Scaling limits of random walks on random graphs: An electrical resistance approach

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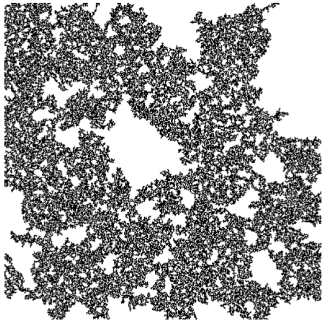
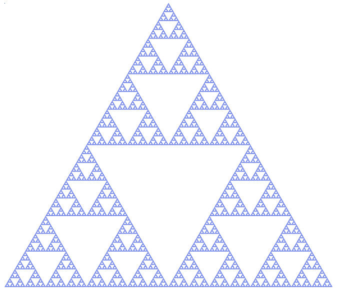
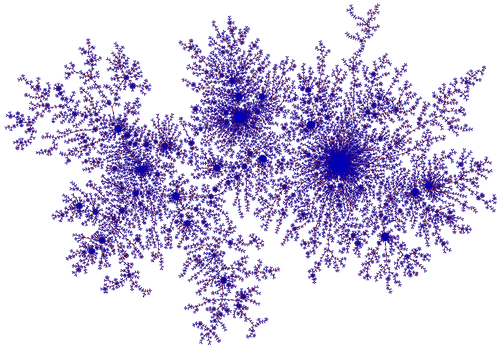
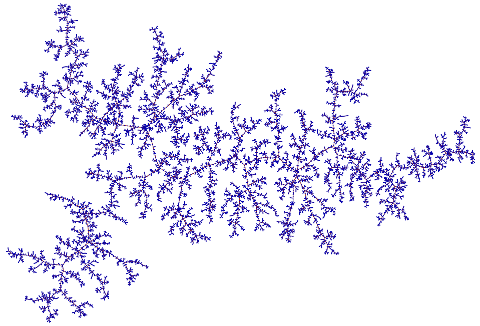
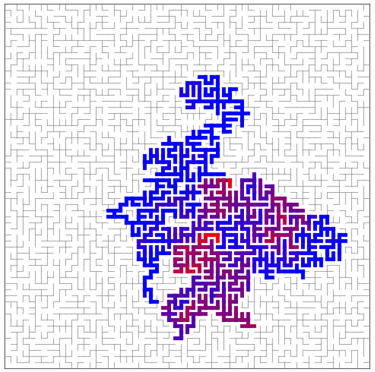
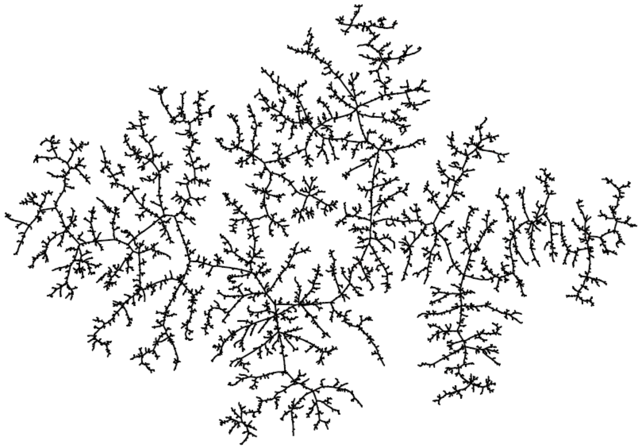
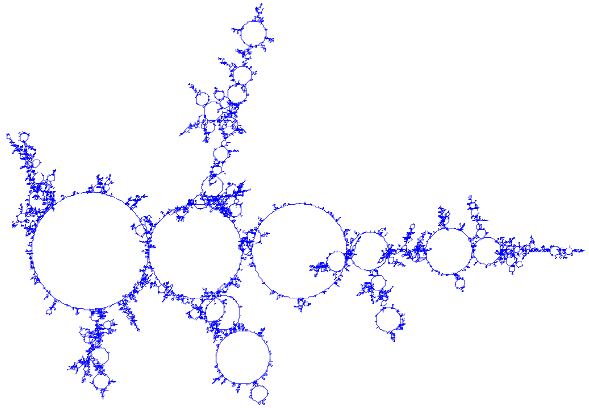
David Croydon (Kyoto)
joint with
O. Angel (UBC), **M. T. Barlow** (UBC),
R. Fukushima (Tsukuba), **B. M. Hambly** (Oxford),
S. Hernandez-Torres (UNAM), **S. Junk** (Tohoku),
T. Kumagai (Waseda) and **D. Shiraishi** (Kyoto).

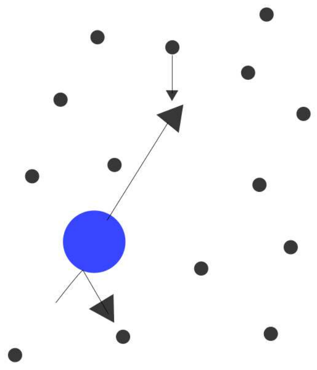





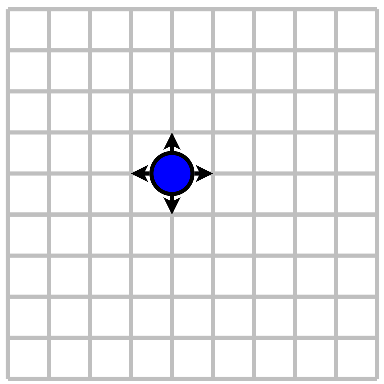
1. MOTIVATION

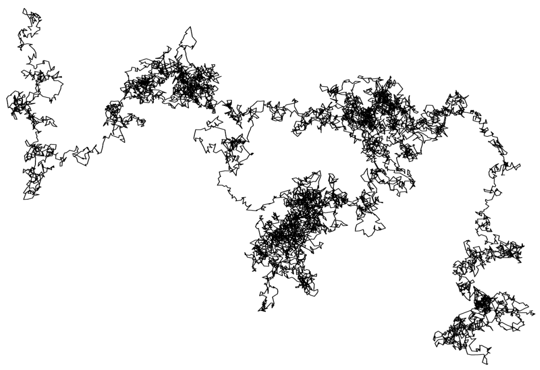




Sources: Ben Avraham/Havlin, Kortchemski, Chhita, Broutin.

RANDOM WALKS AND BROWNIAN MOTION







Source: Mörters/Peres.

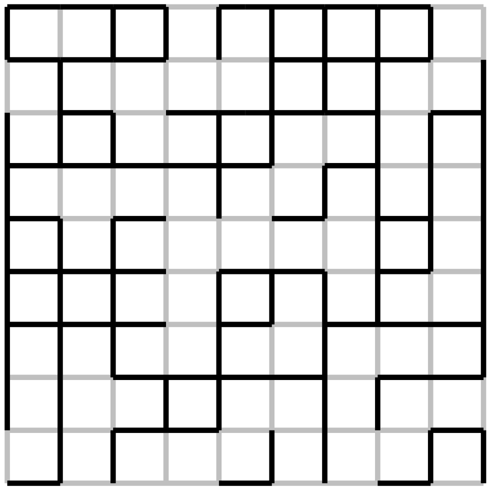
For discrete-time simple symmetric random walk $X = (X_n)_{n \geq 0}$ on integer lattice $\mathbb{Z}^d$ ($d \geq 1$), it holds that

$$\left(n^{-1} X_{tn^2}\right)_{t \geq 0} \to (B_t)_{t \geq 0},$$

where $(B_t)_{t \geq 0}$ is *Brownian motion* [Donsker 1951].

RANDOM WALK ON A PERCOLATION CLUSTER

Bond percolation on integer lattice $\mathbb{Z}^d$ ($d \geq 2$), parameter $p \in (0,1)$. E.g. $p = 0.53$:



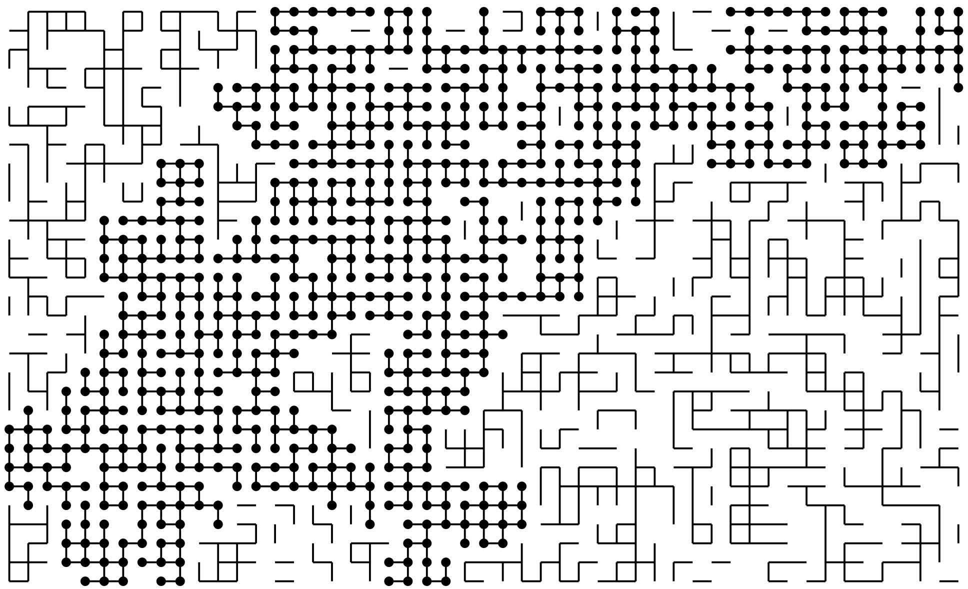
If $p > p_c(d)$, then the random walk is diffusive for $\mathbf{P}$-a.e. environment. In particular,

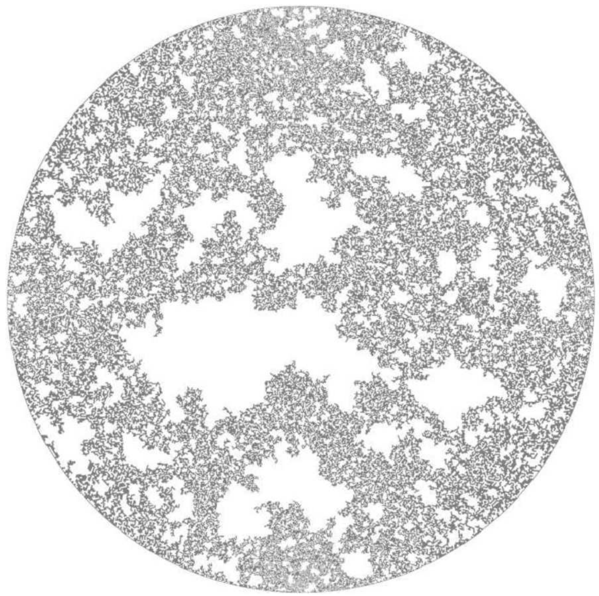
$$\left(n^{-1} X^{\mathcal{C}}_{tn^2}\right)_{t\geq 0} \to \left(B_{c(d,p)t}\right)_{t\geq 0}.$$

See [Sidoravicius/Sznitman 2004, Biskup/Berger 2007, Mathieu/Piatnitski 2007], and heat kernel bounds of [Barlow 2004].

PERCOLATION AT CRITICALITY?

Early physics work [Alexander/Orbach 1982].

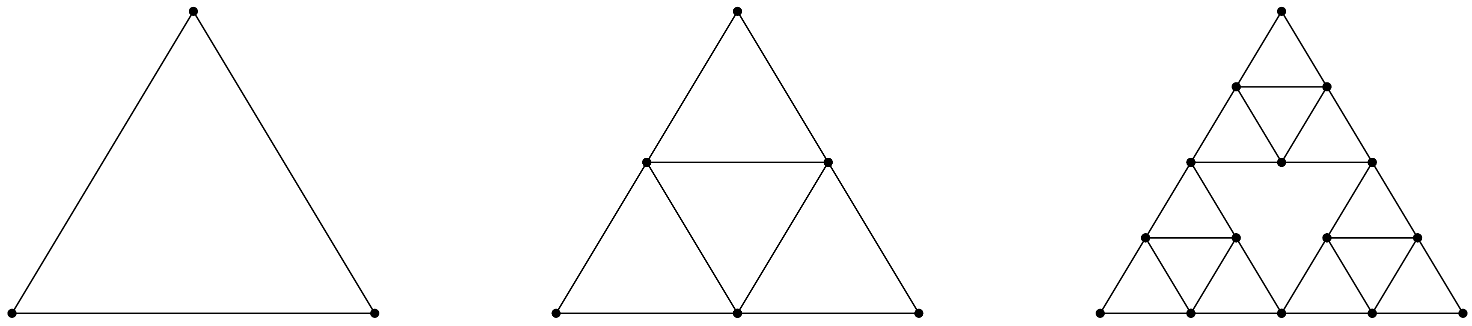




Left: Part of a large critical perc. cluster ($p = p_c(2) = 0.5$).
Right: CLE(6) gasket. *Sources: Barlow/Miller, Sun, Wilson.*

RANDOM WALK ON SELF-SIMILAR FRACTAL GRAPHS

For example, how does random walk behave on the pre-Sierpinski gasket graphs?



Answer. Can be rescaled to 'Brownian motion' on the limiting Sierpinski gasket [Goldstein 1987, Kusuoka 1987, Barlow/Perkins 1988]:

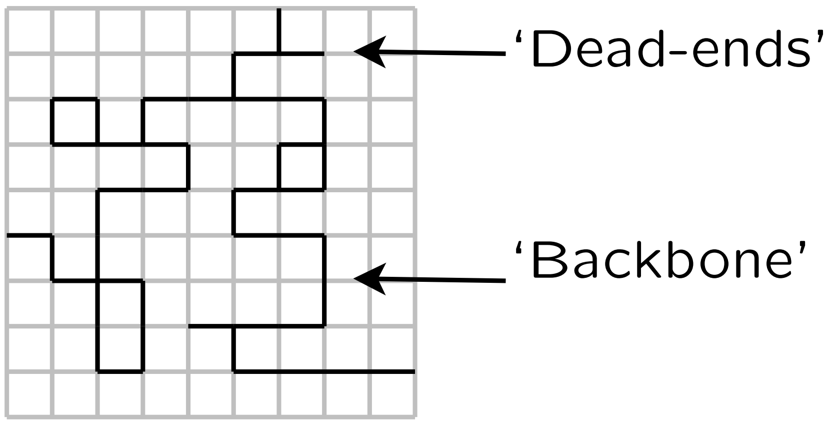
$$\left(X^{(n)}_{5^n t}\right)_{t\geq 0} \to \left(X^{BM}_t\right)_{t\geq 0}.$$

INCIPIENT INFINITE CLUSTER

At $p = p_c(d)$, it is partially confirmed that there is no infinite cluster. Instead, study the random walk on the 'incipient infinite cluster':

$$\mathcal{C}_0 | \{|\mathcal{C}_0| = n\} \to \text{IIC}.$$

Constructed in [Kesten 1986] for $d = 2$, [van der Hofstad/Jarai 2004] for high dimensions.



Tree-like in high dimensions [Hara/Slade 2000], see also [Heydenreich, van der Hofstad/Hulsfhof/Miermont 2017].

SRW ON PERCOLATION AT CRITICALITY?

Random walk is subdiffusive for $d = 2$ and in high-dimensions [Kesten 1986, Nachmias/Kozma 2009], see also [Heydenreich/van der Hofstad/Hulshof 2014].

For example, for almost-every-realisation of the IIC in high-dimensions, we have:

$$\frac{\log E_0^{IIC}\tau(R)}{\log R} \to 3,$$

where $\tau(R) = \inf\{n :\ d_{IIC}(0, X_n^{IIC}) = R\}$, and

$$\frac{\log E_0^{IIC}\tilde{\tau}(R)}{\log R} \to 6,$$

where $\tilde{\tau}(R) = \inf\{n :\ |0 - X_n^{IIC}| = R\}$.

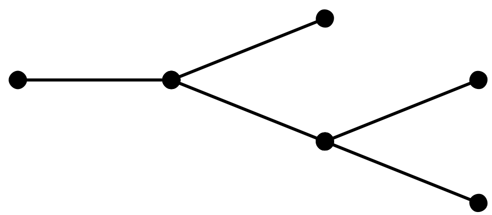
Scaling limit?

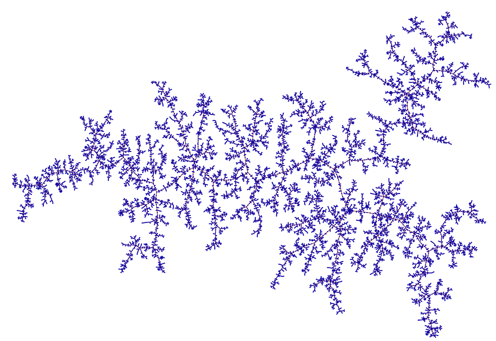
E.G. CRITICAL GALTON-WATSON TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have n vertices, then

$$n^{-1/2} T_n \to \mathcal{T},$$

where $\mathcal{T}$ is (up to a constant) the **Brownian continuum random tree (CRT)** [Aldous 1993], also [Duquesne/Le Gall 2002].





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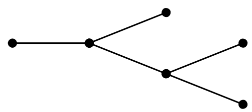
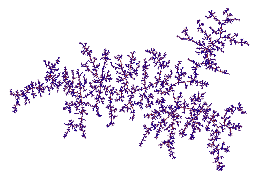


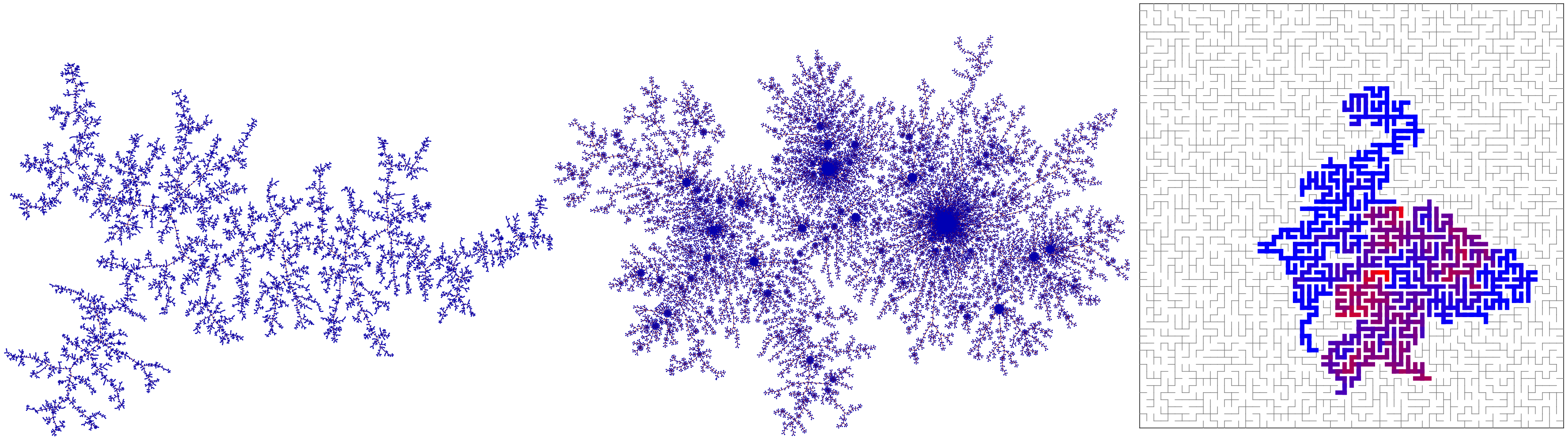


Convergence in Gromov-Hausdorff-Prohorov topology implies

$$\left(n^{-1/2}X^{T_n}_{n^{3/2}t}\right) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0},$$

see [Krebs 1995], [C. 2008] and [Athreya/Löhr/Winter 2014].

EXAMPLES OF 'CRITICAL' TREES



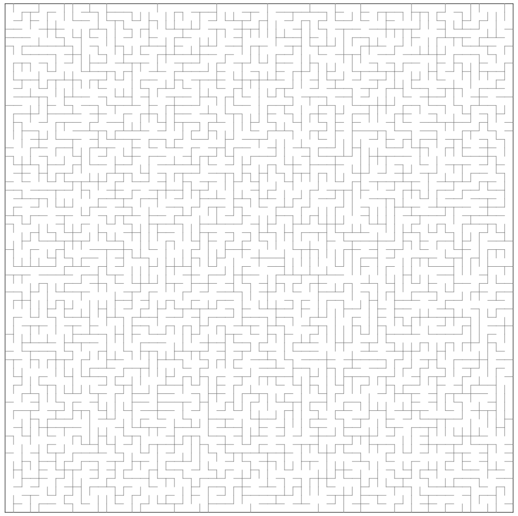
Finite variance Galton-Watson trees: $(n^{-1/2}X^{T_n}_{n^{3/2}t}) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$

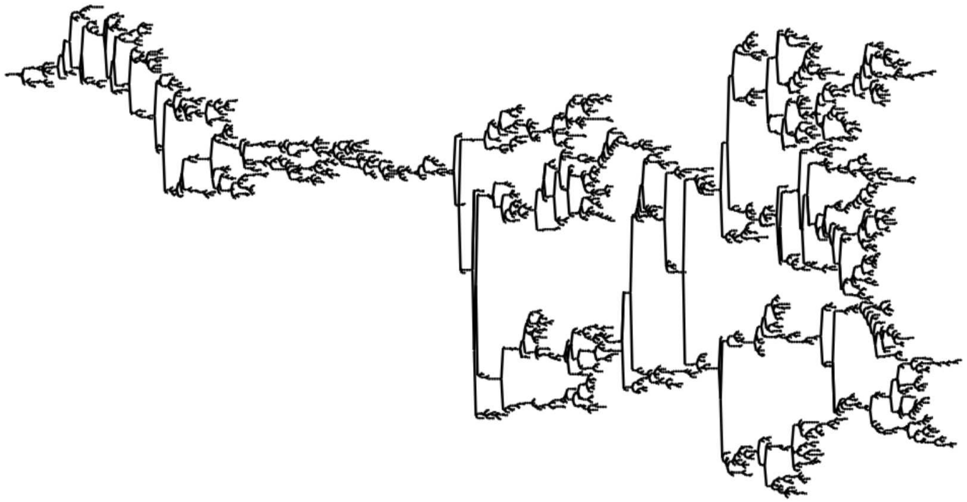
α-stable Galton-Watson trees: $(n^{-1/\alpha}X^{T_n}_{n^{1+1/\alpha}t}) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$

Two-dimensional uniform spanning tree: $(n^{-1}X^{\mathcal{U}}_{n^{13/4}t}) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$

Three-dimensional uniform spanning tree: $(n^{-1}X^{\mathcal{U}}_{n^{4.62\dots}t}) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0}$

EXAMPLES OF ‘CRITICAL’ TREES





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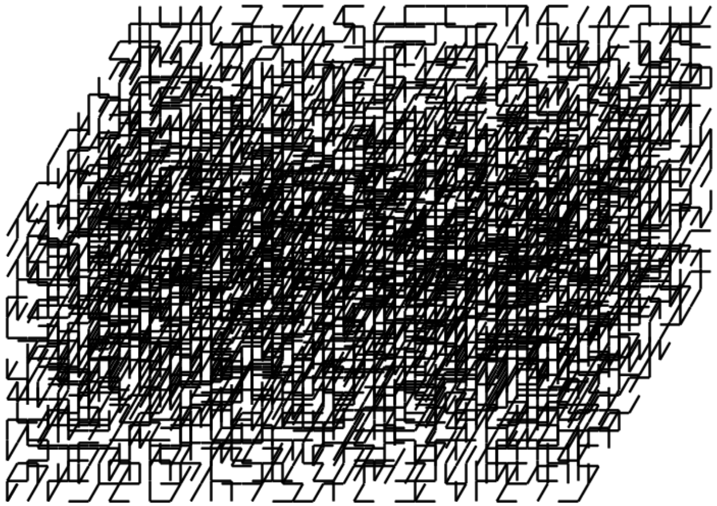
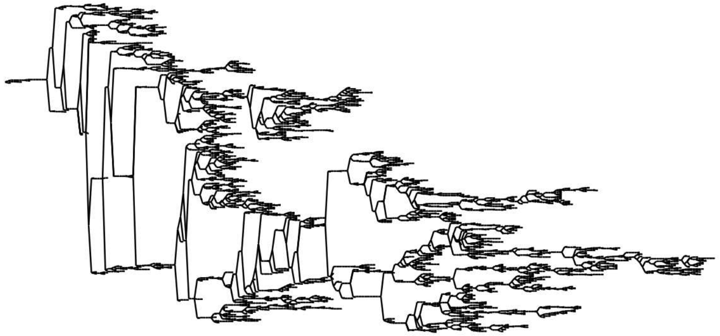
EXAMPLES OF 'CRITICAL' TREES





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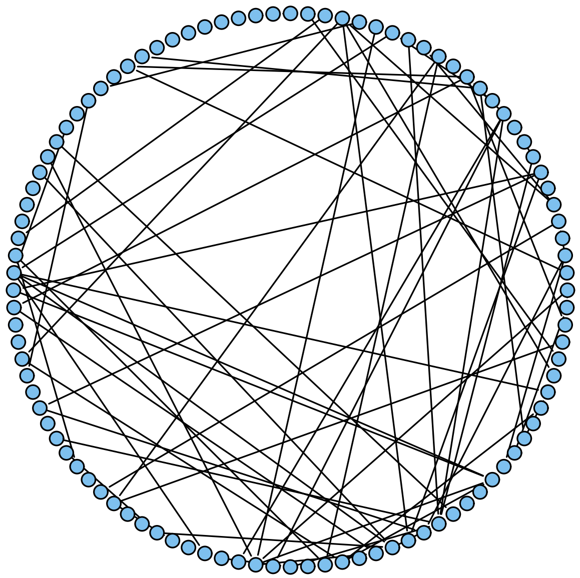
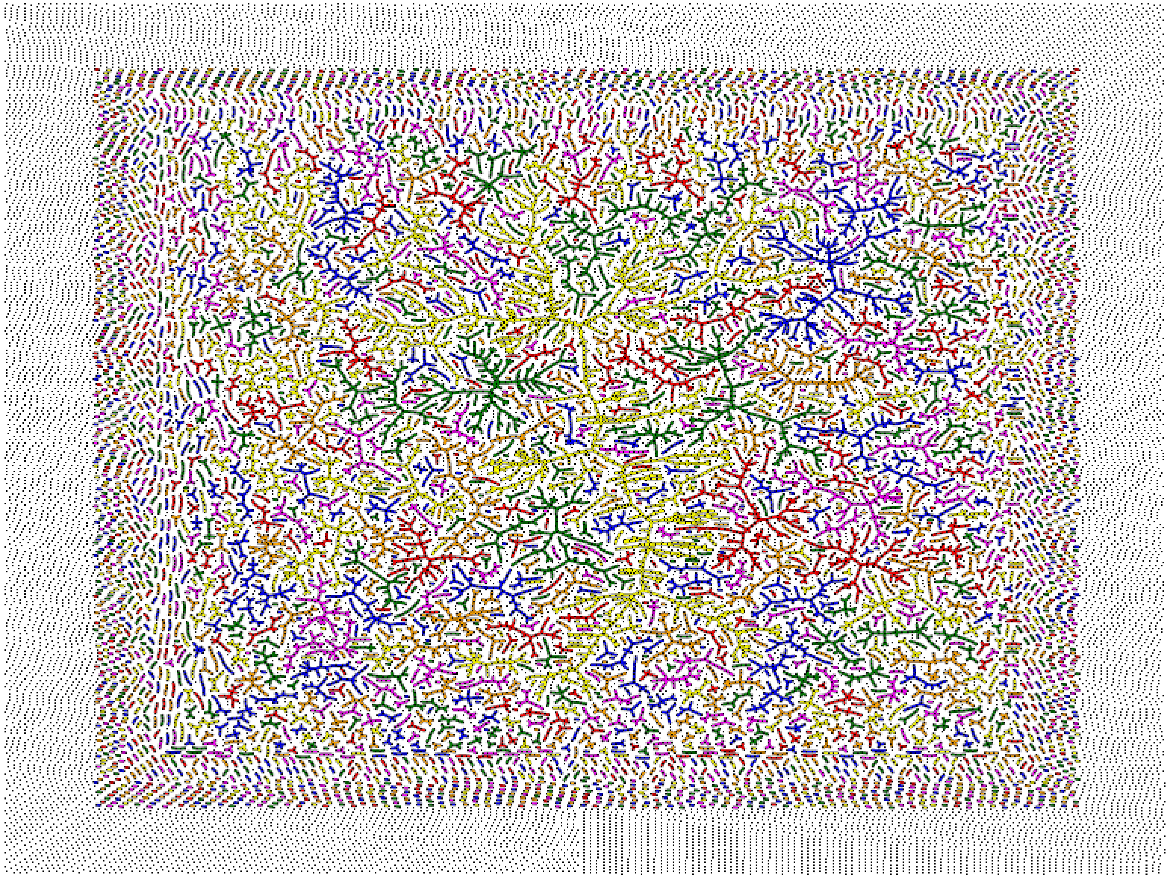
OTHER EXAMPLES OF 'CRITICAL' GRAPHS #1a

If pairs of $\{1, 2, \ldots, n\}$ are independently connected by an edge with probability n^{-1}, then one obtains the critical Erdős-Rényi random graph $G(n, n^{-1})$:



OTHER EXAMPLES OF 'CRITICAL' GRAPHS #1b

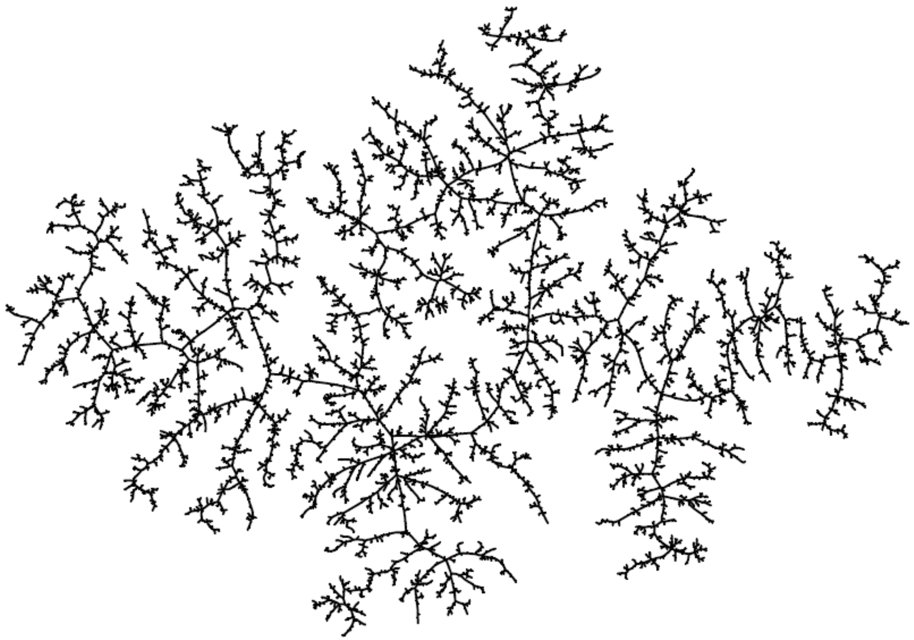
E.g. For $G(n, 1/n)$, the components of the graph have a complex structure:



Source: Broutin, $n = 40,000$.

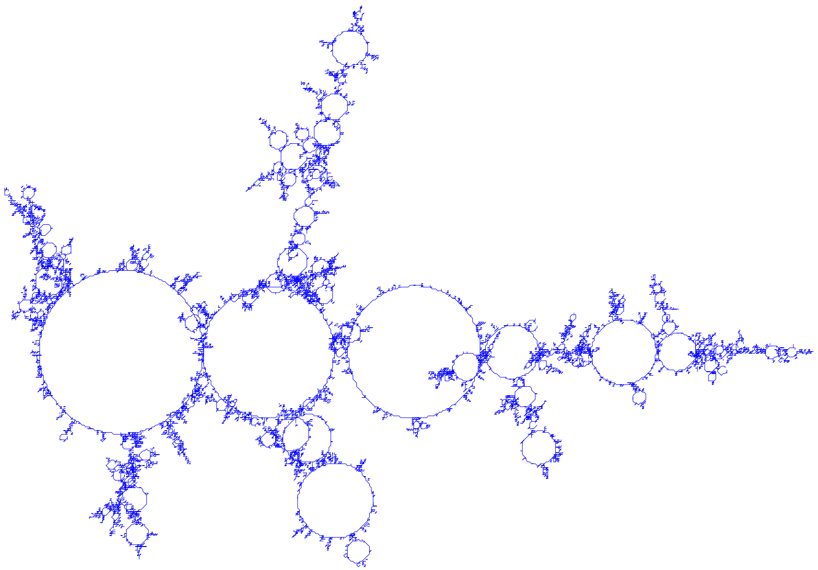
OTHER EXAMPLES OF 'CRITICAL' GRAPHS #1c

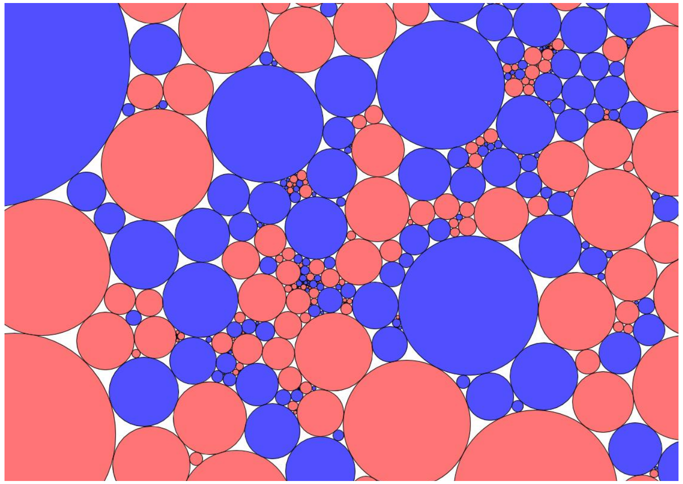
E.g. Largest connected component $\mathcal{C}_1^n$ of $G(n, 1/n)$ has $n^{2/3}$ vertices and rescaling distance by $n^{1/3}$ yields a fractal scaling limit for the space [Addario-Berry/Broutin/Goldschmidt]:



OTHER EXAMPLES OF 'CRITICAL' GRAPHS #2

E.g. Discrete loop-tree given by a critical GW tree, with α-stable offspring distribution, $\alpha \in (1, 2]$.





Relates to boundary of critical percolation cluster on a random planar map [Curien/Kortchemski]. *Source: Budzinski*

RANDOM CONDUCTANCE MODEL AND BOUCHAUD TRAP MODEL

Random conductance model (RCM):

Equip edges of graphs with random weights $(c(x,y))$ such that

$$P(c(x,y) \geq u) = u^{-\alpha}, \qquad \forall u \geq 1,$$

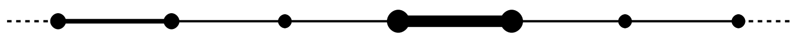
for some $\alpha \in (0,1)$.

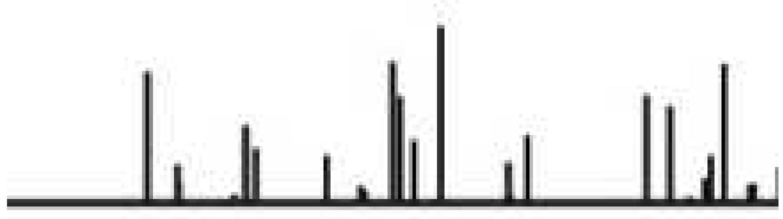
Symmetric Bouchaud trap model (BTM):

Add exponential holding times, mean τ_x, to vertices. In the case where τ is random and heavy-tailed, behaviour similar to RCM.

SUBDIFFUSIVITY OF RCM AND BTM IN 1D

On $\mathbb{Z}$, in the heavy-tailed regime, the conductance/holding time environment remains inhomogeneous in the limit:





The limit measure is described as a Poisson random measure. The associated trapping leads to a subdiffusive scaling limit for the RW [Barlow/Cerny 2011, Cerny 2011]:

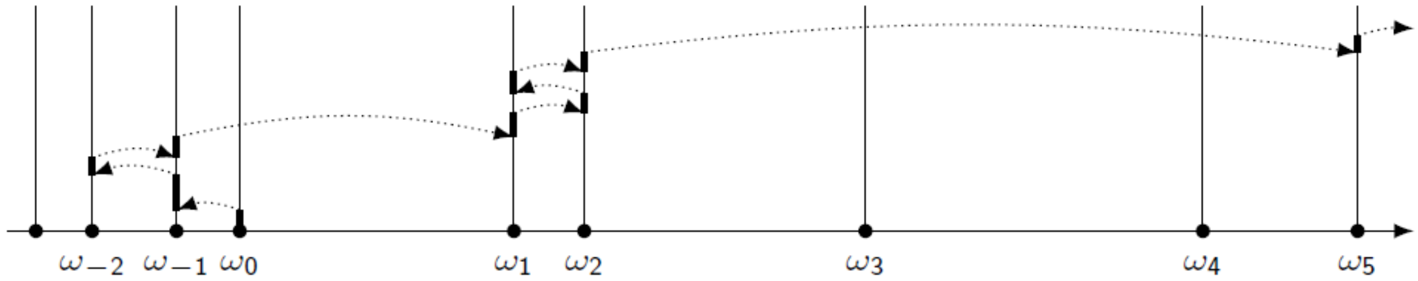
$$\left(n^{-1}X^{RCM/BTM}_{n^{1+\alpha^{-1}}t}\right)_{t\geq 0} \to \left(X^{FIN}_t\right)_{t\geq 0}.$$

THE MOTT RANDOM WALK

Environment $(\omega = (\omega_i)_{i\in\mathbb{Z}}, \mathbf{P})$ jump times of 1-dim. Poisson process, intensity ρ, conditioned on $\omega_0 = 0$. Continuous-time random walk $(X = (X_t)_{t\geq 0}, P_\omega)$ on ω with $X_0 = \omega_0$, jumping at rate 1. Probability of jumping from ω_i to ω_j is

$$\frac{c(\omega_i, \omega_j)}{c(\omega_i)},$$

where $c(\omega_i, \omega_j) := e^{-|\omega_i - \omega_j|}$, $c(\omega_i) := \sum_{j\neq i} c(\omega_i, \omega_j)$.



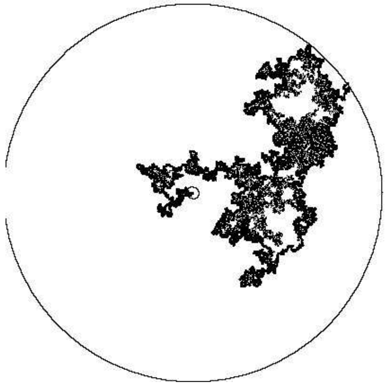
Process is diffusive if $\rho > 1$, subdiffusive if $\rho \leq 1$. Scaling limits known [C./Fukushima/Junk 2021].

2. STOCHASTIC PROCESSES ASSOCIATED WITH RESISTANCE METRICS

PROBABILITY AND POTENTIAL THEORY

To study random walks on general graphs, powerful techniques are provided from the deep connections with potential theory/ electric networks.

If the boundary of a region D is held at potential f, then what is the potential inside the domain? Answer given by solution to 'Dirichlet problem':



$$\begin{aligned} \Delta v &= 0 \text{ inside } D, \\ v &= f \text{ on } \partial D. \end{aligned}$$

Probabilistic solution: $v(x) = E_x f(B_{\tau(\partial D)})$.

[Kakutani 1944]

RANDOM WALKS ON GRAPHS

Let $G = (V, E)$ be a finite, connected graph, equipped with (strictly positive, symmetric) edge conductances $(c(x, y))_{\{x,y\}\in E}$. Let μ be a finite measure on V (of full-support).

Let X be the continuous time Markov chain with generator Δ, as defined by:

$$(\Delta f)(x) := \frac{1}{\mu(\{x\})} \sum_{y:\, y\sim x} c(x, y)(f(y) - f(x)).$$

Transition probabilities: $P(x, y) = c(x, y)/c(x)$, where $c(x) = \sum_{\{x,y\}\in E} c(x, y)$.

Holding times: exponential, mean $\mu(\{x\})/c(x)$.

ELECTRICAL ENERGY AND RESISTANCE METRIC

Suppose we view G as an electrical network with edges assigned conductances according to $(c(x,y))_{\{x,y\}\in E}$.

Define a quadratic form on G by setting

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y:x\sim y} c(x,y)\,(f(x)-f(y))\,(g(x)-g(y))\,.$$

Then $\mathcal{E}(f,f)$ is **electrical energy** dissipated in network if vertices are held at voltages according to f. Also, for any μ, $\mathcal{E}$ is a **Dirichlet form** on $L^2(\mu)$, and

$$\mathcal{E}(f,g) = -\sum_{x\in V} (\Delta f)(x) g(x) \mu(\{x\}).$$

The **effective resistance** between x and y is given by

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f):\ f(x)=1, f(y)=0\right\}.$$

R is a metric on V, e.g. [Tetali 1991], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 1995].

SUMMARY

RANDOM WALK X WITH GENERATOR Δ

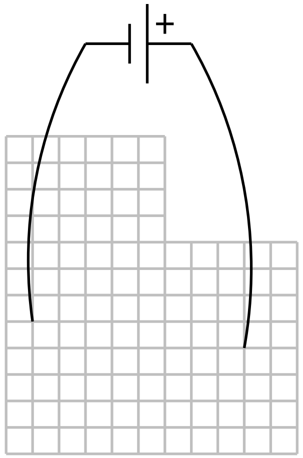
$$\Updownarrow$$

DIRICHLET FORM $\mathcal{E}$ on $L^2(\mu)$

$$\Updownarrow$$

RESISTANCE METRIC R AND MEASURE μ

EXAMPLES OF CONNECTIONS



Voltages and hitting probabilities: Use battery to set $v(a) = 1$, $v(b) = 0$. Voltages in remainder of network given by

$$v(x) = P_x(\tau_a < \tau_b).$$

(Key: both sides solve discrete Dirichlet problem.)

Effective resistance and escape probabilities:

$$P_x(\tau_y < \tau_x^+) = \frac{1}{c(x)R(x,y)}.$$

Effective resistance and Green's function density:

$$\frac{E_y(\text{time in } z \text{ before hitting } x)}{\mu(\{z\})} = \frac{R(x,y) + R(x,z) - R(y,z)}{2}.$$

RESISTANCE METRIC, e.g. [KIGAMI 2001]

Let F be a set. A function $R : F \times F \to \mathbb{R}$ is a **resistance metric** if, for every finite $V \subseteq F$, one can find a weighted (i.e. equipped with conductances) graph with vertex set V for which $R|_{V \times V}$ is the associated effective resistance.

EXAMPLES

- Effective resistance metric on a graph;

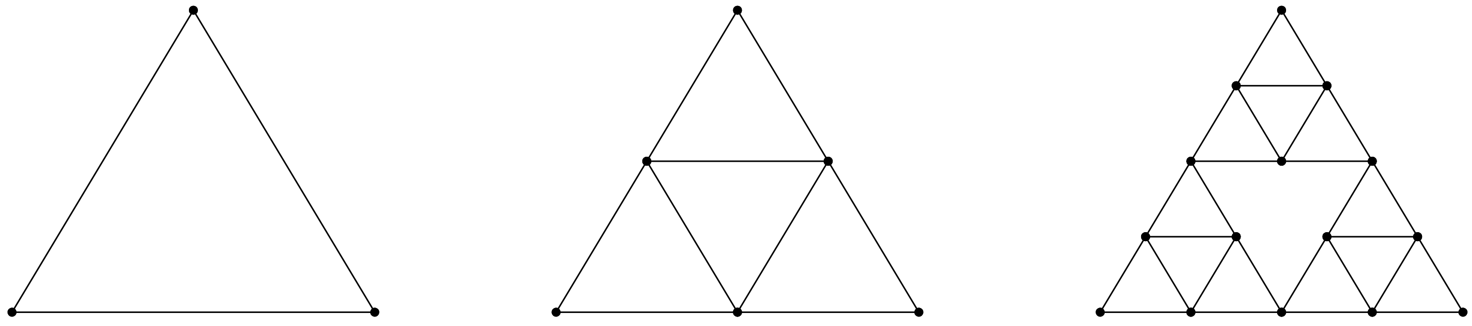
- One-dimensional Euclidean (not true for higher dimensions);

- Any shortest path metric on a tree;

- Resistance metric on a Sierpinski gasket, where for 'vertices' of limiting fractal, we set

$$R(x,y) = (3/5)^n R_n(x,y),$$

then use continuity to extend to whole space.



RESISTANCE AND DIRICHLET FORMS

Theorem (e.g. [Kigami 2001]) There is a one-to-one correspondence between resistance metrics and a class of quadratic forms called **resistance forms**.

The relationship between a resistance metric R and resistance form $(\mathcal{E}, \mathcal{F})$ is characterised by

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f) : \ f \in \mathcal{F}, \ f(x) = 1, \ f(y) = 0\right\}.$$

Moreover, if (F, R) is compact, then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$ for any finite Borel measure μ of full support. (Version of the statement also hold for locally compact spaces.)

A FIRST EXAMPLE

Let $F = [0, 1]$, $R =$ Euclidean, and μ be a finite Borel measure of full support on $[0, 1]$.

Associated resistance form:

$$\mathcal{E}(f, g) = \int_0^1 f'(x)g'(x)dx, \qquad \forall f, g \in \mathcal{F},$$

where $\mathcal{F} = \{f \in C([0, 1]) : f \text{ is abs. cont. and } f' \in L^2(dx)\}$.

Moreover, integration by parts gives:

$$\mathcal{E}(f, g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx).$$

where $\Delta f = \frac{d}{d\mu}\frac{df}{dx}$.

If $\mu(dx) = dx$, then the Markov process naturally associated with Δ is reflected Brownian motion on $[0, 1]$.

SUMMARY

RESISTANCE METRIC R AND MEASURE μ

$$\updownarrow$$

RESISTANCE FORM $(\mathcal{E}, \mathcal{F})$, DIRICHLET FORM on $L^2(\mu)$

$$\updownarrow$$

STRONG MARKOV PROCESS X WITH GENERATOR Δ,
where

$$\mathcal{E}(f,g) = -\int_F (\Delta f) g d\mu.$$

3. CONVERGENCE OF RESISTANCE METRICS AND STOCHASTIC PROCESSES

MAIN RESULT [C. 2018, C./HAMBLY/KUMAGAI 2017]

Write $\mathbb{F}_c$ for the space of marked compact resistance metric spaces, equipped with finite Borel measures of full support. Suppose that the sequence $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1}$ in $\mathbb{F}_c$ satisfies

$$(F_n, R_n, \mu_n, \rho_n) \to (F, R, \mu, \rho)$$

in the (marked) Gromov-Hausdorff-Prohorov topology for some $(F, R, \mu, \rho) \in \mathbb{F}_c$.

It is then possible to isometrically embed $(F_n, R_n)_{n\geq 1}$ and (F, R) into a common metric space (M, d_M) in such a way that

$$P^n_{\rho_n}\left((X^n_t)_{t\geq 0} \in \cdot\right) \to P_\rho\left((X_t)_{t\geq 0} \in \cdot\right)$$

weakly as probability measures on $D(\mathbb{R}_+, M)$.

Holds for locally compact spaces if $\liminf_{n\to\infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c)$ diverges as $r \to \infty$. (Can also include 'spatial embeddings'.)

PROOF IDEA 1: RESOLVENTS

For $(F, R, \mu, \rho) \in \mathbb{F}_c$, let

$$G_x f(y) = E_y \int_0^{\sigma_x} f(X_s) ds$$

be the resolvent of X killed on hitting x. NB. Processes associated with resistance forms hit points.

We have [Kigami 2012] that

$$G_x f(y) = \int_F g_x(y, z) f(z) \mu(dz),$$

where

$$g_x(y, z) = \frac{R(x, y) + R(x, z) - R(y, z)}{2}.$$

Metric measure convergence $\Rightarrow$ resolvent convergence $\Rightarrow$ semigroup convergence $\Rightarrow$ finite dimensional distribution convergence.

PROOF IDEA 2: TIGHTNESS

Using that X has local times $(L_t(x))_{x\in F, t\geq 0}$, and

$$E_y L_{\sigma_A}(z) = g_A(y,z) = \frac{R(y,A)+R(z,A)-R_A(y,z)}{2},$$

can establish via Markov's inequality a general estimate of the form:

$$\sup_{x\in F} P_x\left(\sup_{s\leq t} R(x,X_s)\geq \varepsilon\right) \leq \frac{32N(F,\varepsilon/4)}{\varepsilon}\left(\delta + \frac{t}{\inf_{x\in F}\mu(B_R(x,\delta))}\right),$$

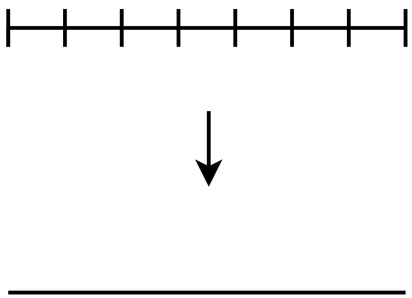
where $N(F,\varepsilon)$ is the minimal size of an ε cover of F.

Metric measure convergence $\Rightarrow$ estimate holds uniformly in n $\Rightarrow$ tightness (application of Aldous' tightness criterion).

Similar estimate also gives non-explosion in locally compact case.

4. APPLICATIONS

AN EASY FIRST EXAMPLE



Discrete space/process:

Space $= \{0, 1, \ldots, n\}$,

Edge resistances $= n^{-1}$,

Vertex masses $= \deg(x)/2n$,

Process = SRW, holding time mean $1/n^2$.

Continuous space/process:

$[0, 1]$ equipped with Euclidean metric,

Lebesgue measure,

Process = BM, reflected at boundary.

TREES

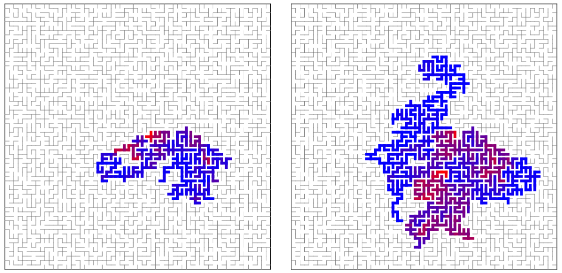
For any sequence of graph trees $(T_n)_{n\geq 1}$ such that

$$(V(T_n), a_n R_n, b_n \mu_n) \to (\mathcal{T}, R, \mu),$$

it holds that

$$\left(a_n^{-1} X_{t a_n b_n}\right)_{t\geq 0} \to (X_t)_{t\geq 0}.$$

- **Critical Galton-Watson trees** with finite variance conditioned on size, $a_n = n^{1/2}$, $b_n = n$. (Also α-stable versions.)
- **2d-UST**, $a_n = n^{5/4}$, $b_n = n^2$ (w/Barlow/Kumagai).

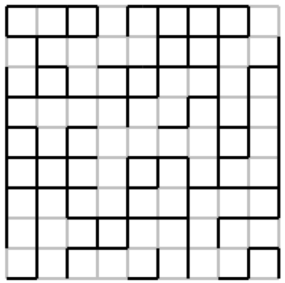


Source: Chhita.

- **3d-UST**, $a_n = n^{1.62...}$, $b_n = n^3$ (w/Angel, Hernandez-Torres, Shiraishi).
- Many other interesting models...e.g. G. Andriopoulos:
- **weakly biased random walk on branching random walk;**
- **RWRE/ERRW on trees.**

CONJECTURE FOR CRITICAL PERCOLATION

Bond percolation on integer lattice $\mathbb{Z}^d$:



At criticality $p = p_c(d)$ in high dimensions, incipient infinite cluster (IIC) conjectured to have same scaling limit as Galton-Watson tree, e.g. [Hara/Slade 2000]. So, expect

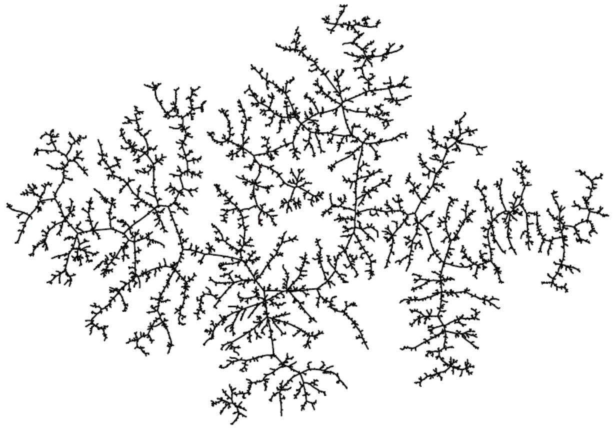
$$\left(\mathrm{IIC}, n^{-2} R_{\mathrm{IIC}}, n^{-4} \mu_{\mathrm{IIC}}\right)$$

to converge, and thus obtain scaling limit for random walks. cf. work of [Ben Arous, Fribergh, Cabezas 2016] for branching random walk. NB. Diffusion scaling limit constructed [C. 2009].

Other graphs that are 'asymptotically trees' for which scaling limit is proved: **1-d Mott random walk** (C./Fukushima/Junk 2021), **RW on range of RW** (C./Shiraishi 2021).

RANDOM WALK SCALING ON CRITICAL RANDOM GRAPH

Consider largest connected component $\mathcal{C}_1^n$ of $G(n, 1/n)$:



It holds that:

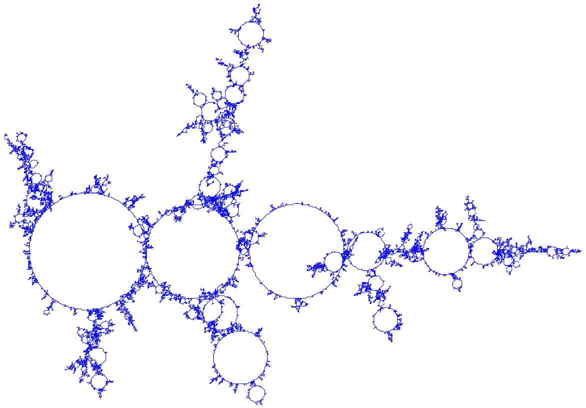
$$\left(\mathcal{C}_1^n, n^{-1/3} R_n, n^{-2/3} \mu_n\right) \to (F, R, \mu),$$

cf. [Addario-Berry, Broutin, Goldschmidt 2012]. Hence, as in [C. 2012],

$$\left(n^{-1/3} X_{tn}^n\right)_{t \geq 0} \to (X_t)_{t \geq 0}.$$

RANDOM WALK SCALING ON CRITICAL RANDOM LOOPTREES

Let L_n be a discrete loop-tree given by a critical GW tree, with α-stable offspring distribution, $\alpha \in (1,2]$.



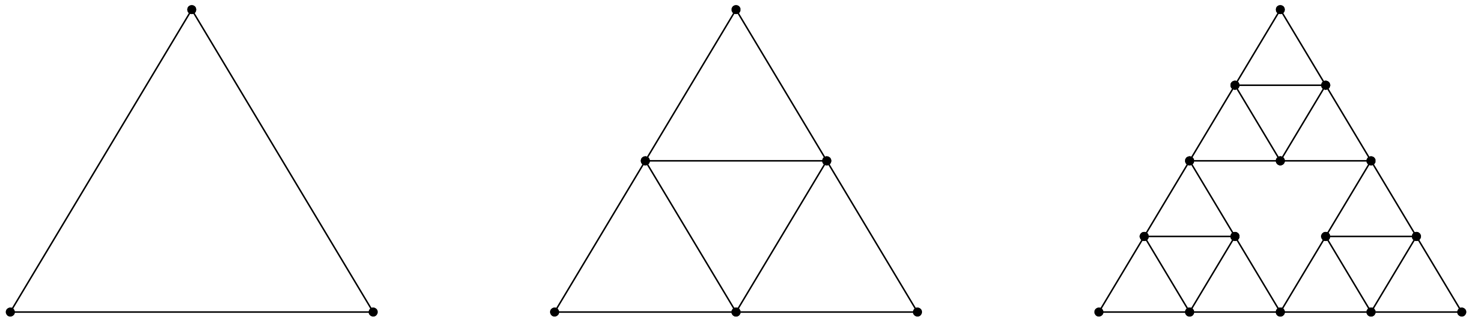
Then

$$\left(L_n, n^{-1/\alpha} R_n, n^{-1} \mu_n\right) \to (F, R, \mu)\,,$$

[Archer 2018], cf. [Curien, Kortchemski 2014]. Hence,

$$\left(n^{-1/\alpha} X^n_{tn^{1+1/\alpha}}\right)_{t\geq 0} \to (X_t)_{t\geq 0}\,.$$

HEAVY-TAILED RCM ON FRACTALS #1



Suppose that $P(c(x,y) \geq u) = u^{-\alpha}$ for $u \geq 1$ and some $\alpha \in (0,1)$. For gaskets, can then check that resistance homogenises [C., Hambly, Kumagai 2016]

$$\left(V_n, (3/5)^n R_n, 3^{-n}\mu_n\right) \rightarrow (F, R, \mu),$$

where:

-(up to a deterministic constant) R is the standard resistance,

- μ is a Hausdorff measure on fractal.

Hence VSRW converges to Brownian motion (spatial scaling assumes graphs already embedded into limiting fractal):

$$(X^n_{t5^n})_{t\geq 0} \rightarrow (X_t)_{t\geq 0}.$$

HEAVY-TAILED RCM ON FRACTALS #2

It further holds that

$$\nu_n := 3^{-n/\alpha} \sum_{x \in V_n} c(x)\delta_x \to \nu = \sum_i v_i \delta_{x_i},$$

in distribution, where $\{(v_i, x_i)\}$ is a Poisson point process with intensity $cv^{-1-\alpha}dv\mu(dx)$. Hence CSRW (and discrete time random walk) converges:

$$\left(X^{n,\nu_n}_{t(5/3)^n 3^{n/\alpha}}\right)_{t\geq 0} \to (X^\nu_t)_{t\geq 0}\,,$$

where the limiting process X^ν is the **Fontes-Isopi-Newman (FIN)** diffusion on the limiting fractal.

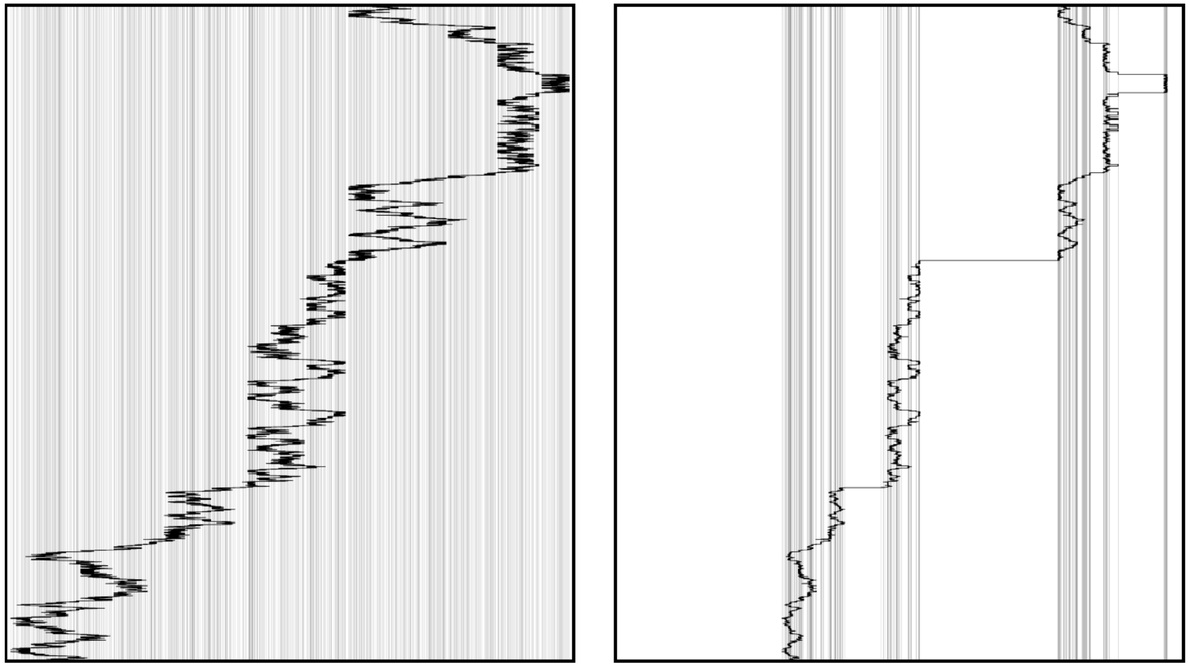
Similarly scaling result for heavy-tailed Bouchaud trap model.

MOTT RANDOM WALK

For $\rho < 1$, the Mott random walk satisfies:

$$\left(n^{-1}X_{n^{1+1/\rho}t}\right)_{t\geq 0} \xrightarrow{d} \left(Z_t^{(\rho)}\right)_{t\geq 0}.$$

with respect to the annealed law [C./Fukushima/Junk 2021].



Ongoing work on ‘aging’ with [Kious/Scali].