EXPLORING THE PHASE DIAGRAM OF OF COMPLEX GMC

HUBERT LACOIN

Abstract. Notes for a course at CIMAT.

2010 *Mathematics Subject Classification: 60F99, 60G15, 82B99.*
Keywords: Random distributions, log*-correlated fields, Gaussian Multiplicative Chaos.*

Contents

The aim of these notes is to provide an introduction to complex Gaussian multiplicative chaos. GMC is formally defined by taking the exponential of a Gaussian log-correlated field, or in other words, which corresponds to the expression

$$M^\gamma(\mathrm{d}x) := e^{\gamma X(x)}\mathrm{d}x, \tag{0.1}$$

where $\gamma \in \mathbb{C}$ and X is a Gaussian field indexed by $\mathbb{R}^d$ whose covariance kernel has the form

$$K(x,y) = \log\frac{1}{|x-y|} + L(x,y)$$

where L is a continuous function.

Since $K(x,x)=\infty$ there is no way to define $(X(x))_{x\in\mathbb{R}^d}$ and we can only make sense of the field X after integrating it against test functions. Hence giving a meaning to (0.1) is not a priori an easy task.

The problem of providing a mathematical construction of M^γ that gives a meaning to (0.1) was first considered by Kahane in [8] in the case where $\gamma\in\mathbb{R}$, we refer to [20, 21] for reviews on the subject. The case of $\gamma\in\mathbb{C}$ was considered only more recently, see for instance [5, 6, 7, 12, 10, 14, 15] and references therein.

One of the main motivation that drove the research on the subject recently is the connection with theoretical physics (Quantum field theory [3, 9]) and statistical mechanics (connection with the Coulomb gaz via the Sine-Gordon representation [16], with the Ising model [7] etc...). Some other application exists for instance GMC was first suggested as a model for three dimensional turbulence [19], we refer to [21] for more details on applications.

The standard procedure to define the GMC is to use a sequence of approximation of the field X, consider the exponential of the approximation and then pass to the limit. In these note we are going to restrict to the case where the covariance kernel of the field is a martingale approximation, $(X_t)_{t\geqslant 0}$.

1. Log-correlated fields

A very brief recall. Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a set $\mathcal{T}$ and random process indexed by $\mathcal{T}$ is a collection of random variables $(Z_t)_{t\in\mathcal{T}}$. We say that a process Z' is *a modification of* Z if

$$\forall t\in\mathcal{T},\quad \mathbb{P}[\,Z_t=Z'_t]=1.$$

1.1. Defining a smooth Gaussian field on $\mathbb{R}^d$ and integrating it against test functions.

Let $J(x,y):\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ be an Hölder continuous positive definite kernel. *Positive definite* means that for every k and $(x_i)_{i=1}^k\in(\mathbb{R}^d)^k$ we have

$$\sum_{i,j=1}^{k}\lambda_i\lambda_j J(x_i,x_j)\geqslant 0 \tag{1.1}$$

The reader can check that (1.1) is equivalent to

$$\int_{\mathbb{R}^{2d}}\rho(x)\rho(y)J(x,y)\mathrm{d}x\mathrm{d}y\geqslant 0. \tag{1.2}$$

for every $\rho\in C_c(\mathbb{R}^d)$ the set of continuous functions with compact support. Using Kolmogorov extension theorem, we can define on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ a centered Gaussian field $(Y(x))_{x\in\mathbb{R}^d}$ indexed by $\mathbb{R}^d$ with covariance J. Using Kolmogorov-Chensov criterion, since J is Hölder continuous, there exists a modification of Y which is such that every realization of the random function $x\mapsto Y(x)$ is continuous in $\mathbb{R}^d$ (in these notes we always consider the most regular modifications of a process when they exist).

Letting $C_c(\mathbb{R}^d)$, we can define

$$\langle Y,\rho\rangle:=\int_{\mathbb{R}^d}Y(x)\rho(x)\mathrm{d}x. \tag{1.3}$$

By the mean of approximation by Riemann sums, one can check that $\langle Y, \rho \rangle_{\rho \in C_c(\mathbb{R}^d)}$ is a centered Gaussian process and that its covariance is given by

$$\widehat{J}(\rho, \rho') := \mathbb{E}\left[\langle Y, \rho\rangle\langle Y, \rho'\rangle\right] = \int_{\mathbb{R}^{2d}} \rho(x)\rho'(y)J(x,y)\mathrm{d}x\mathrm{d}y. \tag{1.4}$$

1.2. **Defining a Gaussian field with a** log**-correlated covariance.** Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ being positive definite kernel that can be written in the form

$$K(x,y) = \log \frac{1}{|x-y|} + L(x,y) \tag{1.5}$$

where $|\cdot|$ - when applied to an element of $\mathbb{R}^d$ - denotes the Euclidean distance, by convention $\log(1/0) = \infty$, and L is a continuous real valued function. The interpretation of positive definite here is that given in (1.2) that is

$$\forall \rho \in C_c(\mathbb{R}^d), \qquad \int_{\mathbb{R}^{2d}} \rho(x)\rho(y)K(x,y)\mathrm{d}x\mathrm{d}y \geqslant 0. \tag{1.6}$$

Since K diverges on the diagonal, it is not possible to define a centered Gaussian field X indexed by $\mathbb{R}^d$ with covariance function K. However, using (1.4), we can define a process indexed by $C_c(\mathbb{R}^d)$ which correspond to how such a field would integrate against test functions. We define $\widehat{K}$, a bilinear form on $C_c(\mathbb{R}^d)$ (the set of compactly supported continuous functions) by

$$\widehat{K}(\rho, \rho') = \int_{\mathbb{R}^{2d}} K(x,y)\rho(x)\rho'(y)\mathrm{d}x\mathrm{d}y. \tag{1.7}$$

Since $\widehat{K}$ is positive definite (in the sense given by (1.1)), one can define a centered Gaussian process $\langle X, \rho \rangle_{\rho \in C_c(\mathbb{R}^d)}$ with covariance $\widehat{K}$.

1.3. **A** log**-correlated field as a distribution.** So far we have only defined $\langle X, \rho \rangle_{\rho \in C_c(\mathbb{R}^d)}$ as a collection of random variables. Given $\rho, \rho' \in C_c(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{R}$ we have almost surely

$$\langle X, \alpha\rho + \beta\rho' \rangle = \alpha\langle X, \rho\rangle + \beta\langle X, \rho'\rangle, \tag{1.8}$$

(the difference between the l.h.s. and the r.h.s. is a centered Gaussian with zero variance). However, this does not imply that $\rho \mapsto \langle X, \rho \rangle$ is a linear application since in (1.8) holds with probability one only for a fixed value of $\rho, \rho', \beta, \beta'$.

It is possible (we do not provide a proof in these notes) that there exist a modification of the process X which takes value in the space of distribution (in the sense of Schwartz). For this modification, (1.8) holds *for every* realization of X and every $\rho, \rho' \in C_c^\infty(\mathbb{R}^d)$.

1.4. **The case of star-scale invariant kernels.** Let $\kappa \in C_c^\infty(\mathbb{R}^d)$ be a smooth compactly supported function which is such that

(a) κ is radial, meaning there exists $\tilde{\kappa} : \mathbb{R}_+ \to \mathbb{R}$ such that $\forall x \in \mathbb{R}^d$, $\kappa(x) = \tilde{\kappa}(|x|)$
(b) $\tilde{\kappa}(0) = 1$, $\tilde{\kappa}(u) = 0$ for $u \geqslant 1$ and $\tilde{\kappa}(u) \geqslant 0$ for $u \in (0,1)$.
(c) The kernel $(x,y) \mapsto \kappa(x-y)$ is definite positive. We have for every $\rho \in C_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^{2d}} \kappa(x-y)\rho(x)\rho(y) \geqslant 0.$$

Remark 1.1. *One possibility to define κ as above is to start with a non-negative radial function $\theta \in C_c^\infty(\mathbb{R}^d)$ whose support is included in $B(0,1/2)$ ($B(x,r)$ denotes the Euclidean ball of center x and radius r) and such that $\int \theta^2(x)\mathrm{d}x = 1$ and define $\kappa(x) = \theta * \theta(x)$.*

We define

$$
\begin{aligned}
Q_t(x,y) &:= \kappa(e^t(x-y)),\\
K_t(x,y) &:= \int_0^t Q_s(x,y)\mathrm{d}s, \\
K(x,y) &:= \int_0^t Q_s(x,y)\mathrm{d}s.
\end{aligned} \tag{1.9}
$$

For $z \neq 0$, regrouping integrals and setting $u = t - \log(1/|z|)$ we have

$$
\overline{L}(z) := K(0,z) - \log(1/|z|) = \int_0^\infty \kappa(e^t z)\mathrm{d}t - \int_0^{\log(\ 1/|z|)} 1\mathrm{d}t
$$

$$
= \int_0^{\log(1/|z|)} [\widetilde{\kappa}(e^t|z|) - 1]\mathrm{d}t = \int_0^{\log(1/|z|)} [\widetilde{\kappa}(e^{-u}) - 1]\mathrm{d}u. \tag{1.10}
$$

Sending $|z|$ to zero we see that $\overline{L}(z)$ can be continuously extended at 0. Since we have

$$
K(x,y) = \log\frac{1}{|x-y|} + \overline{L}(x-y),
$$

the kernel K and log-diverging in the sense of (1.5). From assumption (c) it is positive definite (1.6).

In these notes, we are going to restrict ourselves to the study of GMC for star-scale invariant kernel. This may seem like a big loss of generality, but it turns out that under some small regularity assumption on the function L, kernels of the form (1.5) can be shown to *locally* have *an almost star-scale invariant*. We refer to [6] for the definition of these notion, but the essential point is that this observation allows to extend results proved for star-scale invariant kernels to all sufficiently regular kernels of the form (1.5) (see for instance [11, Appendix C] or [1]).

1.5. **Martingale approximation of a** log**-correlated field.** We can define a centered Gaussian process $(X_t(x))_{t\geqslant 0, x\in\mathbb{R}^d}$, whose covariance is given by

$$
\mathbb{E}\left[X_s(x)X_t(y)\right] = K_{s\wedge t}(x,y). \tag{1.11}
$$

Such a process can be defined since $K_{s\wedge t}(x,y)$ defines a positive definite kernel (in the sense of (1.1)) on $(\mathbb{R}_+ \times \mathbb{R}^d)^2$. Moreover, since the function is Hölder continuous, we can assume (considering an adequate modification of the process) that $(X_t(x))_{t\geqslant 0, x\in\mathbb{R}^d}$ is continuous in space and time. Note that for any $x \in \mathbb{R}^d$ we have

$$
\mathbb{E}\left[X_s(x)X_t(x)\right] = K_{s\wedge t}(x,x) = s \wedge t
$$

Hence (since we have continuity) $(X_t(x))_{t\geqslant 0}$is a standard Brownian motion. Furthermore, if one sets

$$
\mathcal{F}_t := \sigma\left(X_s(x),\ s \in [0,t],\ x \in \mathbb{R}^d\right), \tag{1.12}
$$

the process $(X_t(x))_{t\geqslant 0}$ is a margingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Indeed if $0 \leqslant u \leqslant s \leqslant t$ and $x, y \in \mathbb{R}^d$ we have

$$
\mathbb{E}[(X_t(x) - X_s(x))\, Y_u(y)] = K_{t\wedge u}(x,y) - K_{s\wedge u}(x,y) = 0, \tag{1.13}
$$

so that $X_t(x) - X_s(x)$ is independent of $Y_u(y)$ for $u \in [0, s]$ and $y \in \mathbb{R}^d$ and hence of $\mathcal{F}_s$. The bracket between $(X_t(x))$ and $(X_t(y))$ is simply obtained by observing that

$$\langle X_\cdot(x), X_\cdot(y) \rangle_t = \mathbb{E}[X_t(x)X_t(y)] = K_t(x, y) = \int_0^t Q_s(x, y)\mathrm{d}s \tag{1.14}$$

(we use the same symbol for duality bracket and martingale bracket but the meaning can clearly be inferred from the context). Ultimately, the field $(X_t(\cdot))_{t \geqslant 0}$ converges in the limit to a field with covariance K in the following sense. If one sets

$$\langle X_t, \rho \rangle := \int X_t(x)\rho(x)\mathrm{d}x$$

for $\rho \in C_c(\mathbb{R}^d)$ then one can define

$$\langle X, \rho \rangle := \lim_{t \to \infty} \langle X_t, \rho \rangle, \tag{1.15}$$

where the convergence holds in L^2 and almost surely $((\langle X_t, \rho \rangle)_{t \geqslant 0}$ is a continuous martingale which is bounded in L^2). The process X defined by (1.15) has covariance $\widehat{K}$ defined by (1.7). It can additionally be shown that with probability 1, the convergence $\lim_{t \to \infty} X_t = X$ holds in the space of distribution.

2. Exponentiating a log-correlated field

2.1. Definition of the martingale M_t^γ. Let us assume that $(X_t(x))_{t \geqslant 0, x \in \mathbb{R}^d}$ is a continuous field with covariance (1.11) and let X denote the limit of X_t in the sense given in (1.15). Given $\gamma \in \mathbb{C}$, in an effort to define a random distribution which corresponds to the formal definition

$$M^\gamma(\mathrm{d}x) = e^{\gamma X}\mathrm{d}x, \tag{2.1}$$

we define for every $f \in C_c(\mathbb{R}^d)$

$$M_t^\gamma(f) := \int_{\mathbb{R}^d} f(x) e^{\gamma X_t(x) - \frac{\gamma^2 t}{2}}\mathrm{d}x. \tag{2.2}$$

Since $X_t(x)$ is a Gaussian variable with variance t we have (using Fubini)

$$\mathbb{E}\left[M_t^\gamma(f)\right] := \int_{\mathbb{R}^d} f(x)\mathbb{E}\left[e^{\gamma X_t(x) - \frac{\gamma^2 t}{2}}\right]\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\mathrm{d}x. \tag{2.3}$$

The term $\gamma^2 t/2$ in the exponential in (2.1) is a normalizing factor which is present to make the expectation of M_t^γ independent of t. It also allows to achieve the following property

Proposition 2.1. *Given $\gamma \in \mathbb{C}$ and $f \in C_c(\mathbb{R}^d)$ the process $(M_t^\gamma(f))_{t \geqslant 0}$ is an $(\mathcal{F}_t)$-martingale.*

Proof. Indeed we have (using Fubini)

$$\begin{aligned}\mathbb{E}\left[M_t^\gamma(f) \mid \mathcal{F}_s\right] &= \int_{\mathbb{R}^d} f(x)\mathbb{E}\left[e^{\gamma X_t(x) - \frac{\gamma^2 t}{2}} \mid \mathcal{F}_s\right]\mathrm{d}x \\ &= \int_{\mathbb{R}^d} f(x)^{\gamma X_s(x) - \frac{\gamma^2 s}{2}}\mathbb{E}\left[e^{\gamma(X_t(x) - X_s(x))\frac{\gamma^2(t-s)}{2}} \mid \mathcal{F}_s\right]\mathrm{d}x.\end{aligned} \tag{2.4}$$

Now since $X_t(x) - X_s(x)$ is a Gaussian of variance $t - s$ which is independent of $\mathcal{F}_s$ (cf. (1.13)) we have

$$\mathbb{E}\left[M_t^\gamma(f) \mid \mathcal{F}_s\right] = \int_{\mathbb{R}^d} f(x)^{\gamma X_s(x) - \frac{\gamma^2 s}{2}}\mathrm{d}x = M_s^\gamma(f). \tag{2.5}$$

□

A promising candidate to rigorously define M^γ in (2.1) would be $\lim_{t\to\infty} M_t^\gamma(f)$. The existence and non-triviality of this limit can be established in one proves that $(M_t^\gamma(f))_{t\geqslant 0}$ is uniformly integrable. Indeed in that case $\lim_{t\to\infty} M_t^\gamma(f) = M_\infty^\gamma(f)$ exists almost surely and one has

$$\mathbb{E}\left[M_\infty^\gamma(f)\right] = \int_{\mathbb{R}^d} f(x)\mathrm{d}x. \tag{2.6}$$

2.2. **The L^2 phase.** The easiest way to prove uniform integrability is to show boundedness in L^2 so as an introduction let us perform this computation and see where it leeds us.

Proposition 2.2. *Given $\gamma \in \mathbb{C}$ with $|\gamma|^2 < d$ and $f \in C_c(\mathbb{R}^d)$ the martingale $(M_t^\gamma(f))_{t\geqslant 0}$ is bounded in L^2. In particular this implies that the limit*

$$M_\infty^\gamma(f) := \lim_{t\to\infty} M_t^\gamma(f)$$

exists almost surely, that the convergence also holds in L^2 and that

$$\mathbb{E}[M_\infty^\gamma(f)] = \int_{\mathbb{R}^d} f(x)\mathrm{d}x$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[|M_t^\gamma(f)|^2\right] &= \int_{\mathbb{R}^{2d}} f(x)\overline{f}(y)\mathbb{E}\left[e^{\gamma X_t(x)+\overline{\gamma}X_t-\frac{(\gamma^2+\overline{\gamma}^2)t}{2}}\right]\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^{2d}} f(x)\overline{f}(y)e^{|\gamma|^2K_t(x,y)}\mathrm{d}x\mathrm{d}y \\
&\leqslant \int_{\mathbb{R}^{2d}} |f(x)||f(y)|e^{|\gamma|^2K(x,y)}\mathrm{d}x\mathrm{d}y
\end{aligned} \tag{2.7}$$

We have $e^{|\gamma|^2K(x,y)} \leqslant C|x-y|^{-|\gamma|^2}$ from (1.5) and thus the integral in the last line is finite. □

When $|\gamma| \geqslant d$, the limiting behavior of $M_t^\gamma(f)$ is harder to determine but research efforts have allowed to determine it for a large set of values of γ.

2.3. **Convergence as a distribution.** Note that since $f \mapsto M_t^\gamma(f)$ is a linear application we have for any fixed $\alpha, \beta \in \mathbb{R}$ and $f, g \in C_c(\mathbb{R}^d)$ we almost surely have

$$M_\infty^\gamma(\alpha f + \beta g) = \alpha M_\infty^\gamma(f) + \beta M_\infty^\gamma(g). \tag{2.8}$$

However as it was the case in (1.8) the above is not sufficient to say that M_∞^γ defines a linear application on $C_c(\mathbb{R}^d)$. In fact the result (2.2) does not ensure that $M_\infty^\gamma(f)$ converges simultaneously for every f.

It is in fact possible to show that $M_t^\gamma(\cdot)$ converges in a space of distribution (namely a Sobolev space of negative index), see for instance [12, Section 6].

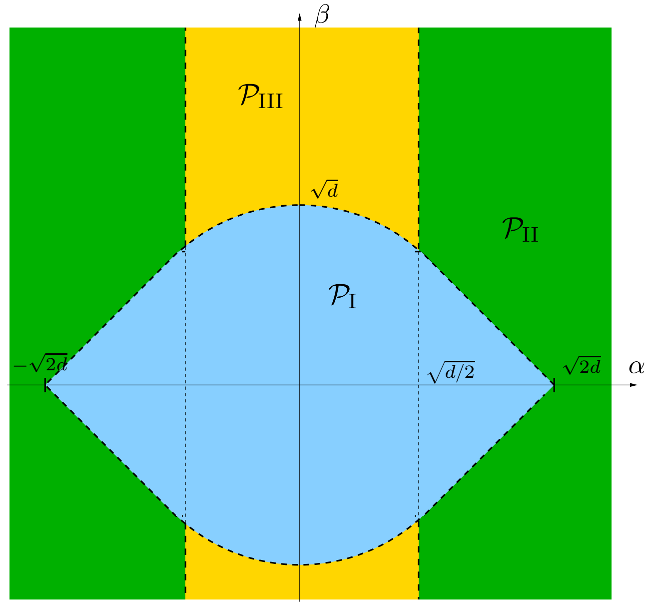

FIGURE 1. The phase diagram of the complex GMC in the complex plane. So far the convergence of M_t^γ has been established everywhere except for $\mathcal{P}_{\mathrm{II}}\backslash\mathbb{R}$.

3. The phase diagram for the complex GMC

In this section, we consider that $\gamma = \alpha + i\beta \in \mathbb{C}$ (with α and β being real numbers). We split the complex plane into three open regions (with overlapping boundary) each corresponding to a different asymptotic behavior for $M_t^\gamma(f)$.

$$\begin{aligned} \mathcal{P}_{\mathrm{I}} &:= \left\{\alpha + i\beta \ : \alpha^2 + \beta^2 < d\right\} \cup \left\{\alpha + i\beta \ : \ |\alpha| \in (\sqrt{d/2}, \sqrt{2d}) \ ; \ |\alpha| + |\beta| < \sqrt{2d} \ \right\}, \\ \mathcal{P}_{\mathrm{II}} &:= \left\{\alpha + i\beta \ : |\alpha| + |\beta| > \sqrt{2d} \ ; \ |\alpha| > \sqrt{d/2}\right\}, \\ \mathcal{P}_{\mathrm{III}} &:= \left\{\alpha + i\beta \ : \alpha^2 + \beta^2 > d \ ; \ |\alpha| < \sqrt{d/2}\right\}. \end{aligned} \tag{3.1}$$

For simplicity in the remainder of the paper, we are going to consider (without loss of generality) that $\alpha, \beta \geqslant 0$.

The region $\mathcal{P}_{\mathrm{I}}$ corresponds to the subcritical phase. For $\gamma \in \mathcal{P}_{\mathrm{I}}$ it has been proved [6, 12] that M_t^γ is uniformly integrable and thus converge to a limit. The region $\mathcal{P}_{\mathrm{I}}$ is the maximal open region on which Proposition 2.2 can be extended.

Theorem 3.1. *If $\gamma \in \mathcal{P}_{\mathrm{I}}$ and $f \in C_c(\mathbb{R})$ then the martingale $M_t^\gamma(f)$ is bounded in L^p for $p \in [1, \frac{\sqrt{2d}}{\alpha} \vee 2)$. In particular it is uniformly integrable and*

$$\lim_{t\to\infty} M_t^\gamma(f) = M_\infty^\gamma(f).$$

The region $\mathcal{P}_{\mathrm{II}}$ corresponds to a phase where the behavior of M_t^γ is dominated by the extreme values of X_t. A different normalization is required to obtain a limit. It is conjectured that $t^{\frac{3\gamma}{2}} e^{t\left(\frac{(\alpha-\sqrt{2d})^2-\beta^2}{2}\right)} M_t^\gamma(\cdot)$ converges in law to an *atomic distribution* (a countable weighted sum of Dirac masses). This result has been proved in the case when $\gamma \in \mathbb{R}$ - that is $\beta = 0$, and $\alpha > \sqrt{2d}$, see [18] - and also in the whole of $\mathcal{P}_{\rm II}$ for other related models [4, 17].

The region $\mathcal{P}_{\rm III}$ corresponds to yet a different asymptotic behavior of M_t^γ. In the limit, rapid the local variations of the argument of $e^{\gamma X}$, due to the term $e^{i\beta X_t(x)}$, produce a complex Gaussian white noise with a random intensity. In order not to avoid the hassle of defining such a white noise, let us state the convergence result only for a fixed function f

Theorem 3.2. *If $\gamma \in \mathcal{P}_{\rm III}$ and $f \in C_c(\mathbb{R})$ then there exists a constant Σ (which depends on d, γ and on the kernel K) such that the following convergence in law occurs*

$$\Sigma^{-1} e^{\frac{(d-|\gamma|)^2}{2}t} M_t^\gamma(f) \overset{t\to\infty}{\Longrightarrow} \sqrt{M_\infty^{2\alpha}(|f|^2)}(\mathcal{N}_1 + i\mathcal{N}_2) \tag{3.2}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are two independent standard Gaussian which are independent of X (and thus of $M_\infty^{2\alpha}$). Furthermore the following joint convergence holds in law

$$(X, \Sigma^{-1} e^{\frac{(d-|\gamma|)^2}{2}t} M_t^\gamma(f)) \overset{t\to\infty}{\Longrightarrow} (X, \sqrt{M_\infty^{2\alpha}(|f|^2)}(\mathcal{N}_1 + i\mathcal{N}_2)) \tag{3.3}$$

Remark 3.3. *In the above theorem, the convergence in the above theorem holds only in law. The equation* (3.3) *actually guarantees that convergence in probability cannot occur. Some how, local variation of X create in the limit Gaussian oscillations which are independent of X.*

Results concerning the phase boundary between two regions as well as the triple points have been proved in [13].

The aim of these notes is to give a flavor of how these results are proved, starting with the case $\gamma \in \mathbb{R}$, presenting a short proof with a presentation similar to the one given in [2]. Then we wish to explain how stochastic calculus can be used to prove convergence, in particular in the case of $\gamma \in \mathcal{P}_{\rm III}$. Aiming for simplicity, we will restrain to the case $\gamma \in i\mathbb{R}$.

4. Tool box

4.1. **Required Gaussian results.** We display two standard results which are used throughout the proof. The first is the standard Cameron-Martin formula which describes how a Gaussian process is affected by an exponential tilt.

Proposition 4.1. *Let $(Y(z))_{z\in\mathcal{Z}}$ be a centered Gaussian field indexed by a set $\mathcal{Z}$. We let H denote its covariance and $\mathbf{P}$ denote its law. Given $z_0 \in \mathcal{Z}$ let us define $\widetilde{\mathbf{P}}_{z_0}$ the probability obtained from $\mathbf{P}$ after a tilt by $Y(z_0)$ that is*

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_{z_0}}{\mathrm{d}\mathbf{P}} := e^{Y(z_0) - \frac{1}{2}H(z_0,z_0)} \tag{4.1}$$

Under $\widetilde{\mathbf{P}}_{z_0}$, Y is a Gaussian field with covariance H, and mean $\widetilde{\mathbb{E}}_{z_0}[Y(z)] = H(z, z_0)$.

Proof. By extending $\mathcal{Z}$ we can always consider that the collection of variable $(Y(z))_{z\in\mathcal{Z}}$ is closed under linear combination. Hence it is sufficient to check that for every z, $(Y(z))_{z\in\mathcal{Z}}$ is a Gaussian random variable with mean $H(z, z_0)$ and variance $H(z, z)$ (the covariance can be recovered from the variances). Using the formula for the expectation of the Gaussian $Z := i\xi Y(z) + Y(z_0)$ for which

$$\mathbf{E}[Z^2] = -\frac{\xi^2}{2} H(z, z) + \frac{1}{2} H(z_0, z_0) + i\xi H(z, z_0)$$

we have

$$\mathbf{E}_{z_0}[e^{i\xi Y(z)}] = \mathbf{E}\left[e^{i\xi Y(z)+Y(z_0)-\frac{1}{2}H(z_0,z_0)}\right] = e^{-\xi^2 H(z,z)+i\xi H(z,z_0)} \tag{4.2}$$

This is the Fourier transform of a Gaussian with the specified mean and variance.

□

The second is the classic gaussian tail bound which says that if Z is a standard Gaussian variable with variance σ^2 ($\sigma > 0$) we have

$$P[Z \geqslant u] \leqslant \frac{\sigma}{u\sqrt{2\pi}} e^{-\frac{u^2}{2}}. \tag{4.3}$$

Finally we include the value of the probability that a Brownian motion remains below an affine function ($a, b > 0$ are fixed number).

$$P[\forall t > 0,\ B_s \leqslant at + b] = 1 - e^{-2ab} \tag{4.4}$$

It can be proved by using the optional stopping theorem for $e^{2a(B_t - at)}$.

4.2. **A kernel estimates.**

Proposition 4.2. *For every $x, y \in \mathbb{R}^d$ with $|x - y| \leqslant 1$ we have*

$$\left| K_t(x,y) - \min\left(\log\left(\frac{1}{|x-y|}\right), t\right)\right| \leqslant C. \tag{4.5}$$

Proof. We only need to treat the case where $y = 0$. Now we have by construction

$$K_t(0,x) = K(0,x) - K(0, e^t x).$$

From (1.5) we can find C such that

$$\left| K(0,z) - \max\left(\log\frac{1}{|z|}, 0\right)\right| \leqslant C/2$$

from which we obtain the desired inequality

□

5. The case of $\gamma \in \mathbb{R}$

Theorem 5.1. *Let $\alpha \in [0, \sqrt{2d})$, $f \in C_c(\mathbb{R}^d)$ (real valued). Then the martingale $M_t^\alpha(f)$ is uniformly integrable*

The main idea for the proof comes from the following observation: When computing the expectation $\mathbb{E}[e^{\alpha X_t(x) - \frac{\alpha^2 t}{2}}]$ (this is what we do using Fubini to compute $\mathbb{E}\left[M_t^\gamma(f)\right]$) most of the contribution comes from $X_t(x)$ which is of order $\alpha t + O(\sqrt{t})$ (this can be seen from Cameron-Martin formula, Proposition 4.1). However, when estimating the second moment, we compute the expectation of $\mathbb{E}[e^{\alpha(X_t(x)+X_t(y)) - \alpha^2 t}]$ which is mostly carried by higher values of $X_t(x)$, for instance when $x = y$, most of the contributiom comes from values $X_t(x)$ which are of order $2\alpha t + O(\sqrt{t})$. By putting a cutoff on the value of X_t somewhere between αt and $2\alpha t$ we may reduce significantly the second moment without affecting much the expectation. To exploit this strategy, we introduce the following proposition.

Proposition 5.2. *Consider* $(Y_t)_{t\geqslant 0}$ *a collection of positive random variables such that* $\sup_{t\geqslant 0}\mathbb{E}[|Y_t|] < \infty$. *Assume that there exists* $Y_t^{(q)}$ *a sequence of approximation of* X_t, *indexed by* $q \geqslant 1$, *which satisfies:*

$$\text{(A)} \qquad \lim_{q\to\infty}\sup_{t\geqslant 0}\mathbb{E}\big[|Y_t^{(q)} - Y_t|\big] = 0\,;$$

$$\text{(B)} \qquad \sup_{t\geqslant 0}\mathbb{E}\big[(Y_t^{(q)})^2\big] < \infty \quad \textit{for every } q \geqslant 1.$$

Then $(Y_t)_{t\geqslant 0}$ *is uniformly integrable.*

Remark 5.3. *The fact that* $(Y_t)_{t\geqslant 0}$ *is bounded in* L^1 *is a consequence of (A) and (B), but taking it as an assumption helps for the presentation.*

Proof. Given $\delta > 0$, we want to find M such that for every $t \geqslant 0$

$$\mathbb{E}\big[|Y_t|\mathbf{1}_{\{|Y_t|>M\}}\big] \leqslant \delta \tag{5.1}$$

We may write, for any $M > 0$ and $t > 0$,

$$\begin{aligned}\mathbb{E}\big[|Y_t|\mathbf{1}_{\{|Y_t|>M\}}\big] &\leqslant \mathbb{E}\big[|Y_t - Y_t^{(q)}|\mathbf{1}_{\{|Y_t|>M\}}\big] + \mathbb{E}\big[|Y_t^{(q)}|\mathbf{1}_{\{|Y_t|>M\}}\big]\\ &\leqslant \mathbb{E}\big[|Y_t^{(q)} - Y_t|\big] + \mathbb{E}\big[(Y_t^{(q)})^2\big]^{1/2}\mathbb{P}\big(|Y_t| > M\big)^{1/2}\\ &\leqslant \mathbb{E}\big[|Y_t^{(q)} - Y_t|\big] + M^{-1/2}\mathbb{E}\big[(Y_t^{(q)})^2\big]^{1/2}\mathbb{E}[|Y_t|]^{1/2}.\end{aligned} \tag{5.2}$$

where we have used Cauchy–Schwarz inequality in the second inequality and Markov's inequality in the third. Using assumption we can take $q = q_0(\delta)$ which is such that

$$\sup_{t\geqslant 0}\mathbb{E}\big[|Y_t^{(q)} - Y_t|\big] \leqslant \delta/2.$$

and then take $M = 4\delta^{-1}\sup_{t\geqslant 0}\mathbb{E}\big[(Y_t^{(q_0)})^2\big]\mathbb{E}[|Y_t|]$, to conclude. □

Proof of Theorem 5.1. We assume without loss of generality that f is nonnegative (if not we consider the positive and negative part separately) and that $\int_{\mathbb{R}^d} f(x)\mathrm{d}x = 1$. Given $x \in \mathbb{R}^d$, $t \geqslant 0$ and $q \in \mathbb{R}$ we define the event

$$\mathcal{A}_{t,q}(x) := \{\forall s \in [0,t],\ X_s(x) \leqslant \sqrt{2d}s + q\}. \tag{5.3}$$

We define

$$M_t^{\alpha,q}(f) = \int_{\mathbb{R}^d} f(x)e^{\alpha X_t(x) - \frac{\alpha^2 t}{2}}\mathbf{1}_{\mathcal{A}_{t,q}(x)}\mathrm{d}x. \tag{5.4}$$

We want to apply Proposition 5.2 to the case $Y_t = M_t^{\alpha}(f)$ and $Y_t^q = M_t^{\alpha,q}(f)$. We need to check assumptions (A) and (B). Let us start with (A) we have

$$\begin{aligned}\mathbb{E}\big[|Y_t - Y_t^{(q)}|\big] &= \mathbb{E}\left[\int_{\mathbb{R}^d} f(x)e^{\alpha X_t(x) - \frac{\alpha^2 t}{2}}\mathbf{1}_{\mathcal{A}^\complement_{t,q}(x)}\mathrm{d}x\right]\\ &= \int_{\mathbb{R}^d} f(x)\mathbb{E}\left[e^{\alpha X_t(x) - \frac{\alpha^2 t}{2}}\mathbf{1}_{\mathcal{A}^\complement_{t,q}(x)}\right]\mathrm{d}x\end{aligned} \tag{5.5}$$

Using Cameron-Martin formula (Proposition 4.1), and then (4.4) we have

$$\begin{aligned}\mathbb{E}\left[e^{\alpha X_t(x) - \frac{\alpha^2 t}{2}}\mathbf{1}_{\mathcal{A}^\complement_{t,q}(x)}\right] &= \mathbb{P}\left[\exists s \in [0,t], X_s(x) + \alpha s \leqslant \sqrt{2ds} + q\right]\\ &\leqslant P[\forall s \geqslant 0, B_s \leqslant (\sqrt{2d} - \alpha) + q] = e^{-2(\sqrt{2d}-\alpha)q},\end{aligned} \tag{5.6}$$

and thus item (A) is satisified. For item (B) we have

$$\begin{aligned}\mathbb{E}\left[M_t^{\alpha,q}(f)^2\right] &= \int_{\mathbb{R}^{2d}} f(x)f(y)\mathbb{E}\left[e^{\alpha(X_t(x)+X_t(y))-\alpha^2 t}\mathbf{1}_{\mathcal{A}_{t,q}(x)\cap\mathcal{A}_{t,q}(y)}\right]\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^{2d}} e^{\alpha^2 K_t(x,y))} f(x)f(y)\widetilde{\mathbb{P}}_{x,y,t}\left[\mathcal{A}_{t,q}(x)\cap\mathcal{A}_{t,q}(y)\right]\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{5.7}$$

where

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{x,y,t}}{\mathrm{d}\mathbb{P}} = e^{\alpha(X_t(x)+X_t(y))-\frac{\alpha^2}{2}\mathbb{E}\left[(X_t(x)+X_t(y))^2\right]}$$

Now let us set $\bar{t}(x,y,t) = \max(t, \log\frac{1}{|x-y|\wedge 1})$ we have

$$\widetilde{\mathbb{P}}_{x,y,t}\left[\mathcal{A}_{t,q}(x)\cap\mathcal{A}_{t,q}(y)\right] \leqslant \widetilde{\mathbb{P}}_{x,y,t}\left[X_{\bar{t}}(x) \leqslant \sqrt{2d}\bar{t} + q\right]. \tag{5.8}$$

We assume that $|x-y| \leqslant 1$ (if no Now using Cameron-Martin formula (Proposition 4.1), $\widetilde{\mathbb{P}}_{x,y,t}$, just shifts the mean of $X_{\bar{t}}(x)$ by an amount $\alpha(\bar{t} + K_{\bar{t}}(x,y))$ so that

$$\begin{aligned}\widetilde{\mathbb{P}}_{x,y,t}\left[X_{\bar{t}}(x) \leqslant \sqrt{2d}\bar{t} + q\right] &= \mathbb{P}\left[X_{\bar{t}}(x) \leqslant \sqrt{2d}\bar{t} - \alpha(\bar{t} + K_{\bar{t}}(x,y)) + q\right] \\ &\leqslant \mathbb{P}\left[X_{\bar{t}}(x) \leqslant (\sqrt{2d} - 2\alpha)\bar{t} + q + C\right].\end{aligned} \tag{5.9}$$

Assuming that $\alpha > \sqrt{d/2}$ using the Gaussian tail bound we obtain that

$$\widetilde{\mathbb{P}}_{x,y,t}\left[\mathcal{A}_{t,q}(x)\cap\mathcal{A}_{t,q}(y)\right] \leqslant e^{-\frac{(2\alpha-\sqrt{2d})^2\bar{t}}{2}} \tag{5.10}$$

Hence we have

$$\begin{aligned}\mathbb{E}\left[M_t^{\alpha,q}(f)^2\right] &\leqslant \int_{\mathbb{R}^{2d}} e^{\left(\alpha^2 - \frac{(2\alpha-\sqrt{2d})^2}{2}\right)K_t(x,y)} f(x)f(y)\mathrm{d}x\mathrm{d}y \\ &\leqslant C\int_{\mathbb{R}^{2d}} |x-y|^{-\zeta(\alpha,d)} f(x)f(y)\mathrm{d}x\mathrm{d}y\end{aligned} \tag{5.11}$$

where $\zeta(\alpha,d) = \alpha^2 - \frac{(2\alpha-\sqrt{2d})^2}{2} = d - (\sqrt{2d}-\alpha)^2 < d$. This conclude the proof of (B) and hence of the theorem. □

6. The case of $\gamma \in i\mathbb{R}$

6.1. A central limit theorem with a random variance.

Theorem 6.1. *Let M_t be a real valued continuous martingale associated with a filtration $(\mathcal{F}_t)$, and let $v(t)$ be an increasing function such that $\lim_{t\to\infty} v(t) = \infty$. We assume that there exists a random variable Z such that the following convergence holds in probability*

$$\lim_{t\to\infty} \frac{\langle M\rangle_t}{v(t)^2} = Z \tag{6.1}$$

Then $M_t/v(t)$ converges in distribution towards a random Gaussian with variance given by Z, that is to say that for any $\mathcal{F}_\infty$ bounded measurable H we have

$$\lim_{t\to\infty} \mathbb{E}\left[He^{i\xi M_t/v(t)}\right] = \mathbb{E}\left[He^{-\frac{\xi^2 Z}{2}}\right] \tag{6.2}$$

This is equivalent to saying that for any $\mathcal{F}_\infty$ random variable Y we have the following convergence in law

$$(Y, M_t/v(t)) \Longrightarrow (Y, \sqrt{Z}\mathcal{N})$$

where $\mathcal{N}$ is a standard Gaussian which is independent of Y and Z.

Corollary 6.2. *Let W_t be a complex valued continuous martingale associated with a filtration $(\mathcal{F}_t)$, and let $v(t)$ be an increasing function such that $\lim_{t\to\infty} v(t) = \infty$. We assume that there exists a random variable Z such that the following convergence holds almost-surely*

$$\lim_{t\to\infty} \frac{\langle W\rangle_t}{v(t)^2} = 2Z \text{ and } \lim_{n\to\infty} \frac{\langle W, W\rangle_t}{v(t)^2} = 0, \tag{6.3}$$

This is equivalent to saying that for any $\mathcal{F}_\infty$ random variable Y we have the following convergence in law

$$(Y, W_t/v(t)) \Longrightarrow (Y, \sqrt{Z}\,(\mathcal{N}_1 + i\mathcal{N}_2)$$

where $\mathcal{N}_1, \mathcal{N}_2$ are independent standard Gaussians which are independent of Y and Z.

Proof. If we let $M^{(1)}$ and $M^{(2)}$ denote the real and imaginary part of W the assumption (6.3) gives us three limits (the limit $\langle W, W\rangle_t$ give one equation for the real part and one for the imaginary one) which can reads like

$$\begin{gathered}\lim_{t\to\infty} \frac{\langle M^{(i)}\rangle_t}{v(t)^2} = Z \text{ for } i \in \{1,2\},\\ \lim_{t\to\infty} \frac{\langle M^{(1)}, M^{(2)}\rangle_t}{v(t)^2} = 0.\end{gathered} \tag{6.4}$$

We need to show that for any ξ', ξ_1 and ξ_2 and Y an $\mathcal{F}_\infty$-measurable variable we have

$$\lim_{t\to\infty} \mathbb{E}\left[e^{i\xi' Y} e^{i(\xi_1 M^{(1)}_t + \xi_2 M^{(2)}_t)/v(t)}\right] = \mathbb{E}\left[e^{i\xi' Y} e^{-\frac{(\xi_1^2+\xi_2^2)Z}{2}}\right]. \tag{6.5}$$

Now setting $\widehat{M}_t := \xi_1 M^{(1)}_t + \xi_2 M^{(2)}_t$ we have

$$\langle \widehat{M}\rangle_t = \xi_1^2\langle M^{(1)}\rangle_t + \xi_2^2\langle M^{(2)}\rangle_t + \xi_1\xi_2\langle M_1 M_2\rangle_t,$$

so that

$$\lim_{t\to\infty} \frac{\langle \widehat{M}\rangle_t}{v(t)^2} = (\xi_1^2 + \xi_2^2)Z := \widehat{Z}.$$

Hence applying (6.2) for $\widehat{M}$ with $\xi = 1$ and $H = e^{i\xi' Y}$ we obtain the desired limit (6.5) □

Proof of Theorem 6.1. We need to show that for any H bounded and $\mathcal{F}_\infty$-measurable and $\xi \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[H\left(e^{i\xi\frac{M_t}{v(t)}} - e^{-\frac{\xi^2 Z}{2}}\right)\right] = 0. \tag{6.6}$$

We first assume that the collection of variables $v(t)^{-2}\langle W\rangle_t$ is uniformly essentially bounded, that is, that there exists M such that for every $t \geqslant 0$

$$\mathbb{P}\left[v(t)^{-2}\langle M\rangle_t > A\right] = 0 \tag{6.7}$$

Note that this implies also that $\mathbb{P}[Z \geqslant A] = 0$. We assume, to simplify notation that $\xi = 1$ (this entails no loss of generality). We set $H_s := \mathbb{E}[H \mid \mathcal{F}_s]$ and $Z_s := \mathbb{E}[Z \mid \mathcal{F}_s]$. We have for any $0 \leqslant s \leqslant t$

$$
\begin{aligned}
&\mathbb{E}\left[H\left(e^{i\frac{M_t}{v(t)}}-e^{-\frac{Z}{2}}\right)\right]\\
&=\mathbb{E}\left[H(e^{-\frac{Z}{2}}-e^{-\frac{Z_s}{2}})\right]+\mathbb{E}\left[(H-H_s)\left(e^{i\frac{M_t}{v(t)}}-e^{-\frac{Z_s}{2}}\right)\right]+\mathbb{E}\left[H_s\left(e^{i\frac{M_t}{v(t)}}-e^{-\frac{Z_s}{2}}\right)\right]\\
&=: E_1(s,t)+E_2(s,t)+E_3(s,t). \qquad (6.8)
\end{aligned}
$$

We prove the convergence (6.6) by showing that for $i=1,2,3$

$$\lim_{s\to\infty}\limsup_{t\to\infty}|E_i(s,t)|=0. \qquad (6.9)$$

Using the fact that $z\mapsto e^z$ is 1-Lipshitz (first line) and has modulus bounded by 1 (second line) in $\{z\in\mathbb{C}\ :\ \mathfrak{Re}(z)\leqslant 0\}$ we have

$$
\begin{aligned}
|E_1(s,t)| &\leqslant \mathbb{E}\left[|H|\left|e^{-\frac{Z}{2}}-e^{-\frac{Z_s}{2}}\right|\right]\leqslant\frac{\|H\|_\infty}{2}\mathbb{E}\left[|Z-Z_s|\right],\\
|E_2(s,t)| &\leqslant \mathbb{E}\left[|H-H_s|\left|e^{i\frac{M_t}{v(t)}}-e^{-\frac{Z}{2}}\right|\right]\leqslant 2\mathbb{E}\left[|H-H_s|\right].
\end{aligned} \qquad (6.10)
$$

Since H_t and Z_t converge respectively to H and Z in L^1, (6.9) holds for $i=1,2$. For $i=3$, we observe that for fixed t the process

$$\left(e^{\frac{iM_u}{v(t)}+\frac{\langle M\rangle_u}{v(t)^2}}\right)_{u\geqslant 0}$$

is a martingale. Hence we have

$$\mathbb{E}\left[e^{\frac{iM_t}{v(t)}+\frac{\langle M\rangle_t}{v(t)^2}}\mid\mathcal{F}_s\right]=e^{\frac{iM_s}{v(t)}+\frac{\langle M\rangle_s}{v(t)^2}}. \qquad (6.11)$$

Using the short-hand notation $U_{[a,b]}=U_b-U_a$ we have

$$\mathbb{E}\left[e^{\frac{iM_{[s,t]}}{v(t)}+\frac{\langle M\rangle_{[s,t]}}{v(t)^2}}\mid\mathcal{F}_s\right]=1. \qquad (6.12)$$

Multiplying by $H_se^{-\frac{Z_s}{2}}$ and taking expectation we obtain that

$$\mathbb{E}\left[H_se^{-\frac{Z_s}{2}}\right]=\mathbb{E}\left[H_se^{-\frac{Z_s}{2}}e^{\frac{iM_{[s,t]}}{v(t)}+\frac{\langle M\rangle_{[s,t]}}{v(t)^2}}\right] \qquad (6.13)$$

Hence we have (using that $\|H_s\|_\infty\leqslant\|H\|_\infty$)

$$
\begin{aligned}
|E_3(s,t)| &= \left|\mathbb{E}\left[H_se^{\frac{iM_{[s,t]}}{v(t)}}\left(e^{\frac{iM_s}{v(t)}}-e^{-\frac{Z_s}{2}+\frac{\langle M\rangle_{[s,t]}}{v(t)^2}}\right)\right]\right|\\
&\leqslant \mathbb{E}\left[|H_s|\left|e^{\frac{iM_s}{v(t)}}-e^{-\frac{Z_s}{2}+\frac{\langle M\rangle_{[s,t]}}{v(t)^2}}\right|\right] \qquad (6.14)
\end{aligned}
$$

Taking the limit when t goes to infinity (using dominated convergence which is not a problem due to (6.7)). We have

$$\limsup_{t\to\infty}|E_3(s,t)|=\mathbb{E}\left[|H_s|\left|1-e^{\frac{Z-Z_s}{2}}\right|\right]. \qquad (6.15)$$

Using dominated convergence again (everything is essentially bounded)

$$\lim_{s\to\infty}\limsup_{t\to\infty}|E_3(s,t)|=0.$$

To remove the boundedness assumption, we need a truncation proceedure to make the Given $A>0$ we set

$$T(A,t):=\inf\{s\ :\ \langle M\rangle_s=Av(t)^2\}\quad\text{ and }\quad M^{(A,t)}_s:=M_{s\wedge T}.$$

$(M^{(A,t)}_s)_{s\geqslant 0}$ is a martingale and we have we have

$$\lim_{t\to\infty}v(t)^{-2}\langle M^{(A,t)}\rangle_t=Z\wedge A. \tag{6.16}$$

Repeating the previous computation with M replaced by $M^{(A,t)}$, we obtain for every $A>0$

$$\lim_{t\to\infty}\mathbb{E}\left[H\left(e^{i\frac{\xi M^{(A,t)}_t}{v(t)}}-e^{-\frac{\xi^2 Z\wedge A}{2}}\right)\right]=0 \tag{6.17}$$

Letting A to infinity we have

$$\lim_{A\to\infty}\sup_{t\geqslant 0}\mathbb{P}\left[M^{(A,t)}_t=M_t;Z\wedge A=A\right]=1, \tag{6.18}$$

we conclude that

$$\lim_{t\to\infty}\mathbb{E}\left[H\left(e^{i\frac{\xi M_t}{v(t)}}-e^{-\frac{\xi^2 Z}{2}}\right)\right]=0. \tag{6.19}$$

□

Let us consider

$$M^{i\beta}_t(f)=\int_{\mathbb{R}^d}f(x)e^{i\beta X_t(x)+\frac{\beta^2 t}{2}}\mathrm{d}x \tag{6.20}$$

We want to show that $M^{i\beta}_t(f)$ converges to a standard complex Gaussian. Hence from Corollary 6.2, setting $W_t:=M^{i\beta}_t(f)$ for better readability, we need to prove that

$$\lim_{t\to\infty}\langle\overline{W},W\rangle_t=\ \text{ and }\ \lim_{t\to\infty}\langle W,W\rangle_t=0. \tag{6.21}$$

We have

$$dW_t=i\beta\int_{\mathbb{R}^d}f(x)e^{\frac{\beta^2 t}{2}}e^{i\beta X_t(x)}\left(dX_t(x)\right)\mathrm{d}x$$

Hence computing the quadratic variation we have

$$\begin{aligned}d\langle W\rangle_t&=\beta^2\int_{\mathbb{R}^{2d}}f(x)f(y)e^{\beta^2 t}e^{i\beta(X_t(x)-X_t(y))}d\langle X_\cdot(x),X_\cdot(y)\rangle\mathrm{d}x\mathrm{d}y\\&=\beta^2\left(\int_{\mathbb{R}^{2d}}Q_t(x,y)f(x)f(y)e^{\beta^2 t}e^{i\beta(X_t(x)-X_t(y))}d\langle X_\cdot(x),X_\cdot(y)\rangle\mathrm{d}x\mathrm{d}y\right)dt\\&=:\beta^2 A_t\end{aligned} \tag{6.22}$$

and similarly

$$\begin{aligned}d\langle W,W\rangle_t&=\beta^2\left(\int_{\mathbb{R}^{2d}}Q_t(x,y)f(x)f(y)e^{\beta^2 t}e^{i\beta(X_t(x)+X_t(y))}d\langle X_\cdot(x),X_\cdot(y)\rangle\mathrm{d}x\mathrm{d}y\right)dt\\&=:\beta^2 B_t\end{aligned} \tag{6.23}$$

We have

$$\lim_{t\to\infty}\mathbb{E}\left[\left|\frac{A(t)}{\phi(t)}-1\right|\right]=1\text{ and }\lim_{t\to\infty}\mathbb{E}\left[\left|\frac{B(t)}{\phi(t)}\right|\right]=0. \tag{6.24}$$

We have

$$\begin{aligned}\mathbb{E}[A_t] &= \int_{\mathbb{R}^{2d}} f(x)f(y)Q_t(x,y)e^{\beta^2 t}\mathbb{E}\left[e^{i\beta(X_t(x)-X_t(y))}\right]\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^{2d}} f(x)f(y)Q_t(x,y)e^{\beta^2 K_t(x,y)}\mathrm{d}x\mathrm{d}y = \phi(t).\end{aligned} \tag{6.25}$$

Hence we just have to evaluate the second moment and show that it is small. We have

$$\widehat{A}_t := A_t - \mathbb{E}[A_t] = \int_{\mathbb{R}^{2d}} \xi_t(x,y)\mathrm{d}x\mathrm{d}y \tag{6.26}$$

where

$$\xi_t(x,y) := f(x)f(y)Q_t(x,y)\left(e^{\beta^2 t}e^{i\beta(X_t(x)-X_t(y))+\beta^2 t} - e^{\beta^2 K_t(x,y)}\right).$$

Setting $\zeta_t(x,y) = e^{i\beta(X_t(x)-X_t(y))+\beta^2 t} - e^{\beta^2 K_t(x,y)}$ we have

$$\mathbb{E}[|\widehat{A}_t|^2] = \int_{\mathbb{R}^{4d}} \mathbb{E}\left[\overline{\xi}_t(x_1,y_1)\xi_t(x_2,y_2)\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2. \tag{6.27}$$

We have

$$\begin{aligned}&\mathbb{E}\left[\overline{\zeta}_t(x_1,y_1)\zeta_t(x_2,y_2)\right] \\ &\quad= \left(e^{\beta^2(K_t(x_1,x_2)+K_t(y_1,y_2)-K_t(x_1,y_2)-K_t(x_2,y_1))} - 1\right)e^{\beta^2(K_t(x_1,y_1)+K_t(x_2,y_2))}\end{aligned} \tag{6.28}$$

Lemma 6.3. *There exists a constant C (which depends on β) which is such that if*

$$|x_i - y_i| \leqslant e^{-t},\ \textit{for } i \in \{1,2\} \tag{6.29}$$

then we have

$$|\mathbb{E}\left[\overline{\zeta}_t(x_1,y_1)\zeta_t(x_2,y_2)\right]| \leqslant Ce^{2\beta^2 t}(e^{-t}|x_1-y_1| \vee 1)^{-1} \tag{6.30}$$

Proof. We have $e^{\beta^2(K_t(x_1,y_1)+K_t(x_2,y_2))} \leqslant e^{2\beta^2 t}$ so we just have to bound the first term in (6.28). Now notice that K_t is Lipshitz with constant Ce^t (this just obtained by integrating the fact that Q_t is Lipshitz with constant Ce^t for a different C). Hence we have

$$(K_t(x_1,x_2) + K_t(y_1,y_2) - K_t(x_1,y_2) - K_t(y_1,x_2)) \leqslant 2Ce^t|x_2-y_2| \leqslant 2C \tag{6.31}$$

where the last inequality follows from (6.29). This implies that

$$\begin{aligned}&|e^{\beta^2(K_t(x_1,x_2)+K_t(y_1,y_2)-K_t(x_1,y_2)-K_t(x_2,y_1))} - 1| \\ &\qquad\leqslant C\,|K_t(x_1,x_2)+K_t(y_1,y_2)-K_t(x_1,y_2)-K_t(x_2,y_1)|\end{aligned} \tag{6.32}$$

and we are left with showing that

$$|K_t(x_1,x_2)+K_t(y_1,y_2)-K_t(x_1,y_2)-K_t(x_2,y_1)| \leqslant C(e^t|x_1-y_1| \vee 1)^{-1}. \tag{6.33}$$

Let us assume that $|x_1-x_2| \geqslant 3e^{-t}$ (of this is not the case then (6.31) allows to conclude). Set $s = \log(\frac{3}{|x_1-x_2|})$. From (6.29), we have

$$\min(|x_1-x_2|,|y_1-y_2|,|x_1-y_2|,|x_2-y_1|) \geqslant e^{-s}, \tag{6.34}$$

and for this reason

$$\begin{aligned}&|K_t(x_1,x_2)+K_t(y_1,y_2)-K_t(x_1,y_2)-K_t(x_2,y_1)| \\ &\qquad= |K_s(x_1,x_2)+K_s(y_1,y_2)-K_s(x_1,y_2)-K_s(x_2,y_1)|\end{aligned} \tag{6.35}$$

and repeating (6.31) we have

$$|K_s(x_1,x_2)+K_s(y_1,y_2)-K_s(x_1,y_2)-K_s(x_2,y_1)| \leqslant 2Ce^s|x_2-y_2| \leqslant 2Ce^{s-t} = \frac{6C}{e^t|x_1-x_2|} \quad (6.36)$$

which is the desired result.

□

Now we have

$$\mathbb{E}[|\widehat{A}_t|^2] \leqslant C\int_{\mathbb{R}^{4d}} \frac{e^{2\beta^2 t}}{1\wedge e^t|x_1-x_2|}\prod_{i=1}^{2} Q_t(x_i,y_i)|f(x_i)f(y_i)|\mathrm{d}x_i\mathrm{d}y_i$$
$$\leqslant C' e^{(2\beta^2-d)t}\int_{\mathbb{R}^{2d}} \frac{|f(x_1)f(x_2)|}{(1\wedge e^t|x_1-x_2|)}\mathrm{d}x_1\mathrm{d}x_2 \leqslant C'' e^{(2\beta^2-d)t}\rho(t) \quad (6.37)$$

where

$$\rho(t) = \begin{cases} te^{-t} & \text{if } d=1,\\ e^{-t} & \text{if } d\geqslant 2.\end{cases}$$

Hence we have

$$\mathrm{Var}(A_t) = \mathbb{E}[|\widehat{A}_t|^2] \leqslant C\rho(t)(\mathbb{E}[A_t])^2,$$

which, since $\rho(t)$ converges to zero, implies that $\mathbb{E}[A_t]$ concentrates around its mean.

In a similar manner we have

$$\mathbb{E}[|B_t|^2] = \int_{\mathbb{R}^{4d}} \mathbb{E}[\zeta'(x_1,y_1)\overline{\zeta}'(x_2,y_2)]\prod_{i=1}^{2} Q_t(x_i,y_i)f(x_i)f(y_i)\mathrm{d}x_i\mathrm{d}y_i \quad (6.38)$$

with $\zeta'(x,y) := e^{i\beta(X_t(x)+X_t(y))+\beta^2 t}$. We have

$$\mathbb{E}[\zeta'(x_1,y_1)\overline{\zeta}'(x_2,y_2)] = e^{\beta^2(K_t(x_1,x_2)+K_t(y_1,y_2)+K_t(x_1,y_2)+K_t(x_2,y_1)-K_t(x_1,y_1)-K_t(x_2,y_2))}. \quad (6.39)$$

Similarly to Lemma 6.3 we can prove that if (6.31) holds then we have

$$\mathbb{E}[\zeta'(x_1,y_1)\overline{\zeta}'(x_2,y_2)] \leqslant Ce^{-2\beta^2 t}(|x_1-y_1|\vee e^{-t})^{-4\beta^2}. \quad (6.40)$$

Altogether this yields

$$\begin{aligned}\mathbb{E}[|B_t|^2] &\leqslant Ce^{-2\beta^2 t}\int_{\mathbb{R}^{4d}} (|x_1-y_1|\vee e^{-t})^{-4\beta^2}\prod_{i=1}^{2} Q_t(x_i,y_i)f(x_i)f(y_i)\mathrm{d}x_i\mathrm{d}y_i\\ &= C'e^{-2(\beta^2+d)t}\int_{\mathbb{R}^{2d}} (|x_1-y_1|\vee e^{-t})^{-4\beta^2}|f(x_1)f(y_1)|\mathrm{d}x_1\mathrm{d}y_1\\ &\leqslant C''e^{(2\beta^2-3d)t}\end{aligned} \quad (6.41)$$

The proof is going to make use of the following proposition

References

[1] Juhan Aru, Antoine Jego, and Janne Junnila. Density of imaginary multiplicative chaos via Malliavin calculus. *Probab. Theory Related Fields*, 184(3-4):749–803, 2022.
[2] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:Paper No. 27, 12, 2017.
[3] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.
[4] Lisa Hartung and Anton Klimovsky. The glassy phase of the complex branching Brownian motion energy model. *Electron. Commun. Probab.*, 20:no. 78, 15, 2015.
[5] Janne Junnila, Eero Saksman, and Lauri Viitasaari. On the regularity of complex multiplicative chaos. *arXiv e-prints*, page arXiv:1905.12027, May 2019.
[6] Janne Junnila, Eero Saksman, and Christian Webb. Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.*, 29(6):3786–3820, 2019.
[7] Janne Junnila, Eero Saksman, and Christian Webb. Imaginary multiplicative chaos: Moments, regularity and connections to the ising model. *Ann. Appl. Probab.*, 30(5):2099–2164, 10 2020.
[8] Jean-Pierre Kahane. Sur le chaos multiplicatif. (On multiplicative chaos). *Ann. Sci. Math. Qué.*, 9:105–150, 1985.
[9] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Integrability of Liouville theory: proof of the DOZZ formula. *Annals of Mathematics*, 191(1):81 – 166, 2020.
[10] Hubert Lacoin. Convergence in law for complex Gaussian multiplicative chaos in phase III. *Ann. Probab.*, 50(3):950–983, 2022.
[11] Hubert Lacoin. Critical Gaussian Multiplicative Chaos revisited. *arXiv e-prints*, page arXiv:2209.06683, September 2022.
[12] Hubert Lacoin. A universality result for subcritical complex Gaussian multiplicative chaos. *Ann. Appl. Probab.*, 32(1):269–293, 2022.
[13] Hubert Lacoin. Convergence for complex gaussian multiplicative chaos on phase boundaries. *arXiv e-prints*, January 2023.
[14] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary Sine-Gordon model. *to appear in Probab. Theory Related Fields*.
[15] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Complex Gaussian multiplicative chaos. *Comm. Math. Phys.*, 337(2):569–632, 2015.
[16] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model. *arXiv preprint arXiv:1903.01394*, 2019.
[17] Thomas Madaule, Rémi Rhodes, and Vincent Vargas. The glassy phase of complex branching Brownian motion. *Comm. Math. Phys.*, 334(3):1157–1187, 2015.
[18] Thomas Madaule, Rémi Rhodes, and Vincent Vargas. Glassy phase and freezing of log-correlated gaussian potentials. *Ann. Appl. Probab.*, 26(2):643–690, 04 2016.
[19] Benoit B. Mandelbrot. Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence. In M. Rosenblatt and C. Van Atta, editors, *Statistical Models and Turbulence*, pages 333–351, Berlin, Heidelberg, 1972. Springer Berlin Heidelberg.
[20] Ellen Powell. Critical Gaussian multiplicative chaos: a review. *arXiv e-prints*, page arXiv:2006.13767, June 2020.
[21] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.

IMPA, Institudo de Matemática Pura e Aplicada, Estrada Dona Castorina 110 Rio de Janeiro, CEP-22460-320, Brasil.