TCC: (Atypical) Behaviour of random walks in random or dynamic environment.

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Notes on the webpage.

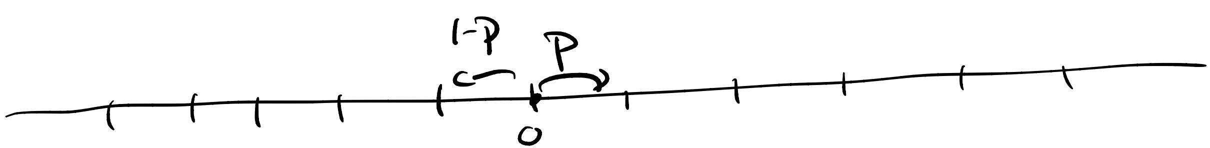
First, what's a typical walk with typical behaviour?

* Simple random walk on $\mathbb{Z}$

Markov Chain $(X_n)_{n\geq 0}$, $X_0 = 0$,

parameter $p \in (0,1)$ bias

$$\mathbb{P}(X_{n+1} = x \underset{-1}{+1} \mid X_n = x) = \underset{1-p}{p}$$



* Law of large numbers

* If $p = \frac{1}{2}$, $\frac{X_n}{n} \longrightarrow 0$ $\mathbb{P}$-a.s.

  (X_n) will be recurrent

  i.e. visit 0 infinitely many times.

* If $p \neq \frac{1}{2}$, $\frac{X_n}{n} \longrightarrow v \neq 0$ | Ballistic

  speed

  (positive speed)

  (X_n) is transient

  i.e. visits 0 finitely many times.

* Central limit Theorem

Donsker's invariance principle

The law $\left(\frac{X_{\lfloor nt \rfloor} - nt v}{\sqrt{n}} \right)_{t \in [0,T]} \Longrightarrow (B_t)_{t \in [0,T]}$.

Typical position at time n is of order $\sqrt{n}$, after recentering.

- $p = \frac{1}{2}$, $\underbrace{\frac{X_n}{\sqrt{n}}}_{}$ $\longrightarrow \mathcal{N}(0,1)$

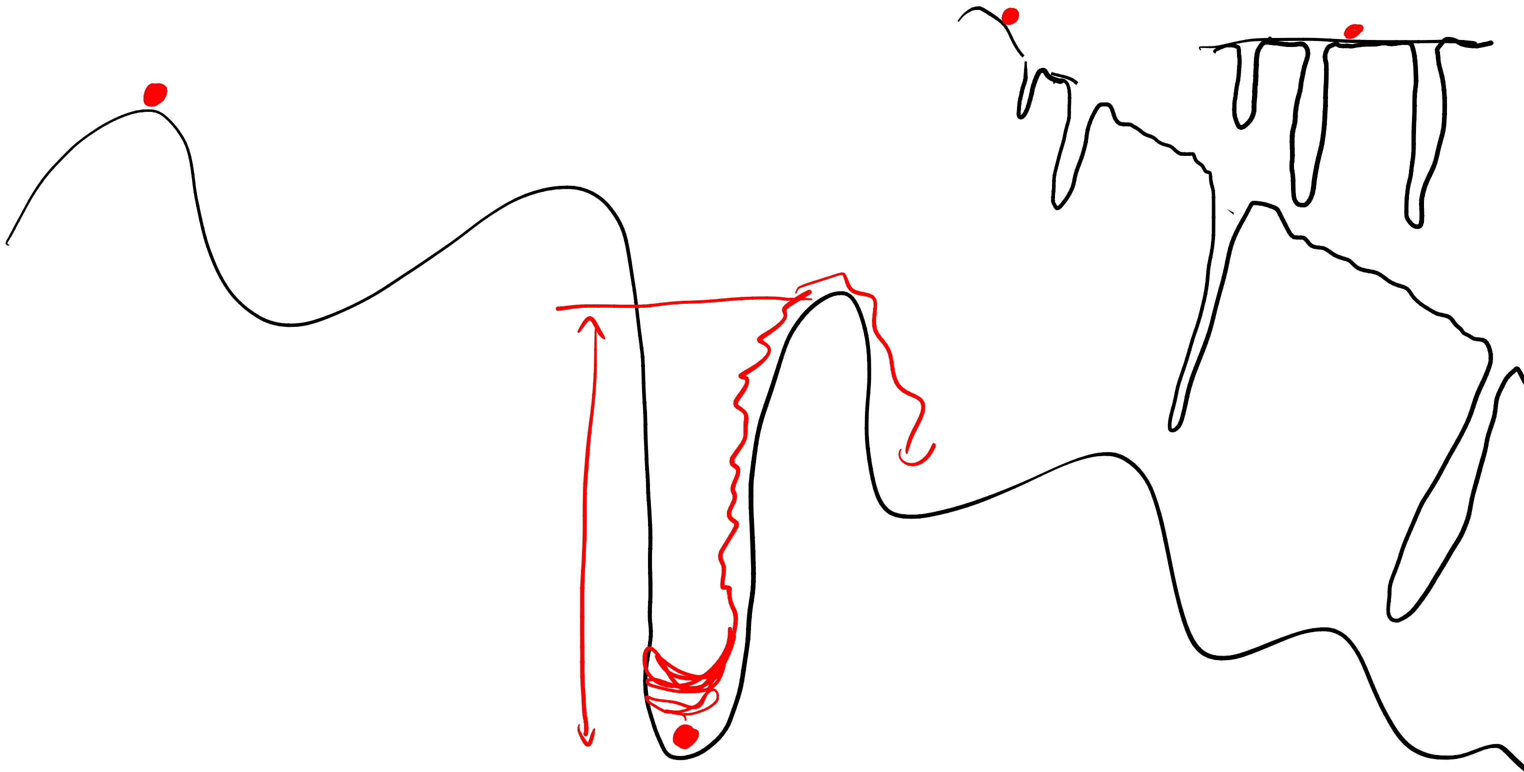
Diffusive rescaling.

This a typical behaviour

Either recurrent with $v = 0$ or transient with $v \neq 0$

What is atypical?

Random walk in random environment, on $\mathbb{Z}$



* How do we create these traps? What are their size? their shape? Their effect?
* Can we have a characterization for the speed of a (trapped) random walk?

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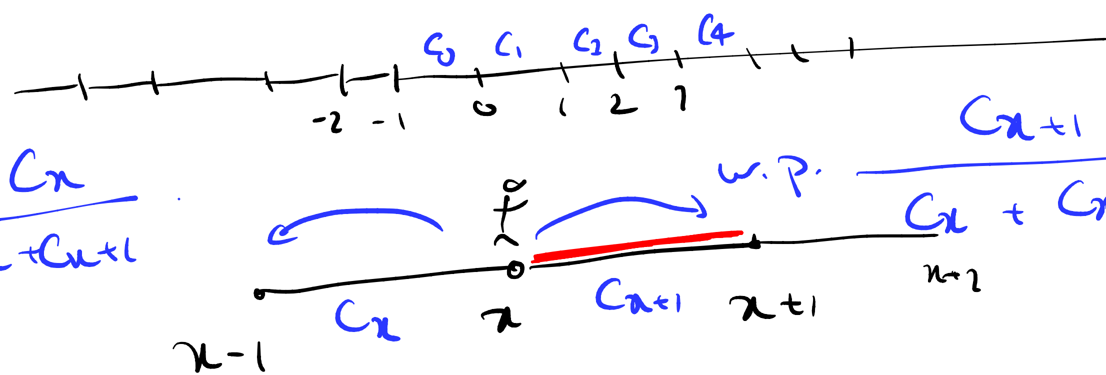
* Random walk on random conductances

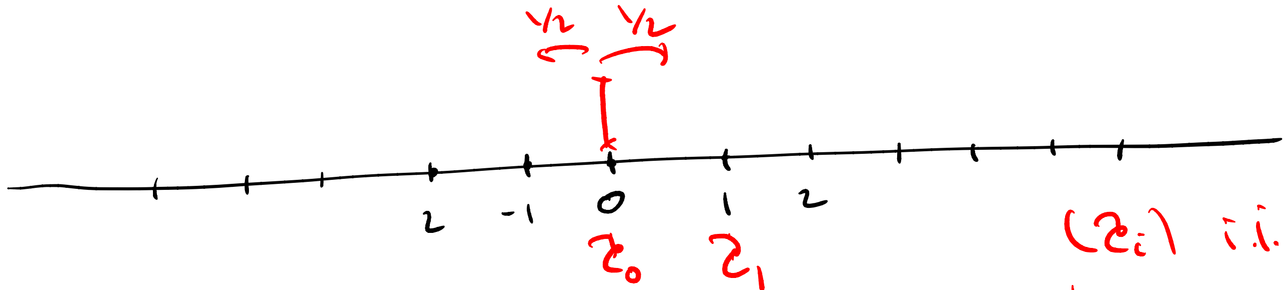
$c_i \in (0, \infty)$



$\dfrac{c_x}{c_x + c_{x+1}}$

w.p. $\dfrac{c_{x+1}}{c_x + c_{x+1}}$

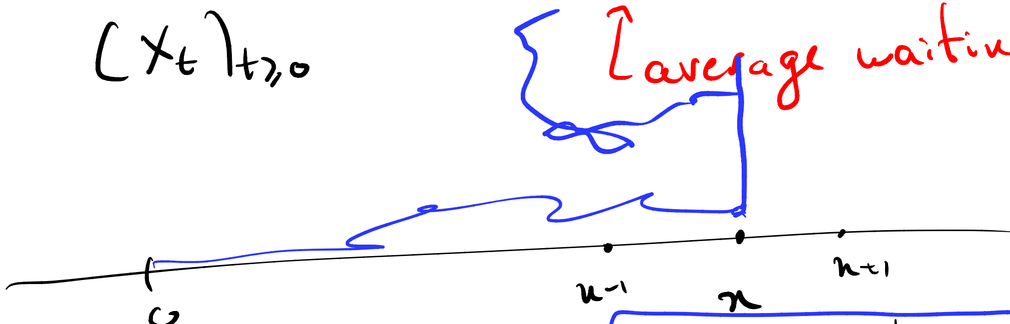
* Bouchard trap model



(τ_i) i.i.d.

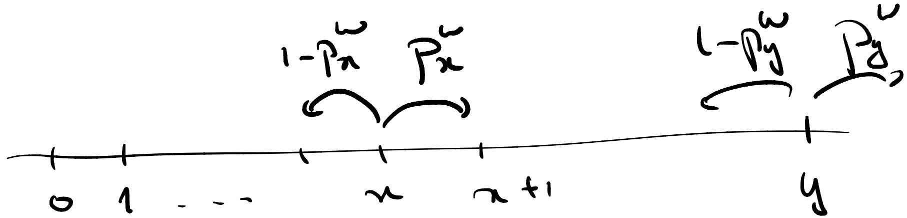
τ average waiting time

$(X_t)_{t \geq 0}$



$\boxed{\tau_n \text{ very large}}$

* Random walk in random environment RWRE



$$(p_x^\omega)_{x\in\mathbb{Z}} \text{ i.i.d in } \left|\begin{array}{l} [\kappa, 1-\kappa] \\ [0,1] \end{array}\right.$$

on $d=1$: Problem $\left|\begin{array}{l}\text{recurrence / transience} \\ \text{ballisticity}\end{array}\right.$

has been fully solved.

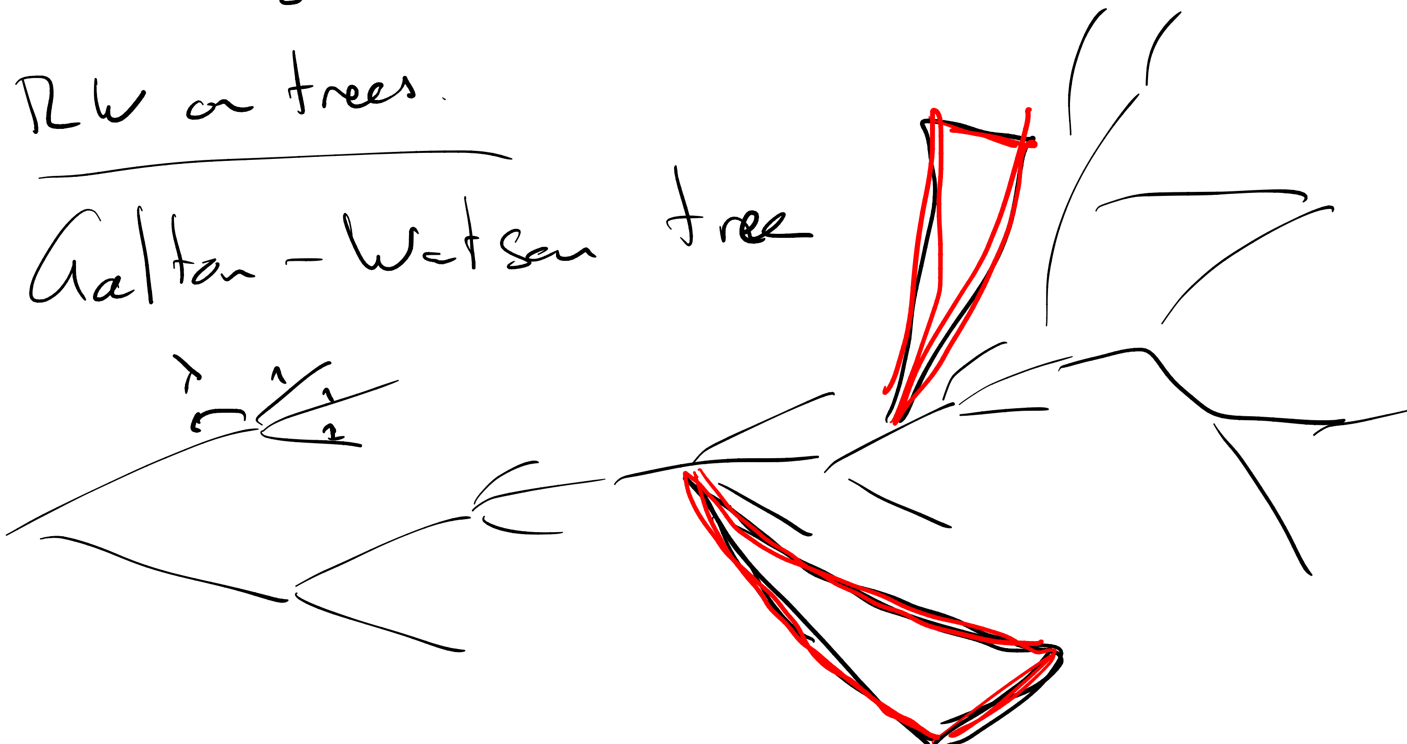
Explicit characterization.

on $d \geq 2$: Much more open

Sznitman's condition (T).

RW on trees.

Galton-Watson tree



Consider a random walk on random conductances

Conductances: $c_x \in [K, +\infty)$ | or $[K, \frac{1}{K}]$

for some $K > 0$;

$\forall x \in \mathbb{Z}$.

Random walk $(X_n)_{n \geq 0}$ on the conductances (c_x) is defined by $X_0 = 0$

$$P^\omega\left(X_{n+1} = x+1 \;\middle|\; X_n = x \right) = \frac{c_{x+1}}{c_x + c_{x+1}}$$

-1

$$= \frac{c_x}{c_x + c_{x+1}}$$

We will choose the c_x's random, i.i.d. over $\mathbb{Z}$, in $[K,+\infty)$ a.s.

$\mathbf{P}$: their law.

If we fix the environment, i.e. collection $\omega := (c_x)$ and at the evolution of X

P_0^ω = quenched law of X

If we average over the environment:

$\mathbb{P}_0 = \mathbf{E}[P_0^\omega(\cdot)]$ annealed law

under $\mathbb{P}_0$, X is not a Markov Chain

Crash course on RW on networks

e edge

c_e : conductances

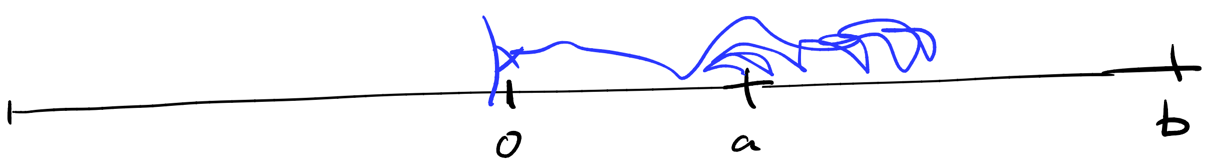
$r_e = \dfrac{1}{c_e}$: resistances

* Electrical network.

Effective resistance / conductance between two points a & z

$$C_{eff}(a \to z) = \pi(a)\,\mathbb{P}(a \to z) = \frac{1}{R_{eff}(a \to z)}$$

$$\sum_{y \sim a} c(a,y)$$



$$\mathbb{P}_a(T_0 < T_b) = \frac{R_{eff}(a \leftrightarrow b)}{R_{eff}(0 \leftrightarrow b)}$$

hitting times

$$R_{eff}(0 \leftrightarrow b) = R_{eff}(0 \leftrightarrow a) + R_{eff}(a \leftrightarrow b)$$

Back on $\mathbb{Z}$: RW X on (C_x) is reversible under P_0^ω.

(left as an exercise)

measure $\pi(x) = C_x + C_{x+1}$ is reversible.

Also true for RWRC on $\mathbb{Z}^d$, $d \geq 1$.

* Assume $c_x \in \left[\kappa, \frac{1}{\kappa}\right]$ $\mathbf{P}$-a.s. for some $\kappa \in (0,1)$.

"Easy" case.

We can prove $\frac{X_n}{n} \to 0$ $\mathbb{P}$-a.s.

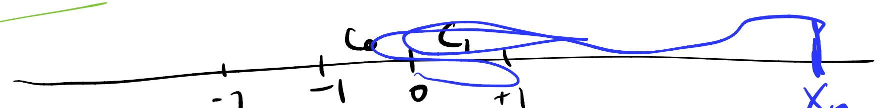
and we can prove diffusive fluctuations

Let's prove these results using the environment seen from the particle / walker.

shift along trajectories

environment

$$\overline{\omega}_n = t_{X_n}\omega$$



$\overline{\omega}_0 = \omega$, $(\overline{\omega}_n)_n$ is a Markov process on the space of the environment

$\leadsto \exists$ a measure $\mathbb{Q}$ ergodic, invariant, probability measure.

Under $\mathbb{Q}$, the law of $\overline{\omega}_1$ is $\mathbb{Q}$

$\mathbb{Q} \sim \mathbb{P}$

$$\mathbb{Q} = \frac{1}{Z}(c_0 + c_1)\mathbb{P},$$ Z normalising cst.

Consider the martingale

$$\Pi_n = X_n - \sum_{k=0}^{n-1} E^{\omega}_{X_k}[X_1] = X_n - \sum_{k=0}^{n-1} d(X_k, \omega)$$

$$d(X_k,\omega) = E[X_{k+1} - X_k \mid F_k] = \frac{c_{1+X_k} - c_{X_k}}{c_{X_k} + c_{1+X_k}}$$

$\leftarrow$ local drift at X_k.

Exercise: Prove $E[\Pi_{n+1} - \Pi_n \mid F_n] = 0$,

$F_n = \sigma(X_0, \ldots, X_n)$.

Azuma's inequality $\Rightarrow$ almost surely,

$\Pi_n \leq n^{\frac{1}{2}+\varepsilon}$ for n large enough.

$\Rightarrow \frac{\Pi_n}{n} \longrightarrow 0$ P_0^{ω}-a.s. for P-a.e. ω

$$0 \xleftarrow{a.c.} \frac{\Pi_n}{n} = \frac{X_n}{n} - \underbrace{\frac{1}{n}\sum_{k=0}^{n-1} d(X_k, \omega)}_{\to 0 \;?}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} d(X_k,\omega) = \underline{\frac{1}{n}\sum_{k=0}^{n-1} d(0, \overline{\omega}_k)}$$

We will use the Birkhoff's Ergodic Theorem

$(Y, \mathcal{B}, T, \mu)$, f: measurable

$E_\mu[|f|] < \infty$

T: measure preserving transformation

$$\mu(T^{-1}(A)) = \mu(A)$$

Then w.p. 1

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = E_\mu[f \mid \mathcal{C}](x)$$

$\mathcal{E}$: σ- field generated by invariant sets of T

$\mu(T^{-1}(E) \Delta E) = 0$

Pointwise E.T.

If T is ergodic (i.e. E invariant $\Rightarrow \mu(E) \in \{0,1\}$)

then $\mathcal{E}$ is trivial

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E_\mu[f](x).$$

This implies, for $\mathbb{Q}$- a.e. ω

$$\frac{1}{n} \sum_{k=0}^{n-1} d(0, \overline{\omega}_k) \xrightarrow{P_0^\omega} \mathbb{Q}[d(0,\omega)]]$$

This also holds for $\mathbf{P}$- a.e. ω. $(\mathbb{Q} \sim \mathbf{P})$.

$\frac{M_n}{n} \to 0$ a.s.

we have that for $\mathbb{P}$-a.e. ω,

$$\frac{X_n}{n} \xrightarrow{P_0^\omega\text{-a.s.}} \mathbb{Q}[d(0,\omega)] = 0$$

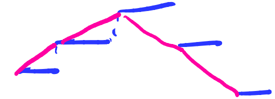
Fluctuations? Use a martingale CLT

Martingale CLT: (M_n) an $(\mathcal{F}_n)$-martingale, square integrable: for each $n \in \mathbb{N}$, $\mathbb{E}[|M_n|^2] < +\infty$.

If (i) $\exists \sigma^2 \in [0, \infty)$ s.t., $\forall t > 0$,

$$\frac{1}{n} \sum_{k=0}^{\lfloor tn \rfloor} \mathbb{E}\left[|M_{k+1} - M_k|^2 \,\middle|\, \mathcal{F}_k \right] \longrightarrow t \cdot \sigma^2 \text{ in probability.}$$

(2) $\forall \varepsilon > 0$,



$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}\left[|\Pi_{k+1}-\Pi_k|^2 \times \mathbb{1}_{\{|\Pi_{k+1}-\Pi_k| > \varepsilon\sqrt{n}\}} \right] \xrightarrow{\mathbb{P}} 0$$

Then

interpolated

the law of $\left(\frac{\Pi_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t\in[0,T]} \xRightarrow{(d)} (B_t)_{t\in[0,T]}$

variance σ^2.

If $|\Pi_{k+1}-\Pi_k| < C^{st}$ then (2) is satisfied.

For us, let consider

$$\tilde{\Pi}_n = \begin{cases} R(0, X_n) & \text{if } X_n \geq 0 \\ -R(0, X_n) & \text{if } X_n < 0 \end{cases}$$

$$R(0,x) = \sum_{i=1}^{x} r_i, \qquad r_i = \frac{1}{c_i}$$

$(\tilde{\Pi}_n)$ is a martingale (left as exercise)

to apply the mart. CLT, we need to prove

2) $(\tilde{\Pi}_n)$ has bounded steps:

true because $r_i < \frac{1}{K}$ $\forall i \in \mathbb{Z}$

1) $$\frac{1}{n}\sum_{k=0}^{\lfloor tn \rfloor} \underbrace{\mathbb{E}\left[|\tilde{\Pi}_{k+1} - \tilde{\Pi}_k|^2 \mid \mathcal{F}_k \right]}_{\frac{c_{X_k+1}}{c_{X_k}+c_{X_k+1}} \cdot r_{X_k+1}^2 + \frac{c_{X_k}}{c_{X_k}+c_{1+X_k}} \cdot r_{X_k}^2} \to ?$$

$$= \frac{r_{X_k} \, r_{1+X_k}^2}{r_{X_k} + r_{1+X_k}} + \frac{r_{1+X_k} \, r_{X_k}^2}{r_{X_k} + r_{1+X_k}} = r_{X_k} r_{X_{k+1}}$$

$\Rightarrow$ Ergodic Theorem

$$\longrightarrow t \times \frac{1}{nt} \times \sum_{k=0}^{\lfloor tn \rfloor} E^{\omega}\left[|M_{k+1} - M_k|^2 \mid \mathcal{F}_k \right]$$

$$\xrightarrow{P_0^{\omega}} t \times Q(r_0 r_1) \quad \text{for} \left| \begin{array}{l} Q\text{-a.e. } \omega \\ \mathbb{P}\text{-a.e.} \end{array} \right.$$

$$\Longrightarrow \left(\frac{R(0, X_{\lfloor nt \rfloor})}{\sqrt{n}} \right) \Longrightarrow (B_t)_{t \in X}$$

$$\Longrightarrow \left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right) \Longrightarrow (B_t) \quad \text{because}$$

n large

$$R(0, x) \sim x \cdot E[r_1]$$

Next week: What happens
if c_n can be very large??