{n,0}} \sum_{\substack{x \in \mathbb{Z} \\ |x| \le K\cdot n}} \delta_{\frac{x}{n}} \cdot r(x,x+1)$$

$d_n \alpha_0 w^{-1-\alpha_0} dw$.

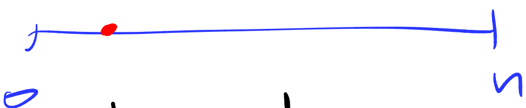
PPP, $d_n w^{-1-\alpha_\infty} dw$, α_∞

to ν^{α_∞} and ν^{α_0} (corresponding S^{α_∞}, S^{α_0}) $\perp\!\!\!\perp$ $\perp\!\!\!\perp$

for vague convergence & p.p. convergence <u>but</u> <u>almost surely</u> under a coupling.

Edges with large conductances or resistances n^{1/α_i}

are typically $\underbrace{\text{far away}}_{n^{1/4}}$ from each other.



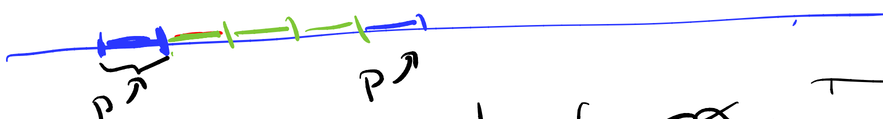
$\Rightarrow$ creates asymptotic independence.

Goal: Couple the discrete environment with the limit env.?

Look at edges with conductances ≥ 1.

V^{α_∞}:

let $p = P(c(0,1) > 1)$



$$\overline{V}^{\alpha_\infty}([a,b]) = p^{-\frac{1}{\alpha_\infty}}\left(S^{\alpha_\infty}(pb) - S^{\alpha_\infty}(pa)\right).$$

Define a set of conductance $\hat{c}(x, x+1) \sim \underline{c(x, x+1) \mid c(x, x+1) \geq 1}$

Define G^{α_∞}:

$$* \overline{\mathbb{P}}\left(S^{\alpha_\infty}(1) > G_{\alpha_\infty}(y) \right) = \mathbb{P}\left(\underline{\hat{c}(0,1) > y} \right).$$

$$\bar{\hat{c}}(x, x+1) = G^{-1}_{\alpha_\infty}\left(n^{\frac{1}{\alpha_\infty}} \left(S^{\alpha_\infty}\left(\frac{x+1}{n}\right) - S^{\alpha_\infty}\left(\frac{x}{n}\right) \right) \right)$$

$$\mathbb{P}\left(\bar{\hat{c}}(x, x+1) > t \right) = \mathbb{P}\left(S^{\alpha_\infty}\left(\frac{x+1}{n}\right) - S^{\alpha_\infty}\left(\frac{x}{n}\right) > n^{-\frac{1}{\alpha_\infty}} \cdot G_\infty(t) \right)$$

$$= \overline{\mathbb{P}}\left(S^{\alpha_\infty}(x+1) - S^{\alpha_\infty}(x) > G_\infty(t) \right)$$

$$= \mathbb{P}\left(S^{\alpha_\infty}(1) > G_\infty(t) \right) = \mathbb{P}\left(\hat{c}(x, x+1) > t \right)$$