D. TSANG & A. NARDUZZO

UNIVERSITY OF BATH

PH10005/53:

VIBRATIONS, WAVES & OPTICS

SEMESTER I, 2021-2022

DEPARTMENT OF PHYSICS, UNIVERSITY OF BATH

Symbol Glossary for Parts I & II

Symbol	Description	SI Units	Page
A	Amplitude (response) of an oscillator	usually m or rad	4, 24
A_{res}	Amplitude at resonance (maximum driven amplitude)	usually m or rad	27
a	Acceleration	m/s^2	4
a_o	Maximum driving acceleration	m/s^2	23
b	Damping force constant	$\text{N}\cdot\text{s/m=kg/s}$	13
β	Logarithmic decrement		17
δ	Initial phase or phase lag	rad	4, 24
δ_{res}	Phase lag at resonance	rad	27
E_{kin}	Kinetic Energy	$\text{J} = \text{kg}\cdot\text{m}^2/\text{s}^2$	9, 18
E_{tot}	Total Energy	$\text{J} = \text{kg}\cdot\text{m}^2/\text{s}^2$	9, 18
f	Frequency	$\text{Hz} = \text{s}^{-1}$	4
f_{beat}	Beat frequency	$\text{Hz} = \text{s}^{-1}$	41
F_o	Maximum driving force	$\text{N} = \text{kg}\cdot\text{m/s}^2$	22
F_r	Restoring force	$\text{N} = \text{kg}\cdot\text{m/s}^2$	2
g	Gravitational acceleration on Earth's surface: 9.80665	m/s^2	7
γ	Damping constant $= b/(2m)$	$\text{N}\cdot\text{s/m=s}^{-1}$	13
k	Spring constant; wavenumber	N/m; rad/m	3, 36
L	A length (usually of a pendulum, string, or pipe)	m	8
ΔL	Path length difference	m	40
λ	Wavelength	m	35
m	Mass	kg	2
μ	linear density: mass per unit length	kg/m	56
Ω	Driving angular frequency for forced oscillations	rad/s	22
$\Delta\Omega$	Bandwidth of a forced oscillator	rad/s	28
$\Omega_{\pm}$	Driving frequencies where amplitude is $A_{\text{res}}/\sqrt{2}$	rad/s	28
ω	Angular frequency	rad/s	3
ω_o	Natural (ang.) freq. of an oscillator (i.e. without damping)	rad/s	13
ω_{res}	Resonant (ang.) freq. (max forced amplitude)	rad/s	27
ω^*	Angular freq. of an underdamped free oscillator	rad/s	16
ϕ	Angular variable	rad	8
Q	Quality factor (underdamped oscillator)		19
Q'	(The other) quality factor (forced oscillator)		28
$\mathcal{R}$	Reflection coefficient		62
s	Arc length position	m	8
T	Period; Tension	s; N	4, 36, 56
T_{beat}	Beat period	s	41
T^*	Oscillation period of an underdamped oscillator	s	17
τ	Decay time of an underdamped oscillator $= 1/2\gamma$	s	19
$\mathcal{T}$	Transmission coefficent		62
U	Potential Energy	$\text{J} = \text{kg}\cdot\text{m}^2/\text{s}^2$	9, 18
v	Velocity or wave velocity	m/s	4, 36
v_g	Group velocity of of a wave	m/s	51
v_p	Phase velocity of of a wave	m/s	49
Z	Impedence of a string	$\text{N}\cdot\text{s/m} = \text{kg/s}$	65
ζ	Trial exponent for char. sol'n of damped harmonic oscillation	s^{-1}	14

PH10005/53: Vibrations, Waves, and Optics
Course Syllabus Semester I, 2021/22

Lecturers:

- Dr. David Tsang, D.Tsang@bath.ac.uk; 3W2.2A.
- Dr. Alex Narduzzo A.Narduzzo@bath.ac.uk; 3W3.11.

Goals: The aims of this unit are to introduce students to the fundamental concepts and mathematical treatment of mechanical vibrations and physical waves, to explore various phenomena arising from the superposition of two or more waves, and to outline some of the general principles governing the propagation of light.

Requisites: While taking this unit you should be taking or have taken PH10007 or equivalent.

Credit and Assessment: 6 Credits. This course is 100% exam.

Online Course Information: The course Moodle `https://moodle.bath.ac.uk/course/view.php?id=1860` is where documents, notes, problem sets and solutions will be posted regularly. There will also be a detailed scheduled of the material covered on the moodle.

Contact Hours: There will be two 50 minute Lectures per week, along with either office hours or a Problem Class/Tutorial. You are expected to attend both the Lectures and the Problem Classes. ***This class is 100% Exam Assessment*** which means that Problem Classes are ***essential*** for providing feedback and guidance for problems similar to the ones on the exam. We will also have a mock exam session in week 6, to provide practice for the exam. Dr. David Tsang will be presenting the first 11 Lectures of material, covering Vibrations and Mechanical Waves, while Dr. Alex Narduzzo will be the lecturer for the second half (Optics). ***The final exam will take place during the Semester 1 exam period (after the winter break).*** See the Moodle or `https://mytimetable.bath.ac.uk/` for exact time and location before the exam. Extra office hour times and lecturer availability after the winter break will be posted to the course page.

Learning Outcomes

After taking this unit the student should be able to:

- analyse oscillating systems under different driving and damping regimes;
- apply the wavefunction for a one-dimensional travelling wave to problems involving mechanical, acoustic, water and electromagnetic waves;
- state the principle of superposition and use it to solve problems involving the superposition of more than one wave;
- define and derive the impedance of a mechanical wave and apply it to reflection and transmission at interfaces;
- construct ray diagrams for use in solving simple geometrical optics problems;
- outline the mathematical analysis of multiple-beam interference;
- derive mathematical expressions for simple diffraction patterns and relate the limits imposed by diffraction to the performance of optical instruments.

Recommended Textbooks

(Library List: `http://www.bath.ac.uk/library/gen/leganto/index.php?course=PH10005`)

1. ***P.A. Tipler & G.P. Mosca,*** **Physics for Scientists and Engineers**, (W.H. Freeman and Company, 4th or 5th Edition), 1999 or later —-This textbook contains most of the basic syllabus, but you will need to supplement it with additional material.

2. ***H.J. Pain*, The Physics of Vibrations and Waves**, (Wiley, Chichetser, 6th or any other edition), 2005 —- This book covers all the required material but in a more mathematical way than Tipler.
3. ***E. Hecht*, Optics**, (Addison Wesley, any edition) —- A very good book on optics.
4. ***D. Halliday, R. Resnick & J. Walker*, Fundamentals of Physics** (Wiley, 8th edition, extended), 2008 —- Similar to Tipler & Mosca, contains most of the basic syllabus.

Course Schedule and Location

	Week Starts	Monday (16:15) CB 1.11	Tuesday (11:15) 2W Univ. Hall	Wednesday (9:15) CB 1.10
1	4 Oct	Lecture 0 (DT)	Lecture 1 (DT)	Lecture 2 (DT)
2	11 Oct	**Prob Class 1 (DT)**	Lecture 3 (DT)	Lecture 4 (DT)
3	18 Oct	Office Hour (DT)	Lecture 5 (DT)	Lecture 6 (DT)
4	25 Oct	**Prob Class 2 (DT)**	Lecture 7 (DT)	Lecture 8 (DT)
5	1 Nov	Lecture 9 (DT)	Lecture 10 (DT)	Lecture 11 (DT)
6	8 Nov	**Prob Class 3 (DT)**	**Mock Exam (DT)**	Lecture 12 (AN)
7	15 Nov	Lecture 13 (AN)	Lecture 14 (AN)	Lecture 15 (AN)
8	22 Nov	**Prob Class 4a (AN)**	**Prob Class 4b (AN)**	Lecture 16 (AN)
9	29 Nov	Lecture 17 (AN)	Lecture 18 (AN)	**Prob Class 5(i) (AN)**
10	6 Dec	**Prob Class 5(ii) (AN)**	Lecture 19 (AN)	Lecture 20 (AN)
11	13 Dec	Lecture 21 (AN)	Lecture 22 (AN)	**Prob Class 6 (AN)**

- CB1.11 and CB1.10 are in the Chancellor's Building.

- Dr. D. Tsang's office hours are in 3 West 2.2A.
- Dr. A. Narduzzo's office hours are in 3 West 3.11.

During Lectures:

- **Lecture Format:** The Lectures will mainly focus on discussing the course material using the ***skeletal notes*** (see below) that will be provided on the Moodle, so that you have a complete set of notes for your reference by the end of semester. There will also be quick comprehension quizzes using online tools, as well as slides, movies, topical examples from real research, and demos as appropriate. Readings will be suggested corresponding to each Chapter of these notes. It is recommended you at least skim the reading before the corresponding class, so that you can identify any areas that need clarification. Questions are highly encouraged! If you have a question, it is very likely that someone else has the same question, so please ask, either on the Moodle Forum, via email, or during class!
- **Skeletal Notes:** The skeletal notes have been written so that you can "fill in the blanks" in the framework, along with the lecturer, so that you can focus on the content without needing to frantically scribble down everything that is said. ***Please make sure you have a copy of the notes at hand when attending the Lectures or watching the Lecture recordings.*** The goal for the skeletal notes is for you to actively fill these in during the Lectures, ending up with a complete set of filled in notes, along with ample margins to write down any extra notes you may want from class. Filling in these notes during lecture is a valuable tool that has been shown to significantly help with material retention and exam preparation.
- **Full Notes:** The full/filled-out version of each week's notes will be made available at the end of that week, so that you can check over your notes/correct anything you missed. If you have a Disability Access Plan (DAP) in place, you will also receive access to the full notes at the beginning of the course. Note that these alone cannot replace attending Lecture, as they are typically

supplemented by class discussion and in-class slides/movies/demonstrations.

- **Accessibility Options:** We have also produced a larger/more easily read version of the course notes which will also be posted online. ***If you would prefer to have a printed version of these more accessible skeletal notes rather than the smaller-print version, please feel free to contact the lecturers privately and we can arrange to have these printed for you, regardless of your official DAP status.*** We are also working on tagged PDF notes for screen-readers but do not have them currently ready. If you require such notes please let us know and we will expedite this.
- **Kahoot! Quizzes:** We will occasionally have interactive quizzes during the LOIL sessions to help assess class-wide comprehension. These are managed through any device that has web access, however participation is purely optional/for fun, and will not contribute to your final mark. If he can figure out the logistics of such, Dr. Tsang will offer prizes to the top participants of each quiz.

Problem Set/Tutorial Sessions:

- **Note that homework is not assessed:** Since this course is 100% Exam, the problem sets are not assessed and do not contribute to your final mark. However, doing the problem sets and attending tutorials/problems classes is ***essential*** preparation for the exam. It is up to you to check your progress by attempting the problems in a timely fashion. If you find that you are struggling with solving the bulk of the problems, for any reason, ***contact the lecturer as soon as you can***, either during office hours or via email. We want you to do well in this course, and will help you as much as we can!
- **Tutorials/Problem Sheets:** Each Tutorial/Problem sheet will be posted at the beginning of the appropriate part of the course. The tutorials will include a few conceptual problems for group discussion, as well as sample problems that comes with short (usually numerical) answers written upside down in the margin, along with some checkboxes for use during your revision. In lieu of marks, a vague "difficulty" of the problem has been given in terms of number of stars, from 1 to 3. This should give you an idea of how much time they would ideally take you (in an exam), though often there may be multiple correct ways to approach a problem that take very different amounts of time. Some 3 star problems will be easy if a certain clever trick is used, while some 1 star problems may take a long time if a brute force approach is taken. Problems are also labeled by type, to give you an indication of what they are meant to assess.
- **Problem Solutions:** Full worked solutions will be posted after the Problems Classes have occurred for you to download. These will provide an example for the amount of workings and the clarity that will be expected in order to obtain full marks on an exam question!
- **Tips for Solving Problems:**
  - ***Read each question carefully*** and try to assess how much time you should spend on each problem (particularly for the exam!).
  - ***Identify what information you are given*** in the question. Circle, highlight, or rewrite the important information.
  - ***Identify what you need to find*** to answer the question. Making sure you understand what is being asked for is the first step to answering it!
  - ***Draw a diagram!*** This is almost always a good idea, particularly if you are a visual learner. Drawing a cartoon or sketch of the physics or problem setup can help make the problem more concrete.
  - ***Use your words!*** Try to write out the key parts of your thought process while you are solving the problem. A stream of pure algebra is hard to give marks to unless we happen to follow your logic. The easier it is to follow your logic, the easier it will be for us to give you full or partial credit on the exam! (Also circle/box your answers!)
  - ***Work in groups!*** For your homework and revision, it is highly encouraged for you to work in groups (in a socially distanced way, either online or during the IPT sessions). Catching

each other's mistakes and complementing each other's strengths and weaknesses is what makes group learning so effective. (Note: Please do not work in groups during the exam!)

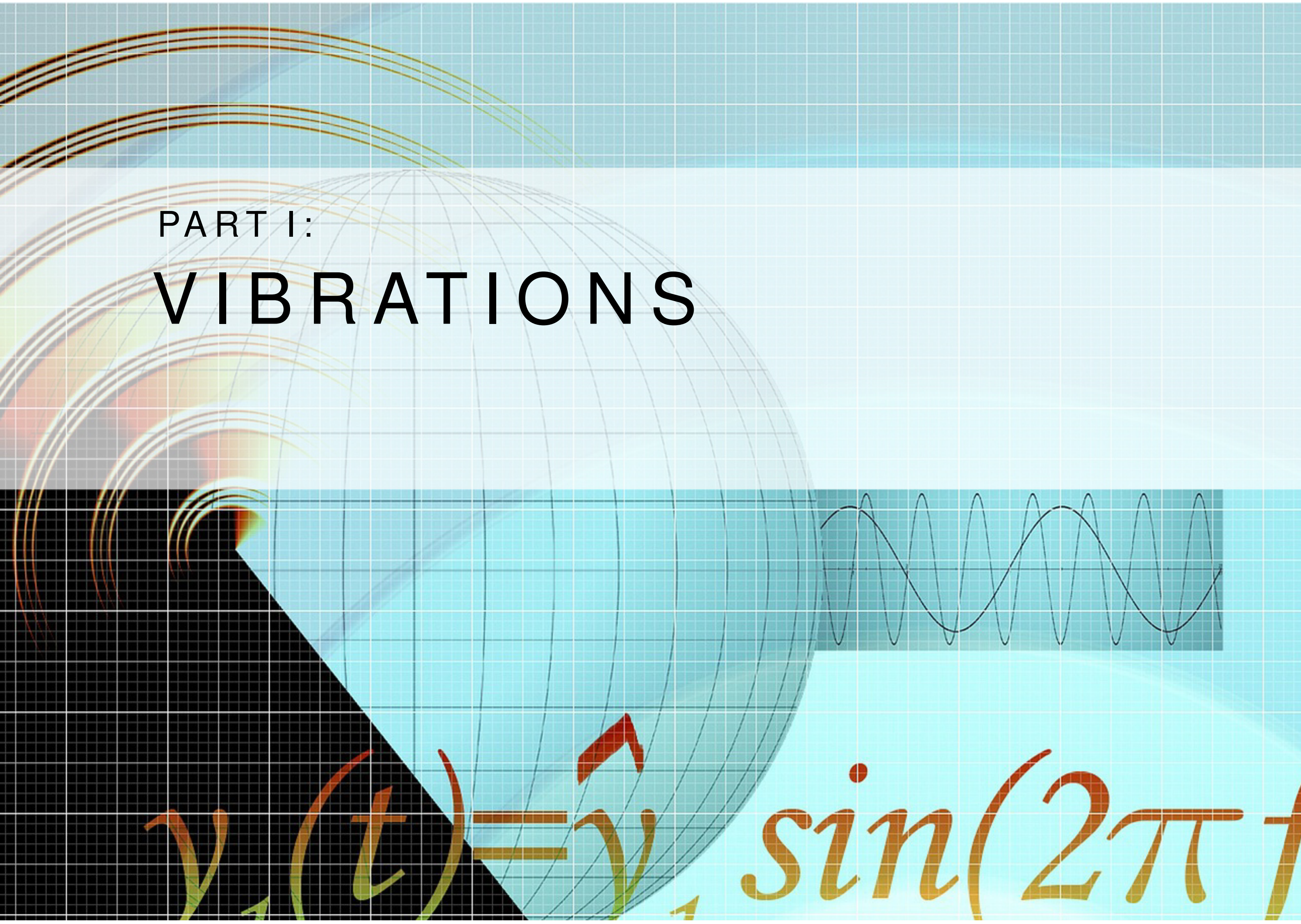
Exam and Revision Advice

- **Revision Resources:**
  - **Previous Exams:** These are one of the best sources for practice during revision, and also to gauge the difficulty that you can expect on this year's exam. ***During the problem classes we will discuss a significant number of previous exam problems that may be particularly relevant***. Those that pay attention during the problem classes should know how to deal with similar problems. You can find previous exam papers via the Moodle, or via `https://www.bath.ac.uk/library/exampapers/index.php?code=PH10005`. You will find that the questions mostly test your understanding rather than memory.
  - **Summaries:** Course summaries will be provided for a condensed overview of the course notes and will posted on the Moodle towards the end of term. These (and the section summaries of the Lecture Notes) will review the most important definitions and formulae. It is not necessary to memorise all this material but you need to understand it. The summaries will also tell you in which cases you need to know derivations of the formula.
  - **Problem Solving Skills:** Your mark will strongly depend on your problem solving skills. You can develop these best through the problem sets, particularly by attempting them first without looking at the solution. Each problem has a checklist to help you keep track of your revision and review. ***Your efforts here will be rewarded as in many cases exam questions are very similar to problems from the problem sheets!*** See the **Tips for Solving Problems** above as well.
  - **Mock Exam:** A mock exam for the first half of the course will posted sometime in week 5/6, with the solution and marking guide discussed during the LOIL session of week 6. You should do the practice exam before that session.
- **Exam Advice:**
  - You may estimate the time required for each exam question by looking at the number of marks it is worth. One mark may be taken to roughly correspond to 2 minutes of time. During your exam ***do not waste your time if you get stuck with a particular question***. Skip it and try another one, and come back if you have time.
  - **Typical Structure of Exam Papers, 6 or 7 questions covering the following topics:**
    - Simple Harmonic Motion (SHM)
    - Energy Conservation in SHM
    - Damped Oscillations
    - Forced Oscillations and Resonance
    - Mechanical Waves (speed superposition, boundary conditions)
    - Phase and Group velocities
    - Geometrical Optics
    - Electromagnetic Waves
    - Interference (Young's experiment, M&M Interferometer)
    - Diffraction and Resolution of Optical Devices

University of Bath Principles of Academic Integrity: The University's principles of academic integrity are set out in `http://www.bath.ac.uk/quality/documents/QA53.pdf`. Any violation of these principles will be referred directly to the appropriate Director of Studies for investigation.

Congratulations! You made it to the end of the syllabus! To demonstrate that you have read this document, please email Dr. Tsang a nice picture of a dog (or animal of your choice) before Lecture 1.

PART I:
VIBRATIONS



1
Simple Harmonic Motion

Introduction

§Suggested Reading:

- ***Pain***, Chapter 1, Simple Harmonic Oscillators
- ***Tipler***, Chapter 14, Oscillations

Periodic motion IS MOTION of an object that regularly repeats, i.e. the object returns to a given position after a fixed time interval.

Example 1.1: Examples of Periodic Motion

-
-
-
-
-
-
-

A very important kind of periodic motion is ***Simple Harmonic Motion (SHM)***. In a certain sense, SHM describes the simplest possible periodic motion. Understanding SHM well can lead to a physical understanding of all kinds of dynamical systems, such as those in the examples above.

1.1 Mechanics of Simple Harmonic Motion

In oscillating mechanical systems there is always a force that acts to reduce displacement of an object and return the system to equilibrium. This force is called the ***restoring force***, F_r.

Simple Harmonic Motion

SHM occurs whenever the restoring force is linearly proportional to the displacement from equilibrium and opposite to the direction of the displacement, i.e.

$$\vec{F}_r \propto -\vec{x}. \tag{1.1}$$

Any (stable) system that oscillates, ***if the oscillation is small enough***, will appear to be undergoing simple harmonic motion.

Consider the mass m attached to a spring, shown in Figure 1.1. When the mass is ***in equilibrium*** there is no (net) force on it. When

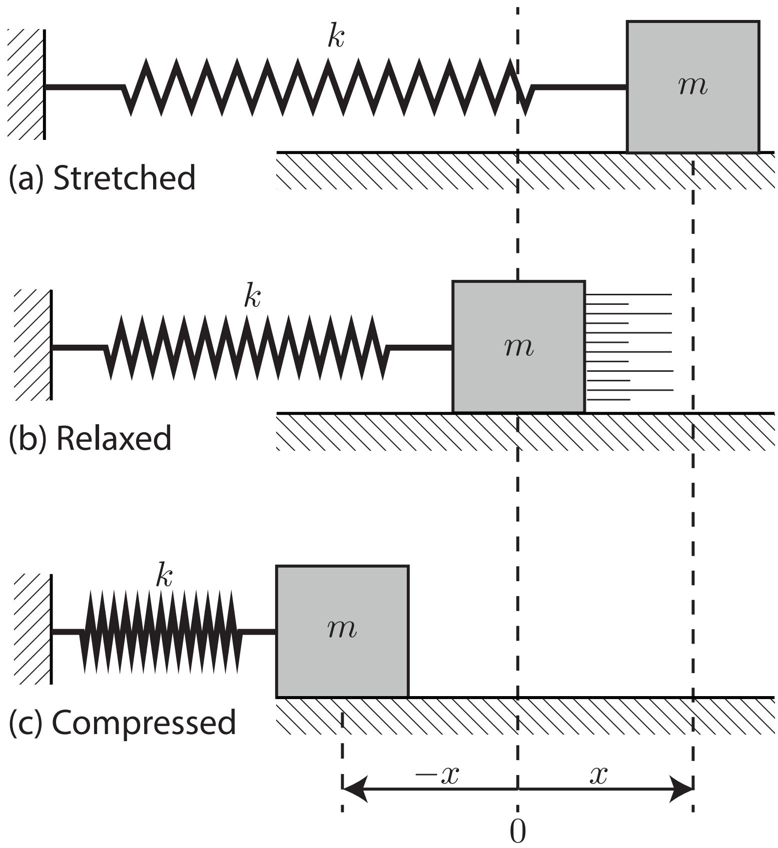


Figure 1.1: In this model our assumptions are: ***(1)*** There is no friction; ***(2)*** The mass of spring is negligibly small in comparison with the mass of the block; ***(3)*** The spring only responds linearly.

the mass is displaced, the spring exerts a ***restoring force*** $-kx$, as given by ***Hooke's Law***

Hooke's Law

(1.2)

where k is the constant of proportionality between the displacement and the restoring force, called the ***spring constant*** (or ***force constant***). The minus sign in Hooke's law arises because the force is in the opposite direction of the displacement.

Using Newton's Second Law of Motion we get

$$\vec{F}(x) = m\vec{a}(x) \qquad \text{or} \qquad -k\vec{x} = m\vec{a} \tag{1.3}$$

This can be written as

(1.4)

If we calculate the dimensions of k/m then we find it has the dimensions of $[\text{Time}]^{-2}$ so we can write,

(1.5)

The differential equation for SHM

where $\omega = \sqrt{k/m}$ is an ***angular frequency*** which has units of radians/second.

Equation (1.5) is a second order differential equation. It is often referred to as the ***differential equation of Simple Harmonic Motion***.

Its solution can always be written in the form,

(1.6)

where A and δ are determined by the initial conditions (i.e. the displacement and velocity at $t = 0$). According to Eq. (1.6), the function $x(t)$ has the form:

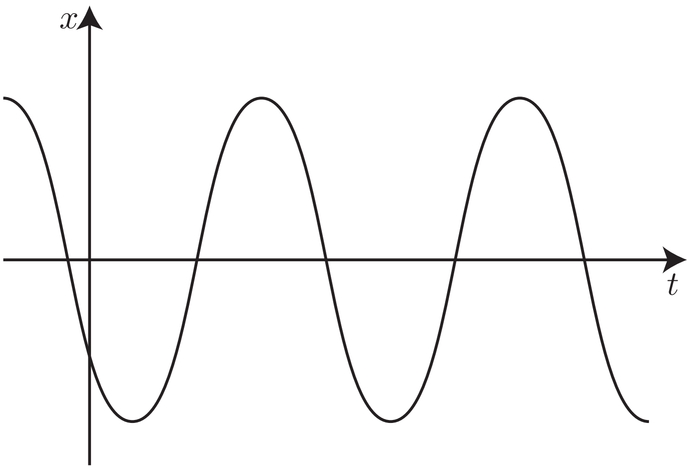


Figure 1.2: ***Units:*** Period T is in 'seconds'; Frequency $f = 1/T$ is in 'Hertz'; Angular frequency $\omega = 2\pi f$ is in 'radians per second'; and initial phase δ is in 'radians'.

The period of the oscillation T is given by

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \tag{1.7}$$

Period in terms of angular frequency

while the frequency f has units of $\mathrm{Hz} = \mathrm{s}^{-1}$.

1.2 *Displacement, Velocity, and Acceleration*

If we have the solution $x = A\cos(\omega t + \delta)$, then the velocity, v, is

SHM solutions for x, v, and a.

(1.8)

and the acceleration, a, is

(1.9)

For initial phase $\delta = 0$, the equations simplify to:

$$x = A\cos\omega t \tag{1.10a}$$

$$v = -\omega A\sin\omega t \tag{1.10b}$$

$$a = -\omega^2 A\cos\omega t. \tag{1.10c}$$

Let's plot out the above solutions for $x(t)$ assuming $\delta = 0$:

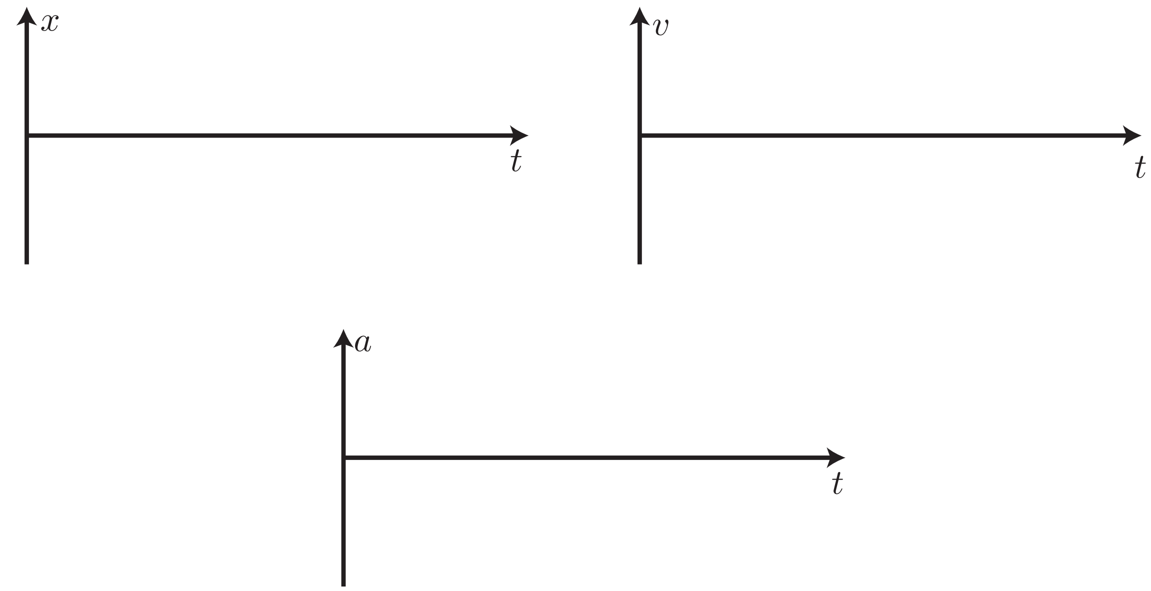


Figure 1.3: Position, Velocity and Acceleration for SHM with $\delta = 0$.

- Velocity is $\pi/2$ out of phase with displacement.
- o - velocity is maximum/minimum when displacement is zero.
- x - Velocity is zero when displacement is maximum/minimum.
- Acceleration is proportional to displacement and acts in the opposite direction to the displacement.

Note that the ***minimum*** isn't the zero point. The displacement, velocity, and acceleration oscillate about zero, ***the equilibrium point***.

The ***frequency*** and ***period*** are related to the ***stiffness*** k of the spring and the ***mass*** of the particle

f and T in terms of k and m for a SHM

$$f = \qquad \text{and} \qquad T = \tag{1.11}$$

Note that the ***spring consant*** k is only a constant for small displacements.

Large displacements could cause (for example) permanent deformation of the spring and the system would not oscillate with SHM. This reminds us that the SHM is typically only found for small displacements.

Example 1.2: An air-track glider

AN AIR-TRACK GLIDER ATTACHED to a spring oscillates with a period of 1.5s. At $t = 0$ the glider is 5cm left of the equilibrium position and moving to the right at 36.3 cm/s.

(a) What are the amplitude and initial phase of the oscillations?

(b) Write down an expression that describes the position of the glider as a function of time

(c) What is the glider's position at $t = 0.5$s?

(a)

(b)

(c)

1.3 Further examples of Simple Harmonic Motion (SHM)

From our basic definition, we observe SHM when the restoring force is linearly proportional to displacement.

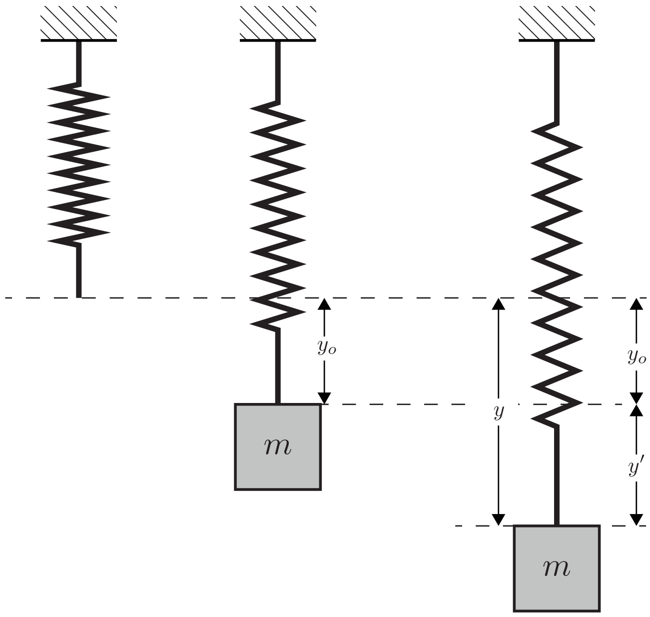
Then for any simple harmonic motion we can write out the expression for the ***force constant*** k,

$$k = -\frac{F_x(t)}{x(t)} \qquad \text{so that} \qquad \omega = \sqrt{\frac{k}{m}} \tag{1.12}$$

where $x(t)$ is measured from the ***equilibrium*** point.

Example 1.3: An Object on a Vertical Spring

Consider a mass hanging on a vertical spring:



We need to take into account the gravitational force mg in addition to the force of the spring $F_s = -ky$ where y is the vertical displacement. In equilibrium with the gravitational force, the spring is extended by a length y_o.
In order to find the equilibrium position, y_o, we need to apply Newton's Second law:

$$\sum F = mg - ky_o = 0 \tag{1.13}$$

So the equilibrium position must be,

(1.14)

Away from equilibrium, we have

(1.15)

This differs from Eq. (1.3) by a constant term mg.
We can handle this extra term by changing to a new variable

$$y' = y - y_o \tag{1.16}$$

such that we have

So that for y' Eq. (1.15) reduces to

(1.17)

and its solution is

(1.18)

Sketch the solution $y(t)$

Example 1.4: The Simple Pendulum

CONSIDER THE SIMPLE PENDULUM, as shown in Figure 1.4.
We model this by assuming that m is a ***point mass*** connected to the anchor point by a ***massless, inextensible string***.
The path of the mass traces out part of a circle. We will call the arc length along this path s.
Let's also call the tension force in the string $\vec{T}$, which pulls the mass towards the anchor point.
The restoring force along the arc of the circle is:

(1.19)

The tangential component of the acceleration is $\frac{d^2s}{dt^2}$, which lets us write the tangential component of Newton's second law as

(1.20)

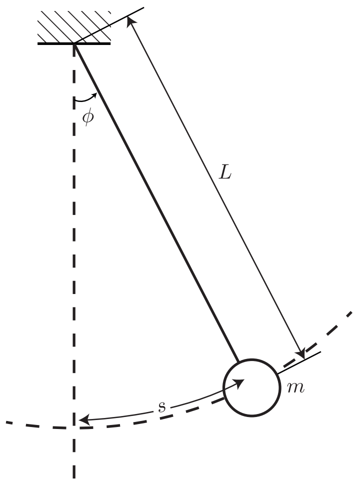


Figure 1.4: The simple pendulum of mass m and length L, at angle ϕ

For small angles $\sin\phi \simeq s/L$:

$$\frac{d^2s}{dt^2} = \tag{1.21}$$

and $s = s_0\cos(\omega t + \delta)$ with $\omega = \sqrt{g/L}$.
Thus for ***small displacements***, the period of the pendulum is ***independent*** of mass of the bob, and depends only on the length of the pendulum.
Notice that the pendulum ***does not*** exhibit true SHM for all angles. If the angle is less than around $10°$, the motion is close to, and can be ***modelled*** as, simple harmonic.

1.4 Energy in Simple Harmonic Motion

We remember from basic mechanics that for any ***conservative*** force that there is a direct link to the potential energy of the system, given by

$$\tag{1.22}$$

For SHM we know that $F_x = -kx$, so we can write

$$U(x) = \int kx\ dx \tag{1.23}$$

$$\tag{1.24}$$

This quadratic/parabolic dependence of potential energy on displacement is a ***general result*** for anything moving with SHM.

The kinetic energy of the oscillator is given by

$$E_{\text{kin}} \tag{1.25}$$

and the total energy at any moment of time is

$$\tag{1.26}$$

For any oscillator we can write

$$x = A\cos(\omega t + \delta)$$
$$v = -A\omega\sin(\omega t + \delta)$$
$$k = m\omega^2$$

therefore

$$E_{\text{tot}} = \frac{1}{2}kA^2\cos^2(\omega t + \delta) + \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \delta)$$

$$= \tag{1.27}$$

Total Energy of SHM

From this we can conclude that ***energy in SHM is conserved***!

1.5 *The General Applicability of Simple Harmonic Motion*

IN SHM, THE RESTORING FORCE is zero ($F_r = 0$) at the equilibrium point. As $F_r = -\frac{dU}{dx}$, that means the potential energy at this point has either a local maximum or a minimum.

For ***small displacements***, we may expand $U(x)$ in a Taylor Series:

$$U(x) = U(0) + x\frac{dU}{dx}(0) + \frac{1}{2}x^2\frac{d^2U}{dx^2}(0) + \ldots \quad (1.28)$$

If we chose the zero point of the potential energy such that $U(0) = 0$, then the first two terms of the equation above are equal to zero. Thus we can write that, approximately,

(1.29)

Potential Energy for SHM

where $k = \frac{d^2U}{dx^2}(0)$.

What happens if $k = \frac{d^2U}{dx^2}(0) < 0$***? Clearly the SHM differential equation and solution no longer apply. Is this a stable equilibrium?***

Why is the Simple Harmonic Oscillator a good approximation for so many systems we encounter?

The small-amplitude motion near almost any point of equilibrium is described approximately by a parabolic potential well. With increasing amplitude of oscillations, contributions of terms with higher derivatives in Eq. (1.28) becomes larger and larger and the motion will be ***anharmonic***.

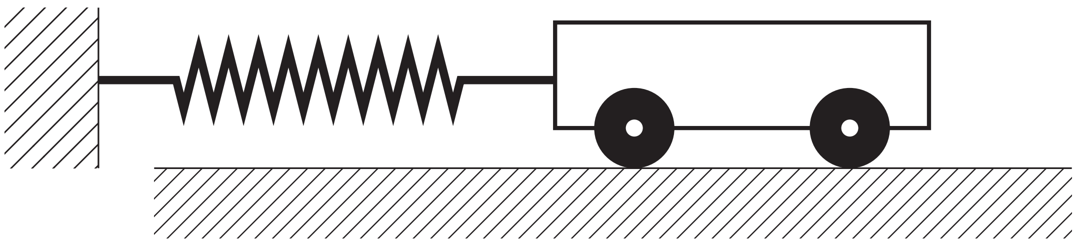
Example 1.5: Energy of an Oscillating Cart

Consider a 0.5 kg cart connected to a light spring for which the force constant is 20 N/m. The cart oscillates on a horizontal, frictionless airtrack.

(a) Calculate the total energy of the system and the maximum speed of the cart if the amplitude of the displacement is 3 cm.

(b) What is the velocity of the cart when the position is 2 cm?

(c) Compute the kinetic and potential energies of the system when the position is 2 cm.



Section Summary

Summary of Simple Harmonic Motion

- ***Simple Harmonic Motion***

  SIMPLE HARMONIC MOTION will occur whenever an object moves from equilibrium under a restoring force that is linearly proportional to the displacement from equilbirium, i.e. it moves with a linear restoring force.

- ***Defining Equation for SHM***

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

- ***General Solution***

$$x = A\cos(\omega t + \delta)$$

- ***Proving that a motion is simple harmonic motion***

  WE CAN DEMONSTRATE that a motion is SHM if we can show that, following the definition above, $F_x = -kx$ where k is a constant – the force constant of the motion.

- ***Universality of Simple Harmonic Motion***

  FOR SMALL DISPLACEMENTS FROM EQUILIBRIUM, in most potential wells, the motion of a displaced mass can be approximated to be simple harmonic motion with a force constant, $k = U''(x = x_{\text{eq}})$.

2

Damped Harmonic Oscillations

Introduction

§***Suggested Reading***
Pain, Chapter 2, Damped Simple Harmonic Motion
Tipler, Chapter 14, Oscillations

IN REAL OSCILLATING SYSTEMS the mechanical energy of the system diminishes in time, and the motion is said to be ***damped***. The mechanism of damping is ***friction*** or ***viscosity***. We will assume that the magnitude of the ***damping force*** can be taken to depend ***linearly*** on the relative speed v between the parts of the system:

A linear damping force

(2.1)

where b is the ***damping force constant***. This is normally a good approximation for low velocities.

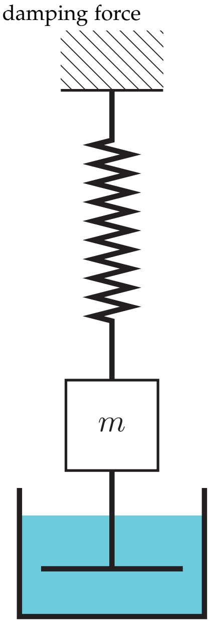


Figure 2.1: A simple example of a damped oscillator. The motion is damped by the plunger immersed in a liquid that experiences viscous drag.

2.1 The Differential Equation for Damped Harmonic Motion

LET'S CONSIDER A SPRING AND DAMPER attached to a mass. Applying Newton's second law we get:

(2.2)

where $\vec{F}_r$ is the restoring force and $\vec{F}_d$ is the damping force.

If the oscillations occur along the x-axis only, the above equation reduces to

(2.3)

from which we can get,

(2.4)

This is called the ***equation of motion of a damped harmonic oscillator*** This equation is often presented in the form,

Equation of motion of a Damped Harmonic Oscillator

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = 0 \qquad (2.5)$$

where $\gamma \equiv b/(2m)$ is the ***damping constant*** and $\omega_o = \sqrt{k/m}$ is the called the ***natural frequency***, which is the angular frequency of oscillations which would have occurred without the damping.

2.2 *Solutions for Damped Oscillations*

EQUATION (2.5) IS a second order differential equation. It's general solution is derived in the maths module.

A reminder of the technique (***you must be able to work through this on your own!***): For the case of $\gamma \neq \omega_o$ the trial solution

(2.6)

gives the ***characteristic equation***,

(2.7)

so ζ has two possible values,

(2.8)

THERE ARE THREE CASES to be considered:

The 3 Types of Damped Harmonic Motion

(a)	$\gamma > \omega_o$	***Heavily Damped*** or ***Overdamped***
(b)	$\gamma = \omega_o$	***Critically Damped***
(c)	$\gamma < \omega_o$	***Lightly Damped*** or ***Underdamped***

2.2.1 **(a) Overdamped ($\gamma > \omega_o$)**

THE CONDITION $\gamma > \omega_o$ means that $b^2 > 4mk$ (show this!) which implies that the viscous friction force term dominates the restoring force causing heavy damping of the oscillator.

The characteristic equation above has two ***real*** distinct roots ζ_+ and ζ_-. The general solution must then be

$$x(t) = A_+ e^{\left(-\gamma+\sqrt{\gamma^2-\omega_o^2}\right)t} + A_- e^{\left(-\gamma-\sqrt{\gamma^2-\omega_o^2}\right)t} \qquad (2.9)$$

where A_+ and A_- are arbitrary constants of integration which have to be determined from the initial conditions, or the positions and velocities at some other instant.

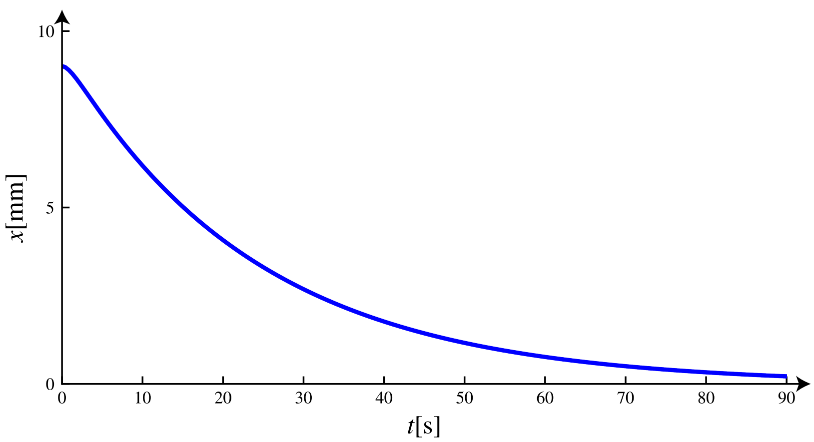


Figure 2.2: Heavily Damped/Overdamped. A mass displaced 9mm and released from rest with $\omega = 0.2\text{s}^{-1}$, $\gamma = 0.5\text{s}^{-1}$, $A_+ = 9.4$mm, $A_- = -0.4$mm. Note that there are ***no oscillations***! The displaced mass creeps back or relaxes, to its equilibrium position very slowly.

2.2.2 (b) Critically Damped ($\gamma = \omega_o$)

When the damping constant matches the natural frequency, this solution leads to a simplification of Equation (2.5), such that

$$ \tag{2.10} $$

However, we have a degenerate solution to the characteristic equation

$$ \tag{2.11} $$

such that $\zeta_+ = \zeta_- = -\gamma = -\omega_o$. But we need 2 independent solutions (or constants of integration) in order to be able to satisfy any initial conditions for a 2nd order ODE. The first solution is the one we are familiar with, $x \propto e^{-\gamma t}$. The second turns out to be of the form,

which gives us the solution in general,

$$x(t) = \tag{2.12}$$

Critical Damping Solution

Here A and B are arbitrary constants determined by the set of initial conditions. The exponential term is never zero. The term $(At + B)$ for some initial conditions, can give one pass of the system through its equilibrium position, but the motion is not recognizable as an oscillation.

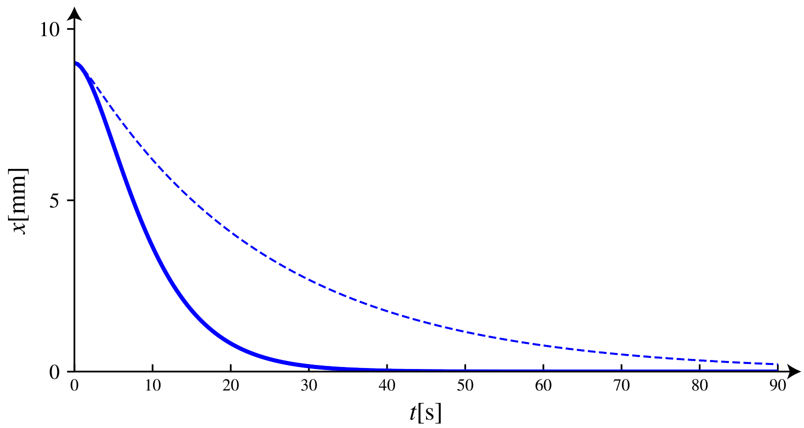


Figure 2.3: ***Critical damping*** of a mass displaced by 9mm and released from rest with $\omega_o = \gamma = 0.2\text{s}^{-1}$, corresponding to $A = 1.8$mm and $B = 9$mm. This is the same setup as in Figure 2.2, but with damping tuned to be critical.

A typical motion of this kind is shown in Fig. 2.3 above. The return to the equilibrium position is faster than in the case of overdamping. ***Critical Damping results in a fast decay*** of the object back to its equilibrium point.

Critical damping is a employed in many practical circumstances, e.g. shock absorbers in suspension systems, in the design of electrical meters, in mechanical gauges etc.

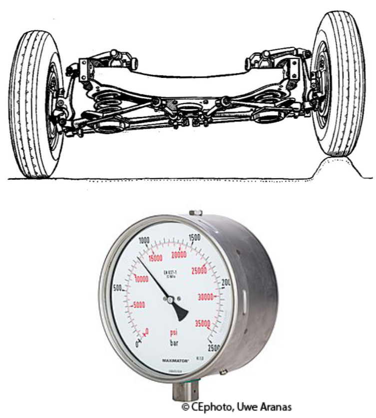


Figure 2.4: Applications of critical damping.

2.2.3 (c) Underdamped or Weak Damping ($\gamma < \omega_o$)

THE MOST PHYSICALLY (AND MATHEMATICALLY) interesting case is weak damping, or underdamping. The overdamped solution (2.9) is still valid, but the exponents are now ***complex quantities***:

$$\begin{aligned} x(t) &= e^{-\gamma t}\left[A_+ e^{i\sqrt{\omega_o^2-\gamma^2}t} + A_- e^{-i\sqrt{\omega_o^2-\gamma^2}t}\right] \\ &= e^{-\gamma t}\left[(A_+ + A_-)\cos\left(t\sqrt{\omega_o^2-\gamma^2}\right)\right. \\ &\qquad \left. + i(A_+ - A_-)\sin\left(t\sqrt{\omega_o^2-\gamma^2}\right)\right] \end{aligned} \tag{2.13}$$

In general, A_+ and A_- must be complex quantities, but since $x(t)$ is ***real***, we must have $(A_+ + A_-)$ and $i(A_+ - A_-)$ both ***real***[1]. Rewriting these, respectively, as C and D we get

[1] This implies that A_+ and A_- are complex conjugates.

(2.14)

which is one way of expressing the general solution.

An alternative form of the solution is obtained by introducing the phase constant δ,

$$x(t) = ae^{-\gamma t}\cos\left[t\sqrt{\omega_o^2-\gamma^2}+\delta\right] \tag{2.15}$$

where a is the "amplitude" of the motion at time $t = 0$ (not to be confused with the acceleration).
THIS SOLUTION is conveniently expressed as

(2.16)

Solution for an Underdamped Oscillator

(2.17)

Underdamped Oscillator Angular Frequency, ω^*

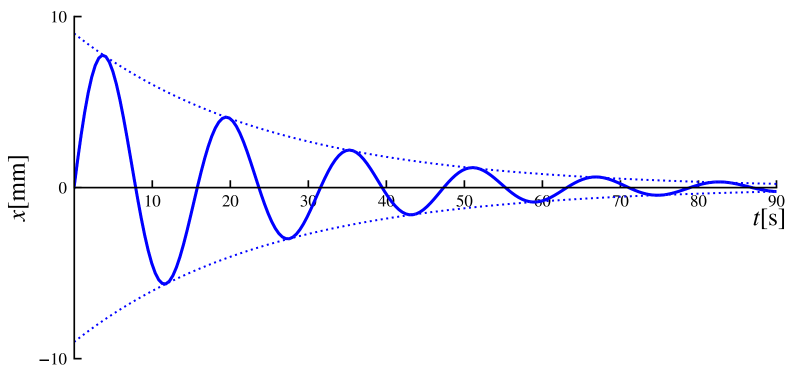


Figure 2.5: ***Underdamped Motion*** of an oscillator with initial amplitude $a = 9$ mm, natural frequency $\omega_o = 0.4\text{s}^{-1}$, damping constant $\gamma = 0.04\text{s}^{-1}$, and initial phase $\delta = -\pi/2$. The mass oscillates within an envelope described by an exponential decay.

Now this is similar to the solution for a undamped simple harmonic oscillator $x(t) = A\cos(\omega t + \delta)$ but the amplitude term A has been replaced by $A \to ae^{-\gamma t}$, and the angular frequency term has become $\omega \to \omega* = \sqrt{\omega_o - \gamma^2}$.

The ***amplitude of the motion*** decreases exponentially with time, while the ***frequency of oscillations*** is smaller than the natural frequency ($\omega^* < \omega_o$). The ***period*** of the motion, T^*, can be defined as the peak-to-peak time, or twice the time between the zero crossings.

(2.18)

Period of an underdamped oscillator

2.3 *Logarithmic Decrement*

THE RATE AT WHICH THE AMPLITUDE of the oscillations dies away is characterised by the ***logarithmic decrement*** β of the decay. This is just the natural logarithm of two successive peaks (i.e. the value of the maxima that are separated by one period):

(2.19)

Logarithmic Decrement

For weak/underdamped oscillations, it can be shown that the maximum displacements to occur at times

$$t_n = nT^* - 2\pi\delta^*/\omega^*,$$

where n is an integer. Here $\delta^* \approx \delta$ is a slightly shifted phase because the peak location is shifted slightly back from where $\cos(\omega^* t + \delta) = 1$ because of the decaying envelope. ***See if you can find this extra phase shift and show that it is constant for a given system!*** The amplitudes at t_n and t_{n+1} are

$$\begin{aligned} a_n &= ae^{-\gamma n T^* + 2\gamma\pi\delta^*/\omega^*} \\ a_{n+1} &= ae^{-\gamma(n+1)T^* + 2\gamma\pi\delta^*/\omega^*} \end{aligned} \tag{2.20}$$

which gives the logarithmic decrement

(2.21)

2.4 *The Energy of the Underdamped Harmonic Oscillator*

WE SAW THAT IN THE CASE of a Simple Harmonic Oscillator the total energy of the system was conserved. Here, we've introduced a damping element, the energy is being removed from the system by the damper, so we should find that the energy decreases over time.

As before we have the total energy in the system, $E_{\text{tot}} = U + E_{\text{kin}}$,

$$\text{where} \qquad U = \frac{1}{2}kx^2 \qquad , \qquad E_{\text{kin}} = \frac{1}{2}m\dot{x}^2.$$

We can substitute in the solutions for $x(t)$ and $\dot{x}(t)$, giving

(2.22)

for the potential energy, and

(2.23)

for the kinetic energy. The total energy, E_{tot} is then

$$E_{\text{tot}}(t) = \qquad (2.24)$$

Energy of a Free Underdamped Harmonic Oscilator

The energy of the oscillator has an overall decay $\propto e^{-2\gamma t}$, with an additional decaying cosine term with twice the frequency of the oscillator. (***Why should we expect this modulation of the energy decay to be at 2× the frequency of the oscillator?***)

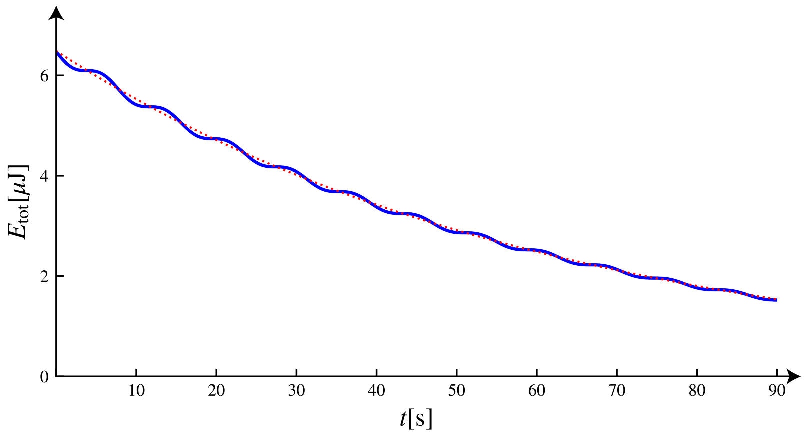


Figure 2.6: Energy evolution of an underdamped oscillator (same specifications and initial conditions as the system above, and assuming a mass $m = 1kg$), which tends to oscillate around the overall exponential decay ($\propto e^{-2\gamma t}$) at twice the damped oscillator frequency $2\omega^*$.

IN THE WEAK DAMPING LIMIT,WHERE $\gamma \ll \omega_0$, the cosine piece is very small, and the energy decays as,

$$E_{\text{tot}}(t) \approx \qquad (2.25)$$

Energy Decay in the Weak Damping Limit

where we can see that as $\gamma \to 0$ we recover the conserved total energy of the simple harmonic oscillator.

2.5 The Decay Time

WE SAW IN THE PREVIOUS SECTION that the total energy in an underdamped harmonic oscillator oscillates around an exponential decay. Indeed, from Eq. (2.24) we can see that the total energy is bounded by an envelope

$$E_+(t) \geq E_{\text{tot}}(t) \geq E_-(t) \tag{2.26}$$

where

$$E_\pm(t) = \tag{2.27}$$

The timescale for this overall exponential decay of the energy is called the ***decay time***, τ, of an underdamped oscillator, which is simply,

Decay Time for the Energy of an Underdamped Oscillator

$$\tau \equiv \tag{2.28}$$

such that the total energy follows the overall decay as $\sim e^{-t/\tau}$.

From Equation (2.24) we can also see that,

$$E_{\text{tot}}(t + T^*) = \tag{2.29}$$

such that,

$$\frac{1}{\tau} = \qquad , \tag{2.30}$$

allowing us to determine the decay time by comparing the energies of the oscillator one period apart. In the weak damping limit,

$$E(t) \approx E_o e^{-t/\tau}.$$

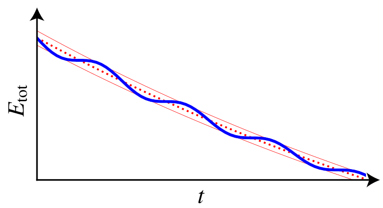


Figure 2.7: Zooming in on the total energy of an underdamped oscillator. The energy oscillates within the decaying envelope, $E_+ > E_{\text{tot}} > E_-$.

2.6 The Q-factor

THE Q-FACTOR OF AN UNDERDAMPED OSCILLATOR has two different common definitions in physics/engineering (one has to do with a forced frequency response which we will discuss later). It gives us ***a comparison of the two main timescales of the damped harmonic oscillator: the oscillator period and the timescale for energy loss***.

Here we will define ***the Quality Factor***, Q, to be ratio of the angular frequency to the fractional (averaged) energy loss rate such that,

Definition of Q-factor

$$Q \equiv \omega^* \times \left[\frac{|\langle dE_{\text{tot}}/dt\rangle_{\text{avg}}|}{E_{\text{tot}}(t)}\right]^{-1} \tag{2.31}$$

$$\tag{2.32}$$

where the power lost is averaged from time t to time $t + T^*$. From Eq. (2.29) above, so that

$$Q = \qquad (2.33)$$

Q-factor for a Free Underdamped Harmonic Oscillator

which is independent of the time t. For weak damping, $\gamma \ll \omega^*$,

$$Q \approx \qquad (2.34)$$

Q-factor for the weak damping limit

Example 2.1: A decaying piano note

WHEN MIDDLE C ON A PIANO (frequency 262 Hz) is struck, it loses half its energy after 4 s.

(a) What is the decay constant, τ?

(b) What is the Q-factor for this oscillation?

(c) What is the fractional loss of energy per cycle?

Summary of Damped Harmonic Motion

Section Summary

- WE ARE ASSUMING a damping force that is proportional to the velocity of the oscillator body,

$$\vec{F}_d = -b\vec{v}$$

- THE DIFFERENTIAL EQUATION for damped harmonic motion has the form

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

- THIS GIVES THREE DIFFERENT TYPES of solutions:

  (a) ***Overdamped*** ($\gamma > \omega_o$)
  Slow decay with solution,

$$x(t) = A_+ e^{\left(-\gamma+\sqrt{\gamma^2-\omega_o^2}\right)t} + A_- e^{\left(-\gamma-\sqrt{\gamma^2-\omega_o^2}\right)t}.$$

  (b) ***Critically damped*** ($\gamma = \omega_o$)
  Fast decay with solution,

$$x(t) = (At + B)e^{-\gamma t}.$$

  (c) ***Underdamped*** ($\gamma < \omega_o$)
  Oscillation within a envelope decaying $\propto \exp(-\gamma t)$, with solution

$$x(t) = ae^{-\gamma t}\cos(\omega^* t + \delta)$$

  where $\omega^* = \sqrt{\omega_o^2 - \gamma^2} < \omega_o$.

- FOR ***Underdamped Motion*** we can also define
  - The ***logarthmic decrement***, β,

$$\beta = \ln\frac{a_n}{a_{n+1}} = \gamma T^*$$

    where a_n and a_{n+1} are two successive amplitudes one period, $T^* = 2\pi/\omega^*$, apart.
  - The ***decay time*** τ, that describes the overall decay of the energy $E(t)$,

$$\tau = \frac{1}{2\gamma}.$$

  - The ***quality factor***, Q, of the oscillator,

$$Q = 2\pi\frac{E(t)}{\Delta E(t)} = \frac{2\pi}{1 - e^{-2\gamma T^*}}$$

    where $\Delta E(t) = E(t) - E(t + T^*)$.

- FINALLY, IN THE ***weak damping limit*** $\gamma \ll \omega_o$, we have

$$E(t) \simeq E_o e^{-t/\tau}, \qquad Q \simeq \frac{\omega^*}{2\gamma} = \omega^*\tau \simeq \frac{\omega_o}{2\gamma}.$$

3
Forced Vibrations and Resonance

Introduction

§*Suggested Reading*
Pain, ***Chapter 3, The Forced Oscillator***
Tipler, Chapter 14, Oscillators

Oscillating systems have ***natural frequencies*** ($\omega_0 = \sqrt{k/m}$) at which they vibrate when excited. (E.g., drums, bells, pendulums, etc.)

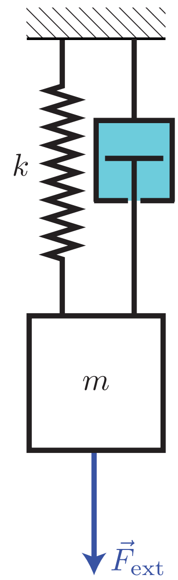
If we want to make something vibrate at a different frequency (Ω), then we must apply a ***periodic driving force*** acting with the required frequency Ω.

If the frequency of the driving force is equal to (or very close to) that of the natural frequency, ω_0, then energy is ***efficiently*** transferred from the driving force to the oscillator, increasing the amplitude. This phenomenon is called ***resonance***.

Examples of resonance:

- A child being pushed on a swing
- An opera singer shattering a glass
- The Tacoma Narrows bridge disaster (Tipler p. 503)
- The Millennium Bridge

The idea of resonance is useful for many branches of physics; scientists and engineers often want to obtain resonance in order to detect something, or avoid resonance to stop something from breaking.



3.1 The Differential Equation for Forced Oscillations

Consider a spring/damper system which is being forced at a fixed frequency. The restoring force and damping force are given by

$$\vec{F}_r = -k\vec{x}, \qquad \vec{F}_d = -b\vec{v} = -b\frac{d\vec{x}}{dt}$$

while we will take the external driving force to be

$$\vec{F}_{\text{ext}} = \vec{F}_0 \sin(\Omega t). \tag{3.1}$$

where F_0 is the maximum driving force. Applying Newton's Second law gives us

$$\tag{3.2}$$

We can make some substitutions (similar to the ones we used in the previous chapter) to rewrite Equation (3.2) into the standard form for the ***Differential Equation of Forced Oscillations*** (make sure you can show this!):

(3.3)

where we have

- Damping constant:
- Natural angular frequency:
- Maximum driving acceleration: a_0

WE ARE GOING TO FIND the ***steady state response*** of the system, ignoring the initial transient response. In fact, the transient motion is a linear combination of solutions to the equation for the undriven oscillator (found in the previous chapter) and the steady state solution that we are going to obtain.

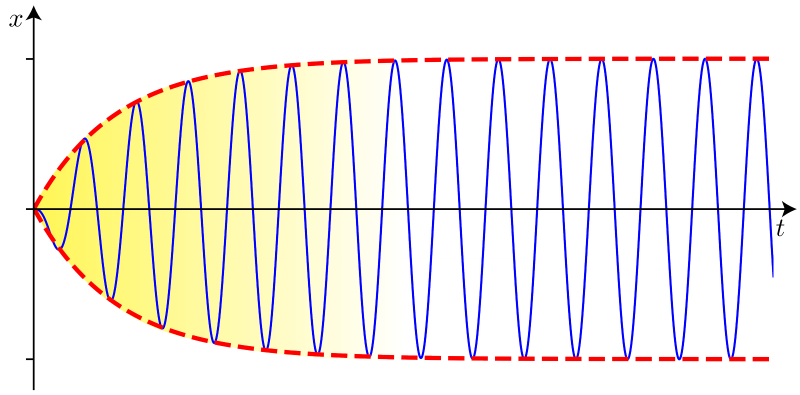


Figure 3.1: Displacement of damped harmonic oscillator, forced at the resonant frequency, starting from rest.

3.2 *The Steady State Response*

We will assume a steady state motion of the form of the trial solution:

(3.4)

The response of the system to the driving will have an amplitude, A, and a phase lag, δ, between the steady state motion and the driving force.

The Steady State Solution

Attention: This proof is examinable!

The final solution must satisfy, Equation (3.3)

If our trial solution is $x(t) = A\sin(\Omega t - \delta)$ then

so that $\ddot{x} = -\Omega^2 x$. Thus (3.3) becomes,

To solve this for A and δ we will again need to make use of a trig identity to find,

$$\sin(\Omega t) = \sin\delta\cos(\Omega t - \delta) + \cos\delta\sin(\Omega t - \delta).$$

Substituting this in for $\sin(\Omega t)$, then gathering terms, we have

(3.5)

Since this needs to hold true for ***all*** t, the coefficients of $\sin(\Omega t - \delta)$ and $\cos(\Omega t - \delta)$ must each vanish giving us the equations,

$$a_0\cos\delta = \qquad (3.6)$$

$$a_0\sin\delta = \qquad . \qquad (3.7)$$

Dividing (3.7) by (3.6) gives,

which gives the ***Phase Lag***

$$\Rightarrow \delta = \qquad . \qquad (3.8)$$

Steady State Phase Lag of a Forced Oscillator

Meanwhile, adding the squares of (3.6) and (3.7) yields

giving us the ***Amplitude Response:***

$$\Rightarrow A = \qquad (3.9)$$

Steady State Amplitude of a Forced Oscillator

3.3 High and Low Frequency Steady State Solutions

We will now investigate the form of the oscillator's response when forced at low frequencies and at high frequencies. We will obtain the general expressions, but mostly focus on the cases where the damping is relatively weak compared to the forcing and natural frequencies.

3.3.1 Low Frequency Limit: $\Omega \to 0$

In the low frequency limit we have the driving frequency much lower than the natural frequency $\omega_o \gg \Omega$, and the damping constant $\gamma \gg \Omega$. From Eq. (3.9) we have

Low Freq. Limit: $\Omega \to 0$

We have $\Omega \ll \omega_o$ and $\Omega \ll \gamma$,

Amplitude: $A = \frac{a_o}{\omega_o^2}$

Phase Lag: $\delta = 0$

Solution: $x(t) \simeq \frac{a_o}{\omega_o^2} \sin(\Omega t)$

$$A =$$

We then obtain steady state amplitude,

$$\Rightarrow A = \tag{3.10}$$

Similarly, for the phase lag we can take Eq. (3.8),

Since $\Omega \to 0$, the steady state phase lag is given by,

$$\Rightarrow \delta = \tag{3.11}$$

3.3.2 *High Frequency Limit: $\Omega \to \infty$*

IN THE HIGH FREQUENCY LIMIT we have the forcing frequency being much larger than the natural frequency ($\Omega \gg \omega_o$) and the damping constant ($\Omega \gg \gamma$). From Eq. (3.9) we have

High Freq. Limit: $\Omega \to \infty$

Amplitude: $A = \frac{a_o}{\Omega^2}$

Phase Lag: $\delta = \pi$

Solution: $x(t) \simeq \frac{a_o}{\Omega^2} \sin(\Omega t - \pi)$

$$A =$$

The steady state amplitude response is then,

$$\Rightarrow A = \tag{3.12}$$

Similarly, for the phase lag we can take Eq. (3.8) with $\Omega \to \infty$,

The steady state phase lag is given by,

$$\Rightarrow \delta = \tag{3.13}$$

3.4 *Near Resonance: ($\Omega \approx \omega_o$)*

AT THE NATURAL FREQUENCY $\Omega = \omega_o$ we can evaluate Eqs. (3.9) and (3.8)

Natural Frequency: ($\Omega = \omega_o$)

Amplitude: $A = \frac{a_o}{2\gamma\Omega}$

Phase Lag: $\delta = \frac{\pi}{2}$

Sol'n: $x(t) = \frac{a_o}{2\gamma\Omega} \sin\left(\Omega t - \frac{\pi}{2}\right)$

$$A =$$

$$\Rightarrow A \tag{3.14}$$

$$\Rightarrow \delta \tag{3.15}$$

THE RESONANCE OF A SYSTEM occurs close to the natural frequency, but not precisely at $\Omega = \omega_o$. ***The resonance is defined to be the frequency at which the amplitude of the response is maximum***, so that when $\Omega = \omega_{\text{res}}$, $dA/d\Omega = 0$:

Which gives us the value of the resonance frequency,

$$\Rightarrow \omega_{\text{res}} \equiv \qquad (3.16)$$

so that the resonance frequency is shifted lower for higher γ.

At resonance, the amplitude is evaluated to be

$$A_{\text{res}} = \qquad (3.17)$$

while the phase lag at resonance is given by,

$$\delta_{\text{res}} = \qquad (3.18)$$

Resonance Freq: $(\Omega = \omega_{\text{res}})$

With $\omega_{\text{res}} = \sqrt{\omega_o^2 - 2\gamma^2} \leq \omega_o$

Amplitude: $A_{\text{res}} = \dfrac{a_o}{2\gamma\omega^*}$

Phase: $\delta_{\text{res}} = \tan^{-1}\left[\dfrac{2\gamma\omega_{\text{res}}}{\omega_o^2 - \omega_{\text{res}}^2}\right]$

Sol'n: $x(t) = \dfrac{a_o}{2\gamma\omega^*}\sin(\Omega t - \delta_{\text{res}})$

.

NOTE THAT FOR WEAK DAMPING $\gamma \ll \omega_o$, $\omega_{\text{res}} \approx \omega_o \approx \omega^*$ and we just get the natural frequency response at the resonance.

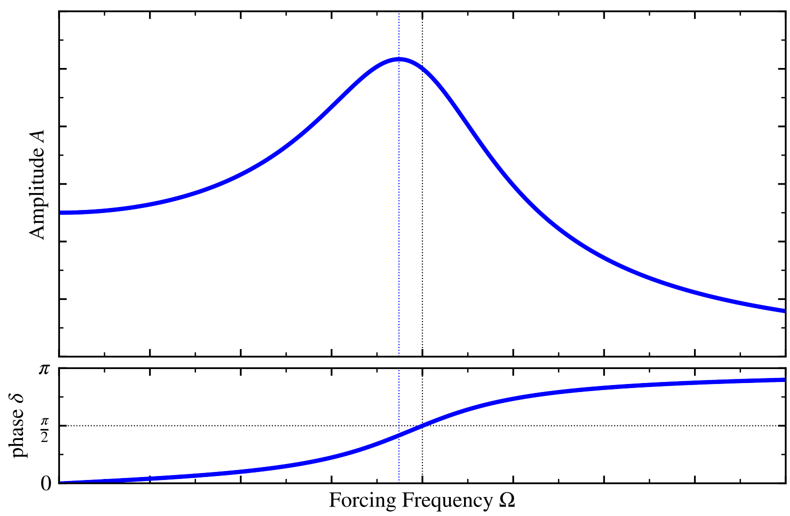


Figure 3.2: The driving frequency response for a damped harmonic oscillator. The resonance frequency ω_{res} is slightly lower than ω_o, but approaches the natural frequency in the weak damping limit $\gamma \ll \omega_o$. The amplitude at low frequencies is $A \simeq a_o/\omega_o^2$, it peaks at the resonance where $A = a_o/(2\gamma\omega^*)$, and falls off at high frequency with $A \simeq a_o/\Omega^2$. The phase is $\delta = 0$ at low frequency as $\Omega \to 0$; $\delta = \pi/2$ at the natural frequency $\Omega = \omega_o$; and $\delta = \pi$ at high frequency as $\Omega \to \infty$.

3.5 *The (Other) Q-factor*

The other common, but slightly different, definition of a Q-factor applies to driven systems. In our course notes we will call this value Q' so we do not confuse it with the other definition. Note,

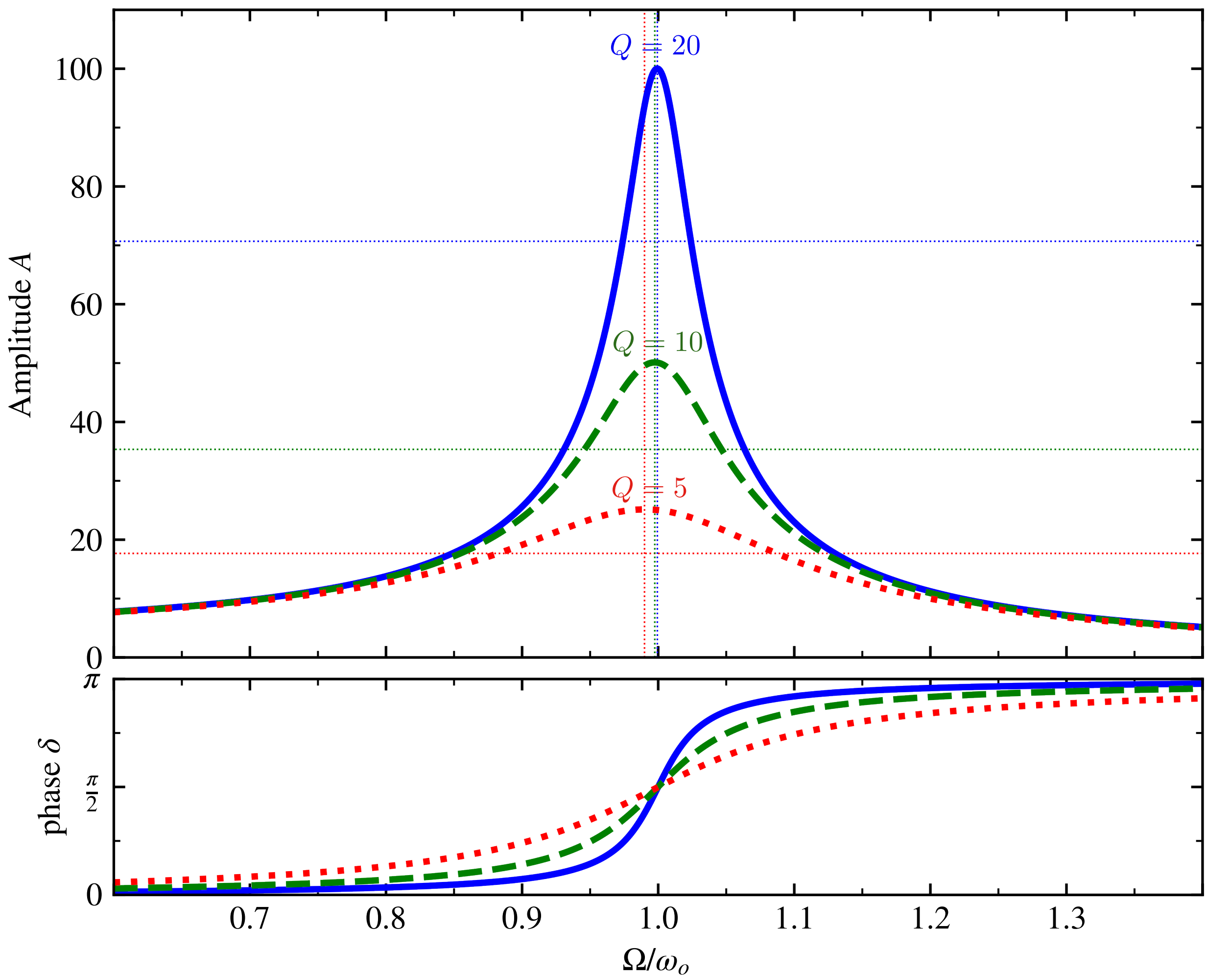


Figure 3.3: The effect of the Q-factor on the frequency response for forced damped harmonic oscillators.

however, that in the ***weak damping limit*** $\gamma \ll \omega_o$, these two definitions give the ***same value*** as we will see below!

The (other) Q-factor, Q' is defined as the resonant frequency, divided by the ***bandwidth*** of the system at resonance,

$$Q' \equiv \frac{\omega_{\text{res}}}{\Delta\Omega} \tag{3.19}$$

where the bandwidth $\Delta\Omega$ is given by the width of the amplitude response where it falls to $A = A_{\text{res}}/\sqrt{2}$. In this way Q' is a ***a direct measure of the sharpness of resonance***.

Remembering, $A_{\text{res}} = a_o/(2\gamma\omega^*)$, and defining $\Omega_\pm$ to be the frequencies where the amplitude fall off by a factor of $1/\sqrt{2}$ from the resonance, we need to have

$$\frac{A}{A_{\text{res}}} = \frac{2\gamma\omega^*}{\sqrt{(\omega_o^2 - \Omega_\pm^2)^2 + 4\gamma^2\Omega_\pm^2}} = \frac{1}{\sqrt{2}}$$

Solving for $\Omega_\pm$ (check this yourself!) we have

$$\Omega_\pm =$$

where

$$\Delta\Omega = \Omega_{+} - \Omega_{-}$$
$$=$$

The (other) Q-factor is then given by,

$$Q' = \frac{\omega_{\text{res}}}{\Delta\Omega} = \tag{3.20}$$

which isn't the same as our original $Q = \frac{2\pi}{1-\exp(-4\pi\gamma/\omega^{*})}$.

HOWEVER IN THE WEAK DAMPING LIMIT, $\gamma \to 0$, we have

$$Q' \simeq$$
$$\simeq \frac{\omega_{\text{res}}^2}{2\gamma\omega^{*}} \simeq \frac{\omega_o}{2\gamma} \approx Q.$$

Thus the two definitions of the quality factor are ***equivalent only in the weak damping limit***!

Another nice fact we have in the weak damping limit is found by comparing the amplitude at resonance to the low frequency response,

$$\frac{A_{\text{res}}}{A_{\Omega\to 0}} = \tag{3.21}$$

Example 3.1: Converting Terms

FIND AN EXPRESSION FOR THE RESONANCE frequency of a driven oscillator in terms of the parameters of the defining differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_o\sin(\Omega t)$$

which gives a solution for the driven amplitude,

$$A = \frac{a_o}{\sqrt{(\omega_o^2 - \Omega^2)^2 + (2\gamma\Omega)^2}}$$

Example 3.2: Shake that mass!

AN OSCILLATOR CONSISTS OF A MASS of 0.02 kg suspended from a spring with a spring constant of 50 N/m. It is subjected to a damping force of the form $-bv$ where v is the velocity and $b = 0.6$ kg s^{-1}, and is driven by a harmonically varying force of amplitude 4 N and frequency 550 rpm. Calculate A and δ of the resulting steady state vibration.

Summary of Driven Damped Harmonic Motion

- THREE IMPORTANT FREQUENCIES for oscillating systems

Simple Harmonic Motion	**Damped Harmonic Motion**	**Driven Harmonic Oscillator**
Natural Frequency $\omega_o = \sqrt{k/m}$	Underdamped Oscillator Frequency $\omega^* = \sqrt{\omega_o^2 - \gamma^2}$	Resonant Frequency $\omega_{\text{res}} = \sqrt{\omega_o^2 - 2\gamma^2}$

- THE DIFFERENTIAL EQUATION for a forced damped harmonic oscillator is:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = a_o \sin(\Omega t)$$

- THE STEADY STATE RESPONSE to the driving can be found using the trial solution:

$$x(t) = Asin(\Omega t - \delta)$$

  – ***Amplitude Response:***

$$A = \frac{a_o}{\sqrt{(\omega_o^2 - \Omega^2)^2 + (2\gamma\Omega)^2}}$$

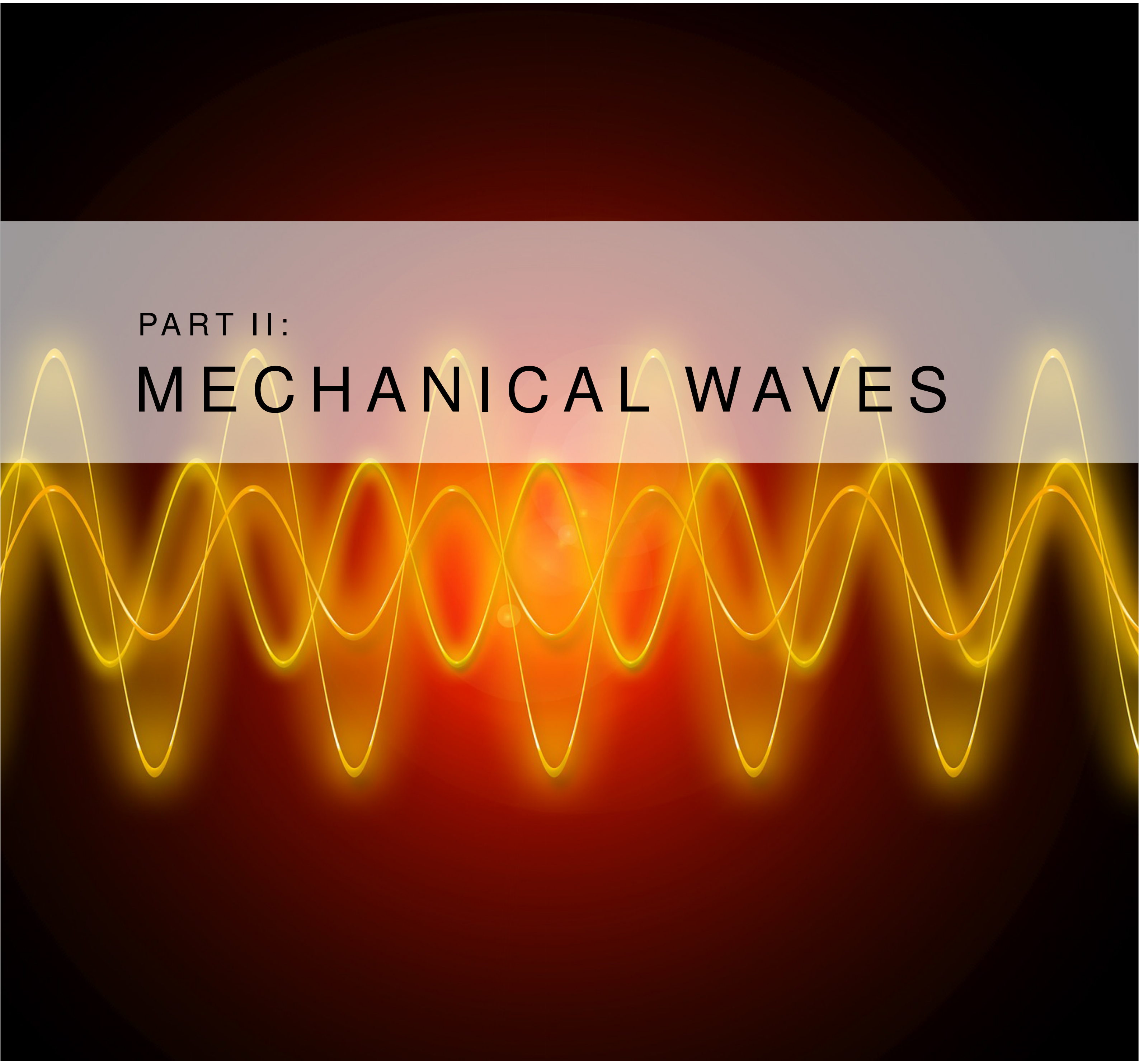
  – ***Phase Response:***

$$\delta = \tan^{-1}\left[\frac{2\gamma\Omega}{\omega_o^2 - \Omega^2}\right]$$

- THE (OTHER) Q-FACTOR, Q' is the ratio of the resonant frequency to the bandwidth of the system, and provides a measure of the sharpness of the resonance peak. ***Only in the weak damping limit is it equivalent*** to the Q-factor discussed for free oscillators (Q). ***In this limit,***

$$Q' \simeq Q \simeq \frac{\omega_o}{2\gamma},$$

  and it is also equal to the ratio of the amplitude at resonance to the low-frequency amplitude response.



PART II:

MECHANICAL WAVES

4
The Introduction to Waves

Introduction

§***Suggested Reading:***
Pain, Chapter 5, Transverse Wave Motion
Tipler, Chapter 15, Wave Motion

We now move away from simple isolated oscillators, and prepare to consider the collective motion of particles of a medium. (Mathematically we move from systems that can be described through Ordinary Differential Equations, to systems that are described by Partial Differential Equations in space and time.)

A ***wave*** is a disturbance that moves progressively in space and time. Mechanical waves are the collective motion of a material that propagates from one part of the medium to the next. Examples of media in which wave motion takes place are:

- water (e.g. sea waves, tidal waves)
- air (sound waves)
- crust of the Earth (seismic waves)

Waves transport energy without transporting particles!

4.1 Transverse and Longitudinal Waves

Transverse (T) waves are those in which the disturbance is perpendicular to the direction of propagation. When a string is disturbed, the displacement is perpendicular to the wave propagation direction. Electromagnetic radiation is another example of a transverse wave, with both the electric and magnetic vectors pointing perpendicular to the propagation direction.

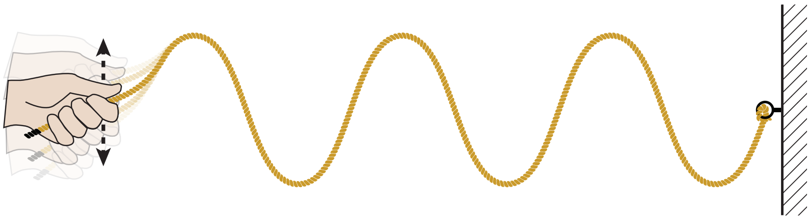


Figure 4.1: A Transverse Wave

Longitudinal (L) waves are those in which the disturbance (displacement of particles) is parallel to the propagation direction (e.g. sound waves and compressional waves in springs).

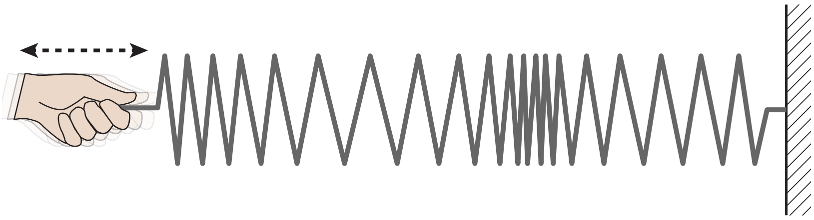


Figure 4.2: A Longitudinal Wave

Some waves, like water waves, are neither completely transverse nor completely longitudinal, but a combination of the two.

When a wave propagates in a single direction it is called a ***plane wave*** (only this kind of wave will be considered in this course, though circular and spherical waves can be considered as plane waves far enough away from the source).

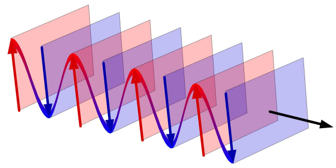


Figure 4.3: A Plane Wave

Figure 4.4: A Circular Wave

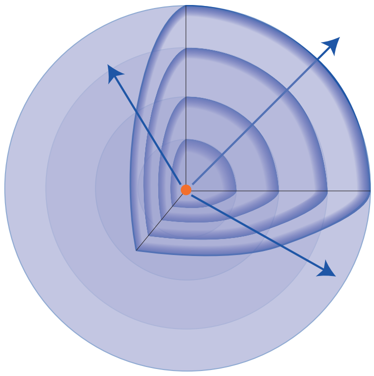
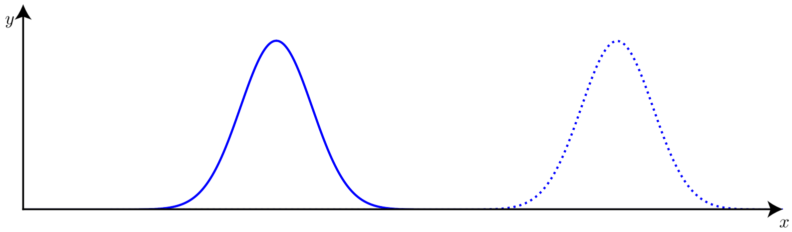


Figure 4.5: A Spherical Wave

4.2 *General Analytic Description of Wave Motion (1-D Waves)*



THE PLOT ABOVE SHOWS A PULSE on a string at two moments of time. At the later time the pulse is further down the string with the same shape. Let us introduce a new co-ordinate system with an origin O' that moves with the speed of the pulse v. In this reference frame the pulse is stationary with a shape $y' = y'(x')$ at all times. The two co-ordinate reference frames are related by

$$y = y', \qquad x = x' + vt. \tag{4.1}$$

Thus the shape of the string in the original frame O can be written,

(for x increasing with time) (4.2)

(for x decreasing with time) (4.3)

These relations define ***wave functions***.

A Wave Function is a mathematical function that describes the shape of a wave as a function of both position and time.

Example 4.1: A traveling wave

A WAVE HAS A FORM $y = f(x - vt)$ like the pulse pictured above, and is peaked at $x = 0$ when $t = 0$, and $v = 5$ m/s. At $t = 2$ s, where is the peak?

Example 4.2: Distance to an Earthquake's Epicentre

THE TRANSVERSE AND LONGITUDINAL WAVES in the Earth's crust that are generated by an earthquake propagate at different velocities.

- The Longitudinal Pressure Waves (also called p-waves in seismology) propagate at about $\sim$ 8 km/s.
- The more dangerous Transverse Shear Waves (also called s-waves in seismology) travel more slowly at $\sim$ 4 km/s (but are more energetic).

Movie star is sitting in Los Angeles mansion and feels the shaking of a p-wave, disturbing . About 6 seconds later, a rolling s-wave knocks over .
How far is the star from the earthquake's epicentre?

Example 4.3: From 2007 Exam

AT $t = 0$ A TRANSVERSE WAVE PULSE in a wire is described by the function $y = \frac{6}{x^2+3}$ where all quantities are in SI units. Write the function $y(x, t)$ that describes this wave if it is travelling in the positive x direction with a speed of 4 m/s.

4.3 Harmonic Waves

THIS IS A SPECIAL CASE of a ***periodic*** wave with the shape of a sine or cosine curve. They are usually generated by a source undergoing SHM.

(4.4)

for a wave traveling in the $+x$-direction and

(4.5)

for a wave traveling in the $-x$-direction.

Here A is the wave amplitude, k, is the wave number and ω is the angular frequency.

More generally, the expression is:

(4.6)

General Form for a Harmonic Wave

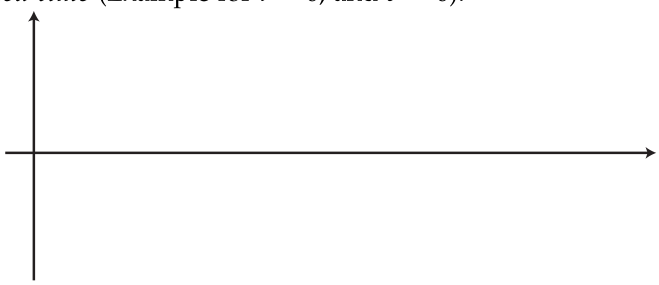
where we also take into account the initial phase of the wave (more about this later).

4.3.1 Graphical Representation of a Harmonic Wave

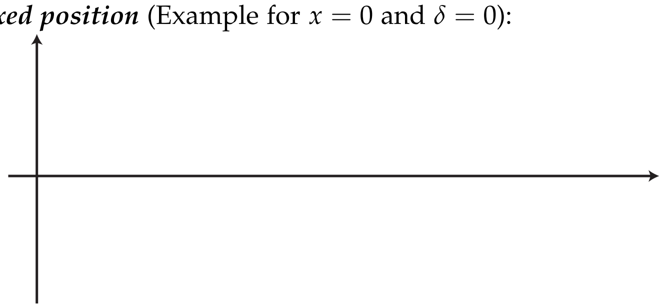
A WAVE IS AN INHERENTLY DYNAMIC OBJECT, so it isn't possible to fully represent a wave by a static, 2-D diagram.

TWO TYPES OF STATIC 2-D plots are commonly used:

(a) ***Fixed time*** (Example for $t = 0$, and $\delta = 0$):



(b) ***Fixed position*** (Example for $x = 0$ and $\delta = 0$):



4.3.2 Wavelength, λ, and Wavenumber, k

HARMONIC WAVES ARE PERIODIC in both time and space. The spatial period is known as the ***wavelength*** (λ) which is the distance between two successive wave crests, such that,

$$y(x+\lambda, t) = y(x,t) \tag{4.7}$$

or, for harmonic waves,

$$A\cos[k(x+\lambda)+\omega t] = A\cos[kx+\omega t] \tag{4.8}$$

Due to the 2π periodicity of the cosine function, this will give

Wavenumber of wave

$$. \tag{4.9}$$

The constant k is called the ***wavenumber***, and has units of inverse distance (m^{-1} or radians/m).

4.3.3 *Period T*

THE PERIOD T IS the time for ***one*** wave to pass a given point (temporal period). Periodicity with respect to t means

or for harmonic waves,

Thus we also have that

Period of a wave

$$\tag{4.10}$$

Here, ω is the angular frequency, with units of radians per second. Make sure you note that this is different than the ***frequency***, which is $f = \frac{1}{T} = \frac{\omega}{2\pi}$ which is measured in Hz.

During one period T, the wave moves a distance of one wavelength, and hence the wave velocity is

$$v = \tag{4.11}$$

This also gives the relation between k and ω:

$$\tag{4.12}$$

Example 4.4: Find the Wave Function

WRITE DOWN THE WAVE FUNCTION of a transverse harmonic wave travelling in the negative x-direction, having an amplitude of 2 mm, frequency of 5 Hz and wavelength of 10 cm, given that the displacement is 1 mm at $x = 0$ and $t = 0$.

Summary of Introduction to Waves

- TRANSVERSE (T) WAVES cause motion of the medium ***perpendicular*** to their direction of propagation
- LONGITUDINAL (L) WAVES cause motion of the medium ***in the direction*** of propagation.
- ANY PLANE WAVE MOVING to the ***right*** along the x-axis with speed v can be described by the wave function

  $$y = f(x - vt)$$

- ANY PLANE WAVE MOVING to the ***left*** along the x-axis with speed v can be described by the wave function

  $$y = f(x + vt)$$

- HARMONIC WAVES ARE PERIODIC SINUSOIDAL WAVES that take the form

  $$y(x,t) = A\cos[kx \pm \omega t + \delta].$$

  where k is the wavenumber, ω is the angular frequency, δ is the initial phase. Waves of the form $y \propto \cos(kx - \omega t)$ propagate in the $+x$ direction, while waves of the form $y \propto \sin(kx + \omega t)$ propagats in the $-x$ direction.
- THE ***period*** OF A HARMONIC WAVE is $T = 2\pi/\omega$, while the ***wavelength*** is given by $\lambda = 2\pi/k$. These are related by the velocity, $v = \lambda/T = f\lambda = \omega/k$.

5
Superposition of Waves

Introduction

§***Suggested Reading:***
Pain, Chapter 5, Transverse Wave Motion
Tipler, Chapter 15, Wave Motion

WAVES ARE A RICH PHENOMENON in physics, in part because of the ways in which waves can combine. Here we will study this ***principle of superposition***: when two disturbances act together, the resulting disturbance is the sum of the two separate disturbances:

$$y = y_1 + y_2 = A_1 \cos[k_1 x \pm \omega_1 t + \phi_1] + A_2 \cos[k_2 x \pm \omega_2 t + \phi_2] \tag{5.1}$$

Note that this principle only applies to ***linear systems*** (in fact it could even be considered to define linear systems). These are the only type of systems we consider in this course.

Using the ***principle of superposition*** we will study three special cases:

- ***Interference:*** Same frequency and wavelength, same (or approximately the same) direction, phase difference.
- ***Beats:*** Slightly different frequencies at a fixed position.
- ***Standing Waves:*** Same frequency and wavelength, opposite directions.

Pro Tip: The Addition to Product Trig Identities

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha \pm \beta}{2}\right)\cos\left(\frac{\alpha \mp \beta}{2}\right) \tag{5.2}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{5.3}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{5.4}$$

These trig identities will come in handy for adding harmonic waves.

We will then explore the general behavior of arbitrary waves constructed through harmonic synthesis and how they propagate in more complicated media.

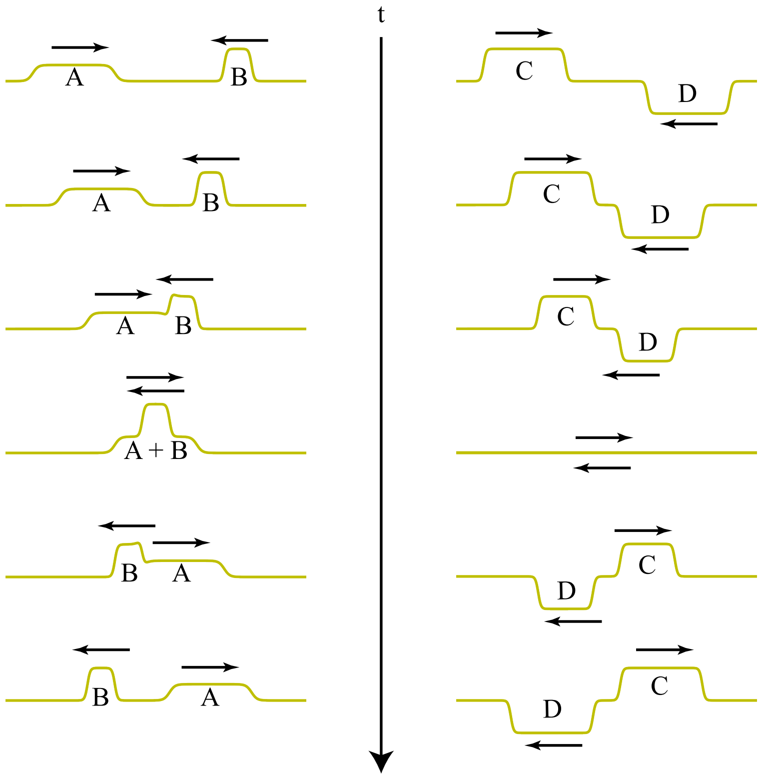


Figure 5.1: Wave pulses moving in opposite directions on a string. The shape of the string when the pulses meet is found by adding the displacements of each separate pulse. In the left figure the pulses have displacements in the same direction. In the right figure the pulses have opposite directions.

5.1 *Interference*

CONSIDER TWO WAVES with the same amplitude, frequency and wavelength, but with a phase difference δ between them,

$$y_1 = A\cos(kx - \omega t), \qquad y_2 = A\cos(kx - \omega t + \delta). \tag{5.5}$$

According to the principle of superposition the resultant wave is

$$\begin{aligned} y' &= \\ &= \end{aligned} \tag{5.6}$$

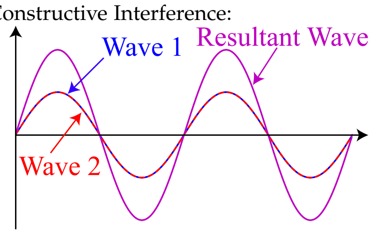
δ	$\lvert A' \rvert$
0	$2A$
$\pi/2$	$\sqrt{2}\,A$
π	0
$3\pi/2$	$\sqrt{2}A$

So this resultant wave has the same frequency and wavelength as the original waves, but its amplitude A' is now $A' = 2A\cos(\delta/2)$. If the phase diffference is $\delta = 2\pi n$ where $n = 0, 1, 2, 3, \ldots$ then we have ***constructive interference*** so that the amplitudes are ***summed***.

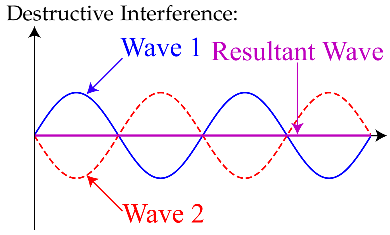
If the phase difference is $\delta = \pi, 3\pi, 5\pi, \ldots, = (2n+1)\pi$ where $n = 0, 1, 2, 3, \ldots$, then we have ***destructive interference*** and the amplitudes are ***subtracted***.

We often get such differences in the phase between two similar waves because of a difference in the path length over which the waves have propagated. This is precisely how laser interferometers measure tiny changes in distance (like in the LIGO experiment).

Constructive Interference:



If one wave has propagated a distance ΔL farther than the other, we can write down the phase difference δ in terms of the ***path difference*** ΔL



(5.7)

The superposition of the two waves that differ only by a path length ΔL can then be written

$$y = \quad (5.8)$$

5.2 *Beats (not by Dre)*

The superposition of two waves of different, but nearly equal frequencies, produces ***beats***. The amplitude of the resultant wave varies with a frequency called the ***beat frequency***.

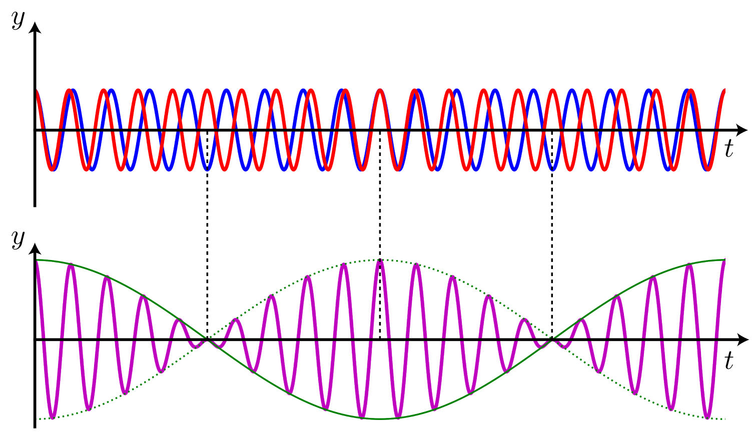


Figure 5.2: Waves of nearly the same frequency that interfere generate beats.

Let us consider two waves of the same amplitude and propagation speed, but with slightly different frequencies:

(5.9)

At a fixed point, say $x = 0$, we have:

$$\begin{aligned} y = y_1 + y_2 &= A[\cos(\omega_1 t) + \cos(\omega_2 t)] \\ &= 2A\cos\left(\frac{\Delta\omega}{2}t\right)\cos(\omega t) \end{aligned} \quad (5.10)$$

where $\omega = \frac{\omega_1+\omega_2}{2}$ is the average angular frequency and $\Delta\omega = \omega_1 - \omega_2$.

The ***beat frequency*** is the frequency at which the amplitude of the wave pulsates. The time between two successive amplitude maxima is

$$T_{\text{beat}} = \frac{2\pi}{\Delta\omega}. \tag{5.11}$$

so that the beat frequency is

$$f_{\text{beat}} = \frac{1}{T_{\text{beat}}} = \frac{\Delta\omega}{2\pi} = \frac{\omega_1 - \omega_2}{2\pi} = f_1 - f_2. \tag{5.12}$$

5.3 *Standing Waves*

WE CAN EXTEND THE ANALYSIS OF THE SUPERPOSITION of harmonic waves to the case with waves of the same frequency, traveling in opposite directions and interfering. (We can consider an infinite string to avoid boundary conditions.)

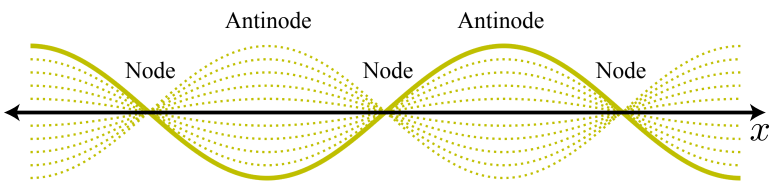
If the amplitude of the two waves propagating in opposite directions is the same ($A_1 = A_2 = A$) then,

(5.13)

The resultant wave is then

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\Big[\cos(kx - \omega t) + \cos(kx + \omega t)\Big] \\ &= \end{aligned} \tag{5.14}$$

This equation generates a fixed spatial pattern that oscillates in time, i.e. the pattern oscillates with angular frequency ω, and amplitude that varies between 0 and $2A$ depending on position.



The minima are called ***nodes*** and the maxima are called ***antinodes***.

At the ***nodes*** the amplitude of the oscillation is zero, therefore $\cos(kx) = 0$, and thus,

$$kx = \pi/2, 3\pi/2, 5\pi/2, \ldots = \tag{5.15}$$

$$\text{or} \qquad x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots = \tag{5.16}$$

where n is an integer.

At ***anti-nodes*** the amplitude of the oscillations is maximum, therefore, $\cos(kx) = \pm 1$, i.e.,

$$kx = 0, \pi, 2\pi, 3\pi, \ldots, n\pi \tag{5.17}$$

$$\text{or} \qquad x = 0, \lambda/2, 3\lambda/2, \ldots, \frac{n\lambda}{2}. \tag{5.18}$$

5.3.1 Standing Waves on a String

Infinite strings are easy to study mathematically, however, they are slightly unphysical. When we study real objects we often have to worry about ***boundary conditions***, the conditions that must be satisfied at the boundaries, or edges, of the system we are interested in.

Example 5.1: A String with Both Ends Fixed

Consider a string that has a wave on it, but has had each end fixed tightly at $x = 0$ and $x = L$. Each endpoint completely reflects the wave, so we have two waves with equal amplitudes traveling in opposite directions.
This forces us to have that

(a) $y = 0$ at $x = 0$, or $y(0, t) = 0$
(b) $y = 0$ at $x = L$, or $y(L, t) = 0$

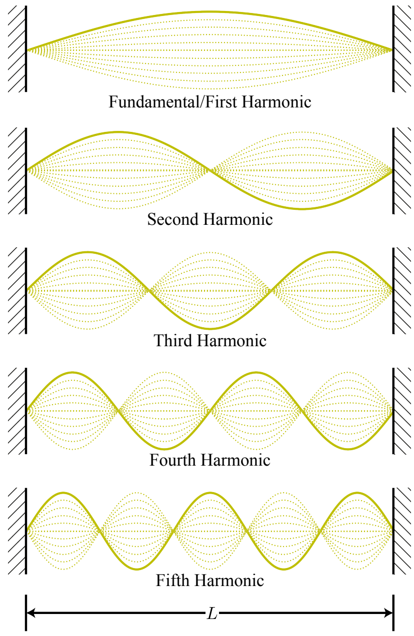
These are called the ***boundary conditions*** that the resulting standing wave must satisfy.
The frequencies that produce these patterns are called ***normal modes*** of the string system. The lowest normal mode is called the ***fundamental mode*** of vibration ($\lambda/2 = L$) or the ***first harmonic***, which has ***no nodes***, except at the boundaries.
The ***second harmonic*** has $\lambda = L$ (with one node), and so on such that ***the*** n***th harmonic (the mode with frequency*** $n\times$ ***the fundamental frequency) has wavelength***

(5.19)

and $n - 1$ nodes (not counting the nodes at the boundaries). This result is known as the ***standing wave condition*** for this system (different boundary conditions will give us different standing wave conditions). In terms of frequencies this is,

(5.20)

where v is the speed of waves on a string.
Thus, we have that $\lambda_n = \lambda_1/n$ and $f_n = n \times f_1$.



Example 5.2: String Fixed at One End

FOR THIS CASE $y = 0$ at $x = 0$ and y is maximum at $x = L$. Note that the free end is an antinode. In the fundamental (longest wavelength, lowest frequency) mode of vibration for a string fixed at one end only, the length L of the string equals $\lambda/4$, so that $kL = \pi/2$.
In the next harmonic, we have to fit another half wavelength (π in phase) before we find another anti-node, so $L = \frac{3\lambda}{4}$, with $kL = 3\pi/2$. The next harmonic that fits the boundary condition has $L = \frac{5\lambda}{4}$ with $kL = 5\pi/2$.
Thus the standing-wave condition for a string with one fixed end and one free end is

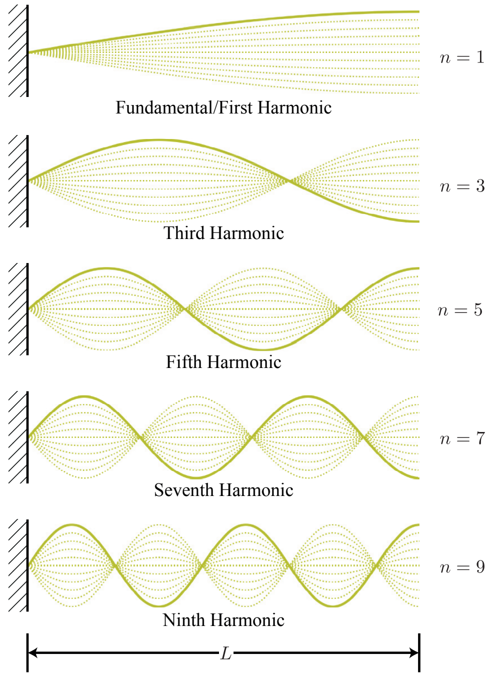
(5.21)

so that only odd harmonics (i.e. odd multiples of the fundamental frequency) are present. In terms of frequencies this can be written as,

$$f_n = \frac{nv}{4L}. \quad (5.22)$$

We can alternatively express these conditions in terms of $m = 0, 1, 2, 3, \ldots$

(5.23)



5.3.2 *Standing Sound Waves in Pipes*

A SOUND WAVE causes ***a longitudinal displacement of air particles, Δx, in the direction of propagation***.

The pressure variation/compression of the air depends on how much this displacement varies over position, x, so it depends on the derivative of the displacement,

$$\Delta P(x) \propto \frac{d}{dx}\Delta x \quad (5.24)$$

Since the derivative of a sinusoid is another sinusoid shifted by a phase $\pi/2$, any ***pressure nodes must occur at displacement antinodes, and vice-versa***.
MUCH OF THE ANALYSIS OF STANDING SOUND WAVES is similar to that of standing waves on strings. If we have air confined in a closed tube of variable length, then the ends must correspond to displacement nodes of the compressional sound wave.

Here instead of vertical displacements of a string, the wave causes longitudinal compression/pressure waves through the pipe. These propagate at ***the speed of sound***.

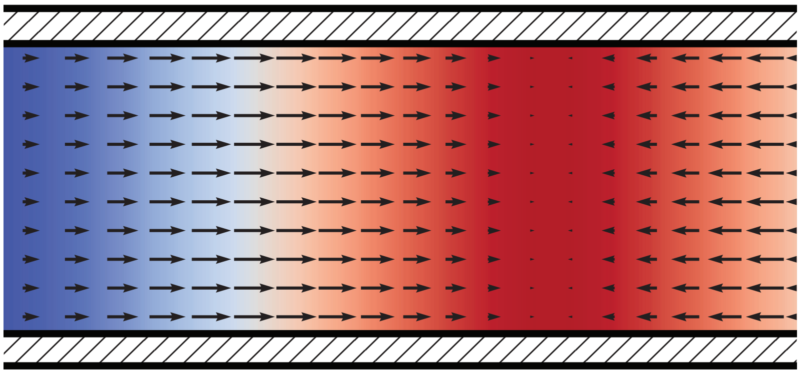


Figure 5.3: For a sound wave, the pressure nodes occur where the displacement antinodes are, and the displacement nodes occur where the pressure antinodes are.

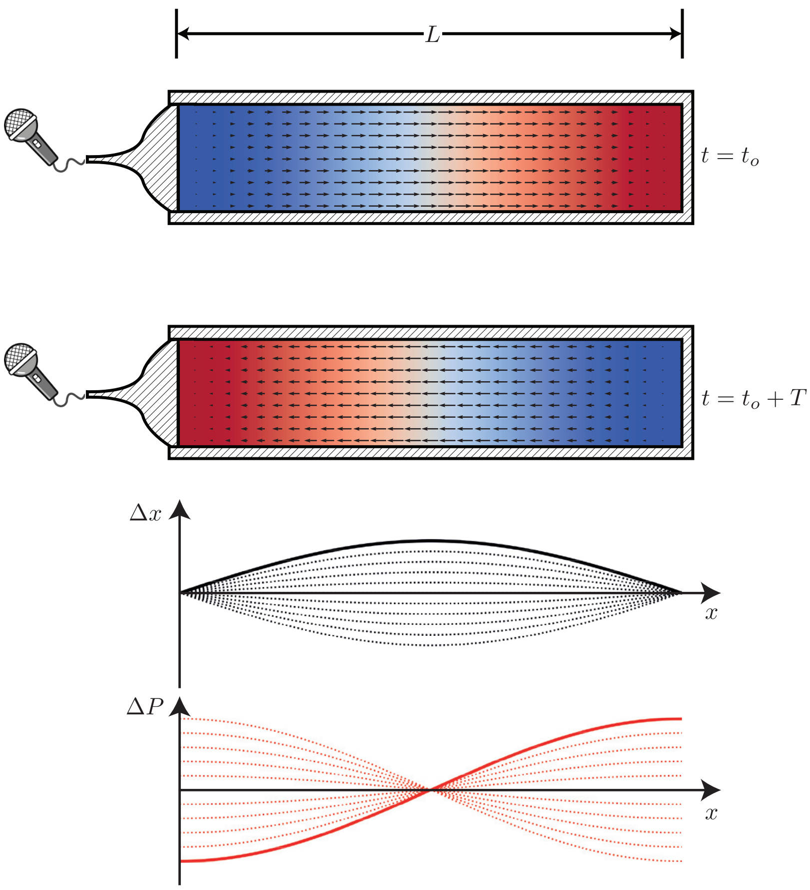


Figure 5.4: A closed air-filled pipe with a moveable piston boundary (i.e. a speaker) that vibrates back and forth to inject sound waves of the fundamental frequency into the pipe. The air particles can't be pushed past the ends of the pipe, so the displacements there are zero. On the other hand, the pressure variations are largest at the closed boundaries of the pipe. Why do we take a node to occur at the ***movable*** piston that makes up our "speaker"? (Hint: If the speaker is playing the fundamental note, of the pipe, it is in ***resonance***.)

Example 5.3: A Closed Pipe

IF WE HAVE AIR CONFINED in a closed tube of variable length, then the ends must correspond to displacement nodes. The standing wave condition must be the same as for a string fixed at both ends. The allowed sound frequencies are then:

(5.25)

Despite the similarity to the condition for the waves on a string, it is important to remember that the v above is the ***sound speed***, while for the other it is the speed of a wave on a string.

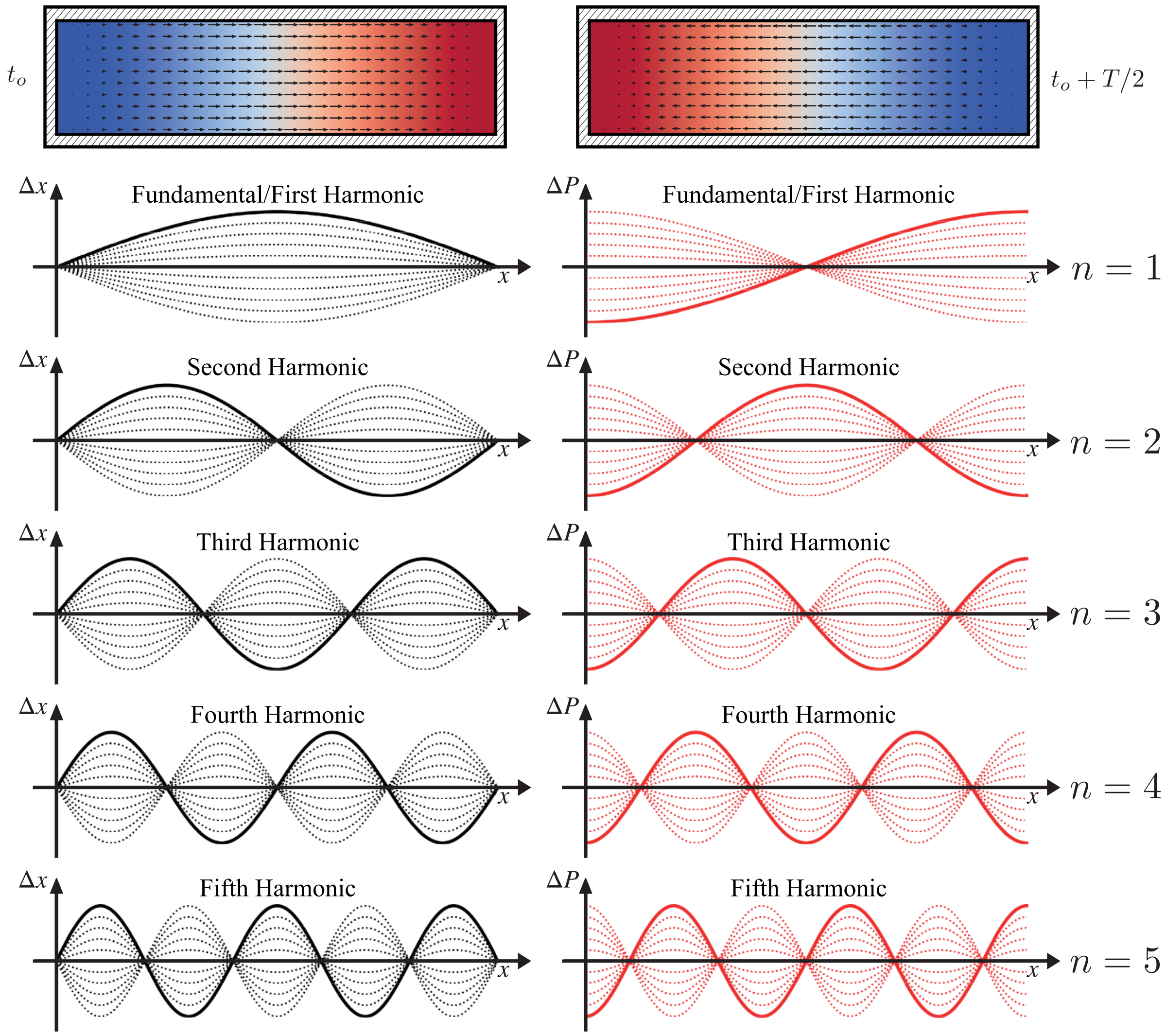


Figure 5.5: For a closed pipe, both ends must have a displacement node/pressure antinode.

Example 5.4: An Open Pipe

THE SAME CONDITIONS (5.25) apply to a tube open at both ends, such that each end corresponds to a displacement ***antinode*** and a pressure ***node***.

You may wonder how a sound wave can reflect from an open end, as there may not appear to be a change in the medium at this point. It is true that the medium through which the sound wave moves is air both inside and outside the pipe. However, sound is a pressure wave and a compression region of sound is constrainted by the sides of the pipe as long as the region is inside the pipe. As the compression region exits at the open end of the pipe, the wave isn't strong enough to change the surrounding atmospheric pressure very much. Thus, there is a change in the ***character*** of the medium between the inside of the pipe and the outside, even though the material itself does not change. This is sufficient to allow some reflection. (We will understand more about how this works when we discuss impedance in the next chapter.)

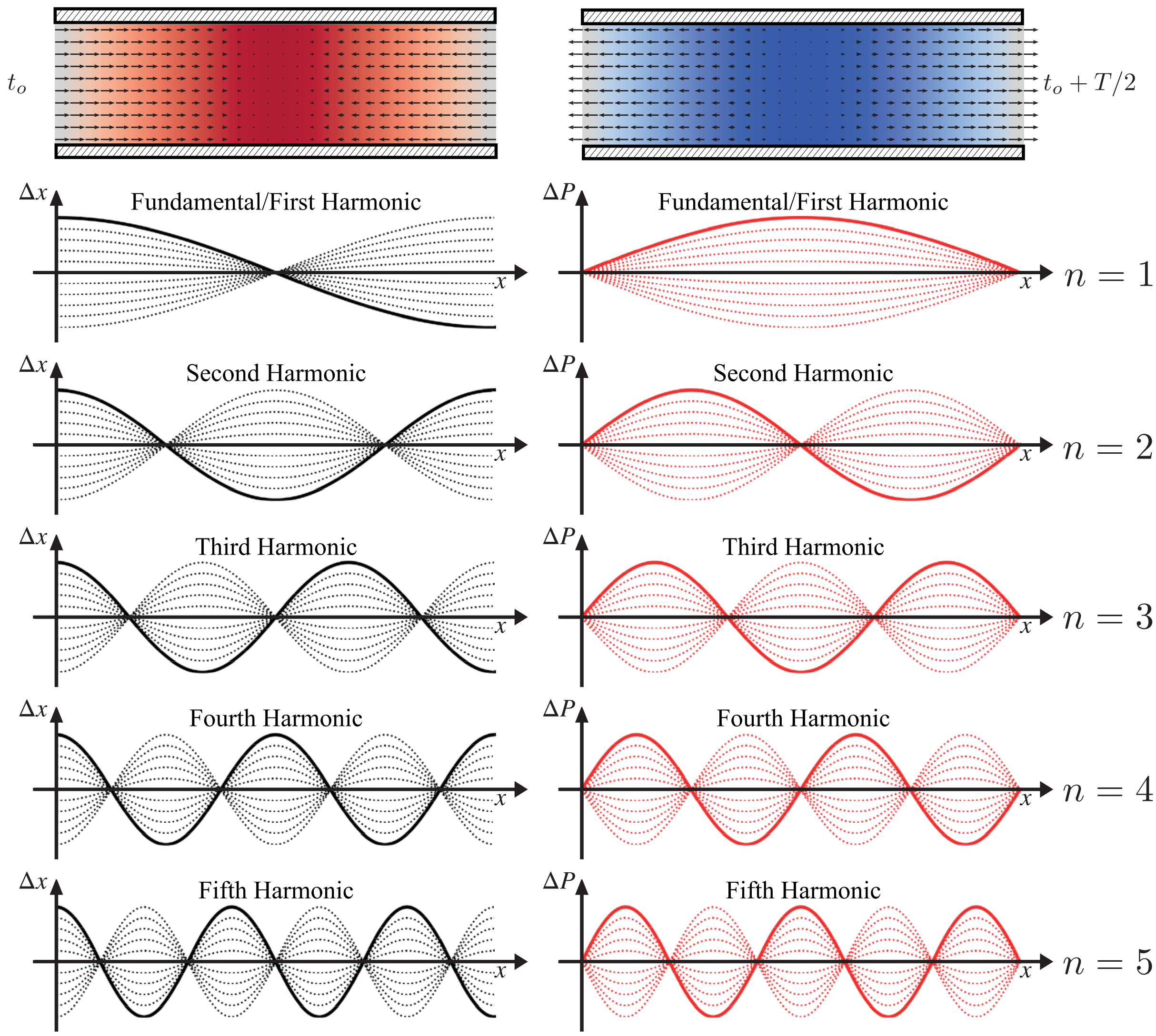


Figure 5.6: For an open pipe both ends must have displacement antinodes/-pressure nodes.

Example 5.5: A Pipe with One End Closed

THE CLOSED END is a displacement node (and pressure anti-node), and the open end is a displacement antinode (and pressure node).
In this case, the condition for allowed wavelengths and frequencies is similar to the case of the string fixed at one end:

(5.26)

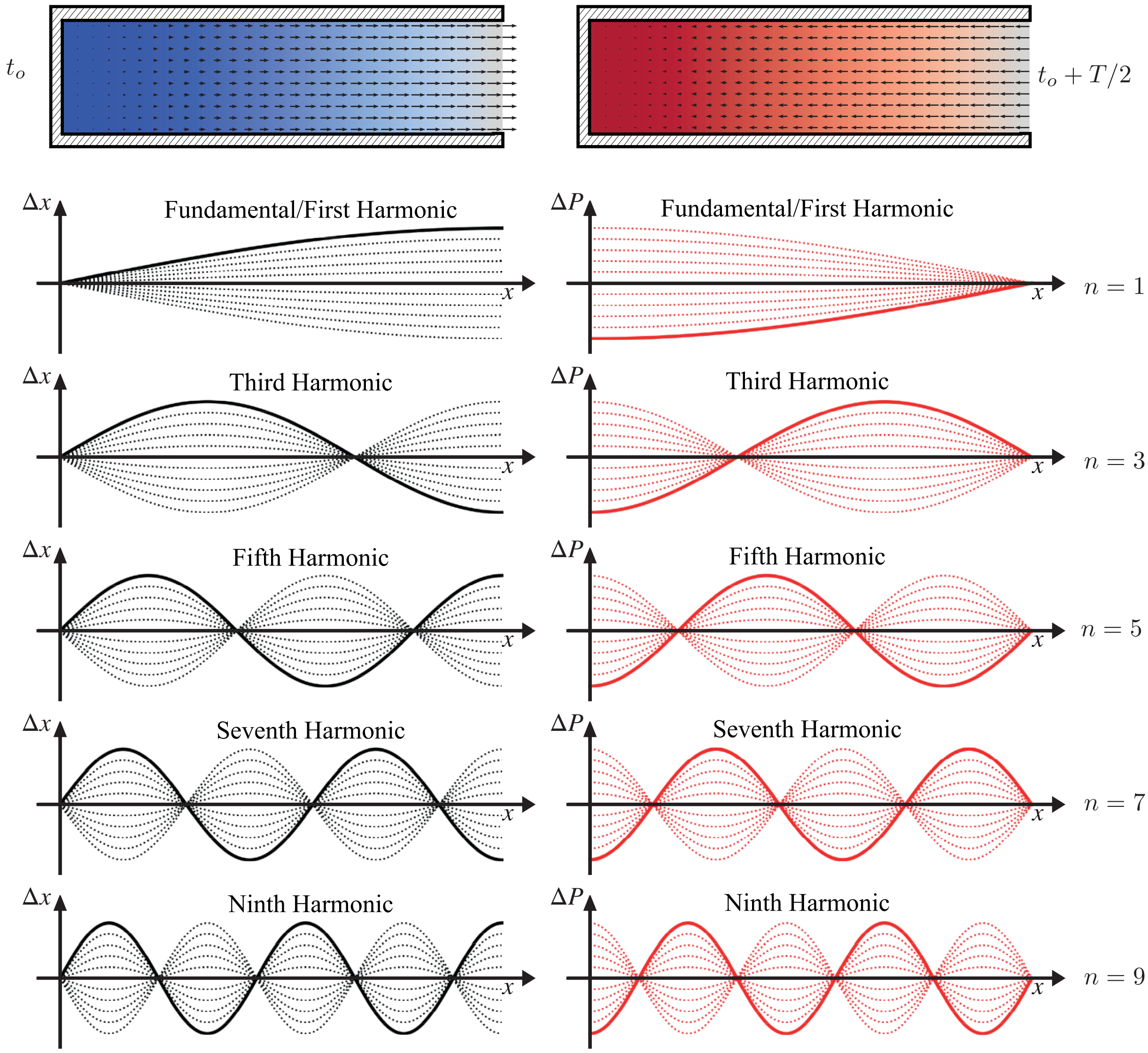


Figure 5.7: For a pipe with one open end, only odd harmonics of the fundamental occur.

5.4 *Harmonic Analysis and Synthesis*

To date we have only considered harmonic waves, which has had the advantage of making the mathematics more tractable. More ***complex waves can be considered as the sum of a number of harmonic waves***.

Fourier analysis is one of the most powerful mathematical tools in engineering and physics. It is foundational in the development of signal processing, radio, image or audio compression.

The different sounds emitted by musical instruments playing the same note is often due to their harmonic content; in addition to the fundamental there are also contributions from higher harmonics. We can analyse the harmonic content by ***harmonic (or Fourier) analysis***. The inverse of harmonic analysis is ***harmonic synthesis*** which is the construction of a complex periodic wave from its harmonic components.

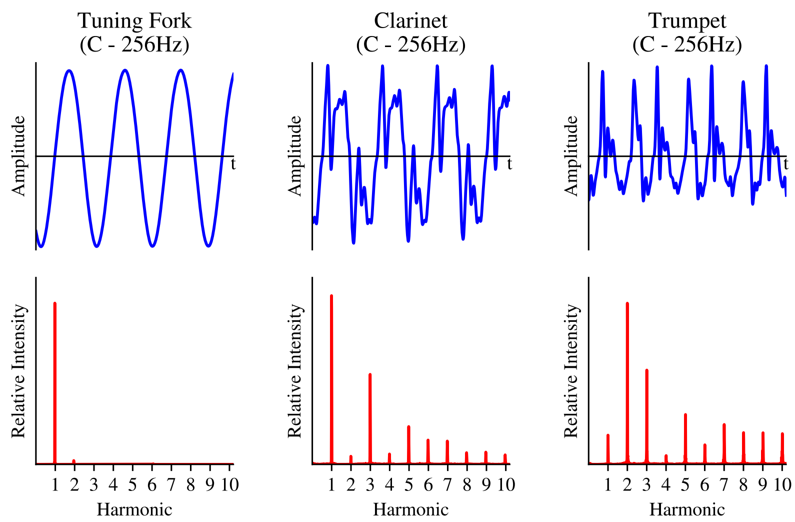


Figure 5.8: Waveforms of a tuning fork, a clarinet, and a trumpet, all playing with fundamental frequency of 256 Hz (also known as the scientific C note, as opposed to the standard C note which is 262 Hz, the rapper's C note which is $100, or the rapper C-note whose 1999 debut album *Third Coast Born* reached #67 on the Billboard charts). Relative intensities of the harmonics are shown in the corresponding Fourier analysis plot.

Example 5.6: Clarinet at 256 Hz

The clarinet playing with a fundamental frequency of 256 Hz, shown in Figure 5.8. If we know the values of the relative intensity we can reconstruct the waveform (up to phase terms, which only change the character of the note slightly).

$$. \tag{5.27}$$

Any ***periodic*** signal, can be constructed as a sum of harmonics that share the same fundamental frequency. This is the basis of Fourier analysis/synthesis.

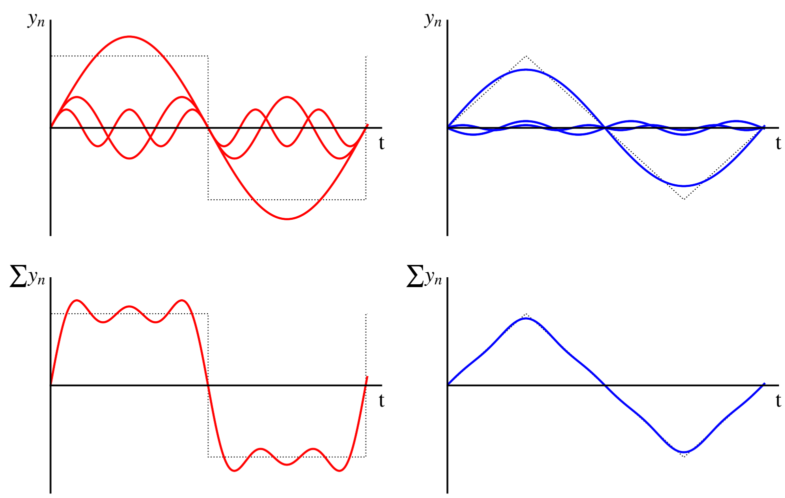


Figure 5.9: Top: A square wave and a triangle wave (dotted black), along with the first 3 odd harmonics used to synthesise them. Bottom: The synthesised approximations to the square and triangle waves using the first three harmonics above.

5.5 Dispersive Media: Group Velocity, Phase Velocity, and Wave Packets

SO FAR WE'VE ONLY STUDIED WAVES in a "non-dispersive" medium such that all waves propagate at the same speed. In general, a medium may respond to different frequencies with differently, resulting in changes in the speed at which harmonic waves will propagate. Remembering that the (phase) velocity[1] of a harmonic wave is given by

$$v_p = \omega/k \tag{5.28}$$

If we have two waves, with frequency ω_1 and ω_2, with wavenumbers k_1 and k_2, these can propagate at different velocities if $\omega_1/k_1 \neq \omega_2/k_2$. In general we can write, for a particular medium, we can express a wave's angular frequency ω as a function of the wavenumber k,

(5.29)

Dispersion Relation

this is known as a ***dispersion relation***, for reasons[2] we will discuss below.

IN ORDER TO BETTER UNDERSTAND THE DIFFERENT KINDS OF VELOCITIES ASSOCIATED WITH A WAVE let's consider our example of two waves that beat, from the previous section, now allowing both k and ω to be slightly different. That is, we have two waves

$$\begin{aligned} y_1(x,t) &= A\cos\left[k_1 x - \omega_1 t\right] \\ y_2(x,t) &= A\cos\left[k_2 x - \omega_2 t\right] \\ \text{where} \qquad v_1 = \omega_1/k_1, &\qquad v_2 = \omega_2/k_2 \end{aligned}$$

Remembering our handy trig identities the resultant wave is,

$$\begin{aligned} y &= y_1 + y_2 \\ &= \end{aligned}$$

[1] By phase velocity we mean the velocity at which the phase of the harmonic wave appears to move in the x-direction. This is in contrast to the ***group*** velocity which we will discuss below.

[2] We will see that some dispersion relations will result in wave pulses being ***dispersed***.

or

$$y = \tag{5.30}$$

where $k = (k_1 + k_2)/2$ and $\omega = (\omega_1 + \omega_2)$ are the average wavenumber and angular frequency, and $\Delta k = k_1 - k_2$, $\Delta\omega = \omega_1 - \omega_2$ are the wavenumber and angular frequency differences between the two waves.

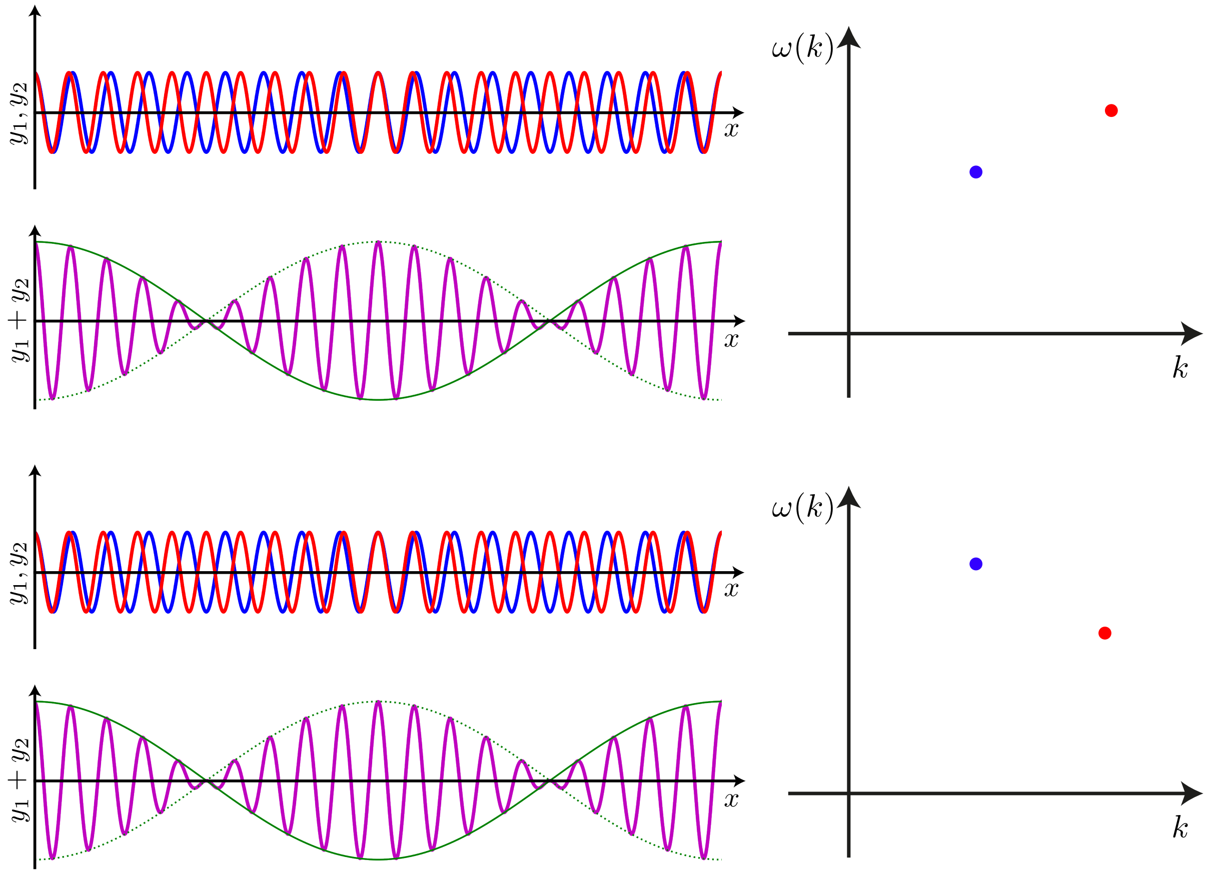


Figure 5.10: Two examples of propagating beating waaves with different ω_1, k_1 and ω_2, k_2. What is the difference in the group/envelope velocities?

Thus we see that the fast oscillation inside the beat propagates at a speed

$$\tag{5.31}$$

where we call this the ***phase velocity*** of the total wave.

We also see that the amplitude envelope propagates with a velocity defined by

$$\tag{5.32}$$

We call this latter velocity the ***group velocity***, because it is the speed at which the group of waves/envelope will propagate.

The complex waveforms discussed above are periodic in time. Functions that are ***not*** periodic, such as pulses, can also be represented by sums of harmonic wave functions, but ***a continuous distribution of frequencies is needed***.

Pulses, unlike harmonic waves, have beginnings and ends. In order to send a signal with a wave, a pulse consisting of a group of waves at different frequencies is needed rather than a harmonic wave. Such a group is called a ***wave packet***. The width of frequencies, $\Delta\omega$, in a pulse is inversely related to the time duration, δt, of the pulse.

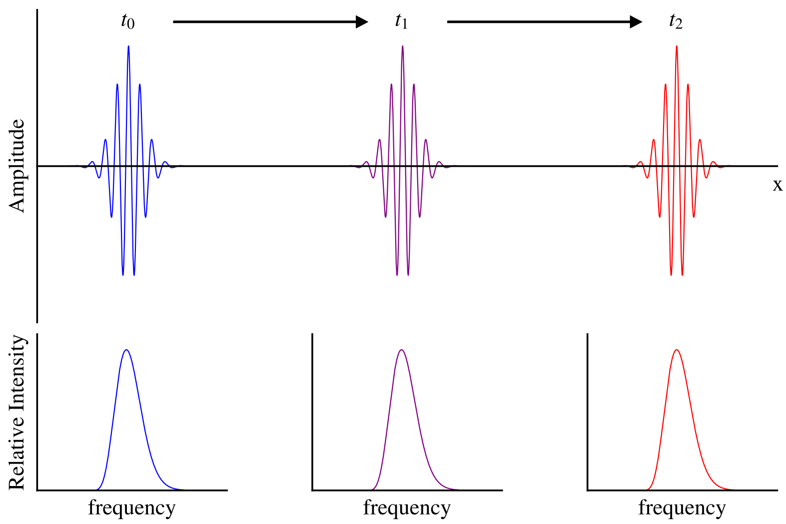


Figure 5.11: A wave packet, built out of a narrow continuous distribution of frequencies, propagates in the x-direction. The wider the wave packet, the narrower the frequency distribution.

Information ***cannot*** be transmitted by a single harmonic wave, which has no beginning or end. Information requires the sending of shorter pulses, which depend on the ability to transmit a wide range of frequencies (hence why bandwidth is used to describe the capacity to transmit information).

When considering the continuum limit on the distributions of frequencies, we must also consider the continuum limit on the differences that define the group velocity. For narrow frequency distributions (wide pulses), we find that the velocity of the pulse (group velocity) is given by

Group Velocity

$$v_g \tag{5.33}$$

while the phase velocity is

Phase Velocity

$$\tag{5.34}$$

where now ω is the average frequency of the pulse, and k is the average wavenumber.

Thus the group velocity of a pulse is given by the ***the slope*** of $\omega(k)$. Let's take a look at a how a pulse will propagate in media with different dispersion relations.

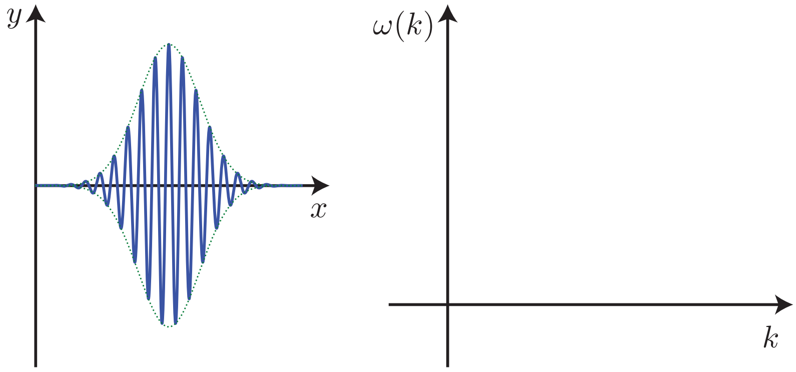


Figure 5.12: The case of $v_g = v_p =$ constant, e.g. light in vacuum. Such a medium is said to be ***non-dispersive***.

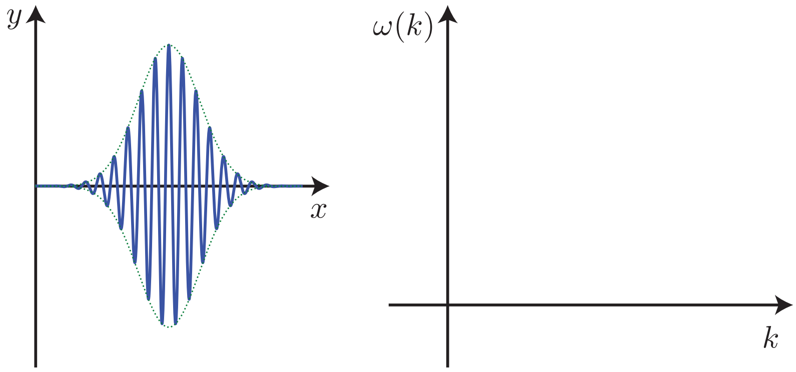


Figure 5.13: The case of $v_g < v_p$, such a medium is said to have ***normal dispersion***.

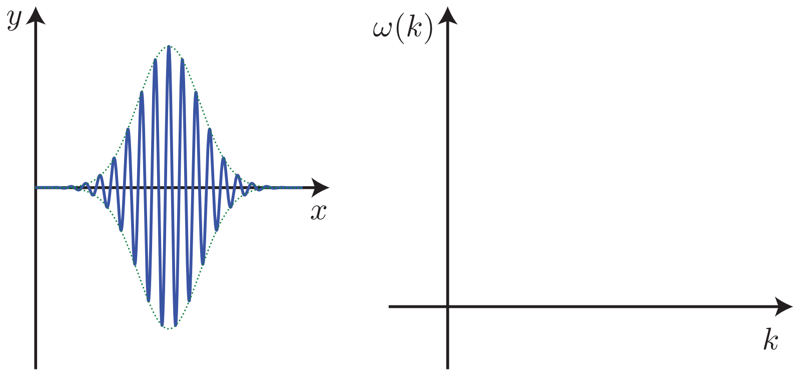


Figure 5.14: The case of $v_g > v_p$, such a medium is said to have ***anomalous dispersion***.

We see above that the ***first derivative*** of angular frequency $\omega(k)$ determines the propagation speed of ***the wave packet***, which is built of waves with a continuous range of frequencies, such that

$$v_g(k) = \frac{d\omega}{dk}.$$

If $v_g = d\omega/dk$ is a ***constant*** for all the wavenumbers/frequencies in our pulse, then the entire pulse has the same group velocity and the medium is called ***non-dispersive*** for this type of wave. However if the group velocity also varies with k, such that

$$v'_g(k) = \frac{d^2\omega}{dk^2} \neq 0, \tag{5.35}$$

then some frequency components of the pulse will travel faster than others, and the pulse gets spread out in time. We call this effect ***dispersion***. Thus in addition to the group velocity and phase velocities for our pulse the ***dispersion relation*** $\omega = \omega(k)$ for a medium also tells us how the pulses will ***disperse*** and spread out over time, through the second derivative $\omega''(k)$.

The example of waves in a dispersive media that you are most familiar with is the refraction of light due to the frequency dependence of the speed of light ($c = \omega/k$) in a prism.

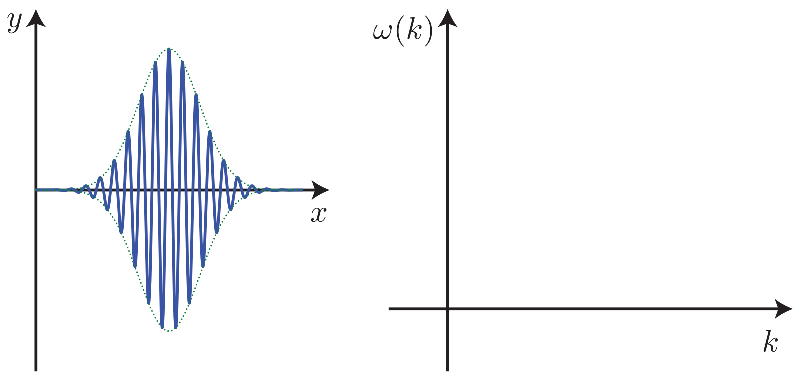


Figure 5.15: The pulse traveling in a dispersive medium, such that $d^2\omega/dk^2 = v'_g(k) \neq 0$.

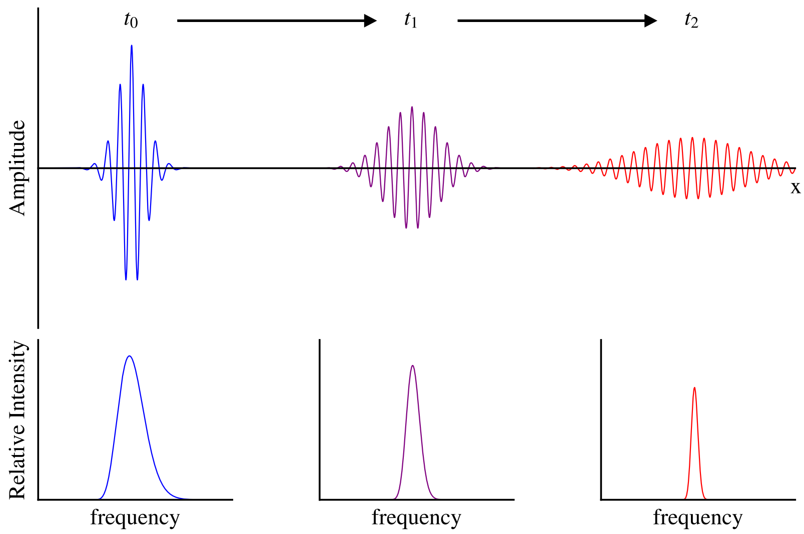


Figure 5.16: A wave pulse in a dispersive medium tends to disperse, spreading out in time as it propagates, while its frequency range narrows.

Summary of Superposition of Waves

- PRINCIPLE OF SUPERPOSITION: Waves can be added together, such that the resulting disturbance is merely the sum of the two disturbances (the waves can be added linearly).
- INTERFERENCE: Occurs at nearly the same frequency and wavenumber, but with a phase difference.
  - ***Constructive interference*** occurs when waves are ***in phase***, such that the amplitudes are added together.
  - ***Destructive interference*** occurs when waves are π radians ***out of phase***, such that the amplitudes are subtracted.
- ***Beats:*** Beats occur when waves with slightly different frequencies pass by some point $x = x_o$. The superposition of the waves gives

$$y(x_o, t) = 2A\cos\left(\frac{\Delta\omega}{2}t\right)\cos(\omega t)$$

  where $\Delta\omega = \omega_1 - \omega_2$ is the beat frequency and $\omega = (\omega_1 + \omega_2)/2$.
- STANDING WAVES: Waves traveling in the opposite direction can superpose and result in a standing wave that does not move in x, but oscillates with the wave frequency ω. The positions which are zeros of the oscillation are called ***nodes***, while the maximum amplitudes occur at the ***antinode*** positions. The standing wave condition depends on the boundary conditions of the system.
  - ***Waves On a String:*** At ***fixed*** boundaries, the transverse displacement has a ***node***; at ***free*** boundaries, the transverse displacement must be at an ***antinode***.
  - ***Sound Waves in a Pipe:*** At ***closed*** ends, the longitudinal ***displacement has a node***|, the ***pressure has an antinode***; at ***open*** ends, the longitudinal ***displacement has an antinode***, the ***pressure has a node.***
- HARMONIC ANALYSIS AND SYNTHESIS: Any complex ***periodic*** wave can be constructed from a sum of harmonic functions with the same fundamental period. Any signal can be ***synthesized*** by adding together harmonics with the appropriate amplitudes (and phases).
- WAVE PACKETS: A ***wave pulse or packet*** has finite extent and must be synthesized from a ***continuous distribution of frequencies***. Information cannot be transmitted via pure harmonic waves, only through wave pulses. The longer the pulse, the narrower the frequency range that makes it up.
- DISPERSION RELATION: The dispersion relation expresses ***the frequency as a function of the wave number***, $\omega = \omega(k)$. It tells how wave packets within the medium ***propagate*** and ***disperse***.
  - ***Phase Velocity:*** $v_p = \omega/k$ is the speed at which the fast oscillations of a wave pulse appear to propagate.
  - ***Group Velocity:*** $v_g = d\omega/dk$ is the speed at which the wave envelope/group propagates. It is the ***speed at which information carried by the wave propagates.***
  - ***Dispersive Medium:*** A wave medium is ***dispersive*** if $d^2\omega/dk^2 \neq 0$, such that the group velocity depends on the wavenumber. Since different components of the wave move at different speeds, the wave pulse tends to spread out.

6
Waves on a String: A Closer Look at Non-Dispersive Waves

Introduction

§*Suggested Reading:*
Pain, Chapter 5, Transverse Wave Motion
Tipler, Chapter 15, Wave Motion

THUS FAR WE HAVE been dealing with non-dispersive waves using only the notion that waves take the form

$$y(x,t) = f(x - vt), \tag{6.1}$$

where f is any unspecified function of a single variable. In this chapter we will examine a transverse wave on a string, in detail, determining the ***partial differential equation*** that it satisfies, which it turns out is the ***wave equation*** that must be satisfied by non-dispersive 1-dimensional waves medium. We will also examine what happens when waves cross a boundary where the properties of the medium (the string) change, and how this depends on the something called the ***impedence***.

Quick Math Note: Partial Derivatives

THIS WILL SERVE as just a quick review/introduction on how to deal with partial derivatives. You have learned (or will soon be learning) that a partial derivative of a function with multiple variables is just the derivative with respect to one of those variables while holding the other variables constant. Partial derivatives are thus defined (using the ∂ symbol instead of d),

$$\frac{\partial f(x,t)}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x, t) - f(x,t)}{\Delta x}.$$

(Operationally, you can just pretend the variables that aren't being differentiated are constants, and take derivatives normally.) The second partial derivative is similarly defined:

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\partial f/\partial x|_{(x+\Delta x,t)} - \partial f/\partial x|_{(x,t)}}{\Delta x}$$

and so on to higher order partial derivatives (and mixed partials, which we won't need).

Using this we will find a general solution to the wave equation, and show how waves reflect and transmit across changes in the medium, and set the stage for the second half of the course, the study of optics.

6.1 The Wave Equation

WE WILL DERIVE THE WAVE EQUATION by consider the particular case of transverse waves on a string, which is held under tension. These waves turn out to be ***non-dispersive***, that is, ω is strictly proportional to k and the wave velocity is constant for any frequency.

Let's make a couple of simplifying assumptions (particularly the last one) to make the maths easier,

(a) the mass of the string per unit length a constant, μ

(b) the tension T (from stretching the string) is the only non-negligible force acting on the elements of the string (we can ignore gravity etc) and it is constant and acts in the direction along the length of the string (ie. the string is perfectly elastic and does not offer resistance to bending);

(c) the transverse motions are small (i.e. $y(x,t) \ll L$), so that the slope/angle of the waves are small.

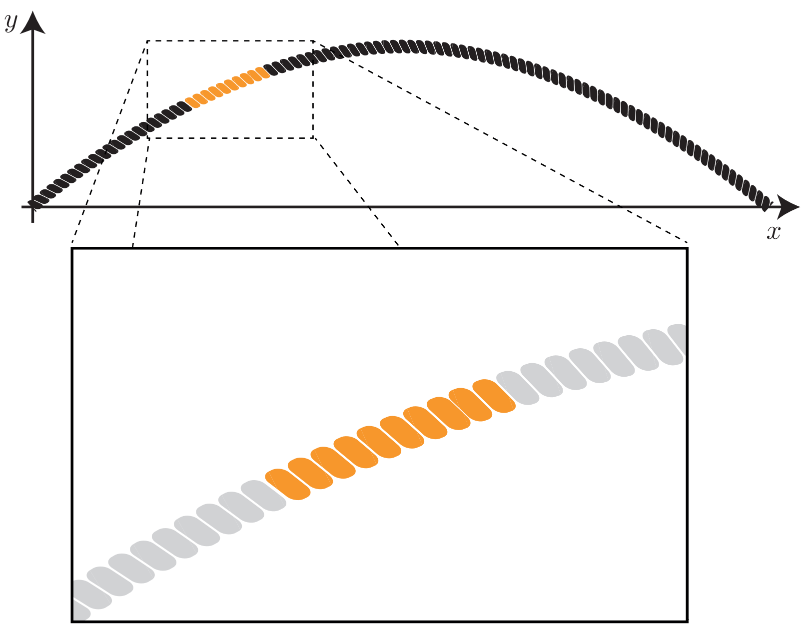


Figure 6.1: Zooming in on a string element of length $\Delta L \approx \Delta x$ we can draw a force diagram to apply Newton's second law.

In the Figure 6.1, we examine the forces that act on a small piece of elastic rope that is under tension (and obeys the assumptions outlined above). The Tension T on either side of the our piece of rope pull at angle α and β relative to the horizontal, where it may be that

$\alpha \neq \beta$ due to the curvature of the transverse wave. Remembering, however, that we are in the ***small angle approximation*** we have that

(6.2)

by our assumptions above, so that Newton's second law in the x-direction has that $\sum F_x = ma_x = 0$, so that the string is not accelerated in the x-direction.

Newton's second law in the y-direction gives us

(6.3)

Applying the small angle approximation gives us

. (6.4)

The values of $\tan\alpha$ and $\tan\beta$ are also given by the slope of $y(x,t)$ (with t held fixed) evaluated at each of the end points of our small section of rope,

(6.5)

thus we have

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x_o+\Delta x,t} - \left.\frac{\partial y}{\partial x}\right|_{x_o,t}\right) = \mu\Delta x\frac{\partial^2 y}{\partial t^2}. \quad (6.6)$$

Dividing both sides by $\mu\Delta x$ and taking the limit as $\Delta x \to 0$,

$$\frac{\partial^2 y}{\partial t^2} =$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \quad (6.7)$$

where $v^2 \equiv T/\mu$. This is the (1-D) wave equation for transverse waves on a string.

All (non-dispersive one-dimensional) waves, with amplitude given by $f(x,t)$, turn out to obey a ***wave equation*** PDE, which is of the form

(6.8)

The Wave Equation

Can you show that any (twice-differentiable) function of the form $f(x,t) = F(x - vt)$ satisfies the wave equation above?

6.2 *Solutions to the Wave Equation*

We have already seen in previous chapters that sinusoidal functions have "wave-like" behaviour. Here we will show that these functions are indeed solutions to the wave equation.

The wave equation is a ***Partial Differential Equation (PDE)***. We won't get into the multitude of techniques that one can use to solve

PDEs in general during this course, but if a PDE is ***linear*** (and "homogeneous" - the 1-D wave equation turns out to satisfy both these conditions), then we can apply the technique of ***separation of variables*** by assuming that the solution takes the form

$$y(x,t) = F(x)G(t)$$

which in turn often allows us to rewrite the PDE as a pair of Ordinary Differential Equations (ODEs) for $F(x)$ and $G(t)$ separately that we are more used to dealing with. As we will see the wave equation is particularly easy to apply this procedure to.

Let's assume that our solution $y(x,t)$ is separable as above. Since $F(x)$ and $G(t)$ are only functions of x and t, respectively, we must have by the chain rule

$$\frac{\partial}{\partial t}y(x,t) = \tag{6.9}$$

Similarly, we also have

$$\frac{\partial}{\partial x}y(x,t) = \tag{6.10}$$

Repeating this procedure we find the second derivatives just as easily,

(6.11)

(6.12)

so that our wave equation now looks like

(6.13)

Dividing both sides by $v^2 F(x)G(t)$, we find

$$\frac{1}{v^2} \tag{6.14}$$

The left hand side is function of t only, while the right hand side is dependent only on x. However, the expression above must hold for any x and t. The only way this can be the case is if it is equal to some arbitrary constant, which, with foresight, we will call $-k^2$. Thus we get two separate (and familiar) ordinary differential equations (ODEs) for $F(x)$ and $G(t)$, in the place of our PDE for $y(x,t)$.

(6.15)

(6.16)

where $\omega^2 \equiv k^2 v^2$. These are both the same as the differential equation for simple harmonic motion!

Note that since k is arbitrary, then, a general solution to the wave equation $f(x,t)$ may be a sum of solutions for several different ks!

Complex Exponential Notation for Waves

OFTEN IN ANALYZING WAVES, IT IS CONVENIENT to exploit ***Euler's identity***,

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{6.17}$$

Here we can do this by taking

$$y(x,t) = \text{Re}[F(x)G(t)] \tag{6.18}$$

where F and G are now complex functions, that solve Eqs. (6.15) and (6.16)

$$\tag{6.19}$$

and $\text{Re}[z] \equiv (z+z^*)/2$ denotes taking only the real part of a complex number. Remembering that handy fact that $e^X e^Y = e^{X+Y}$ we see that the wave equation must then have solutions of the form

$$\begin{aligned} y(x,t) &= \text{Re}[Ae^{i(kx\pm\omega t)}] \\ &= [Ae^{i(kx\pm\omega t)} + A^* e^{-i(kx\pm\omega t)}]/2 \\ &= |A|\cos(kx \pm \omega t + \delta) \end{aligned} \tag{6.20}$$

where A is a complex constant with $|A| \equiv \sqrt{AA^*}$ is the real-valued amplitude and $\tan^{-1}(\text{Im}[A]/\text{Re}[A]) = \arg(A) = \delta$ is the initial phase at $x = 0, t = 0$.
Often we will see the Re[...] dropped, and the solution written in a purely complex form,

$$y(x,t) = Ae^{i(kx\pm\omega t)}. \tag{6.21}$$

This is almost always just a shorthand (until you learn quantum mechanics), and it is understood to mean only the real part of the complex expression.
While this complex exponential form makes it easy to do certain manipulations of the waves (as we will see below), ***you should make sure to remember to take the real part in the end***!

6.3 *Waves Incident on a Boundary*

SO FAR WE HAVE ONLY examined a string of a single material. Let's imagine we have a heavier string attached to our lighter string. What happens when a wave traveling down the string encounters the attachment point?

Let's assume that our perfect string/rope system is described by a density μ_1 for $x < 0$, and density μ_2 for $x > 0$. Even though our combined string is no longer uniform, we can assume that the tension $T = T_1 = T_2$ is uniform across both the string and the rope, such that there is no horizontal acceleration.

We also know that in the region $x < 0$, that waves propagate with speed speed $v_1 \equiv \sqrt{T/\mu_1}$, and similarly for $x > 0$, waves propagate with speed $v_2 \equiv \sqrt{T/\mu_2}$. We also know that waves can be written in the form

$$y(x,t) = g(x \pm vt), \tag{6.22}$$

where v is whatever the wave speed is in that material. However it will turn out to be much more convenient to recast this into the form

$$y(x,t) = f(t \mp x/v) \tag{6.23}$$

where $f(s) = g(-s/v)$.

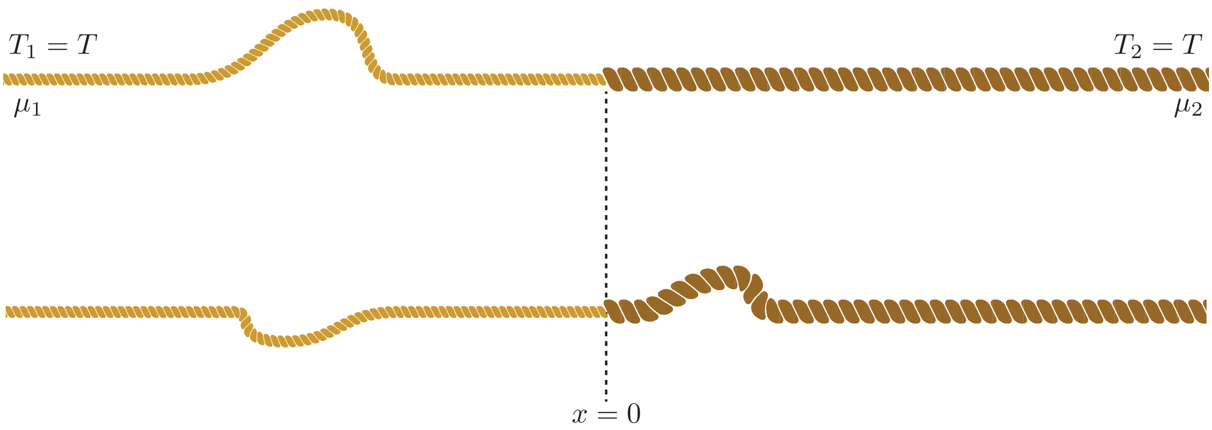


Figure 6.2: Two strings with different densities μ_1 and μ_2 connected at $x = 0$. A wave $y_i(x,t)$ is incident on the boundary point between the two strings, and a wave $y_r(x,t)$ is reflected, while a wave $y_t(x,t)$ is transmitted.

To examine what happens let's break the total solution $y(x,t)$ up into several parts.

Let's assume that we start with some wave in the region $x < 0$ which we will call the ***incident*** wave, that is moving towards the boundary such that it is described by wave propagating to the ***right***,

$$y_i(x,t) = \begin{cases} & \text{for } x \le 0 \\ & \text{for } x > 0 \end{cases} \tag{6.24}$$

where since this is only considering the part of the solution that is a wave moving ***towards*** the boundary from the ***left***, we ignore any contributions for $x > 0$.

What happens when this wave encounters the boundary at $x = 0$? In general, some of it gets ***transmitted***, and some of it gets ***reflected***. A reflected wave must have the form, for $x < 0$,

$$y_r(x,t) = \begin{cases} & \text{for } x \le 0 \\ & \text{for } x > 0 \end{cases} \tag{6.25}$$

since it is propagating to the ***left***, on the ***left*** of the boundary.

Meanwhile, transmitted wave is moving to the ***right***, and must have the form,

$$y_t(x,t) = \begin{cases} & \text{for } x < 0 \\ & \text{for } x \geq 0 \end{cases} \tag{6.26}$$

where this part of the solution propagates to the ***right*** with speed $v_2 = \sqrt{T/\mu_2}$, on the ***right*** side of the boundary where $x > 0$.

The total solution is then given by

$$y(x,t) = \tag{6.27}$$

for any point x and any time t!

What we'd like to do is solve for the total solution $y(x,t)$ in terms of the shape of the incident wave f_i. Let's take a closer look at the boundary point and see what it tells us about how the functions f_i, f_r and f_t are related.

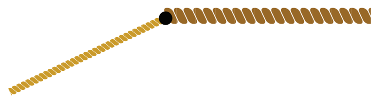


Figure 6.3: A kink.

Figure 6.4: The Kinks.

First off, we can assume that the string is ***connected*** and does not detached at the boundary, so that $y(x,t)$ must be continuous across that point (and indeed at all points in the string). Therefore for $x = 0$, we can written,

$$\Rightarrow \boxed{f_i(t) + f_r(t) = f_t(t)}. \tag{6.28}$$

Equation (6.28) is our ***first boundary condition***.

Our second boundary condition can be obtained by examining the slope of the string at $x = 0$. If the slope dy/dx was discontinuous across $x = 0$, then the rope would have a ***kink*** at the boundary. Examining the tiny piece of mass at the boundary point, this would give us a ***net force***, which would cause (nearly) infinite acceleration of our (nearly) massless piece of string at the boundary. Thus we can take as our ***second boundary condition***:

$$\left.\frac{\partial y}{\partial x}\right|_{x=0^-} = \left.\frac{\partial y}{\partial x}\right|_{x=0^+}$$

$$. \tag{6.29}$$

This can be rewritten,

$$\frac{d}{dt}\left[v_1 f_t(t) - v_2 f_i(t) + v_2 f_r(t)\right] = 0, \tag{6.30}$$

which implies that the thing in square brackets is a constant, giving us,

$$v_2 f_i(t) - v_2 f_r(t) = v_1 f_t(t) + \text{const} \tag{6.31}$$

If this constant of integration were non-zero it would mean that the right hand side would be permanently shifted by a certain transverse (y) displacement. However, since we are free to measure y from wherever we want, we can just assume that this constant of integration is zero with no loss of generality. This gives the expression,

$$\boxed{v_2 f_i(t) - v_2 f_r(t) = v_1 f_t(t).} \tag{6.32}$$

6.3.1 *Reflection and Transmission Coefficients*

USING EQUATIONS (6.28) and (6.32), we can solve for $f_r(t)$ and $f_t(t)$ in terms of $f_i(t)$.

$$f_r(t) = \tag{6.33}$$

(where we've multiplied (6.28) by v_1 and subtracted it from (6.32)), and

$$f_t(t) = \tag{6.34}$$

(where we've multiplied (6.28) by v_2 and added it to (6.32)).

Thus we find that the reflected and transmitted waves have the same overall shape as the incident wave, just multiplied by constant factors. We call these factors the ***Reflection Coefficient***, $\mathcal{R}$, and the ***Transmission Coefficient***, $\mathcal{T}$,

$$\boxed{\mathcal{R} \equiv \frac{v_2 - v_1}{v_2 + v_1} = \frac{f_r}{f_i},} \qquad \boxed{\mathcal{T} \equiv \frac{2v_2}{v_2 + v_1} = \frac{f_t}{f_i}.} \tag{6.35}$$

WE SEE ABOVE THAT THE SHAPES of the reflected and transmitted wave are related to the shape of the incident wave by a simple scaling, given by Reflection and Transmission coefficients, such that our total solution for any x and t is now

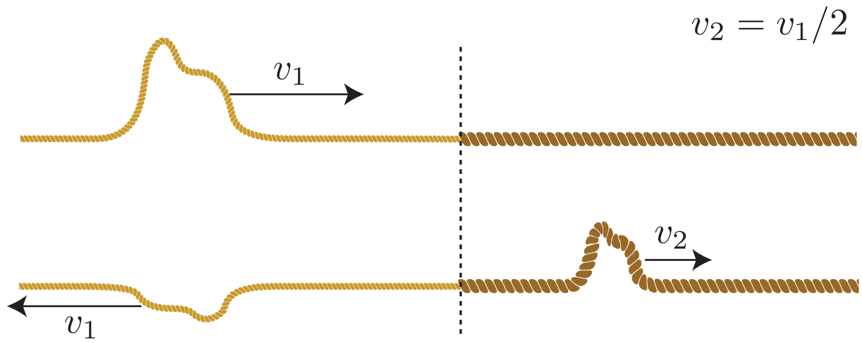
$$y(x,t) = y_i(x,t) + y_r(x,t) + y_t(x,t) \tag{6.36}$$

where,

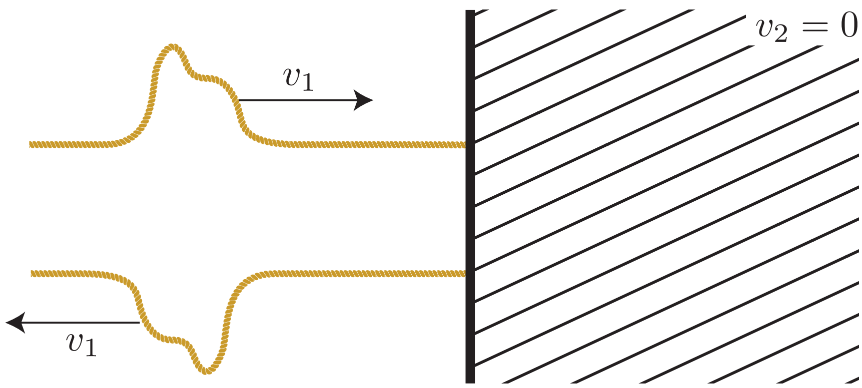
$$y_i(x,t) = \begin{cases} f_i(t - \frac{x}{v_1}), & \text{for } x \leq 0 \\ 0, & \text{for } x > 0 \end{cases}$$
$$y_r(x,t) = \begin{cases} \mathcal{R} f_i(t + \frac{x}{v_1}), & \text{for } x \leq 0 \\ 0, & \text{for } x > 0 \end{cases}$$
$$y_t(x,t) = \begin{cases} 0, & \text{for } x < 0 \\ \mathcal{T} f_i(t - \frac{x}{v_2}), & \text{for } x \geq 0. \end{cases} \tag{6.37}$$

Various Cases of Transmission and Reflection

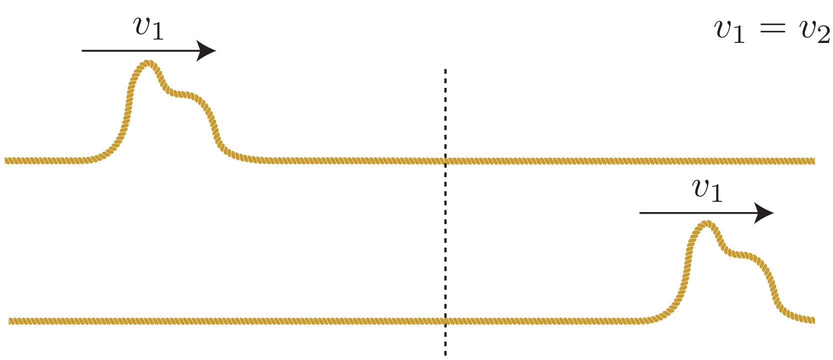
- LIGHT STRING → HEAVY STRING: Here $\mu_1 < \mu_2 < \infty$ and $v_2 < v_1$. This gives us $1 < \mathcal{R} < 0$ and $0 < \mathcal{T} < 1$.



- STRING → SOLID WALL: Here we have $\mu_2 = \infty$ (i.e. $v_2 = 0$) so that $\mathcal{R} = -1$ and $\mathcal{T} = 0$. Therefore nothing is transmitted, and the reflected wave is the same size but is ***inverted***.



- UNIFORM STRING: Here we just have two strings of the same density attached to each other so that $\mu_1 = \mu_2$ and $v_1 = v_2$. We see that $\mathcal{R} = 0$ and $\mathcal{T} = 1$. Thus, as expected, the wave propagates without interruption.



6.4 Impedance

IN THE PREVIOUS SECTION, we examined a system where the density of the string changed at $x = 0$, but we assumed that the tension was uniform across this boundary ($T_1 = T_2$ otherwise the (massless) knot would have been (infinitely) accelerated horizontally, violating our basic assumptions). In this section, we relax this condition, and allow the tension to be different in the two strings. This is only possible if we change the way in which the strings are attached.

Consider two strings tied to a massless ring which is threaded onto a fixed pole at $x = 0$, as shown in Figure 6.5. The ring is prevented from accelerating horizontally by the force from the pole, therefore the tensions can be different $T_1 \neq T_2$ without infinitely accelerating the massless ring horizontally since

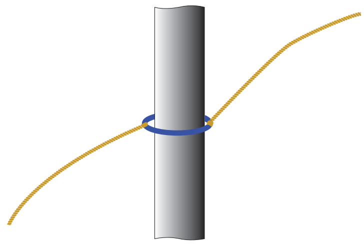


Figure 6.5: Two strings attached to a massless ring, which is threaded on a fixed pole. The tensions on the strings can be different since the pole provides a horizontal force to prevent the ring from accelerating in the x-direction.

$$\sum F_x = \tag{6.38}$$

However the pole cannot a provide transverse/vertical (y-direction) force on the ring, and we also don't want the massless ring to have infinite acceleration in the y-direction! Therefore we need the forces in the y-direction to cancel out too!

$$\sum F_y = \tag{6.39}$$

which can be written (since we only ever deal with small angles/y-displacements)

$$T_1 \tan\theta_1 = T_2 \tan\theta_2 \tag{6.40}$$

so that if T_1 and T_2 are different, the angles must also be different.

As before we can we know that the tangents are just the slopes of the strings on either side of the ring,

$$\tan\theta_1 = \left.\frac{\partial y}{\partial x}\right|_{x=0^-} \tag{6.41}$$

$$\tan\theta_2 = \left.\frac{\partial y}{\partial x}\right|_{x=0^+}. \tag{6.42}$$

Assuming an incident wave from the left, with reflected and transmitted waves as in the previous section,

Applying the chain rule we see this gives us

$$\tag{6.43}$$

which looks almost the same as the "no-kinks" boundary condition we had above, except now we have the tensions included (allowing for kinks at the ring).

We can repeat the steps from the previous section (make sure you can do this yourself!) and we will find almost the same expressions for $\mathcal{R}$ and $\mathcal{T}$

$$\mathcal{R} \equiv \frac{\frac{v_2}{T_2} - \frac{v_1}{T_1}}{\frac{v_2}{T_2} + \frac{v_1}{T_1}} = \frac{f_r}{f_i} \quad (6.44)$$

$$\mathcal{T} \equiv \frac{2\frac{v_2}{T_2}}{\frac{v_2}{T_2} + \frac{v_1}{T_1}} = \frac{f_r}{f_i} \quad (6.45)$$

except that v_1 and v_2 have been replaced by v_1/T_1 and v_2/T_2!

It is convenient here to define a new quantity for each string, called ***the impedance*** of that string, ,

$$Z \equiv T/v = T/\sqrt{T/\mu} = \sqrt{\mu T} \quad (6.46)$$

The Impedance of a String

such that the ***Reflection and Transmission coefficients*** are:

(6.47)

The Reflection and Transmission Coefficients in Terms of Impedance

. (6.48)

We saw in the previous section that for two strings tied together with matching tension, in order to get total transmission we needed to have $\mu_1 = \mu_2$. Now we see that If we allow the tensions to be different by coupling the string together as described above, then ***total transmission*** ($\mathcal{T} = 1$) (and no reflection, $\mathcal{R} = 0$) occurs if the ***impendances match***, $Z_1 = \sqrt{\mu_1 T_1} = \sqrt{\mu_2 T_2} = Z_2$.

If we tie no string (or a massless or limp string) to the right hand side of the ring, then $Z_2 = 0$. This gives a reflection coefficient of $\mathcal{R} = +1$, so that the wave is reflected without inverting. (The transmission coefficient $\mathcal{T} = 2$ is kind of meaningless in this case since it describes either a wave with no velocity (the limp string case $T = 0$) or a massless string with infinite velocity ($\mu = 0$). In either case the transmitted wave can't carry away any any of the wave energy!

Note: The mechanical impedance of a medium depends on the its physical properties, but also what ***type*** of wave disturbs it. For example, a string can support ***transverse***, ***longitudinal***, and ***torsional*** waves, each of these has a different impedance. We have only derived it for the string in this section. The general definition for mechanical impedance is ***the ratio of the wave restoring force to the displacement velocity in the direction of the force.***

6.4.1 What does Impedance mean?

ABOVE WE'VE DEFINED a new term $Z = \sqrt{\mu T} = T/v$ and called it by a special name, the impedance. Why did we do this? What is the physical significance of this quantity?

To explain this, let's imagine that someone is holding one end of a string and moving their arm to reproduce the motion of an arbitrary wave $y(x_o, t) = f(t \pm x/v)$. What is the rate of work (power) that their hand has to do to the string in order to make it perform this wave motion?

As before we can assume the horizontal forces cancel such that there is no horizontal acceleration (i.e. the hand holds the string taught). The vertical force exerted by the hand ***in order to match***

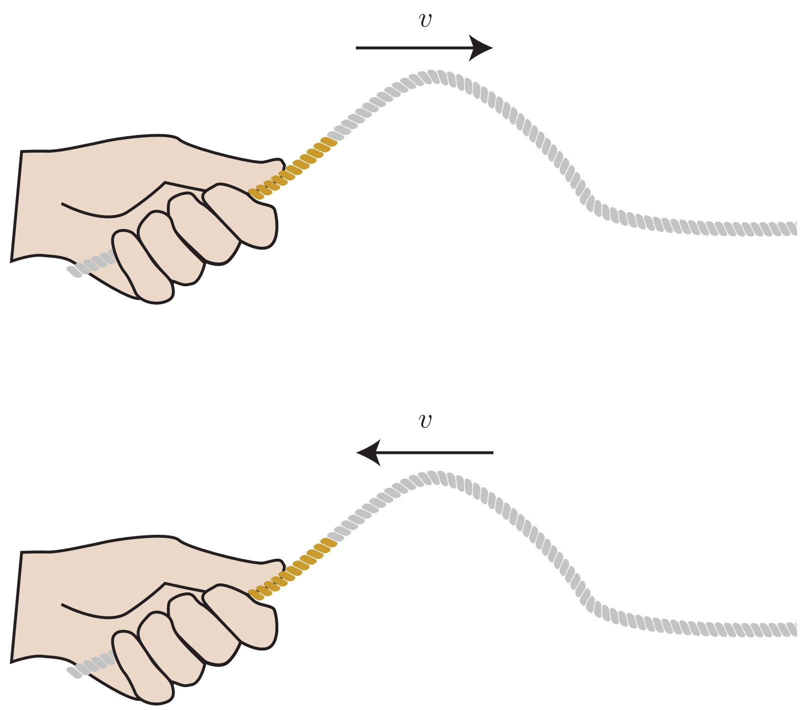


Figure 6.6: What is the power transferred to the string by the hand in each situation?

the wave is given by

$$F_{\text{hand,y}} = \tag{6.49}$$

where we've again used the small angle approximation.

Meanwhile the transverse/vertical velocity of the wave is

$$v_y \equiv \frac{\partial y}{\partial t}. \tag{6.50}$$

The power input into the string by the hand is just the rate of work done by the hand, $P = \vec{F} \cdot \vec{v}$

$$P(\text{hand} \rightarrow \text{string}) = \tag{6.51}$$

Let's examine a wave moving away from the hand (top figure), with the form

$$y(x, t) = f(x - vt) \tag{6.52}$$

By the chain rule we have

$$\tag{6.53}$$

such that the power is

$$P_{(\text{hand}\rightarrow\text{string})} = \tag{6.54}$$

Now let's consider a wave moving ***towards*** the hand, what is the power the hand put into the string? Our wave solution is now of the form

$$y(x,t) = f(x + vt) \tag{6.55}$$

such that

$$\tag{6.56}$$

The power is then

$$\begin{aligned} P_{(\text{hand}\to\text{string})} &= -T\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t} \\ &= \\ &= \end{aligned} \tag{6.57}$$

which is ***negative***! This means that the string actually transmits power into the hand!

This means that waves transmit energy in the direction of their propagation at a rate given by

$$|P| = Z v_y^2 \tag{6.58}$$

We see that the the ***impedance*** $Z \equiv F_y/v_y = |P|/v_y^2$ ***acts as a measure of how much power a wave transmits in the direction of propagation*** for a given transverse velocity v_y!

Now imagine we have two strings connected together via some coupler, as in the Figure 6.5. Energy is transmitted by an incident wave into the ring/coupler at a rate

$$|P_i| = Z_1\left(\frac{\partial y_i}{\partial t}\right)^2 \tag{6.59}$$

The power carried ***right*** by the ***transmitted*** wave is

$$|P_t| = Z_2\left(\frac{\partial y_t}{\partial t}\right)^2 = Z_2\mathcal{T}^2\left(\frac{\partial y_i}{\partial t}\right)^2 \tag{6.60}$$

while the power carried ***back*** to the left by the ***reflected*** wave is

$$|P_r| = Z_1\left(\frac{\partial y_r}{\partial t}\right)^2 = Z_1\mathcal{R}^2\left(\frac{\partial y_i}{\partial t}\right)^2 \tag{6.61}$$

(See if you can show that the ***energy is conserved*** i.e. that $|P_i| - |P_r| = |P_t|$.)

Comparing the transmitted power to the incident power we see

$$\frac{P_t}{P_i} = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1+Z_2}\right)^2 = \tag{6.62}$$

which can only be 1 if $Z_1 = Z_2$.

This gives us alternative viewpoint on why reflections arise when the impedances are mismatched: The transmitted wave is able to match the power transmitted into the ring by the incident wave ***only if*** the impedances are matched ($Z_1 = Z_2$).

Otherwise we ***must*** have $4Z_1Z_2 < (Z_1+Z_2)^2$ (check this yourselves!) and the transmitted power cannot keep up with the incident power, so ***some power must be reflected*** instead.

Summary of Non-Dispersive Waves

- THE 1-D NON-DISPERSIVE WAVE EQUATION can be derived by considering the forces on an element of a tensioned string:

$$\frac{\partial^2 f(x,t)}{\partial t^2} = v^2 \frac{\partial^2 f(x,t)}{\partial x^2}.$$

  which has general solution $f(x,t) = F(x \pm vt)$, where $F(s)$ is an arbitrary function of a single variable.

- THE COMPLEX EXPONENTIAL NOTATION FOR WAVES makes use of the Euler identity, and allows us to write 1-D waves in the form:

$$\begin{aligned} y(x,t) &= \mathrm{Re}[Ae^{i(\pm kx \pm \omega t)}] \\ &= |A| \cos(kx \pm \omega t + \delta) \end{aligned}$$

  where $\omega = kv$ and $\delta = \mathrm{Arg}(A)$. In general, waves can be the sum of many waves of the above form, with many different k.

- TWO TENSIONED STRINGS TIED TOGETHER must have equal tension across the knot. Waves are transmitted across and reflected back from this interface with the transmission and refelection coefficients (respectively):

$$\mathcal{T} \equiv \frac{2v_2}{v_2 + v_1}, \qquad \mathcal{R} \equiv \frac{v_2 - v_1}{v_2 + v_1}$$

- WE CAN ALLOW THE TENSIONS TO BE DIFFERENT by tying the strings to a ring mounted on a fixed pole. The transmission and reflection coefficients then become:

$$\mathcal{T} \equiv \frac{2Z_1}{Z_1 + Z_2}, \qquad \mathcal{R} \equiv \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

  where Z_1, Z_2 are the ***impedances*** of the two different strings.

- The impedance for a string is given by the ratio of the tangential force to the tangential velocity (or the ratio of the power transmitted to the square of the tangential velocity).

$$\begin{aligned} Z &= \frac{|F_y|}{|v_y|} = \frac{P}{v_y^2} \\ &= \frac{T}{v} = \sqrt{T\mu} \end{aligned}$$

  The impedance depends only on the string properties and the type of wave but ***not*** on the wave shape. It tells us how power is transmitted down a medium, and has the general form of the ratio of the wave's restoring force to the displacement velocity in that direction.

PART III:
OPTICS