Introduction to the Gaussian free field

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Notes based on "Lecture Notes on the Gaussian Free Field" by Wendelin Werner and Ellen Powell

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Abstract

One simple way to think of the Gaussian Free Field (GFF) is that it is the most natural and tractable model for a random function defined on either a discrete graph (each vertex of the graph is assigned a random real-valued height, and the distribution favours configurations where neighbouring vertices have similar heights) or on a subdomain of Euclidean space. The goal of these lectures is to give an elementary, self-contained introduction to both of these models, and highlight some of their main properties. We will start with a gentle introduction to the discrete GFF, and discuss its various resampling properties and decompositions. We will then move on to the continuum GFF, which can be obtained as an appropriate limit of the discrete GFF when it is defined on a sequence of increasingly fine graphs. We will explain what sort of random object (i.e, generalised function) it actually is, and how to make sense of various properties that generalise those of the discrete GFF.

These notes form the basis of a mini-course given by EP at the BUC meeting at CIMAT, Guanajuato, from the 16th-19th January 2023. They are a based on part of the book "Lecture notes on the Gaussian free field" by Wendelin Werner and EP.

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LECTURE 1

Warm Up

We first recall some features of random walks and Brownian motion that will guide us as we try to construct the Gaussian free field.

1.1 Random walks and Brownian motion

Definition 1.1 (Random walk). *Let μ be the law of a real-valued random variable with expectation 0 and finite variance. For $x \in \mathbb{R}$, we define the random walk starting at x and with step distribution μ by*

$$S_n := x + \sum_{i=1}^{n} X_i \quad \textit{for} \quad n \geq 0,$$

where $(X_i)_{i\geq 1}$ are independent and each have law μ.

This means that $(S_n)_{n\geq 0}$ has the Markov property: for all $n \geq 0$, conditionally on $(S_0, \ldots, S_n)$, the conditional law of $(S_{n+k} - S_n)_{k\geq 0}$ is simply the law of the random walk $(S_k)_{k\geq 0}$ started from 0.

Remark 1.2. *It also has the* strong *Markov property. If τ is a stopping time for $(S_n)_{n\geq 0}$ (that is, $\{\tau > n\}$ is $\sigma(S_0, \ldots, S_n)$ measurable for each n), then $(S_{\tau+k} - S_\tau)_{k\geq 0}$ has the law of the random walk $(S_k)_{k\geq 0}$ started from 0, independent of $(\tau, (S_n)_{n\leq\tau})$.*

Recall, that the *central limit theorem* tells us that

$$\frac{S_n}{\sqrt{n}} \Rightarrow Z \quad , \quad Z \sim \mathcal{N}(0, \sigma^2)$$

as $n \to \infty$, where $\Rightarrow$ denotes convergence in distribution, and σ^2 is the variance of a random variable with law μ.

Here we write $\mathcal{N}(m, \sigma)$ for the law of a *Gaussian* or *normal* random variable with mean m and variance σ^2. That is, if $Z \sim \mathcal{N}(m, \sigma)$, then

$$\mathbb{P}(Z \in A) = \int_A \tfrac{1}{\sqrt{2\pi\sigma^2}} \exp(-\tfrac{(x-m)^2}{2\sigma^2}) \, \mathrm{d}x$$

for $A \subset \mathbb{R}$.

So, the central limit theorem is a result about the law of the random walk (rescaled appropriately) at a *single* time. *Donsker's invariance principle* is a much stronger version of this, which describes the law of the random walk path in the limit, when time is sped up, and steps are rescaled appropriately.

Quick recap: stochastic processes Recall that a random real-valued process indexed by some set $\mathcal{A}$ is just a collection of random variables $(X_a)_{a\in\mathcal{A}}$ defined on the same probability space. The *law of the process* is a measure on $\mathbb{R}^{\mathcal{A}}$ (endowed with the product σ-field) and is characterised by its finite-dimensional distributions (i.e., the law of the finite-dimensional vector $(X(a_1),\dots,X(a_n))$ for each $a_1,\dots,a_n\in\mathcal{A}$).

When all finite-dimensional distributions are those of (centered) Gaussian vectors, we say that the process is a centered Gaussian process. In other words, a stochastic process $(X_a)_{a\in\mathcal{A}}$ is a centred Gaussian process if and only if for any n, for any $a_1,\dots,a_n$ in $\mathcal{A}$ and any real constants $\lambda_1,\dots,\lambda_n$, the random variable $\lambda_1 X_{a_1}+\cdots+\lambda_n X_{a_n}$ is a centred Gaussian random variable. The law of a centered Gaussian process $(X_a)_{a\in\mathcal{A}}$ is fully described by its covariance function $\Sigma(a,a') := \mathbb{E}[X_a X_{a'}]$ defined on $A\times A$.

A consequence of *Kolmogorov's extension theorem* is that when $\mathcal{A}$ is a given set and Σ is a real-valued symmetric bilinear form (defined on $A\times A$) such that for all n, for all $a_1,\dots,a_n$ in $\mathcal{A}$ and all $\lambda_1,\dots,\lambda_n$ in $\mathbb{R}$, $\sum_{i,j}\lambda_i\lambda_j\Sigma(a_i,a_j)\geq 0$ (we say that Σ is non-negative definite) then it is possible to construct a probability space and a process $(X_a)_{a\in\mathcal{A}}$ on this probability space, such that X is a centred Gaussian process with covariance function Σ.

We say that a sequence of stochastic processes (with the same indexing set for each process) converges to a limiting stochastic process, if and only if all the finite dimensional distributions converge.

Exercise 1. *Let V be a finite set and let $(X_v^{(n)})_{v\in V}$ be a centred Gaussian process for every $n\in\mathbb{N}$ with $E[X_v^{(n)}X_w^{(n)}] =: \Sigma_n(v,w)$. Suppose that for every $v,w\in V$, $\Sigma_n(v,w)\to\Sigma(v,w)$, where Σ is a symmetric, non-negative definite bilinear form on $V\times V$. Show that, in the sense of stochastic processes, $X^{(n)}$ converges in distribution to X: the centered Gaussian process $(X_v)_{v\in V}$ with covariance Σ.*

Theorem 1.3 (Donsker's invariance principle). *Let $(S_n)_{n\geq 0}$ be as in Definition 1.1. Then*

$$\Big(\frac{S_{[Nt]}}{\sqrt{\sigma^2 N}}\Big)_{t\geq 0}\Rightarrow \big(B_t\big)_{t\geq 0},$$

in the sense of stochastic processes (where $[u]$ denotes the integer part of the real number u). The limiting process $(B_t)_{t\geq 0}$ is called a standard (linear) Brownian motion.

Definition 1.4 (Brownian motion on $\mathbb{R}$). *The standard (linear) Brownian motion $(B_t)_{t\geq 0}$ is the (unique) centered Gaussian process indexed by $t\geq 0$, with covariance given by*

$$\mathbb{E}(B_s B_t)=\min(s,t)\quad s,t\geq 0.$$

The proof of Theorem 1.3 is not terribly difficult, but does not fall into the scope of the present lectures. It does illustrate, however, that Brownian motion is some kind of natural *universal* objects describing the fluctuations of a random path or function.

Remark 1.5. Kolmogorov's continuity criterion *provides the existence of a random* almost surely continuous *function* $(B_t)_{t\geq 0}$*, whose finite dimensional distributions are as in Definition 1.4. In practice, when working with Brownian motion, we will usually work with this object rather than the (more abstract) stochastic process.*

In fact, if we take this continuous version *of Brownian motion to be the limit, the convergence in Theorem 1.3 can be extended to weak convergence in the space of right-continuous functions equipped with the topology of uniform convergence on compact subsets of time.*

Just like the random walk, Brownian motion possesses the following important property.

Proposition 1.6 (Markov property of Brownian motion)**.** *For all* $s \geq 0$*, conditionally on* $\sigma((B_u)_{u\leq s})$*,* $(B_{t+s} - B_s)_{t\geq 0}$ *has the law of a standard Brownian motion. In other words, the process* $(B_{t+s} - B_s)_{t\geq 0}$ *is independent of* $(B_u)_{u\leq s}$ *and has the same law as* $(B_t)_{t\geq 0}$*.*

Brownian motion also has the strong *Markov property. If* τ *is a stopping time for* B *(that is,* $\{\tau > t\}$ *is* $\sigma((B_s)_{0\leq s\leq t})$ *measurable for each* t*), then* $(B_{t+\tau} - B_\tau)_{t\geq 0}$ *has the law of a standard Brownian motion, independent of the stopped sigma-algebra* [1] *at* τ*.*

Remark 1.7 (Brownian motion on $\mathbb{R}^d$)**.** *For* $d \geq 2$*, we can extend the definition of linear Brownian motion to* d*-dimensions by taking a vector of linear Brownian motions. More precisely, the standard* d*-dimensional Brownian motion* $(\mathbf{B}_t)_{t\geq 0}$ *is defined by setting*

$$\mathbf{B}_t := (B_t^{(1)}, \ldots, B_t^{(d)}) \quad , \quad t \geq 0,$$

where the $(B_t^{(i)})_{t\geq 0}$ *for* $1 \leq i \leq d$ *are independent standard linear Brownian motions.*

1.2 Conditioned random walks and Brownian bridges

Definition 1.8 (Brownian bridge)**.** *Let* $(B(t))_{t\in[0,1]}$ *be a one-dimensional Brownian motion, restricted to the interval* $[0, 1]$*. The process* $(\beta_t := B_t - tB_1)_{t\in[0,1]}$ *is called a* Brownian bridge *(from* 0 *to* 0*, of length* 1*).*

- β *is a Gaussian process with mean zero and covariance* $E(\beta_t\beta_s) = t(1 - s)$ *when* $0 \leq t \leq s \leq 1$*.*
- β *is independent of the random variable* B_1*, so that its law can be interpreted as the law of Brownian motion "conditioned to be equal to* 0 *at time* 1*".*

Analogously to the previous section, the Brownian bridge is known to be the scaling limit of a rather large class of random walks, when they are *conditioned* to be back at 0 after a large number of steps.

[1] this consists of events A such that $A \cap \{\tau \leq t\} \in \sigma((B_s)_{s\leq t})$ for all t

Example 1.9. *Choose a path $(S(0), \dots, S(N))$ with N steps (N even) with values in $\mathbb{Z}$, uniformly from the set $\mathcal{S}_N$ of walks such that*

$$S(0) = S(N) = 0 \text{ and } |S(j) - S(j-1)| = 1 \text{ for all } 1 \leq j \leq N.$$

Then the law of $(S_{[Nt]}/\sqrt{N})_{t\in[0,1]}$ converges weakly (for the topology of the sup-norm on the space of real-valued right-continuous functions on $[0,1]$) to the law of the Brownian bridge.

Example 1.10. *Take a symmetric density function $h(x)$ on $\mathbb{R}$ such that $\int xh(x)dx = 0$ and $\int x^2 h(x) = 1$, and consider the random vector $(S(1), \dots, S(N-1))$ with density (with respect to Lebesgue measure on $\mathbb{R}^{N-1}$) proportional to*

$$\prod_{j=1}^{N} h(\gamma_j - \gamma_{j-1})$$

at $(\gamma_1, \dots, \gamma_{N-1})$ (with the convention $\gamma_0 = \gamma_N = 0$). Then again, the law of $(S_{[Nt]}/\sqrt{N})_{t\in[0,1]}$ converges to the law of the Brownian bridge.

As with Brownian motion, these results illustrate that Brownian bridges (and constant multiples of the Brownian bridge) are natural universal objects describing the fluctuations of a random function f on $[0,1]$, constrained to satisfy $f(0) = f(1) = 1$.

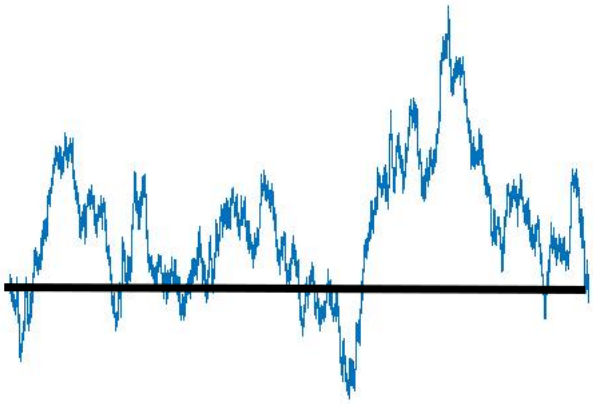

Figure 1.1: A Brownian bridge from zero to zero.

There is one special case of type (2) conditioned walks that is worth highlighting. This is when one takes h to be the Gaussian distribution function with variance 1 i.e., $h(x) = \exp(-x^2/2)/\sqrt{2\pi}$.

Exercise 2. *Suppose that $h(x) = \exp(-x^2/2)/\sqrt{2\pi}$ and S is defined as in Exercise 1.10. Show that for each N, $(S(1), \dots, S(N-1))$ is a centred Gaussian vector with covariance*

$$E[S(j)S(j')] = j(N-j')/N$$

for $1 \leq j \leq j' < N$. Show that, if $\beta = (\beta_t, t \in [0,1])$ is itself a Brownian bridge, then the vector

$$(\sqrt{N}\beta(1/N), \dots, \sqrt{N}\beta((N-1)/N))$$

is distributed exactly like $(S(1), \dots, S(N-1))$. Use this to prove the convergence in distribution of the conditioned walk to the Brownian bridge in this case.

Remark 1.11 (Stationary distribution of a Markov chain)**.** *It is worth noticing that for each given N, the laws of conditioned random walks of the type (1) or (2) can be viewed as the unique stationary measures of simple Markov chains on the space of "admissible" paths. For instance, in case (1) and when $N \geq 4$, the natural dynamics on the space $\mathcal{S}_N$ can be described as follows. When we are given a path γ in $\mathcal{S}_N$, the Markovian algorithm to produce a new path γ' is the following.*

(a) Choose a point J uniformly at random in $\{1, \dots, N-1\}$. The new path γ' will then be equal to γ except possibly at time J.

(b) • If $\gamma(J-1) = \gamma(J+1)$, toss a fair coin. If the coin lands heads, set $\gamma' = \gamma$. If the coin lands tails, set γ' to be equal to γ except at time J, and set

$$\gamma'(J) = \gamma(J-1) - (\gamma(J) - \gamma(J-1)).$$

• If $\gamma(J-1) \neq \gamma(J+1)$ (which means that $|\gamma(J+1) - \gamma(J-1)| = 2$), then keep γ unchanged, i.e., set $\gamma' = \gamma$.

It is then a simple exercise to check that this Markov chain is irreducible, aperiodic and that the uniform measure on $\mathcal{S}_N$ is reversible (indeed, if the probability to jump from γ to γ' when $\gamma' \neq \gamma$ in one step is positive, then it is equal to $1/(N-1)$, and equal to the probability to jump from γ' to γ). Hence the law of the conditioned random walk in case (1) is equal to the unique stationary law of this Markov chain.

If $[r,s] \subset [0,\infty)$ and $a, b \in \mathbb{R}$, then we define the Brownian bridge from a to b on $[r,s]$ (so, of length $s-r$) by

$$\beta_t = (\frac{s-t}{s-r}a + \frac{t-r}{s-r}b) + \sqrt{s-r}\,\tilde{\beta}_{\frac{t-r}{s-r}} \quad , \quad t \in [r,s],$$

where $\tilde{\beta}$ is a Brownian bridge (of length 1 from 0 to 0). Again this can be interpreted as a Brownian motion on the interval $[r,s]$, conditioned to be equal to a at time r and b at time s. Observe that the first term is simply the deterministic linear function equal to a at time r and b at time s, while the second is a rescaled version of the Brownian bridge. The scaling by $\sqrt{s-r}$ (so-called *Brownian scaling*) appears, because this is the natural scaling that makes Brownian motion invariant: namely, for any $c \in \mathbb{R}$, $(\frac{1}{\sqrt{c}}B_{ct})_{t \geq 0}$ simply has the law of a standard Brownian motion.

With this definition in hand, we can explain the Markov property of the Brownian bridge, that it in turn inherits from that of Brownian motion.

Proposition 1.12 (Markov property of the Brownian bridge). *Let $(\beta_t)_{t\in[0,1]}$ be a Brownian bridge of length 1 from 0 to 0. Let $[r,s] \subset [0,1]$: then the following equivalent statements hold.*

(v1) The conditional law of $(\beta_t)_{t\in[r,s]}$ given $(\beta_t)_{t\in[0,r]\cup[s,1]}$, is that of a Brownian bridge in $[r,s]$ from β_r to β_s.

(v2) Let $(F(t))_{t\in[r,s]}$ be the unique linear function equal to a at time r and b at time s. Then $(\beta_t - F(t))_{t\in[r,s]}$ is a Brownian bridge from 0 to 0 on $[r,s]$ (i.e., a Brownian bridge of length 1 from 0 to 0, translated and scaled using Brownian scaling).

(v3) Write $\beta_{[r,s]}$ for the function that is equal to β on $[0,1]\setminus[r,s]$ and F on $[r,s]$. Set $\beta^{[r,s]} = \beta - \beta_{[r,s]}$. Then $\beta_{[r,s]}$ and $\beta^{[r,s]}$ are independent, and $\beta^{[r,s]}$ is a Brownian bridge from 0 to 0 on $[r,s]$.

In other words, if we condition a Brownian bridge on its values outside of a subinterval, then the conditional law of the bridge inside the sub-interval can be written as a sum of two independent functions:

- a linear function interpolating its values at the endpoints of the subinterval;
- a Brownian bridge from 0 to 0 inside the subinterval (this is a rescaled version of the Brownian bridge of length 1).

Note that it is only the first function above whose conditional law depends on the values of the Brownian bridge outside of the subinterval.

1.3 Towards the GFF

What is the corresponding object describing fluctuations, when instead of considering a one-dimensional string, one looks at some tambourine skin? In other words, what happens in the previous subsection when one replaces the one-dimensional time-segment $[0,1]$ by a two-dimensional set D (that plays the role of the shape of the tambourine), and tries to look at random functions from D into $\mathbb{R}$?

We start by considering functions on a discrete (grid) approximation to $D = [0,1]^2$ (in a similar spirit to Examples 1.9, 1.10). For $N \geq 2$, we use the following notation.

- $\overline{\Lambda}_N := \{0,\dots,N\}^2$ = closed $N \times N$ discrete square.
- $\Lambda_N := \{1,\dots,N-1\}^2$ = *inside* of the square.
- $\partial_N := \overline{\Lambda}_N \setminus \Lambda_N$ = boundary of the square.
- E_N = set of (unoriented) edges that join two neighbouring points (i.e., at distance 1) in $\overline{\Lambda}_N$. Write xy for the edge between $x, y \in \bar{\Lambda}_N$.
- If $(\gamma_x)_{x\in\Lambda_N}$ is a vector indexed by Λ_N, we write $|\nabla\gamma(xy)| = |\gamma_x - \gamma_y|$ for $xy \in E_N$, with the convention that $\gamma_x = 0$ for $x \in \partial_N$.

We will consider functions f from the discrete square $\overline{\Lambda}_N$ into $\mathbb{R}$, with the constraint that f is equal to zero on ∂_N. Here are some concrete ways to choose such a function f at random.

(1) **Analogue of Example 1.9:** Choose f uniformly among the finite set of all integer-valued functions f such that $f = 0$ on ∂_N and for any $xy \in E_N$, $f(x) - f(y) \in \{-1, 0, 1\}$.

(2) **Continuous version of (1):** Choose f function uniformly (i.e., with respect to the Lebesgue measure on $\mathbb{R}^{\Lambda_N}$) in the set of all *real-valued* functions f such that $f = 0$ on ∂_N and for any $xy \in E_N$, $|f(x) - f(y)| \leq 1$.

(3) **Analogue of Example 1.10:** More generally, suppose that h is the density function of a symmetric L^2 random variable with zero mean. Choose f with $f = 0$ on ∂_N in such a way that the random vector $(f(x))_{x \in \Lambda_N}$ has density at $(\gamma_x)_{x \in \Lambda_N}$, (with respect to Lebesgue measure on $\mathbb{R}^{\Lambda_N}$) proportional to

$$\prod_{e \in E_N} h(|\nabla \gamma(e)|).$$

One way to think about it is that each edge $e \in E_N$ consists of a little spring (so that the tambourine skin is actually made of a little trampoline web of springs). Each point x in Λ_N (in the horizontal plane) is allowed to move vertically (in some third direction perpendicular to $\overline{\Lambda}_N$) to the position $(x, \gamma(x))$ in three-dimensional space, while the boundary points $x \in \partial_N$ are stuck to height 0. The spring on the edge e puts some constraints on the height-difference between the two extremities of e, and in particular tends to prevent this difference from being very large.

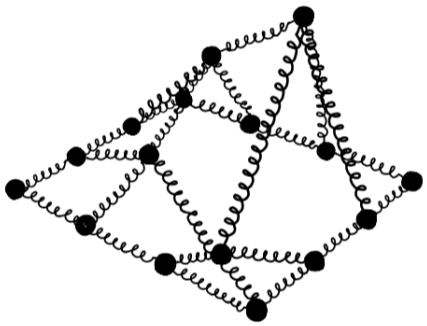

Figure 1.2: An illustration when $N = 3$ and $d = 2$.

Remark 1.13. *As in the previous one-dimensional case, each of these measures can be viewed as the stationary measure of some rather simple Markov chain on the state space of*

functions from Λ_N into $\mathbb{R}$, where at each step of the chain, one resamples the value (height) of the function at at most one site, according to the conditional distribution of that height given those of its neighbours.

Motivated by the example of the Brownian bridge, we ask:

Question. Suppose that for each N, the function f_N on $\bar{\Lambda}_N$ is chosen randomly according to procedure (1), (2) or (3) above, and turn each f_N into a function defined on $[0,1]^2$ by rescaling (and making the function constant on each square):

$$\hat{f}_N(x_1, x_2) := f_N([Nx_1], [Nx_2]).$$

For some good choice of sequence ϵ_N, does $\epsilon_N \hat{f}_N$ converge to that of some "universal random function" f from $[0,1]^2$ to $\mathbb{R}$?

As we will see very soon, the story turns out to be a little more subtle due to the actual nature of this universal random function f, but the conjecture is roughly that this should be correct (and actually we will see that in this two-dimensional case ϵ_N should be constant).

This is actually still a conjecture for most of the examples mentioned above! There exist a couple of cases where this is known to be true (for instance when h is the exponential of a uniformly concave function), but for case (1), this is (to our knowledge) an open problem.

In these lectures, we will actually not discuss these universality questions at all. Rather, we will first focus on the special Gaussian subcase of example (3), for which one can:

- say a lot in the discrete case, which already gives rise to combinatorially very rich mathematical objects;
- show very easily that (when suitably rescaled), the discrete models converge in distribution as $N \to \infty$ to their counterparts in the continuum.

This particular example is that of the *discrete Gaussian Free Field* (we will use the acronym GFF for Gaussian Free Field throughout these notes). This corresponds to case (3) above, when $h(u)$ is the distribution function of a Gaussian random variable i.e., $\exp(-u^2/2\sigma^2)$ for some choice of σ^2.

So, the discrete GFF is the probability measure on $\mathbb{R}^{\Lambda_N}$ with density at $(\gamma_x)_{x\in\Lambda_N}$ a constant multiple of

$$\exp(-\sum_{e\in E_N} |\nabla\gamma(e)|^2/(2\sigma^2))$$

with the convention that $\gamma = 0$ on ∂_N.

In this case, the obtained random function f_N is a centred Gaussian vector. Hence, its law is fully described via its covariance function, and if one controls this covariance function well in the limit when $N \to \infty$, one will obtain convergence to some Gaussian object in the continuum space (with covariances given by limit of the covariances). Hence, we can determine what the continuous object that we are looking for should be.

LECTURE 2

Discrete Gaussian free field

2.1 Definition via density function

Before defining the discrete GFF, let us first introduce some notation that we will use throughout these notes. We suppose that $d \geq 1$.

When f is a function from $\mathbb{Z}^d$ into $\mathbb{R}$, we define $\overline{f}(x)$ to be the average value of f at the $(2d)$ neighbours of x. In other words,

$$\overline{f}(x) = \frac{1}{2d} \sum_{y:y\sim x} f(y),$$

where here and in the sequel, $\sum_{y:y\sim x}$ means that we sum over the $2d$ neighbours of x in $\mathbb{Z}^d$.

Definition 2.1 (Discrete Laplacian). *We define the discrete Laplacian Δf of f to be the function*

$$\Delta f(x) := \overline{f}(x) - f(x).$$

When D is a subset of $\mathbb{Z}^d$, we define its (discrete) boundary

$$\partial D := \{x \in \mathbb{Z}^d : d(x, D) = 1\} \text{ and } \overline{D} := D \cup \partial D.$$

We will denote by $\mathcal{F}_{(D)}$ the set of functions from $\mathbb{Z}^d$ into $\mathbb{R}$ that are equal to 0 outside of D. When D is finite and has n elements, then $\mathcal{F}_{(D)}$ is of course a real vector space of dimension n.

We define the set $E_{\overline{D}}$ to be the set of edges of $\mathbb{Z}^d$ such that at least one end-point of the edge is in D. For each $F \in \mathcal{F}_{(D)}$ and each unoriented edge $e \in E_{\overline{D}}$, we define $|\nabla F(e)| := |F(x) - F(y)|$ as before, where x and y are the two endpoints of e. Note that to decide about the sign of ∇F, we would need to consider oriented edges, but that $|\nabla F(e)|$ and its square do not depend on the orientation of e. Similarly, when F_1 and F_2 are in $\mathcal{F}_{(D)}$, we can define unambiguously the product $\nabla F_1(e) \times \nabla F_2(e)$. Finally, when D is finite we define

$$\mathcal{E}_D(F) := \sum_{e\in E_{\overline{D}}} |\nabla F(e)|^2.$$

This quantity (or half of this quantity) is often referred to as the Dirichlet energy of the function F.

Definition 2.2 (Discrete GFF via its density function). *The discrete GFF in D with Dirichlet boundary conditions (also sometimes referred to as zero boundary conditions) on ∂D is the*

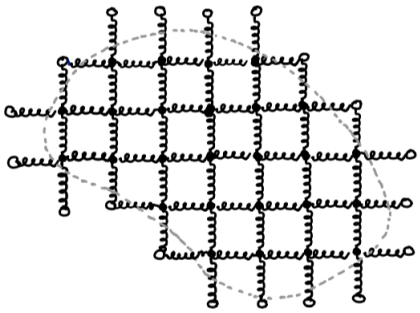

Figure 2.1: A domain $D \subset \mathbb{Z}^2$, formed by taking all $z \in \mathbb{Z}^2$ that lie inside a domain $\Omega \subset \mathbb{R}^2$ (the boundary of Ω is represented by the dotted line). Solid discs represent points of D, and open discs points of ∂D. Each edge in $E_{\overline{D}}$ is depicted as a "spring".

centred Gaussian vector $(\Gamma(x))_{x\in D}$ *whose density function on* $\mathbb{R}^D$ *at* $(\gamma_x)_{x\in D}$ *is a constant multiple of*

$$\exp(-\frac{1}{2} \times \frac{\mathcal{E}_D(\gamma)}{2d}) = \exp(-\frac{1}{2} \times \frac{1}{2d} \sum_{e\in E_{\overline{D}}} |\nabla\gamma(e)|^2)$$

with the convention that $\gamma = 0$ *on* ∂D.

Remark 2.3. *We use the notation* $(\gamma_x)_{x\in D}$ *rather than* $(\gamma(x))_{x\in D}$ *to distinguish it as a fixed vector. Recall that the quantity* $|\nabla\gamma(e)|$ *when* e *has endpoints* $\{x, y\}$ *is equal to* $|\gamma_x - \gamma_y|$.

Note that by definition $(\gamma_x)_{x\in D} \mapsto \mathcal{E}_D(\gamma)$ is a bilinear form, and it is also positive definite (indeed if $\mathcal{E}_D(\gamma)$ is 0, it means that $|\nabla\gamma(e)| = 0$ on all edges, so that γ is identically 0). Thus, the exponential above is indeed a multiple of the density function of some Gaussian vector on $\mathbb{R}^D$, and this definition makes sense.

Recall that the law of a centred Gaussian vector is completely determined by its covariance function. It will turn out that the covariance function of the Gaussian Free Field is very nice, and we will come back to this later.

2.2 Resampling procedure and consequences

Suppose that x is a given point in D. What is the conditional distribution of $\Gamma(x)$ given $(\Gamma(y))_{y\in D\setminus\{x\}}$? An inspection of the density function of Γ shows that the conditional distribution of $\Gamma(x)$ given $(\Gamma(y))_{y\in D\setminus\{x\}} = (h(y))_{y\in D\setminus\{x\}}$ has a density at $(\gamma_x)_{x\in D}$ that is proportional to

$$\exp\big(-\frac{1}{2\times(2d)} \sum_{y:y\sim x} |\gamma_x - h(y)|^2\big).$$

Expanding this sum over y, we get that this is equal to

$$\exp(-\frac{1}{2}(\gamma_x - \overline{h}(x))^2)$$

times some normalising function that depends only on h. In other words, this conditional law is that of the Gaussian distribution $\mathcal{N}(\overline{h}(x), 1)$.

Remark 2.4. *Two important features.*

- *This conditional distribution depends only on the values $h(y)$ at the neighbours y of x (which is due to the interaction via nearest-neighbours only).*
- *The conditional law of $\Gamma(x) - \overline{h}(x)$ is a standard normal Gaussian (for all choices of $\overline{h}(x)$). This means that, for all x, $\Gamma(x) - \overline{\Gamma}(x)$ is a standard Gaussian random variable that is independent of $(\Gamma(y))_{y \in D \setminus \{x\}}$.*

The second point above has a number of important consequences.

- It indicates what the natural Markov chain (on the space of functions) is, for which the law of the GFF is stationary. For this chain, the Markovian step can be described as follows: if we are given a function h in $\mathcal{F}_{(D)}$, then we choose a point $x \in D$ uniformly at random, and replace the value of $h(x)$ by $\overline{h}(x) + N$ where N is a standard Gaussian random variable.
- It allows us to derive some interesting properties of the covariance function of Γ. For all x and y in D, let us denote this covariance function by

$$\Sigma(x, y) = \Sigma_x(y) := \mathbb{E}[\Gamma(x)\Gamma(y)].$$

Note that for each given x, $y \mapsto \Sigma_x(y)$ is a function in $\mathcal{F}_{(D)}$. When $x \neq y$ are both in D,

$$\begin{aligned}\Sigma_x(y) &= E[\Gamma(x)\overline{\Gamma}(y)] + E[\Gamma(x)(\Gamma(y) - \overline{\Gamma}(y))] \\ &= E[\Gamma(x)\overline{\Gamma}(y)] = \frac{1}{2d} \sum_{z : z \sim y} E[\Gamma(x)\Gamma(z)] = \overline{\Sigma_x}(y).\end{aligned}$$

Similarly,

$$\begin{aligned}\Sigma_x(x) &= E[\Gamma(x)\Gamma(x)] = E[\Gamma(x)\overline{\Gamma}(x)] + E[(\Gamma(x) - \overline{\Gamma}(x))\Gamma(x)] \\ &= (2d)^{-1} \sum_{z : z \sim x} E[\Gamma(z)\Gamma(x)] + E[(\Gamma(x) - \overline{\Gamma}(x))^2] + E[(\Gamma(x) - \overline{\Gamma}(x))\overline{\Gamma}(x)] \\ &= \overline{\Sigma_x}(x) + 1 + 0.\end{aligned}$$

In other words, the function Σ_x satisfies

$$\Delta\Sigma_x(y) = -\mathbf{1}_{\{y=x\}}$$

for all y in D. Note that (for each given x) this provides as many linear equations as there are entries for $\Sigma_x(\cdot)$; these equations are clearly linearly independent, so that these relations fully determine Σ_x.

2.3 The discrete Green's function

The previous analysis leads us naturally to quickly review and browse through some basic definitions and properties related to the discrete Laplacian and Green's function.

The discrete Laplacian Recall that for all $F \in \mathcal{F}_{(D)}$, we defined for $x \in D$,

$$\Delta F(x) := \frac{1}{2d} \sum_{y : y \sim x} (F(y) - F(x)) = \overline{F}(x) - F(x)$$

By convention, we will denote by $\Delta_D F$ the function that is equal to ΔF in D and is equal to 0 outside of D (mind that here we do not care about the value of ΔF outside of D, in particular on ∂D).

Clearly, we can then view Δ_D as a linear operator from $\mathcal{F}_{(D)}$ into itself. Δ_D is injective using the maximum principle: if $\Delta_D F = 0$, then choose $x_0 \in D$ so that $|F(x_0)| = \max_{x \in D} |F(x)|$, and because $\Delta_D F(x_0) = 0$, this implies readily that the value of F on all the neighbours of x_0 are all equal to $F(x_0)$ (as otherwise, their mean value could not be equal to $F(x_0)$); this also holds for all neighbours of neighbours of x_0 as well, and eventually, since D is finite, this means that we will find a boundary point y for which $F(y) = F(x_0)$; finally, since $F = 0$ on the boundary, it follows that $\max_{x \in D} |F(x)| = |F(x_0)| = 0$.

Hence, Δ_D is a bijective linear map from the vector space $\mathcal{F}_{(D)}$ into itself. One can therefore define its linear inverse map: for any choice of function $u : D \to \mathbb{R}$, there exists exactly one function $F \in \mathcal{F}_{(D)}$ such that $\Delta_D F(x) = u(x)$ for all $x \in D$.

If we apply this to the previous analysis, it shows that indeed, $y \mapsto \Sigma_x(y)$ is the unique function in $\mathcal{F}_{(D)}$ such that its Laplacian Δ_D in D is the function $y \mapsto -\mathbf{1}_{\{y=x\}}$. This function has a name...

The Green's function Let $(X_n)_{n \geq 0}$ be a simple random walk in $\mathbb{Z}^d$, with law denoted by P_x when it is started at x. (That is, $X_n = x + Y_1 + \cdots + Y_n$ where $(Y_i)_{i \geq 1}$ are independent and identically distributed, equal to $+\mathbf{e}_j$ or $-\mathbf{e}_j$ each with probability $1/2d$, where $(\mathbf{e}_j)_{1 \leq j \leq d}$ are the standard basis of unit vectors in $\mathbb{Z}^d$). Let $\tau = \tau_D := \inf\{n \geq 0 : X_n \notin D\}$ be its first exit time from D.

Definition 2.5 (Green's function). *We define the Green's function G_D in D to be the function defined on $D \times D$ by*

$$G_D(x, y) := E_x\Big[\sum_{k=0}^{\tau - 1} \mathbf{1}_{\{X_k = y\}}\Big].$$

By convention, we will set $G_D(x, y) = 0$ as soon as one of the two points x, y is not in D.

Proposition 2.6. *The Green's function G_D is the inverse of $-\Delta_D$, and it is equal to Σ.*

Proof. We will use a slightly convoluted, but hopefully instructive, strategy to prove this (see the exercise below for a more direct approach). The idea is that the Markov property of the simple random walk immediately enables us to determine the Laplacian of the function $g_{D,x}(\cdot) = G_D(\cdot, x)$ in D (note that $g_{D,x} \in \mathcal{F}_{(D)}$, as $g_{D,x}$ is equal to zero outside of D). Indeed, we have that for all $y \neq x$ in D, $\Delta_D g_{D,x}(y) = 0$, simply because

$$G_D(y,x) = E_y\Big[\sum_{k\geq 1} \mathbf{1}_{\{X_k = x, k<\tau\}}\Big] = \sum_{z:z\sim y} \frac{1}{2d} G_D(z,x),$$

where we have used the Markov property at time 1 in the first identity. Also, the very same observation (but noting that at time 0, the random walk starting at x is at x) shows that $\Delta_D g_{D,x}(x) = -1$. Hence, $g_{D,x}$ is a function in $\mathcal{F}_{(D)}$ satisfying

$$\Delta_D g_{D,x}(y) = -\mathbf{1}_{\{x=y\}}$$

for all y in D. Since Δ_D is a bijection of $\mathcal{F}_{(D)}$ onto itself, the function $g_{D,x}$ is in fact the unique function in $\mathcal{F}_D$ with this property. We therefore conclude that for all x and y in D, $\Sigma(x,y) = G_D(x,y)$. □

This provides the following equivalent definition of the discrete Gaussian Free Field:

Definition 2.7 (Discrete GFF via the covariance function)**.** *The discrete Gaussian Free Field in D with Dirichlet boundary conditions on ∂D is the centred Gaussian process $(\Gamma(x))_{x\in D}$ with covariance function $G_D(x,y)$ on $D \times D$.*

Remark 2.8. *We see that, as opposed to the first definition, this second equivalent definition actually also works when D is infinite, so long as G_D is well-defined. That is, as long as the random walk in D, killed when it reaches ∂D, is transient. In other words, the definition can also be used for any infinite subset of $\mathbb{Z}^d$ when $d \geq 3$ (because the simple random walk on $\mathbb{Z}^d$ is transient), or for any infinite subset $D \neq \mathbb{Z}^d$ when $d = 1, 2$.*

Remark 2.9. *The two definitions are equivalent. It is a matter of taste which to use, when one wants to derive properties of the GFF.*

Note that if D has n points $\{x_1, \ldots, x_n\}$, we can view the functions G_D, $-\Delta_D$ and Σ defined on $D \times D$ as $n \times n$ matrices.

Exercise 3. *(i) Show that*

$$G_D(x,y) = \sum_{k\geq 0} \#\{\text{paths } x \to y \text{ in } k \text{ steps within } D\} \times \Big[\frac{1}{2d}\Big]^k.$$

Deduce that $G_D(x,y) = G_D(y,x)$

(ii) Consider the matrix $P_D := I + \Delta_D$ Show that

$$P_x[X_k = y, k < \tau] = (P_D)^k(x, y),$$

where $(P_D)^k$ is the k-th power of the matrix P^D, and by writing $G_D(x, y)$ as a sum in terms of the $(P_D)^k$, deduce that G_D is equal to the inverse of $(I - P_D) = -\Delta_D$.

Now we would like to ask: is there an analogue of the Markov property for the simple random walk that extends to the setting of the discrete GFF? In this section we will use the more hands-on definition of the GFF via density functions, as it provides a little more insight.Note that we have defined the discrete GFF in any finite subset of $\mathbb{Z}^d$ (in particular, this set does not need to be connected).

2.4 The GFF with non-zero boundary conditions

In view of our intuitive description of the GFF, it is natural to generalise our definition to the case of non-zero boundary conditions. More precisely, suppose that f is some given real-valued function defined on ∂D. Then, the definition of the GFF via its density function can be extended as follows:

Definition 2.10 (Discrete GFF with non-zero boundary conditions, via its density function). *The discrete GFF in D* with boundary condition *f on ∂D is the Gaussian vector $(\Gamma(x))_{x\in D}$ whose density function on $\mathbb{R}^D$ at $(\gamma_x)_{x\in D}$ is a constant multiple of*

$$\exp(-\frac{1}{2} \times \frac{\mathcal{E}_D(\gamma)}{2d}),$$

with the convention that $\gamma = f$ on ∂D. Note that the values of f on ∂D are implicitly used in the expression of $\mathcal{E}_D(\gamma)$ via the terms $|\nabla\gamma(e)|$ for those edges $e \in E_{\overline{D}}$ having one endpoint in ∂D.

In other words, instead of fixing the height of Γ on ∂D to be 0, we now fix it to be f. Then Γ is still a Gaussian process, but it is not necessarily centered.

Looking at the expression of the density function for Γ, we can deduce the following: suppose that $(\Gamma(x))_{x\in D}$ is a GFF in D with boundary condition f on ∂D and that O is some given subset of D. Then, the conditional law of $(\Gamma(x))_{x\in O}$ given $(\Gamma(x))_{x\notin O}$ will be a GFF in O with boundary conditions given by the (random) function f_O on ∂O that is equal to the observed values of Γ on ∂O. We can rephrase this in a form that will be reminiscent of the simple Markov property of random walks

Proposition 2.11 (Markov property, version 1). *The conditional law of $(\Gamma(x))_{x\in O}$ given that $(\Gamma(x))_{x\notin O}$ is equal to $(f(x))_{x\notin O}$ is that of a GFF in O with boundary condition $f|_{\partial O}$.*

From this we see why it is so natural to consider the GFF with non-zero boundary conditions.

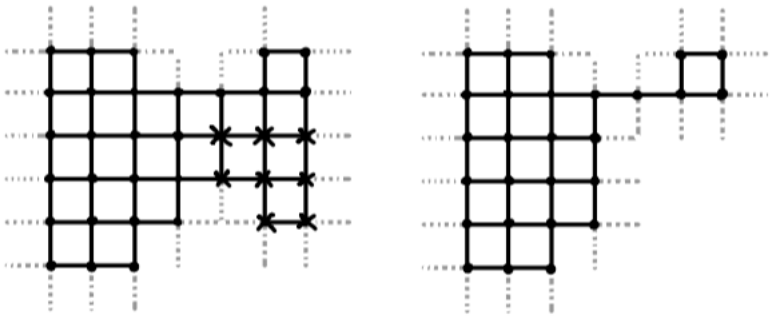

Figure 2.2: The left-hand side is an example of $D \subset \mathbb{Z}^2$ and $O \subset D$, where the vertices of $D \setminus O$ are marked with a cross, and the vertices of O are marked with a disc. The edges of $\mathbb{Z}^2$ joining two points in D are represented by solid lines, and the edges with one endpoint in D and one endpoint in ∂D are represented by dotted lines. The right-hand side illustrates O, where here solid lines are edges joining two vertices in O and dotted lines are edges with one endpoint in O and one endpoint in ∂O. The Markov property says that if Γ is a GFF on the left graph, and we are given the values of Γ "on the crosses", then Γ restricted to the right graph has the law of a GFF in that graph with non-zero boundary conditions.

Definition 2.12. *When F_1 and F_2 are two real-valued functions defined on $\mathbb{Z}^d$ and with finite support, then we define*

$$(F_1, F_2) = \frac{1}{2} \times \frac{1}{2d} \times \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d, y \sim x} (F_1(y) - F_1(x))(F_2(y) - F_2(x)).$$

Notice that

$$\begin{aligned}(F_1, F_2) &= \frac{1}{2d} \times \sum_{x \in \mathbb{Z}^d} \sum_{y : y \sim x} \left[-F_1(x)(F_2(y) - F_2(x))\right] \\ &= -\sum_{x \in \mathbb{Z}^d} F_1(x) \Delta F_2(x) = -\sum_{x \in \mathbb{Z}^d} F_2(x) \Delta F_1(x),\end{aligned}$$

where we have deduced the last equality by symmetry.

We will also use the following definition: when f is a real-valued function defined on ∂D, we define *the harmonic extension F of f to D* to be the unique function defined in $D \cup \partial D$ such that $F = f$ on ∂D and $\Delta F = 0$ in D.

Proposition 2.13. *If $(\Gamma(x))_{x \in D}$ is a GFF with Dirichlet boundary conditions in D, and if F is the harmonic extension to D of some given function f on ∂D, then $(\Gamma(x) + F(x))_{x \in D}$ is a GFF in D with boundary condition f on ∂D.*

Equivalently, one can of course restate this as:

Proposition 2.14 (Markov property, version 2). *If $(\Gamma(x))_{x\in D}$ is a GFF in D with boundary conditions f on ∂D, and if F is the harmonic extension to D of f, then $(\Gamma(x) - F(x))_{x\in D}$ is a GFF in D with Dirichlet boundary conditions.*

Hence, the Gaussian vector $(\Gamma(x))_{x\in D}$ is characterised by its expectation $(F(x))_{x\in D}$ and its covariance function $\Sigma(x, y) = G_D(x, y)$. The effect of the non-zero boundary conditions is only to tilt the expectation of the GFF, but it does not change its covariance structure.

Proof of Proposition 2.14. Let us consider a GFF Γ in D with Dirichlet boundary conditions, and let F be the harmonic extension of f to D. Then if we define $\tilde{\Gamma} = F + \Gamma$, by a simple change of variables, $\tilde{\Gamma}$ will have a density at $(\gamma_x)_{x\in D}$ which is a multiple of

$$\exp(-(\gamma - F, \gamma - F)),$$

with the convention that $\gamma = f$ on ∂D. This (given that F is deterministic and harmonic, so $(\gamma - F, \gamma - F) = (\gamma, \gamma) - 2(\gamma, F) + (F, F) = (\gamma, \gamma) + 2\sum_x \gamma_x \Delta_D F(x) - \sum_x F(x)\Delta_D F(x) = (\gamma, \gamma)$) is a multiple of

$$\exp(-(\gamma, \gamma)) = \exp(-\frac{1}{2} \times \frac{\mathcal{E}_D(\gamma)}{2d})$$

(using the same convention on γ). In other words, Γ is indeed a GFF in D with boundary conditions f on ∂D. □

Let us now introduce some notation. Suppose that Γ is a GFF in a finite subset D of $\mathbb{Z}^d$ with boundary conditions given by some real-valued function f on ∂D. Suppose that B is some finite subset of D. We define $O = O(B) := D \setminus B$ and then define the following two new processes:

Definition 2.15. (The processes Γ_B and Γ^B)

- *$(\Gamma_B(x))_{x\in D}$ is the process that is equal to Γ in B and in $O(B)$, it is defined to be the harmonic extension to O of the values of Γ on ∂O. So the process Γ_B can be constructed in a deterministic way given f and the values of Γ on B.*
- *The process $(\Gamma^B(x))_{x\in D}$ is then defined to be equal to $\Gamma - \Gamma_B$. Clearly, $\Gamma^B(x) = 0$ as soon as $x \notin O$, and $\Gamma_B + \Gamma^B = \Gamma$.*

Combining our previous observations readily implies the following alternative statement of the Markov property:

Proposition 2.16 (Markov property, version 3). *The processes Γ_B and Γ^B are independent, and Γ^B is a GFF in $O = D \setminus B$ with Dirichlet boundary conditions.*

One main feature in the statement above is the independence of Γ^B from Γ_B, i.e., that fact that Γ^B does not depend on the values of Γ in B. Another equivalent way to reformulate this result is therefore that conditionally on $(\Gamma(x))_{x\in B}$, the conditional law of $(\Gamma(x))_{x\in D\setminus B}$ is that of a GFF in $D \setminus B$ with boundary conditions given by the values of Γ on $\partial(D \setminus B)$.

Note that the special case where $D \setminus B$ is a singleton point $\{x\}$ is exactly the resampling property of the GFF that we mentioned earlier: the conditional law of the GFF at x given its values at all other points is equal to a Gaussian random variable with variance 1 and mean given by the mean value of the GFF at the neighbours of x.

Remark 2.17. *Since Γ_B and Γ^B are independent, and since we know that the covariance functions of Γ and Γ^B are G_D and G_O respectively, we get that*

$$G_D(x,y) = E[\Gamma(x)\Gamma(y)] = E[\Gamma_B(x)\Gamma_B(y)]+E[\Gamma^B(x)\Gamma^B(y)] = E[\Gamma_B(x)\Gamma_B(y)]+G_O(x,y),$$

so that the covariance function of Γ_B is

$$E[\Gamma_B(x)\Gamma_B(y)] = G_D(x,y) - G_O(x,y) \quad x,y \in D.$$

Exercise 4 (Deterministic and algorithmic discoveries of the GFF)**.** *Suppose that Γ is a GFF in D with Dirichlet boundary conditions. Suppose that $D = \{x_1, \dots, x_n\}$, and for each j, define $B_j = \{x_1, \dots, x_j\}$ and $O_j = \{x_{j+1}, \dots, x_n\}$.*

By exploring the values of the GFF at the points $\{x_1, x_2, \dots\}$ in turn, and iteratively applying the Markov property described above, conclude that we can write

$$\Gamma(\cdot) = \sum_{j=1}^{n} N_j \times \sqrt{G_{O_{j-1}}(x_j, x_j)} \times v_j(\cdot)$$

for n independent identically distributed centred Gaussian random variables $N_1, \dots, N_n$ and some functions $(v_j)_{1\le j\le n}$.

Exercise 5. *Consider the subset $\Lambda_N = [1, N-1] \times [1, N-1]$ of $\mathbb{Z}^2$ for $N \in \mathbb{N}$. Show that for suitable $(m_1, m_2) \in \mathbb{N}^2$*

$$\psi_{m_1,m_2}(x_1, x_2) = \sin(\frac{\pi}{N} x_1 m_1) \sin(\frac{\pi}{N} x_2 m_2)$$

is an eigenvector of Δ_{Λ_N}, and determine its eigenvalue. Use this to write an expression for a Gaussian free field in Λ_N with Dirichlet boundary conditions, as a sum of the form $\sum_{i=1}^{n} \alpha_i f_i$ where $(\alpha_1, \cdots, \alpha_n)$ are i.i.d $\mathcal{N}(0,1)$ random variables and the f_i's are multiples of an appropriate collection of the ψ_{m_1,m_2}'s.

Challenge (optional!): *use this to show that $G_{D_N}((N/2, N/2), (N/2, N/2)) \asymp \log N$ as $N \to \infty$.*

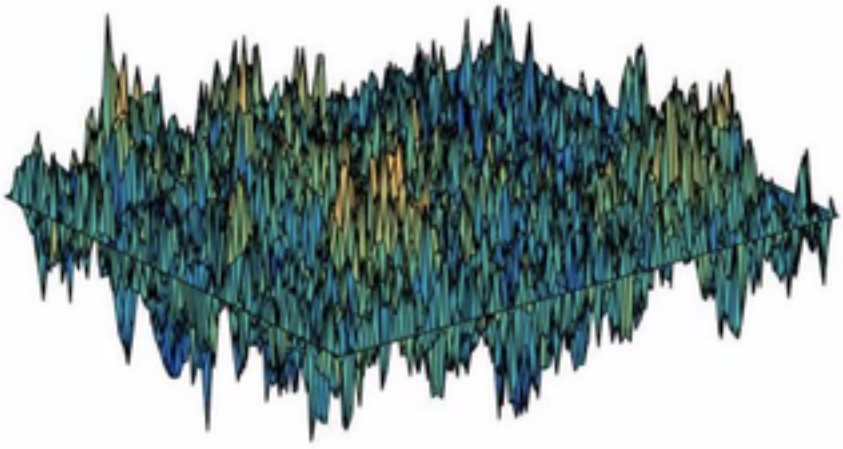

Figure 2.3: A simulation of $\tilde{\Gamma}_\delta$ on a square.

2.5 Informal comments about the possible scaling limit

In this section, we use the above definition of the discrete Gaussian free field to formulate some heuristics about how a "continuum Gaussian free field" on a subset of $\mathbb{R}^d$ could be defined. This discussion is non-rigorous: a proper study of the continuum GFF will come in the next lecture.

Suppose that d is some open subset of $\mathbb{R}^d$ for $d \geq 1$. The idea is to approximate the continuum process $(\Gamma(x))_{x\in D}$ that we want to define, using the GFF on a fine grid approximation of D. For instance, one can do the following for $\delta > 0$:

- define $\tilde{D}_\delta = \delta\mathbb{Z}^d \cap D$ to be (so an approximation of D on the fine grid $\delta\mathbb{Z}^d$ when δ is small;
- define $D_\delta = \delta^{-1}\tilde{D}_\delta = \mathbb{Z}^d \cap (\delta^{-1}D)$ to be the $(1/\delta)$ blow-up of $\tilde{D}_\delta$: a subset of $\mathbb{Z}^d$;
- define the GFF Γ_δ on D_δ as in the previous section, and a GFF $\tilde{\Gamma}_\delta$ on $\tilde{D}_\delta$ by setting $\tilde{\Gamma}_\delta(x) = \Gamma_\delta(x\delta^{-1})$;
- extend this random function $\tilde{\Gamma}_\delta$ to all of $\mathbb{R}^d$ by (for instance) choosing $\tilde{\Gamma}(y) = \tilde{\Gamma}(x)$ for all $y = (y_1, \ldots, y_d) \in [x_1, x_1 + \delta) \times \ldots \times [x_d, x_d + \delta)$ when $x \in \delta\mathbb{Z}^d$.

In other words, $\tilde{\Gamma}_\delta$ is a (extrapolated) GFF on the grid approximation $\tilde{D}_\delta$ of D in $\delta\mathbb{Z}^d$, normalised in such a way that the variance of the difference between $\tilde{\Gamma}_\delta(x)$ and the mean value of its $2d$ neighbours in $\tilde{D}_\delta$ is equal to 1 for all $x \in \tilde{D}_\delta$.

Now the philosophy is the following: when a centred Gaussian process converges in law (which is exactly when all its finite-dimensional distributions converge), then the limiting law is bound to be a centred Gaussian process as well, and the covariances of the limit are the limits of the covariances (recall Exercise 1).

So, it is natural to look at what happens to the covariance function of $\tilde{\Gamma}_\delta$ as $\delta \to 0$. Let us collect here some observations and facts, leaving out any detailed proof:

1. When $x \neq y$ in D, then it turns out that as $\delta \to 0$,

$$G_{D_\delta}(x\delta^{-1}, y\delta^{-1}) \sim \delta^{d-2} G_D(x, y),$$

where $G_D(x, y)$ is some positive function of x and y (called the continuum Green's function, see later. The main point to note is that this quantity converges when $d = 2$, but tends to 0 when $d > 2$. A simple way to understand the formula above is to note that the mean number of steps spent by the random walk before exiting a compact portion of D is of the order of δ^{-2} (this 2 comes from the central limit theorem renormalisation). On the other hand, in expectation, this time is spread rather regularly among all points y (when y is not too close to x), and the number of such points y is of the order of δ^{-d}. Hence, we should not be surprised by the coefficient δ^{d-2}.

2. As a consequence, when $d = 2$, we see that the covariances converge to something non-trivial *without any rescaling*. In other words, one would like to simply take the limit of $(\tilde{\Gamma}_\delta(x))_{x \in D}$ to define the continuum GFF in D. We already see that such a limit is unlikely to be a continuous function, because the variance of the difference between the values of $\tilde{\Gamma}_\delta$ at two points that are δ apart in D will be of order 1, and in particular will not go to 0. In fact, $\mathbb{E}[(\tilde{\Gamma}_\delta(x))^2]$ will grow like $\log(1/\delta)$ as $\delta \to 0$: see Exercise 5 for an example.

3. When $d \geq 3$, things are even worse! In order to get a limit for the covariance function, point (1) implies that we need to rescale $\tilde{\Gamma}_\delta$ and to look instead at $\delta^{1-d/2}\tilde{\Gamma}_\delta$. This time, it means that the variance between the value of $\delta^{1-d/2}\tilde{\Gamma}_\delta$ at a point x and its mean-value among the $2d$ neighbours of x in $\tilde{D}_\delta$ is not only going to stay positive as $\delta \to 0$, but will actually blow up. It therefore seems that in the limit, any obtained process must be unbounded everywhere, and equal to $\pm\infty$ simultaneously at each point of D!

4. We finally observe that for $x \in D$ the variance of $\delta^{1-d/2}\tilde{\Gamma}_\delta(x)$ tends to infinity as $\delta \to 0$ (when $d = 2$, this follows from recurrence of random walk in $\mathbb{Z}^2$). So, any limiting process cannot possibly be defined as a random function, as it would then be a centered Gaussian with infinite variance. We could try to fix this by renormalising $\tilde{\Gamma}_\delta$ by some constant $\epsilon(\delta)$, so that the variance of $\epsilon(\delta)\tilde{\Gamma}_\delta(x)$ converges to something finite, and the process has a proper Gaussian limit. However, the covariance function of the limit would then be 0 on $\{(x, y) \in D \times D,\ x \neq y\}$, so that the limiting process would consist of a collection of independent Gaussian random variables (one for each point in the domain D). This is clearly not the interesting process that we are looking for!

As we shall see, in a later lecture, the proper way to define the Gaussian free field in the continuum will be to view it as a random *generalised function* rather than as a normal (point-wise defined) function.

LECTURE 3

Continuum Gaussian free field

We now turn to the continuum world, and we start our study of the *continuum GFF* (in open subsets of $\mathbb{R}^d$, $d \geq 2$). Recall from the first lecture that our natural "universal function" with zero boundary conditions (the scaling limit of constrained random walks) on an interval I of $\mathbb{R}$ (i.e., when $d = 1$), was the Brownian bridge. Our goal here is to describe the analogue of the Brownian bridge when the parameter-space is higher dimensional. As we progress with this, the reader may find it interesting to draw analogies with (a) the corresponding properties of Brownian bridges and (b) the corresponding properties of the discrete GFF.

From now on, D will denote an open subset of $\mathbb{R}^d$, $d \geq 2$, with a regular [2] boundary (for example, a smooth boundary will certainly be enough) and when $d = 2$, we will require that $D \neq \mathbb{R}^2$ (this will ensure that the continuum Green's function in D is finite). Sometimes, we will add further conditions on D, such as requiring it to be bounded or connected.

3.1 Warm-up and heuristics

Recap. The object that we would like to define should be some sort of random function, or process, $(\Gamma(x))_{x\in D}$. The process Γ should be a centered Gaussian process, and should correspond to the (appropriately normalised) limit of the discrete GFF on a lattice approximation $\tilde{D}_\delta \subset \delta\mathbb{Z}^d$ to D. Recall that the covariance function of the discrete GFF on $\tilde{D}_\delta$ (defined by rescaling the discrete GFF on $D_\delta = \delta^{-1}\tilde{D}_\delta \subset \mathbb{Z}^d$ as in the previous lecture) is the (rescaled) discrete Green's function on $\tilde{D}_\delta$. We noted that the only way to take $\delta \to 0$ in order to get a limiting process with some non-trivial correlation structure appears to be to first normalise the discrete GFF (at each level δ) in such a way that the (level δ) discrete Green's function converges to a non-trivial function in D as $\delta \to 0$. This limiting covariance function should still be *harmonic* away from the diagonal and positive. In fact, this already essentially characterises it as the *continuum Green's function* G_D in D.

So, given that the weak limit of Gaussian processes is a Gaussian process, it looks like we are trying to define a centred Gaussian process $(\Gamma(x))_{x\in D}$ with covariance function $E[\Gamma(x)\Gamma(y)] = G_D(x, y)$. As we have already pointed out in the warm-up chapter, this does not appear to be possible, due to the fact that $G_D(x, x) = \infty$. Formally, this would mean that $\Gamma(x)$ is a Gaussian with infinite variance for every x...

What can we do? In a different direction, if we suppose that D is bounded, and Γ has covariance structure as described in the previous paragraph, we could formally consider

[2] meaning that for each z on the boundary and for B a d-dimensional Brownian motion started from z, we have $\inf\{t > 0,\ B_t \notin D\} = 0$ almost surely

the "integral" $I_\Gamma(1)$ of $\Gamma(x)$ over D. Then, by Fubini, we would have

$$\mathbb{E}[I_\Gamma(1)^2] = \int_{D\times D} \mathrm{d}x\mathrm{d}y\, \mathbb{E}[\Gamma(x)\Gamma(y)] = \int_{D\times D} \mathrm{d}x\mathrm{d}y\, G_D(x,y).$$

Now, as we will see in a moment, even if $G_D(x,y)$ explodes as $y \to x$, it is easy to see that for each given x, $\int_D G_D(x,y)\mathrm{d}y$ is finite (one can write this in terms of the expected exit time of D by a Brownian motion started from x). Thus the formal variance of $I_\Gamma(1)$ is actually finite. So even if for each given x, $\Gamma(x)$ does not make sense as a Gaussian random variable, it seems that $I_\Gamma(1)$ should be a Gaussian random variable with finite variance.

More generally, for any given continuous test function f with compact support in D, it turns out that the integral

$$G_D(f,f) := \int_{D\times D} \mathrm{d}x\mathrm{d}y\, f(x)f(y)G_D(x,y)$$

is absolutely convergent. This in turn indicates that one should be able to define a quantity $I_\Gamma(f)$, that is a centred Gaussian random variable with variance given by $G_D(f,f)$, and can be formally interpreted as $\int_D f(x)\Gamma(x)\mathrm{d}x$.

Finally, if f_1 and f_2 are two continuous functions with compact support in D, then the same argument indicates (formally) that

$$\mathbb{E}[I_\Gamma(f_1)I_\Gamma(f_2)] = \int_{D\times D} \mathrm{d}x\mathrm{d}y f_1(x)f_2(y)G_D(x,y) =: G_D(f_1,f_2) < \infty.$$

In summary, it seems that it should be possible to define a family of random variables $I_\Gamma(f)$ (indexed by the family of continuous functions f with compact support in D) as a centred Gaussian process with covariance function $\mathbb{E}[I_\Gamma(f_1)I_\Gamma(f_2)] = G_D(f_1,f_2)$.

This formal heuristic conclusion will be the starting point of our definition of the continuum GFF. We will essentially *define* the GFF to be this Gaussian process I_Γ (with the specified covariance structure). In fact, we will just use the notation $\Gamma(f)$ instead of $I_\Gamma(f)$. In other words, while the value of the continuum GFF at given points will not make sense, quantities that one can interpret as "mean" values of the GFF on bounded open domains U (i.e., $\Gamma(I_U)$) will be well-defined Gaussian random variables.

3.2 Basics on the continuum Green's function

Here we state without proofs some properties of the continuum Green's function. Some proofs are left as an exercise (see below).

Suppose that $D \subset \mathbb{R}^d$ satisfies the conditions that we stated at the beginning of this lecture, and that it is connected.

Let $y \in D$. Then, up to a multiplicative constant, there exists exactly one positive harmonic function $\tilde{H}_y$ in $D \setminus \{y\}$ such that $\tilde{H}_y(x)$ tends to 0 as $x \to \partial D$ or $x \to \infty$. For

example, when $d \geq 3$ and $D = \mathbb{R}^d$, the function $\tilde{H}_y$ must be equal to a constant times $H_y(x) := |x-y|^{2-d}$. In general, if we also set $H_y(x) = -\log(|x-y|)$ when $d = 2$, then for $D \subset \mathbb{R}^d$ $\tilde{H}_y$ must be a constant multiple of $H_y - h_y^D$, where h_y^D is the harmonic extension into D of the values given by $H_y(\cdot)$ on ∂D. That is, $h_y^D(x)$ is the unique harmonic function in D that is equal to $H_y(x)$ for $x \in \partial D$.

Definition 3.1. *The Green's function is defined to be this function*

$$G_D(x, y) := a_d^{-1}(H_y(x) - h_y^D(x)),$$

where for all $d \geq 2$, a_d is the $(d-1)$-dimensional Lebesgue measure of the unit $(d-1)$-dimensional sphere in $\mathbb{R}^d$.

This function then turns out to have all the properties that one would expect from the continuum analogue of the discrete Green's function. In particular :

(1) it can be interpreted as an integral operator that turns out to be the inverse of $-\Delta$, for $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ the continuum Laplacian; in other words, for f smooth and compactly supported in D,

$$F(y) := \int_D G_D(x, y) f(x) \mathrm{d}x$$

is smooth on D, tends to 0 on ∂D, and satisfies $-\Delta F = f$ on D;

(2) it is non-negative definite, in the sense that for f as above, $G_D(f, f) \geq 0$;

(3) it is a symmetric function, i.e., $G_D(x, y) = G_D(y, x)$;

(4) it can be interpreted in terms of expected occupation times by a d-dimensional Brownian motion B stopped when exiting D. More precisely, if τ is the time at which B first exits D, and $\mathbb{E}_x$ denotes the law of B when started from $x \in D$, then $\frac{1}{2}\mathbb{E}_x[\int_0^\tau f(B_t)] = \int_D G_D(x, y) f(y) \mathrm{d}y$ for any smooth compactly supported function f on D.

Exercise 6. *Prove item (1) above using the definition of G_D via H_y and h_y. Show further that for f_1, f_2 smooth and compactly supported in D,*

$$G_D(f_1, f_2) = \int_{D \times D} G_D(x, y) f(x) f(y) \, \mathrm{d}x \mathrm{d}y = \int_D \nabla F_1(x) \cdot \nabla F_2(x) \, \mathrm{d}x$$

where F_i is defined for each f_i in the same way as F is from f. Deduce (2) and (3).

Exercise 7. *Give explicit expressions for the Green's function in the case when $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is the unit disc, and when $D = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper half plane.*

We define $\mathcal{M}_D^+$ to be the set of finite measures that are supported in D, and such that

$$\int_{D\times D} G_D(x,y)d\mu(x)d\mu(y) < \infty.$$

We also define $\mathcal{M}_D$ to be the vector space of signed measures $\mu^+ - \mu^-$, where μ^+ and μ^- are in $\mathcal{M}_D^+$. We will omit the subscript D and simply write $\{\mathcal{M}, \mathcal{M}^+\}$, unless we are discussing various domains simultaneously.

One class $\mathcal{M}_c$ of measures $\mu \in \mathcal{M}$ is given by the set of measures of the form $f(x)dx$, when f is continuous with compact support in D (and dx denotes the Lebesgue measure). It is easy to check that such measures do lie in $\mathcal{M}$, simply by considering the rate at which $G_D(x,y)$ explodes as $|x-y| \to 0$.

3.3 Definition of the continuum GFF

We are now ready to define the continuum GFF. As in the discrete case, there are several possible ways to do this. We choose here to first define it as a random process.

Definition 3.2 (Continuum GFF). *We say that the process $(\Gamma(\mu))_{\mu\in\mathcal{M}}$ is a Gaussian Free Field in D if it is a centred Gaussian process with covariance function*

$$\Sigma(\mu,\nu) := \int_{D\times D} G_D(x,y)d\mu(x)d\nu(y).$$

In order to check that this definition makes sense, it suffices to check that this function $\Sigma(\mu,\nu)$ is indeed a well-defined covariance function, i.e., that for any $\mu_1,\dots,\mu_n$ in $\mathcal{M}$ and any real $\lambda_1,\dots,\lambda_n$,

$$\sum_{i,j\le n} \lambda_i\lambda_j\Sigma(\mu_i,\mu_j) \ge 0.$$

Note that the left-hand side is equal to $\Sigma(\mu,\mu)$ for $\mu = \lambda_1\mu_1 + \ldots + \lambda_n\mu_n$, and the fact that this quantity is non-negative follows from Exercise 6. So, the GFF in D does indeed exist.

When f is a measurable function in D such that

$$\int_{D\times D} |f(x)||f(y)|G_D(x,y)dxdy < \infty,$$

then $\mu_f = f(x)dx$ lies in $\mathcal{M}$, and we will often write $\Gamma(f)$ as a shorthand notation for $\Gamma(\mu_f)$. We can immediately note that for all $\lambda \in \mathbb{R}$ and all μ and ν in $\mathcal{M}$, one has

$$\Gamma(\lambda\mu) = \lambda\Gamma(\mu) \text{ and } \Gamma(\mu+\nu) = \Gamma(\mu) + \Gamma(\nu) \text{ almost surely}$$

(by simply noticing that in both cases the second moment of the differences between left and right-hand sides vanish). It readily follows that the law of the GFF is characterised by

this linearity relation and the fact that for each given μ in $\mathcal{M}$, $\Gamma(\mu)$ is a centred Gaussian random variable with variance $\Sigma(\mu, \mu)$.

Let us stress once again that when dealing with processes with uncountable index sets (such as $\mathcal{M}$) one has to pay close attention to the positioning of "for all $\mu \in \mathcal{M}$" or "for each given $\mu \in \mathcal{M}$" in statements such as the above. As an illustration, we would like to mention straight away that for general *random* measures $\mu \in \mathcal{M}$ (coupled to the GFF), $\Gamma(\mu)$ is not necessarily a well-defined random variable (i.e. it is not necessarily measurable), and may not make any sense at all. This contrasts with Brownian motion, where B_t can be defined for all t simultaneously, because one can choose it to be a continuous function). So, the GFF cannot be viewed as a random *function* from $\mathcal{M}$ into $\mathbb{R}$: this would mean that one is able to define $\Gamma(\mu)$ for all $\mu \in \mathcal{M}$ "simultaneously", which turns out not to be possible. On the other hand, we will be able to define it simultaneously for all μ in certain nice subsets of $\mathcal{M}$ (see the next lecture!)

3.4 Other boundary conditions

Suppose now that D is bounded and that H is a given harmonic function in D. Note that the function H may be unbounded near ∂D (this will typically happen when we will discuss the Markov property of the continuum GFF). However, the harmonic function is fully determined by its value on any neighbourhood of ∂D, so that one can (at least informally) think of it as the harmonic extension of "its trace on ∂D". In other words, we can view the information of a "boundary conditions" as the same information as the knowledge of the entire harmonic function. This (and of course the corresponding features of the discrete GFF) leads to the following definition:

Definition 3.3 (GFF with non-constant boundary conditions). *We say that $\hat{\Gamma}$ is a GFF in D with boundary conditions given by H if $\hat{\Gamma} = H + \Gamma$, where Γ is a Dirichlet GFF in D.*

The equation $\hat{\Gamma} = H + \Gamma$ should be understood in the sense that

$$\hat{\Gamma}(\mu) = \int H(x)\mu(dx) + \Gamma(\mu)$$

for each $\mu \in \mathcal{M}$. If H is unbounded in the neighbourhood of ∂D, one can restrict the definition to the set of measures μ in $\mathcal{M}$ with compact support in D to be on the safe side (in order to be sure that $\int H(x)\mu(dx)$ is well-defined).

After having collected all these basic facts about the Green's function, we can proceed to study the GFF as introduced in Definition 3.2. *We assume that D is a connected domain of $\mathbb{R}^d$ with regular boundary (and $D \neq \mathbb{R}^2$ when $d = 2$).*

3.5 The GFF as a random Fourier series, the GFF as a random generalised function

It is a classical fact from functional analysis that if D is bounded, then there exists an orthonormal basis $(\varphi_j)_{j\geq 1}$ of $L^2(D)$ that consists of the eigenfunctions of $-\Delta$, vanishing

on the boundary of D. We denote the associated eigenvalues by $(\lambda_j)_{j\geq 1}$.

Weyl's law tells us that the number $N(\lambda)$ of eigenvalues smaller than λ satisfies

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = c_d \mathsf{vol}(D)$$

for some finite dimension-dependent constant c_d.

Note that that (if Σ denotes the covariance of the GFF in D, as in Definition 3.2) then one has

$$\Sigma(\varphi_i, \varphi_j) = \int_{D\times D} G_D(x,y)\varphi_i(x)\varphi_j(y)\,\mathrm{d}x\mathrm{d}y = -\int \varphi_i(x)\Delta\varphi_j(x)\,dx = \lambda_i^{-1}\mathbf{1}_{i=j}.$$

In particular, this means that $(\mathcal{N}_j := \sqrt{\lambda_j}\,\Gamma(\varphi_j))_{j\geq 1}$ is a sequence of independent standard Gaussian random variables.

Conversely, one can actually start from such a family of i.i.d. centred normal variables $(\mathcal{N}_j)_{j\geq 1}$ and (re)construct the GFF. For instance, for any given L^2 function f with compact support in D, we can decompose f using the orthonormal basis $(\varphi_j)_j$ as $f(\cdot) = \sum_{j\geq 1} f_j\varphi_j(\cdot)$, where $f_j := \int_D f(x)\varphi_j(x)\,dx$ and the sum is converging in L^2. Then, we can simply define

$$\Gamma(f) := \sum_{j\geq 1} \frac{\mathcal{N}_j}{\sqrt{\lambda_j}} f_j \tag{1}$$

(this sum converges in L^2 as $\lambda_j \to \infty$ and $\sum_j f_j^2 < \infty$).

In fact, for any fixed $\mu \in \mathcal{M}$, if we set $\mu_j := \int_D \varphi_j(x)d\mu(x)$ then the defining property of $\mathcal{M}$ implies that $\sum_{j\geq 1}\lambda_j^{-1}\mu_j^2 < \infty$. Thus we can set

$$\Gamma(\mu) := \sum_{j\geq 1} \lambda_j^{-1/2}\mu_j\mathcal{N}_j,$$

where the sum also converges in L^2. The obtained process $(\Gamma(\mu))_{\mu\in\mathcal{M}}$ is easily seen to be a GFF. Conversely, we can note that if we start with a GFF Γ, we can also recover the variables $\mathcal{N}_j = \Gamma(\varphi_j)/\sqrt{\lambda_j}$.

We may wonder whether it is actually possible to use this description of the GFF to define $(\Gamma(f))_{f\in\mathcal{S}}$ for all f in some class $\mathcal{S}$ of smooth functions *simultaneously*. Recall that we formally interpret $\Gamma(f)$ as "$\int f(x)\Gamma(x)\,dx$". Then the above expressions suggest that (formally),

$$\Gamma(f) = \sum_{j\geq 1}\Big[\frac{\mathcal{N}_j}{\sqrt{\lambda_j}}\int_D f(x)\varphi_j(x)dx\Big] = \int_D f(x)\Big[\sum_{j\geq 1}\frac{\mathcal{N}_j}{\sqrt{\lambda_j}}\varphi_j(x)\Big]dx$$

so that one could try to say, in some appropriate space of generalised functions, that

$$\Gamma(\cdot) = \sum_{j\geq 1}\frac{\mathcal{N}_j}{\sqrt{\lambda_j}}\varphi_j(\cdot). \tag{2}$$

To make sense of this, let us take some $s > d/2 - 1$ and consider the set of functions $f \in L^2(D)$ that satisfy

$$\|f\|_{\mathcal{H}^s}^2 := \sum_{j \geq 1} \lambda_j^s f_j^2 < \infty$$

(note that the set $\mathcal{H}^s$ of such functions equipped with the corresponding inner product is a Hilbert space; it can alternatively be characterised, when s is even, as the space of L^2 functions in D which have $s/2$ generalised derivatives also lying in L^2). Defining $\Gamma(f)$ by (1) as above, we see, using Cauchy–Schwarz that

$$\sum_{j \geq 1} |\frac{f_j}{\sqrt{\lambda_j}} \mathcal{N}_j| \leq \Big[\sum_{j \geq 1} \lambda_j^s f_j^2\Big]^{1/2} \times \Big[\sum_{j \geq 1} \frac{\mathcal{N}_j^2}{\lambda_j^{1+s}}\Big]^{1/2}.$$

Moreover, by Weyl's law we know that $\sum_{j \geq 1} \lambda_j^{-\beta}$ is finite as soon as $\beta > d/2$. We can therefore deduce that almost surely

$$C(s) := \sum_{j \geq 1} \frac{\mathcal{N}_j^2}{\lambda_j^{1+s}} < \infty,$$

because

$$\sum_{j \geq 1} \frac{E[\mathcal{N}_j^2]}{|\lambda_j^{1+s}|} = \sum_{j \geq 1} \frac{1}{\lambda_j^{1+s}} < \infty.$$

Hence, we can control the absolute convergence of the sum in (1) for all $f \in \mathcal{H}^s$ simultaneously. In other words, we can almost surely define $\Gamma(f)$ for all $f \in \mathcal{H}^s$ at once. Furthermore, we see that for all f, g in $\mathcal{H}^s$,

$$\big|\Gamma(f) - \Gamma(g)\big| = \big|\Gamma(f - g)\big| \leq \sum_{j \geq 1} \Big|\frac{f_j - g_j}{\sqrt{\lambda_j}} \mathcal{N}_j\Big| \leq C(s)^{1/2} \times \|f - g\|_{\mathcal{H}^s}$$

(with the obvious definition for g_j). This shows that Γ can be viewed as a random generalised function, when acting on the space $\mathcal{H}^s$ of test functions, and the map $f \mapsto \Gamma(f)$ is then continuous on $\mathcal{H}^s$. In fact, this exactly says that Γ can be viewed as a random element of a Sobolev space of negative exponent.

3.6 Translation/scale/conformal invariance of the GFF

When $d > 2$, since for any connected D, $r > 0$ and x, y in D we have $G_{rD}(rx, ry) = r^{2-d} G_D(x, y)$, it follows that

$$r^{d/2-1} \Gamma_{rD} \overset{\text{(law)}}{=} \Gamma_D.$$

Moreover, for any $a \in \mathbb{R}^d$, it is clear that $G_{D+a}(x + a, y + a) = G_D(x, y)$, and so

$$\Gamma_{D+a} \overset{\text{(law)}}{=} \Gamma_D.$$

Observe here that $d = 2$ plays a special role: it is the only dimension in which the Gaussian free field is scale invariant.

In fact, when $d = 2$, the GFF has a stronger property than scale invariance, known as *conformal invariance*. More precisely, suppose that D and $\tilde{D}$ are two conformally equivalent domains in the plane (i.e. there exists an angle-preserving bijection Φ from D onto $\tilde{D}$). Then one can show that $G_D(x, y) = G_{\tilde{D}}(\Phi(x), \Phi(y))$. Hence, if the GFF were an actual function, then the law of this function would be conformally invariant. In reality, it is conformally invariant "as a generalised function", which means that for any $\mu \in \mathcal{M}$, $\Gamma_D(\mu)$ is distributed like $\tilde{\Gamma}_{\tilde{D}}(\tilde{\mu})$ (where this $\tilde{\Gamma}_{\tilde{D}}$ is a GFF in $\tilde{D}$), for $\tilde{\mu}$ the push-forward measure defined by

$$\tilde{\mu}(\Phi(A)) := \int_A \mu(dx)|\Phi'(x)|^2.$$

In other words, if Γ is a GFF in D and if we *define* for each $\tilde{\mu}$ in $\mathcal{M}_{\tilde{D}}$ the random variable

$$\tilde{\Gamma}(\tilde{\mu}) = \Gamma_D(\mu), \quad \mu(A) := \int_{\Phi(A)} \tilde{\mu}(dy)|(\Phi^{-1})'(y)|^2,$$

then $\tilde{\Gamma}$ is a GFF in $\tilde{D}$.

LECTURE 4

The Markov property

The goal of this section is to describe the continuum analogue of the Markov property for the discrete GFF. There are as usual several ways to tackle this, and we will present one route, which is possibly not the most elegant one!

Let us fix some compact subset A of $\overline{D}$, such that the boundary of $O := D \backslash A$ is regular as well. Let Γ denote a continuum GFF in D. Our goal, inspired by the corresponding results in the discrete case, is to decompose Γ into the sum of two independent processes Γ_A and Γ^A, i.e., $\Gamma(\mu) = \Gamma_A(\mu) + \Gamma^A(\mu)$ for all $\mu \in \mathcal{M}_D$, where:

- the process Γ^A is a continuum GFF in $O = D \setminus A$ (with zero boundary conditions);
- the field Γ_A should be thought of as "equal to Γ in A" and to be defined in O as the harmonic extension h_A of the "values of Γ on ∂O".

Let us first comment on why it is possible to make sense of the harmonic function h_A described above, even though Γ is not defined pointwise (so it is a priori not so clear what this "harmonic extension" should mean).

The idea is the following: if g is a (nice) function defined on ∂O, then one can define the harmonic extension F of g into O, by setting $F(z) = \int_{\partial O} \nu_{z,\partial O}(\mathrm{d}y)g(y)$ for each $z \in O$ where $\nu_{z,\partial O}$ is a measure on ∂O called "harmonic measure" seen from z. (One

way to define $\nu(z, \partial O)$ is as the law of the first point on ∂O that a (two-dimensional) Brownian motion started from z hits). With this in mind, it makes sense to define

$$h_A(z) := \Gamma(\nu_{z,\partial O}) \tag{3}$$

(it is quite straightforward to check that $\nu_{z,\partial O} \in \mathcal{M}_D$ so that this makes sense for the GFF Γ). Note that this definition immediately implies that $(h_A(z))_{z\in O}$ is a centred Gaussian process. Moreover, setting $H_{D,O}(x, y) = G_D(x, y) - G_O(x, y)$ for $x, y \in D$ (with $G_O(x, y) \equiv 0$ outside of O) we have

$$\mathbb{E}[h_A(z)h_A(z')] = \mathbb{E}[\Gamma(\nu_{z,\partial O})\Gamma(\nu_{z',\partial O})] = \int_{\partial O\times\partial O} G_D(x, y)\nu_{z,\partial O}(\mathrm{d}x)\nu_{z',\partial O}(\mathrm{d}y)$$
$$= \int_{\partial O\times\partial O} H_{D,O}(x, y)\nu_{z,\partial O}(\mathrm{d}x)\nu_{z',\partial O}(\mathrm{d}y) + \int_{\partial O\times\partial O} G_O(x, y)\nu_{z,\partial O}(\mathrm{d}x)\nu_{z',\partial O}(\mathrm{d}y),$$

and since $G_O(x, y)$ is 0 on ∂O and $H_{D,O}(x, y) = H_x(y) - h^D_x(y) - H_x(y) + h^O_x(y)$ defines a harmonic function in O when one (either) of the arguments is fixed, we obtain that

$$\mathbb{E}[h_A(z)h_A(z')] = H_{D,O}(z, z'). \tag{4}$$

One would now like to say that h_A can actually be realised as a harmonic function, and indeed this turns out to be the case (using the definition of $\nu_{z,\partial O}$ and the fact that any continuous function satisfying the mean value property - that its average on any circle is equal to its value at the centre - is harmonic).

Lemma 4.1 (Defining the harmonic extension)**.** *There exists a continuous version of the process $(h_A(z))_{z\in O}$, and this continuous version is a harmonic function in O.*

It is important to note that when $\partial O \subset D$ is a deterministic set, then the harmonic function h_A will not be bounded in any neighbourhood of ∂O. It will typically start oscillating pretty wildly: this corresponds to the fact that Γ is not defined pointwise on ∂O.

The next step in our quest for the Markov property is to define the random variable $\Gamma_A(\mu)$ for $\mu \in \mathcal{M}$. Remembering the initial goal, that Γ_A should be equal to Γ on A and equal to h_A on O, we might first try to define it as

$$\Gamma(\mu\mathbf{1}_A) + \int_O h_A(x)\mu(dx).$$

However, care is required, because it is not clear whether the integral of h_A is well-defined in the usual sense. As we have already mentioned, the function h_A will not be bounded near ∂A and indeed, in general, it might happen that $\int |h_A|\mu(dx) = \infty$ for some measure μ. One way around this is to instead define another measure $\nu_{\mu,\partial O}$ which is the integral with respect to $\mu(dx)1_O$ of $\nu_{x,\partial O}$, and to then define

$$\Gamma_A(\mu) := \Gamma(\mu\mathbf{1}_A) + \Gamma(\nu_{\mu,\partial O}), \tag{5}$$

(which seems a good alternative to $\Gamma(\mu\mathbf{1}_A)+\int_O h_A(x)\mu(dx)$ given that $h_A(z) = \Gamma(\nu_{z,\partial O})$).

To justify this definition, we need the following.

Lemma 4.2. *For each $\mu \in \mathcal{D}$, we have that $\nu_{\mu,\partial O} \in \mathcal{M}_D$.*

Proof. For this, first assume that μ is non-negative, and note that by the same reasoning giving (4):

$$\begin{aligned}
&\int_{\partial O\times\partial O} \nu_{\mu,\partial O}(\mathrm{d}z)\nu_{\mu,\partial O}(\mathrm{d}z')G_D(z,z') \\
&= \int_{O\times O} \mu(\mathrm{d}x)\mu(\mathrm{d}x')\,[\int_{\partial O\times\partial O} \nu_{x,\partial O}(\mathrm{d}z)\nu_{x',\partial O}(\mathrm{d}z')G_D(z,z')] \\
&= \int_{O\times O} \mu(dx)\mu(dx')H_{D,O}(x,x') \le \int_{D\times D} \mu(dx)\mu(dx')G_D(x,x') < \infty.
\end{aligned}$$

The justification for general $\mu \in \mathcal{M}$ follows by splitting μ into positive and negative parts. □

Since $\mu\mathbf{1}_A$ and $\nu_{\mu,\partial O}$ are deterministic measures (i.e., deterministic functions of μ), it follows that the process Γ_A defined by (5) is a centred Gaussian process. It also fulfils the properties that we are looking for: it is nothing else than Γ when restricted to measures supported on A, and when restricted to measures μ with *compact support* in O, it is exactly the integral $\int h_A(z)d\mu(z)$. Similarly to (4) one easily checks that

$$E[\Gamma_A(\mu)\Gamma_A(\mu')] = \int_{D\times D} H_{D,O}(x,y)\mu(\mathrm{d}x)\mu'(\mathrm{d}y) = \int_{D\times D} (G_D(x,y)-G_O(x,y))\mu(\mathrm{d}x)\mu'(\mathrm{d}y)$$

for $\mu,\mu' \in \mathcal{M}_D$. This is also exactly what we want because if we finally define

$$\Gamma^A := \Gamma - \Gamma_A,$$

then *if* we can show that the processes Γ^A and Γ_A are independent (we will do this shortly), then the covariance function of Γ^A must be given by the difference between that of Γ and that of Γ_A. That is, for any $\mu \in \mathcal{M}$,

$$E[\Gamma^A(\mu)^2] = \iint d\mu(x)d\mu(y)G_O(x,y),$$

i.e., the process Γ^A is a GFF in O.

Finally, to show that $(\Gamma_A(\mu))_{\mu\in\mathcal{M}}$ and $(\Gamma^A(\mu))_{\mu\in\mathcal{M}}$ are independent, observe that for all ν supported in A and all μ supported in O, we have

$$\begin{aligned}
\mathbb{E}[\Gamma(\nu)\Gamma_A(\mu)] &= \int_{A\times O} d\nu(x)d\nu_{\mu,\partial O}(y)G_D(x,y) \\
&= \int_{A\times O} d\nu(x)d\mu(y)G_D(x,y) = E[\Gamma(\nu)\Gamma(\mu)].
\end{aligned}$$

That is, $\mathbb{E}[\Gamma(\nu)\Gamma^A(\mu)]$ for all such ν,μ. This implies the independence between $\Gamma(\nu)$ and $\Gamma^A(\mu)$. On the other hand, the definition of Γ_A means that it is measurable with respect

to $\{\Gamma(\nu) : \nu \text{ supported in } A\}$. Thus we obtain the independence between $\Gamma^A(\mu)$ and Γ_A, for all μ supported in O. Since Γ^A vanishes on all measures supported in A, we get the desired independence of Γ^A and Γ_A.

Let us summarise the above discussion with the following proposition.

Proposition 4.3 (Markov property of the continuum GFF). *Let D and A satisfy the assumptions stated at the beginning of this section, and Γ be a GFF in D. Then defining*

$$\Gamma_A(\mu) = \Gamma(\mu \mathbf{1}_A) + \Gamma(\nu_{\mu,\partial O}); \quad \Gamma^A(\mu) = \Gamma(\mu) - \Gamma_A(\mu)$$

for all $\mu \in \mathcal{M}$ as above, one has that:

- *Γ^A and Γ_A are independent Gaussian processes;*
- *Γ^A has the law of a Gaussian free field in O;*
- *there exists a version of Γ_A such that $\Gamma_A|_O$ is almost surely equal to a harmonic function h_A in O.*

Remark 4.4. *One can also derive the Markov property using the definition of the GFF as the limit of a sum of weighted eigenfunctions - as discussed at the start of this lecture. Roughly speaking, it is a consequence of the fact that the space $\mathcal{H}^1$ admits an orthogonal decomposition into the space of functions that are harmonic in O, and the space of functions that are supported in O.*

Exercise 8. *Let $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ be the unit disc of $\mathbb{R}^2$ and for $r < 1$, let ρ_r denote uniform measure on the circle centered at 0 with radius r. If Γ is a Dirichlet GFF in D, show that the process $(\Gamma(\rho_{e^{-t}}))_{t\geq 0}$ is well defined and that its law is that of a linear Brownian motion.*