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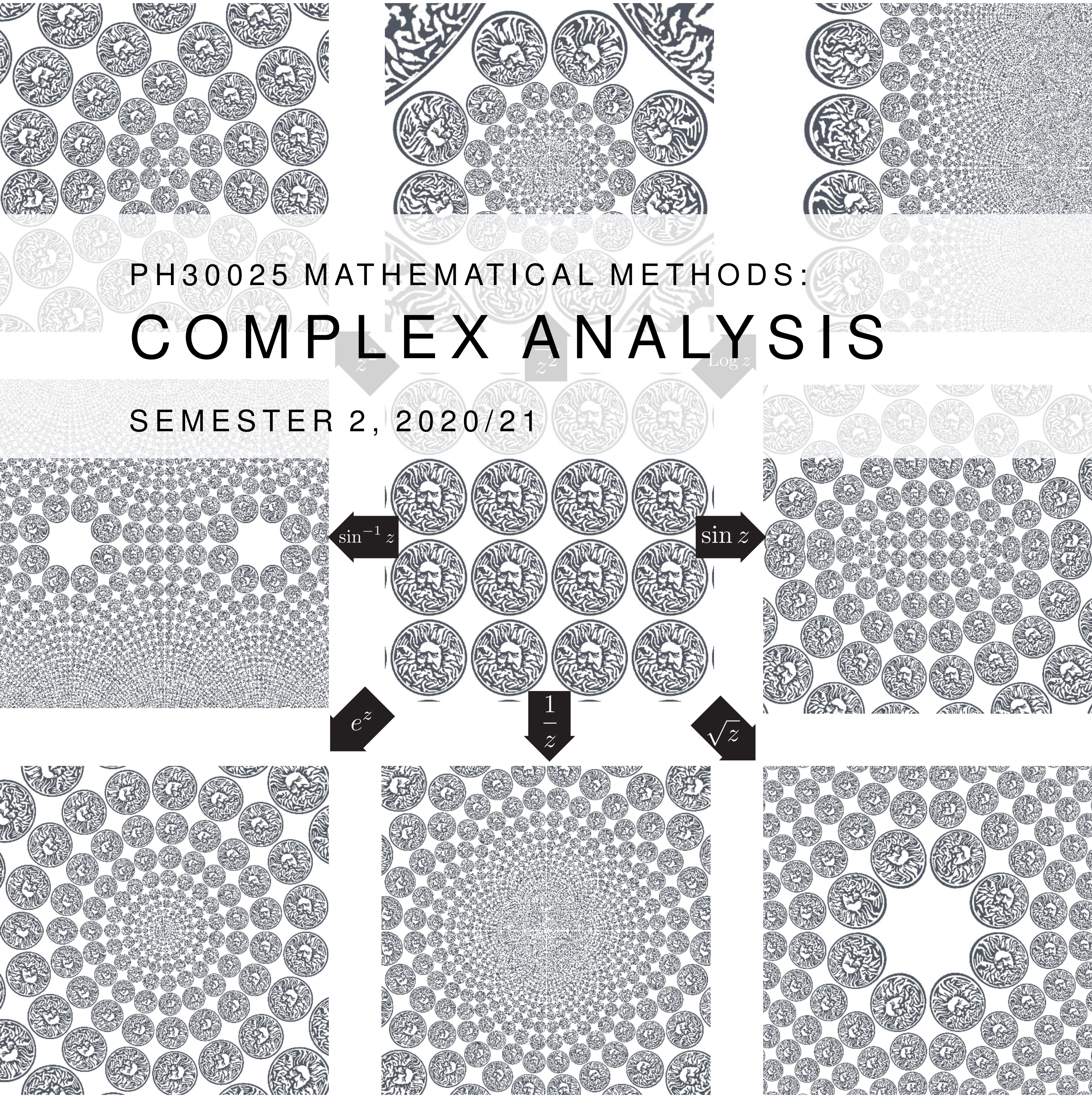
UNIVERSITY OF BATH

PH30025 MATHEMATICAL METHODS:

COMPLEX ANALYSIS

SEMESTER 2, 2020/21



DEPARTMENT OF PHYSICS, UNIVERSITY OF BATH

1
Review of Complex Numbers and Functions

1.1 Complex Numbers

1.1.1 Mathematical Fields

Mathematical fields are sets of "numbers" that possess the usual sense of addition, subtraction, multiplication and division. They serve as a foundational concept for algebra, number theory, and real and complex analysis. For example, any mathematical field may act as the set of scalars for a vector space, due to the basic properties that fields possess.

Mathematical Fields

Definition 1.1. *In mathematics, a* **field** *is defined to be a* **set** *of mathematical objects that comes equipped with an* **addition** *operation, and a* **multiplication** *operation, which both take two elements of the field and produce another element of the field. For a field $\mathcal{F}$, equipped with $+$ and $\cdot$,*

$$+ : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$$
$$\cdot : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$$

$\mathcal{F}$ must be **closed** *under $+$ and $\cdot$, and must also obey the axioms $\forall a, b, c \in \mathcal{F}$:*

($\mathbf{C}^+$) $a + b = b + a$, (**C**ommutativity of $+$)
($\mathbf{A}^+$) $a + (b + c) = (a + b) + c$, (**A**ssociativity of $+$)
($\mathbf{N}^+$) $\exists$ an element $0 \in \mathcal{F}$ such that $a + 0 = a$, (**N**eutral element under $+$)
($\mathbf{I}^+$) $\exists$ an element $-a \in \mathcal{F}$ such that $a + (-a) = 0$, (**I**nverse under $+$)

($\mathbf{C}^\cdot$) $a \cdot b = b \cdot a$, (**C**ommutativity of $\cdot$)
($\mathbf{A}^\cdot$) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, (**A**ssociativity of $\cdot$)
($\mathbf{N}^\cdot$) $\exists$ an element $1 \in \mathcal{F}$ such that $1 \cdot a = a$, (**N**eutral element under $\cdot$)
($\mathbf{I}^\cdot$) $\forall a \neq 0, \exists$ an element $a^{-1} \in \mathcal{F}$ such that $a^{-1} \cdot a = 1$, (**I**nverse under $\cdot$)

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, (**D**istributivity of $\cdot$ over $+$)

The above definition may seem very intimidating, but really all it says is that addition and multiplication (as well as subtraction and division) behave as "usual" for a field.

Maths Shorthand
$\exists$: "there exists".
$\forall$: "for all".
$\in$: "in" or "is an element of".

The simplest example of a field is the set of ***rational*** numbers with the usual addition and multiplciation (and therefore subtraction and division). However, the rational numbers don't contain the solution to some equations, such as

$$x \cdot x = 2. \tag{1.1}$$

To solve such equations, we must add the ***irrational*** numbers (such as $\sqrt{2}$) to the set of rational numbers to get the set of ***real*** numbers, $\mathbb{R}$, that you are used to.

Similarly the real numbers cannot solve certain equations, like

$$x \cdot x = -1. \tag{1.2}$$

To "correct" this, we can add a number denoted by i to the reals, defined such that

$$i^2 = i \cdot i = -1 \tag{1.3}$$

Note that engineers often use j instead of i to avoid confusion with current. We are not engineers.

1.1.2 Complex Numbers and their Representations

The Complex Numbers, $\mathbb{C}$

Definition 1.2. *We can then define the set of complex numbers $\mathbb{C}$ as a field such that*

1. *Every real number is also a complex number*
2. *i is a complex number*
3. *Every complex number can be written in the form $a + ib$ where $a, b \in \mathbb{R}$, and any such number is a member of $\mathbb{C}$.*
4. *Addition is defined* **componentwise**, *as inherited from the reals, such that for $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$, then*
$$\alpha + \beta = (a_1 + b_1) + i(a_2 + b_2)$$
5. *Multiplication is defined using multiplication inherited from the reals and $i^2 = -1$, such that*
$$\begin{aligned}\alpha \cdot \beta &= (a_1 + ia_2) \cdot (b_1 + ib_2) \\ &= a_1 b_1 - a_2 b_2 + i(a_1 b_2 + a_2 b_1)\end{aligned}$$

We will adopt the normal convention and write $ab = a \cdot b = a \times b$ interchangeably for the multiplication operation

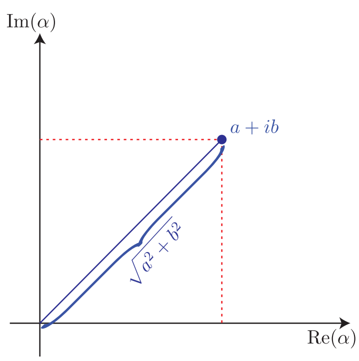


Figure 1.1: The point (cartesian/rectangular) representation of a complex number.

The Complex Conjugate

Definition 1.3. *Let $\alpha = a + ib$ be a complex number. We define $\overline{\alpha}$ to be*

$$\overline{\alpha} \equiv a - ib \in \mathbb{C} \tag{1.4}$$

The complex number $\overline{\alpha}$ is called the **complex conjugate** *of α.*

We will always use the overbar to denote complex conjugation. Other common notation includes e.g. α^*.

From the definition of the complex conjugate we can immediately see that

$$\alpha\overline{\alpha} = a^2 + b^2 \in \mathbb{R}. \tag{1.5}$$

Interpreting the real and imaginary "components" as vector components in the 2-D plane, we see that $\alpha\overline{\alpha}$ corresponds to the square of the distance from the origin, also known as the absolute value (or modulus) squared.

Absolute Value

Definition 1.4. *The* **absolute value** *or* **modulus** *of the complex number* $\alpha = a + ib$ *is denoted* $|\alpha|$ *and is given by*

$$|\alpha| \equiv \sqrt{a^2 + b^2} = \sqrt{\alpha\overline{\alpha}}$$

Does the set of complex numbers satisfy the properties of a field? The set $\mathbb{C}$ is clearly closed since the real numbers we use to build them are closed, and we see that by basing the addition and multiplication operations on those inherited from the real numbers, we obviously satisfy most of the $\mathrm{C^+A^+N^+I^+\ C^{\cdot}A^{\cdot}N^{\cdot}I^{\cdot}\ D}$ axioms.

The one tricky axiom we may wish to verify is $\mathrm{I^{\cdot}}$, the existence of the multiplicative inverse for any complex number $\neq 0$. For any $\alpha = a + ib \neq 0$ we can use the notion of a complex conjugate to construct the inverse

$$\alpha^{-1} \equiv \frac{\overline{\alpha}}{a^2 + b^2} \tag{1.6}$$

We can immediately see that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = 1$ since

$$\frac{\alpha\overline{\alpha}}{a^2 + b^2} = \frac{\alpha\overline{\alpha}}{\alpha\overline{\alpha}} = 1 \tag{1.7}$$

hence every non-zero $\alpha \in \mathbb{C}$ has a multiplicative inverse also in $\mathbb{C}$.

Exercise 1.1:

Verify the following:

Theorem 1.1. *Let* α *and* β *be any complex numbers. Then*

$$\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}, \qquad \overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}, \qquad \overline{\overline{\alpha}} = \alpha.$$

Since we can associate each complex number with a vector in the 2-D plane, we can also see that the component-wise addition of complex numbers also corresponds to the component-wise addition of vectors. Recalling the geometric fact that the length of any side of triangle is less than or equal to the sum of the lengths of the other two sides, we can, by analogy, deduce ***the triangle inequality***.

The Triangle Inequality

Theorem 1.2 (The triangle inequality). *For any two complex numbers* α *and* β*, we have*

$$|\alpha + \beta| \leq |\alpha| + |\beta|.$$

Just as 2-D vectors can be expressed in either cartesian form (x, y), or polar form (r, θ), we can also express complex numbers in a "polar" form.

We know that we can clearly express cartesian coordinates in terms of polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1.8}$$

However, we must be careful in expressing the polar coordinates in terms of the cartesian coordinates. While the expression for r is clear,

$$r = \sqrt{x^2 + y^2}, \tag{1.9}$$

we have for $\tan\theta = y/x$, however if we specify

$$\theta = \tan^{-1}(y/x), \tag{1.10}$$

this expression is invalid for points in the second and third quadrants (for the normal definition of the arctangent, $\tan^{-1}$). In practice we must add or subtract π in order to correctly place points in those quadrants. We can instead specify a related function

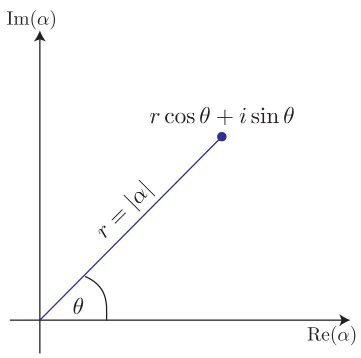


Figure 1.2: The polar representation of complex number α.

$$\theta = \arctan(x,y) \equiv \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0, \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, y \geq 0, \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0, y < 0, \\ +\pi/2 & \text{if } x = 0, y > 0, \\ -\pi/2 & \text{if } x = 0, y < 0, \\ \text{undefined} & \text{if } x = 0, y = 0. \end{cases} \tag{1.11}$$

Definition of $\arctan(x,y)$. In many programming languages, this 2 variable function is often called "atan2".

However, even using this approach, there is ambiguity, as the angle θ is only determined up to an integer multiple of 2π. To accommodate this, we will refer to the value of any of these "angles" for complex number $z = x + iy$, as the ***set*** of angles, called the ***argument*** of z,

$$\arg z \equiv \{\arctan(x,y) + 2k\pi : k = 0, \pm1, \pm2, ...\}. \tag{1.12}$$

Definition of $\arg z$

For example, $\arg i = \{\frac{\pi}{2} + 2k\pi, \text{ for } k = 0, \pm1, \pm2, \ldots\}$.

It is convenient to have a notation for a specific angle from the set of $\arg z$. Any half open interval of length 2π will contain one and only one value of the argument. Specify a particular choice of this range, is called choosing a particular ***branch*** of $\arg z$. (Note that $\arg(0)$ cannot be reasonably defined for any branch.)

Normal convention defines the selection of the branch of $\arg z$ from $(-\pi, \pi]$ to be the ***principal value*** of the argument, which is usually written as $\operatorname{Arg} z$ with a capital A. The principal value is particularly important in complex arithmetic in numerical codes, and is inherently discontinuous, with value jumping by 2π as z crosses the negative real axis. This line of discontinuity is called ***the branch cut*** of $\operatorname{Arg} z$.

One handy notation is $\arg_\tau(z)$ which is used for the branch of $\arg(z)$ taking values from the interval $(\tau, \tau + 2\pi]$. Thus $\arg_{-\pi}(z)$ is the principal value $\operatorname{Arg}(z)$.

With all these conventions in hand, we can write $z = x + iy$ in the ***polar form***

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.13}$$

where $r = |z| = \sqrt{z\bar{z}}$ and $\theta \in \arg(z)$,

In many circumstances, the rectangular form $(x + iy)$ or the polar form $(r[\cos\theta + i\sin\theta])$ may be more suitable than the other. The rectangular form, for example is very convenient for addition or subtraction, whereas in the polar form this can be very difficult. However, multiplication and division of complex numbers in polar form provides a very useful geometric interpretation of multiplication and division.

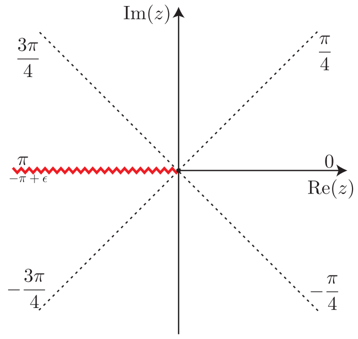


Figure 1.3: Arg (z), where the principal value of $\arg(z)$ has a branch cut (denoted by the jagged line) along the negative real axis. Along this axis Arg $(z) = \pi$, while slightly below the negative real axis Arg $(z) \to -\pi$.

Exercise 1.2: Polar Multiplication

Show the following using trigonometric identities:

Theorem 1.3. The modulus of the product is the product of the moduli*:*

$$|\alpha\beta| = |\alpha||\beta|$$

and **the argument of the product is the sum of the arguments***:*

$$\arg(\alpha\beta) = \arg(\alpha) + \arg(\beta).$$

(Where the above is to be interpreted as saying that if particular values are assigned to the arguments on the left hand side, one can find a value for the right hand side that satisfies the relation.)

With the properties in Theorem 1.3, we may begin to see a connection to the exponential function, where multiplication becomes an addition of the exponents. Indeed we can formalise this by providing a suitable definition for $\exp(z)$ where $z = x + iy \in \mathbb{C}$ preserves the basic identities satisfied by the real exponential.

We want the complex exponential to share the product-to-sum-of-the-exponents property of the real exponential such that

$$e^{\alpha}e^{\beta} = e^{\alpha+\beta}, \qquad \forall \alpha, \beta \in \mathbb{C} \tag{1.14}$$

This allows us to simplify the problem of defining the complex exponential considerably, since it allows the decomposition

$$e^{z} = e^{x+iy} = e^{x}e^{iy} \tag{1.15}$$

such that all that remains is to define the exponential of a ***purely imaginary*** number!

Another property that will help us proceed is the differentiation property of the exponential function,

$$\frac{d}{dz}e^{z} = e^{z} \tag{1.16}$$

which we would like to be true, as in the real version.

Differentiation of a complex function is a bit tricky, (which we will cover later), but we can just focus on the derivative of the purely imaginary exponent which we want to be

$$\frac{d}{d(iy)}e^{iy} = e^{iy} \tag{1.17}$$

which by the chain rule gives

$$\frac{d}{dy}e^{iy} = ie^{iy}. \tag{1.18}$$

Taking the second derivative, using this rule, we see

$$\frac{d^2}{dy^2}e^{iy} = \frac{d}{dy}(ie^{iy}) = i^2e^{iy} = -e^{iy} \tag{1.19}$$

but we know that the differential equation $f''(y) = -f(y)$ is solved by functions of the form $f(y) = A\cos y + B\sin y$, where A and B are constants.

We can determine these constants by noting that $f(0) = e^{i0} = e^0 = 1 = A\cos 0 + B\sin 0$, and $f'(0) = ie^{i0} = i = -A\sin 0 + B\cos 0$. Hence $A = 1, B = i$, and we must have

Euler's Equation

Theorem 1.4 (Euler's equation)**.**

$$e^{iy} = \cos y + i\sin y, \qquad \forall y \in \mathbb{R}.$$

This naturally leads to the definition:

The Complex Exponential

Definition 1.5. *If $z = x + iy \in \mathbb{C}$, for $x, y \in \mathbb{R}$, then e^z is defined to be the complex number*

$$e^z \equiv e^x(\cos y + i\sin y).$$

which turns out to be the "right choice", as it obeys all the same properties as the real exponential, and is ***holomorphic*** which we will define and demonstrate later when we look in detail at the differentiability of complex functions.

Exponential Polar Form

Euler's equation allows us to write the polar form of a complex number particularly compactly,

$$z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta} = |z|e^{i\operatorname{Arg} z} \tag{1.20}$$

The complex conjugate $\overline{z}$ can be written as

$$\overline{z} = r(\cos\theta - i\sin\theta) = re^{-i\theta} = |z|e^{-i\operatorname{Arg} z}. \tag{1.21}$$

Exercise 1.3: Taylor Expansion of Euler's Equation

Show that Euler's equation is formally consistent with the usual Taylor series expansions

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ \sin x &= 1x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \end{aligned}$$

Exercise 1.4: De Moivre's theorem

Prove the following,

Theorem 1.5 (De Moivre's thoerem).

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \qquad n = 1, 2, 3, \dots$$

1.1.3 *Powers and Roots of Complex Numbers*

Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ be the polar form of the complex number z. By Theorem 1.3 (or the properties of the exponential function), we have

$$z^2 = z \cdot z = r^2 e^{i2\theta}. \tag{1.22}$$

Multiplying by z again, we have

$$z^3 = r^3 e^{i3\theta}. \tag{1.23}$$

Continuing this procedure provides us with the generalisation of De Moivre's theorem for powers of z,

$$z^n = r^n e^{i\theta} = r^n(\cos n\theta + i\sin n\theta) \tag{1.24}$$

Exercise 1.5: Generalised De Moivre's Theorem

The expression above is a useful formula for raising a complex number to a positive integer power n. Show that the above equation also holds for ***negative*** n.

The question arises whether this generalised De Moivre's theorem will work for $n = 1/m$, so that that $\zeta = z^{1/m}$ is the mth root of z satisfying

$$\zeta^m = z. \tag{1.25}$$

Certainly if we define

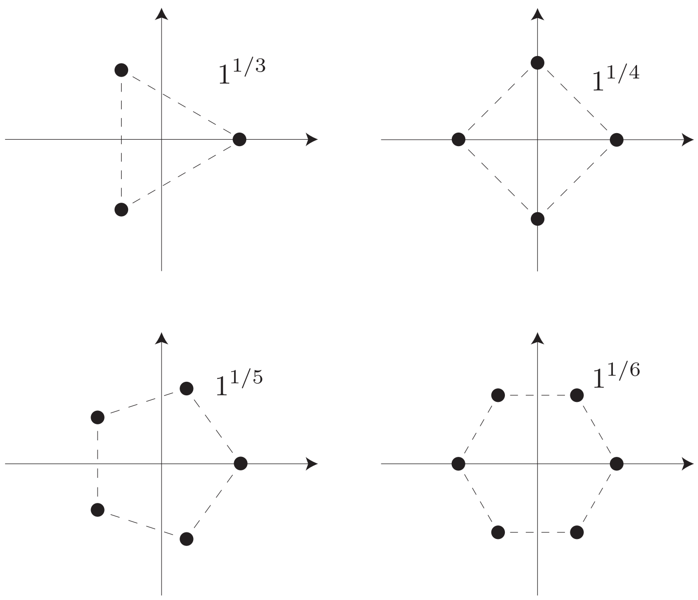
$$\zeta = \sqrt[m]{r}\, e^{i\theta/m}, \tag{1.26}$$

then ζ satisfies the equation above. However, the the presence of ***additional roots*** due to ***multiple branches*** of the argument $\theta \in \arg(z)$ complicates matters, as θ is then defined only up to a value of 2π.

Example 1.1: Roots of Unity

There are exactly m distinct mth roots of unity, denoted by $1^{1/m}$, and they are given by

$$e^{i2k\pi/m} = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m} \qquad (k = 0, 1, 2, \ldots, m-1) \tag{1.27}$$



Let $\omega_m \equiv e^{i2\pi/m}$. The complete set of mth roots of unity is

$$1, \omega_m, \omega_m^2, \ldots, \omega_m^{m-1}. \tag{1.28}$$

We can also prove that

$$1 + \omega_m + \omega_m^2 + \ldots + \omega_m^{m-1} = 0. \tag{1.29}$$

This result is obvious from a physical point of view, since by symmetry the centre of "mass" $(1 + \omega_m + \omega_m^2 + \ldots + \omega_m^{m-1})/m$ of the system of m unit masses located at the mth roots of unity must be at the origin.

1.1.4 *The Complex Logarithm*

We want to define the complex logarithm, $\log z$, as the "inverse" of the exponential function; i.e.

We will almost always take the logarithm with base e, so log or Log will always refer to this. In other texts the notation ln or Ln is often used.

$$w = \log z \qquad \text{if} \qquad z = e^w. \tag{1.30}$$

Since e^w is never zero, we presume that $z \neq 0$. To find $\log z$ explicitly, let us write z in polar form as $z = re^{i\theta}$ and write w in the standard form $w = u + iv$. Then the equation $z = e^w$ becomes

$$re^{i\theta} = e^{u+iv} = e^u e^{iv}. \tag{1.31}$$

From this we can deduce that $u = \operatorname{Log} r$, where this is the usual (real) logarithm of a real number, and $v = \arg z = \theta$ can take multiple values.

The Complex Log

Definition 1.6 (The Complex Logarithm). *If $z \neq 0$, then we define $\log z$ to be any of the infinitely many values*

$$\begin{aligned}\log z &\equiv \operatorname{Log}|z| + i \arg z \\ &= \operatorname{Log}|z| + i\operatorname{Arg} z + i2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)\end{aligned}$$

The multiple-valued nature of $\log z$ simply reflects that the imaginary part of the logarithm is the polar angle θ; the real part is single-valued. We can choose a single-value by using, once again, the notion of a ***branch cut*** to resolve the ambiguity, as we did for $\operatorname{Arg} z$.

Definition 1.7. *We can define the* **principal value of** *of the logarithm, $\operatorname{Log} z$, to be the value inherited from the principal value of $\operatorname{Arg} z$,*

$$\operatorname{Log} z \equiv \operatorname{Log}|z| + i\operatorname{Arg} z$$

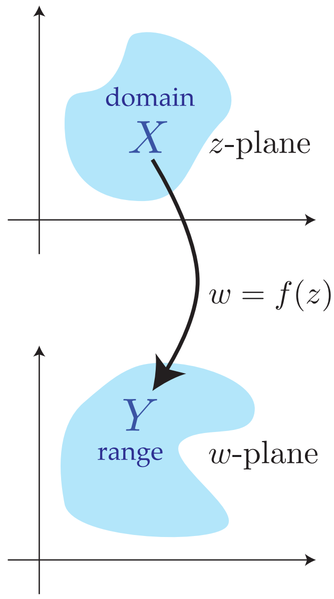


Figure 1.4: A complex function $f(z)$ with domain X, and range Y.

1.2 *Complex Functions*

Let S be a set of complex numbers. A mapping which associates each element of S to a complex number is called a ***complex valued function***, or ***function*** for short. We denote such a function by symbols like

$$f : S \to \mathbb{C} \tag{1.32}$$

where S is a subset of $\mathbb{C}$. If $z \in S$, we write the association of the ***value*** $f(z)$ to z by the special arrow

$$z \mapsto f(z) \tag{1.33}$$

If a function f is defined $f : X \to Y$, (where $X, Y \subseteq \mathbb{C}$) we call the set X the ***domain*** of f, and the set Y the ***range*** of f.

Usually, when we use the word "function" to describe f we mean that f assigns a single value w in the range to each permissible value of z in the domain. Sometimes we explicitly state that "f is a single-valued function". Of course there are equations that do not define single-valued functions, for example $w = \arg z$, $w = z^{1/2}$, and $w = \log z$ as we saw in the previous section. In general, if for some values of z there corresponds more than one value of

$f(z)$, then we say that $f(z)$ is a ***multi-valued function*** (note that technically these are not considered functions). We commonly obtain multiple-valued functions by taking "inverses" of single valued functions that are are not one-to-one (see discussion below), such as in the case of the complex logarithm we discussed above.

1.2.1 *Sets on the Complex Plane*

Here we will briefly cover notions of point sets in the plane, in order to be able to more carefully define the ***domains*** and ***ranges*** which complex functions map between.

Open and Closed Discs

Definition 1.8. *Let α be a complex number. An* **open disc**, *denoted by $D(\alpha, r)$, of (real) radius $r > 0$ centred at α is the set of all complex numbers z, such that*

$$D(\alpha, r) \equiv \{z : |z - \alpha| < r\}.$$

Similarly the set $\overline{D}(\alpha, r) \equiv \{z : |z - \alpha| \leq r\}$ is called the **closed disc** *centred at α, of radius r.*

We will make frequent reference to the neighbourhood $D(0, 1)$, the ***unit open disc***.

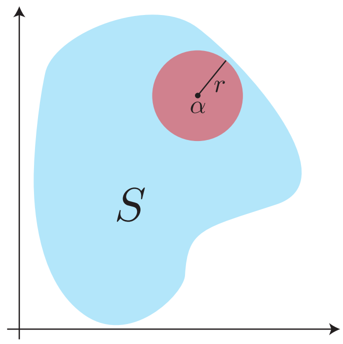


Figure 1.5: The point α is an interior point of S if you can find some $r > 0$ such that disc $D(\alpha, r) \subset S$. S is open if it only contains interior points.

Open and Closed Sets

Definition 1.9 (Open Sets)**.** *Let S be a subset of the complex plane. S is called an* **open set** *if for every point α in S, there is a disc $D(\alpha, r)$ centred at α, and of some radius $r > 0$ such that this disc $D(\alpha, r)$ is completely contained in S.*

Each point α that obeys the condition above is called an ***interior point*** of set S. Open sets contain ***only*** interior points.
A point β is called a ***boundary point*** of set S if any $\forall r > 0$, the disc $D(\beta, r)$ contains at least one point in S, and at least one point ***not*** in S. The set of all boundary points is called the ***boundary*** or ***frontier*** of S.

Definition 1.10 (Closed Sets)**.** *A set $\overline{S}$ is called a* **closed set** *if it contains all of its boundary points.*

Note that the set $\{z : 0 < |z| \leq 1\}$ is not closed since it does not contain the boundary point 0. Nor is it open since points in the set where $|z| = 1$ are boundary points. (Such part open part closed sets are sometimes called ***clopen*** sets. Unfortunately.)
Also note that the same set can be considered an open subset of one space, but not of another. For example, the interval $(0, 2\pi)$ is an open subset of the real line $\mathbb{R}$, but it is ***not*** an open subset of $\mathbb{C}$ (all points in the interval are boundary points in this case).

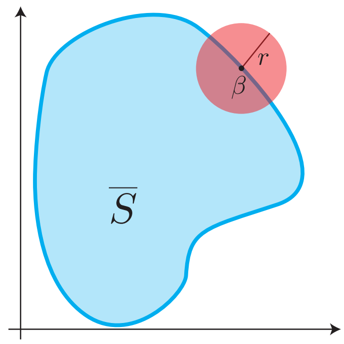


Figure 1.6: The point β is a boundary point of $\bar{S}$ if any disc $D(\beta, r)$ contains points both inside and outside $\bar{S}$. $\bar{S}$ is closed if it contains all its boundary points. The set of all boundary points of a set S is called the boundary of S and is sometimes denoted ∂S.

The ***closure*** of a set S is defined to be the union of S and all its boundary points. We denote this closure by $\overline{S}$.

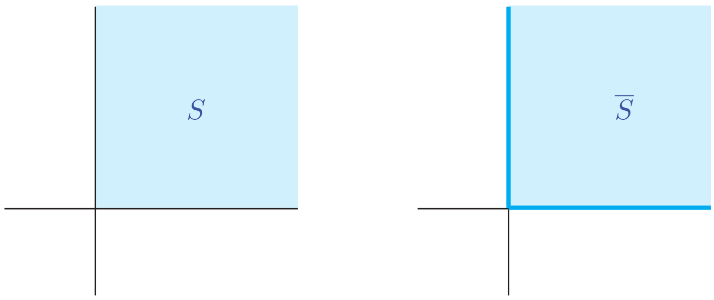


Figure 1.7: ***Left:*** The set S is the ***open*** first quadrant on the complex plane. ***Right:*** The set $\overline{S}$ is the ***closed*** first quadrant which includes the positive real, and positive imaginary axes. (Neither of these sets is compact.)

A set S is said to be ***bounded*** if there exists a real number $C > 0$ such that

$$|z| \leq C, \qquad \forall z \in S \tag{1.34}$$

There are a few other definitions of compactness, particularly that every sequence of elements of S has a convergent subsequence whose limit is in S. This can be shown to be equivalent to our definition, at least on the complex plane.

A set S is said to be ***compact*** if and only if it is both closed and bounded.

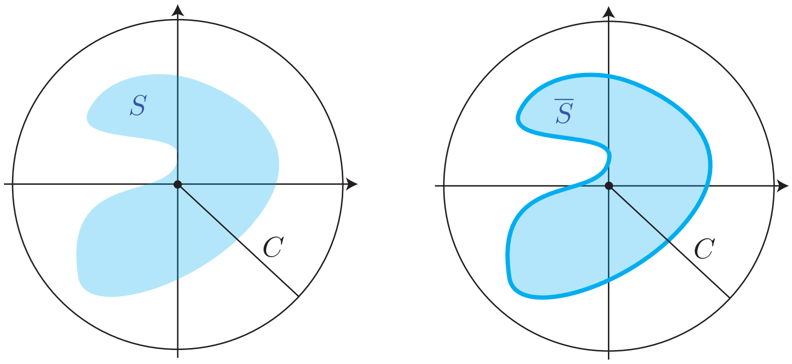


Figure 1.8: ***Left:*** The open set S is ***bounded***, but is not compact. ***Right:*** The closed set $\overline{S}$ is also bounded, therefore it is ***compact***.

A set S, is said to be ***connected*** if given any two points α and β in S, there exists a path within S that joins α to β. (This relatively strong definition of connectedness is sometimes called ***pathwise connectedess***; there are other notions of connectedness that have to do with topology, which we will not get into in this module.)

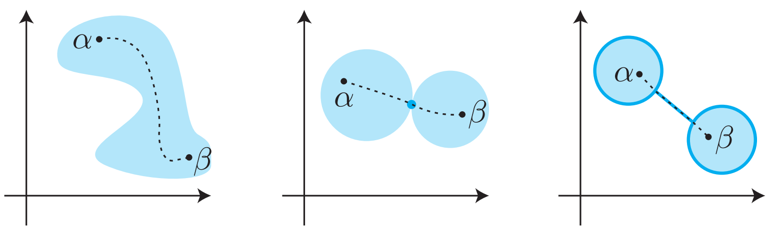


Figure 1.9: Examples of connected sets.

A (path-wise) connected set is said to be ***simply connected*** if ***any*** closed curve (i.e. a loop) inside the set can be continuously shrunk down to a point.

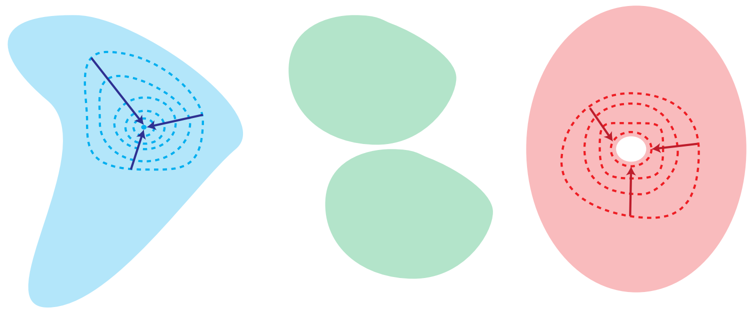


Figure 1.10: Simply connected sets are path-wise connected sets where ***any*** closed loop can be shrunk to a point.

1.2.2 *Limits, Sequences, and Continuity*

The definition of the absolute value can be used to designate the distance between two complex numbers (this is sometimes called ***a metric***). This allows us to easily define "open" sets (though it is not the only way to do so, see e.g. any introductory text on topology, where "openness" is defined by fiat). Having the concept of distance we can also proceed to introduce notions of limits and convergence, analogous to those you are used to with the real numbers.

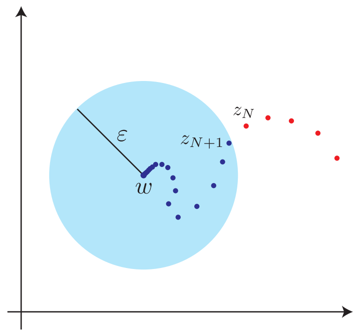


Figure 1.11: The convergence of a sequence $\{z_n\}$ to w. Each term z_n for $n > N$ lies inside the open disc of radius ε about w.

Informally, when we have an infinite sequence of complex numbers, $z_1, z_2, z_3, \ldots$, we say that the number w is the limit of the sequence if the z_n eventually (i.e. for large enough n) stay arbitrarily close to w. More precisely,

Limit of a Complex Sequence

Definition 1.11. *A sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is said to have the* **limit** *$w \in \mathbb{C}$ or to* **converge** *to w, i.e.*

$$\lim_{n\to\infty} z_n = w, \qquad or \qquad z_n \to w \text{ as } n \to \infty$$

if $\forall \varepsilon > 0$, $(\varepsilon \in \mathbb{R})$ $\exists N$ such that $|z_n - w| < \varepsilon, \forall n > N$.

There is an additional notion of convergence that is very useful, known as the ***Cauchy Criterion***. In fact, a necessary and sufficient condition for a complex sequence to converge is that it obeys this condition

---

Definition of a Cauchy Sequence

Theorem 1.6 (Cauchy Criterion)**.** *A sequence $\{z_n\}_{n=1}^{\infty}$ will converge* **if and only if** *$\forall \varepsilon > 0$, $\exists$ integer N such that*

$$|z_n - z_m| < \varepsilon, \qquad \forall m > N, \forall n > N.$$

(We will demur from rigorously proving this, for now, as sufficiency is slightly tricky.) A sequence that obeys the Cauchy Criterion is called a ***Cauchy Sequence***. Note that this provides an alternative and very useful definition of convergence that doesn't require reference to the notion of limits.

A RELATED CONCEPT is the limit of a complex-valued function $f(z)$. Roughly speaking, we say that the number w_o is the limit of the function $f(z)$ as z approaches z_o if $f(z)$ stays close to w_o ***whenever*** z is sufficiently near z_o. More precisely we have,

Limit of a Complex Function

Definition 1.12. *Let f be a function defined on some (open set) neighborhood of z_o, with the possible exception of the point z_o itself. We say that the* **limit of $f(z)$ as z approaches z_o is the complex number** w_o *i.e.*

$$\lim_{z \to z_o} f(z) = w_o,$$

$$\text{or} \qquad f(z) \to w_o \text{ as } z \to z_o$$

if $\forall \varepsilon > 0, \exists \delta > 0, (\varepsilon, \delta \in \mathbb{R})$ *such that*

$$|f(z) - w_o| < \varepsilon, \text{ if } |z - z_o| < \delta.$$

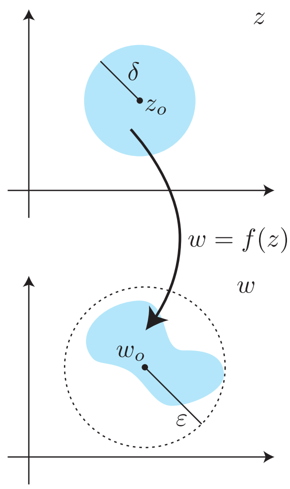


Figure 1.12: The limit of a function is defined such that any open disc of w_o contains all the values assumed by f in some open disc around z_o (except possibly the value $f(z_o)$).

THERE IS AN OBVIOUS RELATION between the limit of a function and the limit of a sequence; namely, if $\lim_{z \to z_o} f(z) = w_o$, then for every sequence $\{z_n\}$ converging to z_o, the sequence $\{f(z_n)\}$ converges to w_o. The converse of this statement is also true.

With these notions in hand, we can define the concept of continuity of a function:

Continuous Functions

Definition 1.13 (Continuity at a point). *Let f be a function defined in an open disc centred at z_o. Then f is called* **continuous at** z_o *if*

$$\lim_{z \to z_o} f(z) = f(z_o).$$

In other words, for f to be continuous at z_o, it must have a limiting value at z_o and this limiting value must be $f(z_o)$.

Definition 1.14 (Continuity on a set). *A function f is said to be* **continuous on a set** S *if it is continuous at each point of S.*

Cultural Side Note: A more general (topological) notion of continuity is that a mapping $f : X \to Y$ is continuous if for any ***open*** subset of the range, $W \subset Y$, the ***preimage*** of $f(W)$ (i.e. the domain set that corresponds to the range set W, also known as $f^{-1}(W)$ if f is invertible), is an ***open*** subset of X. For $X, Y \subseteq \mathbb{C}$, this definition is equivalent to ours.

CLEARLY THE DEFINITIONS for continuity and limits are directly analogous to those used for normal calculus and analysis on the real numbers. Because of this analogy, many of the familiar theorems and lemmas on real sequences, limits, and continuity remain valid in the complex case, such as,

Limits of Complex Functions

Theorem 1.7. *If $f(z)$ and $g(z)$ are complex functions for $z \in \mathbb{C}$ and $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then*

$$\lim_{z \to z_0} (f(z) \pm g(z)) = A \pm B,$$
$$\lim_{z \to z_0} f(z)g(z) = AB,$$
$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \qquad \text{if } B \neq 0.$$

Lemma 1.1. *If $f(z)$ and $g(z)$ are continuous complex functions at z_0, then so are $f(z) \pm g(z)$ and $f(z)g(z)$. The quotient $f(z)/g(z)$ is also continuous at z_0 provided $g(z_0) \neq 0$.*

Exercise 1.6: Limits of Real and Complex Functions

Let $f(z) = u(x,y) + iv(x,y)$, for $z = x + iy$, and $z_0 = x_0 + iy_0$, and $w_0 = u_0 + iv_0$. Prove that

$$\lim_{z \to z_0} f(z) = w_0$$

if, and only if,

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0, \qquad \text{and} \qquad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0,$$

i.e., a complex function has a limit if and only if its real and imaginary components have the appropriate limits.
[Hint: Use the triangle inequality and the facts that $|\text{Re}\, w| \leq |w|$ and $|\text{Im}\, w| \leq |w|$.]

1.2.3 *Injective, Surjective and Bijective Functions*

Finally, we review the notion of (single-valued) functions that are ***one-to-one*** (also called ***injective***), ***onto*** (also called ***surjective***), and ***invertible*** (also called ***bijective***). Note that these notions are applicable to more than just complex-valued functions.

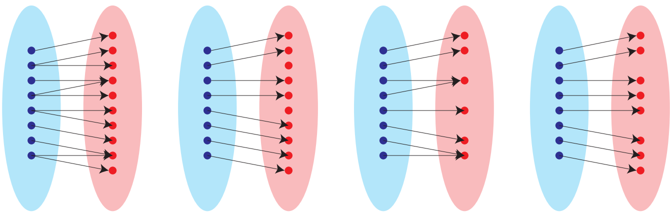
One-to-One, Onto, and Invertible Functions

For single valued functions (mutli-valued functions, are, technically, not considered functions):

Definition 1.15. *An* **injective** *function is one for which every element of the domain points to a unique element of the range. Injective functions are also called* **one-to-one***.*

Definition 1.16. *A* **surjective** *function is one for which every element of the range has (at least) one element in the domain which maps to it. Surjective functions are also called* **onto**.

Definition 1.17. *A* **bijective** *function is one that is both injective* **and** *surjective. A bijective function perfectly pairs up members of the domain and range with no overlaps or omissions. Bijective functions are often called* **one-to-one and onto**, *or* **invertible**, *since they admit* **single-valued inverse functions**.



Multi-Valued Function (Not a function) — Injective/One-to-one (Not surjective) — Surjective/Onto (Not injective) — Bijective/One-to-one & Onto (Invertible)

2

Complex Differentiation and Holomorphic Functions

2.1 *The Complex Derivative*

Now that we have a reviewed the notions of complex numbers and complex functions, we turn to the main topic of this part of the module: the study of functions that are ***differentiable*** in (most of) the complex plane. In studying differentiable functions of real variables, we took such functions defined on intervals. For complex variables, we have to select domains of definition in an analogous manner.

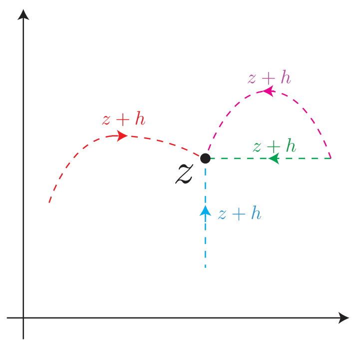


Figure 2.1: A function is only complex differentiable at a point z if $f'(z)$ is the same well-defined value regardless of which path the complex number h takes to 0. In other words the "direction" the derivative is taken in on the complex plane does not matter.

Derivative of a Complex Function

Definition 2.1 (The Complex Derivative). *Let U be an open subset of $\mathbb{C}$, and let z be a point of U. Let f be a function on U. We say that f is* **complex differentiable** *at z if the limit*

$$f'(z) \equiv \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}, \tag{2.1}$$

exists (where $h \in \mathbb{C}$). This limit is the **derivative of f at z**, *and can also be denoted by $\mathrm{d}f/\mathrm{d}z$.*

Unless otherwise specified, for us, ***differentiable*** will always mean ***complex differentiable***. This is almost the same definition of derivative as for real variables, with one crucial difference. On the real line there are two "directions" from which h can approach zero. But on the complex plane, there are an infinite number of directions for which this can occur. Indeed, in order for a function to be considered complex differentiable, this limit must approach only ***a single value, regardless of which path h takes***.

Example 2.1: The Complex Conjugate

Show that $\bar{z}$ is nowhere differentiable.

Solution: The finite difference approximation for $f(z) = \bar{z}$ takes the form

$$\frac{f(z+h)-f(z)}{h} = \frac{\overline{(z+h)}-\bar{z}}{h} = \frac{\bar{h}}{h}$$

where $h \in \mathbb{C}$. Now if $h \to 0$ through real values, then

$h = \Delta x$, and $\overline{h} = h$, and we must have the above quotient be 1. On the other hand if $h \to 0$ from above, then $h = i\Delta y$ and $\overline{h} = -h$, so the quotient is -1. Thus there is no simple way of assigning a unique value to the derivative of $\overline{z}$ at any point, and it is not differentiable.
Can you show $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ ***and*** $|z|$ ***are not differentiable?***

Holomorphic (Analytic) Functions

Definition 2.2 (Holomorphic Functions)**.** *Let U be an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is called* **holomorphic** *on U if $\forall z \in U$, f is* **complex differentiable** *at z.*

NOTE THAT DIFFERENTIABILITY may be defined at a single point, whereas a function can only be considered ***holomorphic***[1] (or ***analytic***) on ***open sets***. A function that is holomorphic on the whole complex plane is called an ***entire*** function.

Definition of an entire function.

[1] The term ***holomorphic*** comes from the Greek, "holos" for "entire", and "morphe" for "form" or "shape". This is sometimes used interchangeably with ***analytic***, though these mean slightly different things. We will discuss this more later.

Example 2.2: $\mathrm{d}z^n/\mathrm{d}z$

Show that for any positive integer n,

$$\frac{\mathrm{d}}{\mathrm{d}z}z^n = nz^{n-1}$$

Solution: Using the binomial theorem (Worksheet 1) we find

$$\frac{(z+h)^n - z^n}{h} = \frac{nz^{n-1}h + \frac{n(n-1)}{2}z^{n-2}h^2 + \ldots + h^n}{h}.$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}z}z^n = \lim_{h\to 0}\frac{(z+h)^n - z^n}{h} = nz^{n-1}$$

THE USUAL PROOFS OF THE CALCULUS of real variables concerning the basic properties of differentiability are valid for complex differentiability[2].

We note that if f is ***differentiable*** at z then it is also ***continuous*** at z because

$$\lim_{h\to 0}(f(z+h) - f(z)) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}\cdot\lim_{h\to 0} h = 0$$
$$\implies \lim_{h\to 0} f(z+h) = f(z).$$

Differentiable implies continuous.

since (from Thm. 1.7) the limit of the product is the product of the limits.

[2] We will review them here, for your reference, but will not go over them in detail in lecture.

Sum and Product Rules for Complex Derivatives

Let f, g be functions defined on the open set $U \subset \mathbb{C}$, and differentiable at some point $z \in U$.

Theorem 2.1 (Sum Rule). *The sum $f+g$ is differentiable at z and*

$$(f+g)'(z) = f'(z) + g'(z)$$

Proof: This follows immediately from Theorem 1.7, that the limit of the sums is the sum of the limits.

Theorem 2.2 (Product Rule). *The product fg is differentiable at z, and*

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z).$$

Proof: We want to find the limit as $h \to 0$ of

$$\frac{f(z+h)g(z+h) - f(z)g(z)}{h}.$$

We can write the numerator in the form

$$f(z+h)g(z+h) \underbrace{-f(z)g(z+h) + f(z)g(z+h)}_{0} - f(z)g(z),$$

so that the derivative takes the form

$$\begin{aligned}(fg)'(z) &= \lim_{h\to 0}\left[\frac{f(z+h)-f(z)}{h}g(z+h) + f(z)\frac{g(z+h)-g(z)}{h}\right] \\ &= f'(z)g(z) + f(z)g'(z).\end{aligned}$$

The Quotient Rule

Theorem 2.3 (Quotient Rule). *If $g(z) \neq 0$, then the quotient of f/g is differentiable at z, and*

$$(f/g)'(z) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}$$

Proof: First we have to show that $1/g$ is differentiable. As for real calculus we have,

$$\frac{\frac{1}{g(z+h)} - \frac{1}{g(z)}}{h} = -\frac{g(z+h) - g(z)}{h}\frac{1}{g(z+h)g(z)}.$$

Applying the limit gives,

$$\frac{\mathrm{d}(1/g(z))}{\mathrm{d}z} = -\frac{g'(z)}{g(z)^2}$$

which exists if $g(z) \neq 0$ and if $g'(z)$ exists. The quotient rule is then proved using the above result and the product rule.

The Chain Rule

Theorem 2.4 (Chain Rule). *Let $w = f(z)$. Assume that f is differentiable at z and function $g(w)$ is differentiable at w. Then $g \circ f$, is differentiable at z and*

$$(g \circ f)'(z) = g'(f(z))f'(z)$$

Proof: Since $f(z)$ is differentiable at z, for $h \in \mathbb{C}$, we define the function $\varphi_f(h)$ to be

$$\varphi_f(h) \equiv \frac{f(z+h) - f(z)}{h} - f'(z)$$

such that it is the "error" of the "finite difference" approximation to $f'(z)$.
Then we have both

$$f(z+h) - f(z) = f'(z)h + h\phi_f(h) \quad \text{and} \quad \lim_{h \to 0} \varphi_f(h) = 0.$$

The existence of such a function $\varphi_f(h)$ obeying the above equations can be shown to be ***equivalent*** to differentiability of f at z.
We can then use this alternate definition of differentiability and let $w = f(z)$ and

$$k = f(z+h) - f(z)$$

so that

$$g(f(z+h)) - g(f(z)) = g(w+k) - g(w)$$

Since $g(w)$ is differentiable at w we have a function $\varphi_g(k)$ such that

$$g(w+k) - g(w) = g'(w)k + k\varphi_g(k), \quad \text{and} \quad \lim_{k \to 0} \varphi_g(k) = 0.$$

Substituting in for k, and dividing by h we find

$$\begin{aligned} \frac{g \circ f(z+h) - g \circ f(z)}{h} &= g'(w)\frac{f(z+h) - f(z)}{h} \\ &\quad + \frac{f(z+h) - f(z)}{h}\varphi_g(k). \end{aligned}$$

Taking the limit as $h \to 0$, we note that $k \to 0$ by continuity of f at z. Then, since $\varphi_g(k) \to 0$ as $k \to 0$, the last term vanishes, and we recover the chain rule.

The function $(g \circ f)(z)$ is pronounced "g circ f", "g composed with f", or sometimes "g after f". It means to apply function g to the result of $f(z)$. Obviously, the domain of g must overlap with the range of f in order for $g \circ f$ to have a non-null domain.

IT FOLLOWS FROM THE SUM AND PRODUCT RULES and Example 2.2 that any ***polynomial*** in z,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0, \tag{2.2}$$

is differentiable on the whole plane (i.e. $P(z)$ is an entire function), and has derivative given by

$$P'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \ldots + a_1. \qquad (2.3)$$

Consequently, from the Quotient Rule, any ***rational function*** of z, $R(z) \equiv P_N(z)/P_D(z)$, is differentiable at every point of its domain where the denominator $P_D(z) \neq 0$. For the purposes of differentiation, polynomial and rational functions in z can be treated as if z was a real variable.

2.2 *Holomorphic functions on* $\mathbb{C}$ *vs functions on* $\mathbb{R}^2$

In the previous chapter we viewed complex functions of a complex variable (e.g. $f(z)$), as somewhat arbitrary mappings from the xy-plane to the uv-plane. We have individual names for the real (x) and imaginary (y) parts of z, and for the real (u) and imaginary (v) parts of f. ***Any*** pair of two-variable functions $u(x,y)$ and $v(x,y)$ gives us a complex function $(u+iv)$ in this sense. But notice that there is something special about the pair

$$u_1(x,y) = x^2 - y^2, \qquad v_1(x,y) = 2xy, \qquad (2.4)$$

as opposed to (e.g.)

$$u_2(x,y) = x^2 - y^2 \qquad v_2(x,y) = 3xy. \qquad (2.5)$$

That is, the complex function $u_1 + iv_1$ treats $z = x + iy$ as a single "unit" because it can be written $x^2 - y^2 + i2xy = (x+iy)^2$, and thus respects the complex structure of $z = x + iy$. However, the formulation $u_2 + iv_2$ requires us to break apart[3] the real and imaginary parts of z.

[3] In real calculus we don't deal with functions that look at a number like $3 + 4\sqrt{2}$ and square the 3 but cube the 4! The interesting calculus functions treat the number as an indivisible module.

We want to consider functions that are functions of z, rather than functions of x and y separately. Thus $z^2 = x^2 - y^2 + i2xy$ is a function we consider "admissible", while functions like $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $\bar{z}$ are not, as they require explicit separation the real and imaginary parts of z. Similarly $|z|$ is not the kind of function we are looking for since $\bar{z} = |z|/z$, while the power series expansion of e^z shows us that it can be written in terms of only functions of z in its entirety.

It will turn out that this vague criteria we are looking for, which apparently respects the complex structure of z, can be expressed simply in terms of ***differentiability***. Thus the functions we want to consider, that treat z as a complex number, rather than a pair of coordinates, are functions that are ***holomorphic*** (thus the "holos" etymology, since such a function respects the complex number in its entirety). We will explore why these notions are equivalent more in later chapters, but we can get a sense of this by considering how functions approach a particular value $f(z_o) = w_o$.

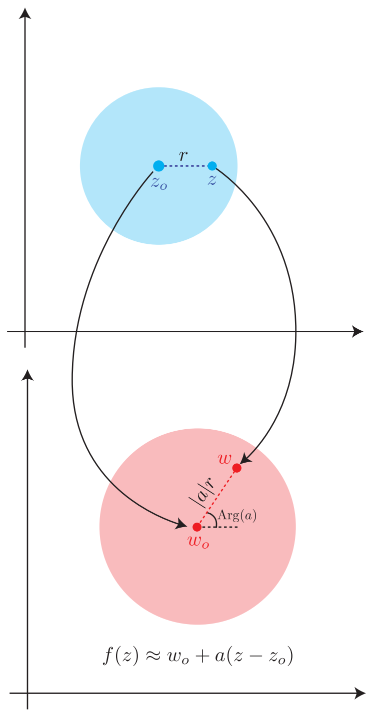


Figure 2.2: Close to a differentiable point z_o, the ***holomorphic*** function $w = f(z)$ maps the ***circular domain to a circular range***, since it can only scale r by $|a|$ and rotate by Arg (a) about point $w_o = f(z_o)$, where $a = f'(z_o)$ and $r = |z - z_o|$.

In the domain $D(z_o, \varepsilon)$, the circular disc centred at z_o, a ***holomorphic*** function $w = f(z)$ at $z \in D(z_o, \varepsilon)$ has the well defined approximation as $\varepsilon \to 0$,

$$f(z) \approx w_o + a(z - z_o) \qquad \Longrightarrow \quad w - w_o \approx a(z - z_o) \tag{2.6}$$

where $w_o = f(z_o)$ and $a = f'(z_o) \in \mathbb{C}$ is the well defined complex derivative at z_o. If we write $a = |a|e^{i\operatorname{Arg} a}$, we see that if we zoom in close enough to a differentiable point z_o, a circular domain centred at z_o must map to a circular range centred at w_o, since a only provides an overall scaling and rotation about w_o (see Figure 2.2). Hence holomorphic functions ***map locally circular domains to locally circular ranges*** (with no reflections).

LET'S CONSIDER A GENERAL (NON-HOLOMORPHIC) FUNCTION on the complex plane, $g(z) = u(x, y) + iv(x, y)$, where $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$, which is equivalent to a 2-dimensional function $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$, such that $\tilde{g}(\{x, y\}) = \{u(x, y), v(x, y)\}$. By analogy with $\tilde{g}$, we can see that near a point $z_o = x_o + iy_o$ where $g(z_o) = w_o$ and the partial derivatives are well defined, we can approximate $w = g(z)$ by

$$\begin{aligned} g(z) \approx w_o &+ \left[\Delta x \frac{\partial}{\partial x} u(x, y) + \Delta y \frac{\partial}{\partial y} u(x, y)\right] \\ &+ i\left[\Delta x \frac{\partial}{\partial x} v(x, y) + \Delta y \frac{\partial}{\partial y} v(x, y)\right]. \end{aligned} \tag{2.7}$$

where $\Delta x = \operatorname{Re}(z - z_o)$ and $\Delta y = \operatorname{Im}(z - z_o)$.

Since for a ***non-holomorphic function*** there is no particular relationship between the partial derivatives of $u(x, y)$ and $v(x, y)$, we see (from Figure 2.3) that "small" ***circular domains centred at*** z_o ***must map to elliptical ranges centred at*** w_o. That is, a function approaching along the y direction has a different derivative than when approaching along the x direction, which leads to a "elliptical" mapping, that treats x and y differently. Hence, non-holomorphic functions locally map circular domains to elliptical ones.

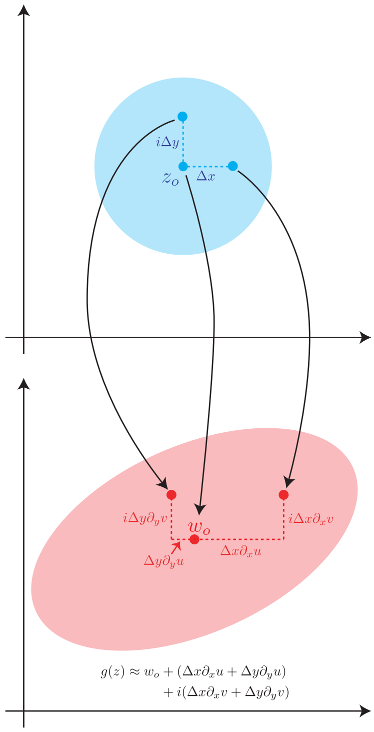


Figure 2.3: For a ***general non-holomorphic function*** $g(z) = u(x, y) + iv(x, y)$ (where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$) and close to some point $z_o = x_o + iy_o$ where the partial derivatives of $u(x, y)$ and $v(x, y)$ exist, ***circular domains*** are generally mapped to ***elliptical ranges***, since the partial derivatives are, in general, unrelated.

THE ABOVE DISCUSSION provides us with a few insights:

- To treat the complex input z as a unit, we want functions that locally map circular domains to circular ranges with only scaling and rotations allowed (no reflections – ***why?***).
- The above condition is exactly satisfied by holomorphic functions.
- Holomorphic functions are actually quite restricted compared to general functions on $\mathbb{R}^2$.

We can focus a bit more carefully on this last point, that ***holomorphic*** functions $f(z)$ are special compared to general functions $u(x, y) + iv(x, y)$. Indeed it turns out that the existence of a complex derivative is a very strong condition to place on a function, and that ***any*** holomorphic function must also be ***infinitely differentiable***[4] on the same domain, that is it has well defined (complex) derivatives at all orders.

[4] We will not prove this yet, but can do so once we develop a few more tools, like Cauchy's Integral Theorem.

2.3 *The Cauchy-Reimann Equations*

Figures 2.2 and 2.3 gives us a hint that holomorphic functions must have a special relationship between $u(x, y)$ and $v(x, y)$ that has to be satisfied. Here we will find this relationship, by calculating the derivative along two different paths.

Derivatives from different directions

If the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_o = x_o + iy_o$, the complex derivative can be found using Definition 2.1. Approaching from the horizontal direction (see Figure 2.4), we take $h = \Delta x \to 0$,

$$\begin{aligned} f'(z_o) &= \lim_{\Delta x \to 0} \frac{u(x_o + \Delta x, y_o) + iv(x_o + \Delta x, y_o) - u(x_o, y_o) - iv(x_o, y_o)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[\frac{u(x_o + \Delta x, y_o) - u(x_o, y_o)}{\Delta x} \right] + \lim_{\Delta x \to 0} i \left[\frac{v(x_o + \Delta x, y_o) - v(x_o, y_o)}{\Delta x} \right] \end{aligned}$$

We see that the terms in the square brackets are the definitions of the ***partial derivatives***,

$$f'(z_o) = \frac{\partial u}{\partial x}(x_o, y_o) + i \frac{\partial v}{\partial x}(x_o, y_o). \tag{2.8}$$

Similarly if we approach z_o from the vertical direction we take $h = \Delta y \to 0$,

$$\begin{aligned} f'(z_o) &= \lim_{\Delta y \to 0} \left[\frac{u(x_o, y_o + \Delta y) - u(x_o, y_o)}{i\Delta y} \right] + \lim_{\Delta y \to 0} i \left[\frac{v(x_o, y_o + \Delta y) - v(x_o, y_o)}{i\Delta y} \right] \\ &= -i \frac{\partial u}{\partial y}(x_o, y_o) + \frac{\partial v}{\partial y}(x_o, y_o). \end{aligned} \tag{2.9}$$

In order for the value of the derivative to be the same no matter which direction the limit was taken in, we require the real and imaginary components of Equations 2.8 and 2.9 to be the same, resulting in the following theorem.

The Cauchy-Reimann Equations

Theorem 2.5 (The Cauchy-Reimann Equations). *Let $f(z) = u(x, y) + iv(x, y)$ be defined on some open set S containing the point z_o. The first partial derivatives of $u(x, y)$ and $v(x, y)$* **exist**, *are* **continuous**, *and satisfy* **the Cauchy-Reimann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{2.10}$$

at $z_o = x_o + iy_o$, if and only if $f(z)$ is **differentiable** *at z_o. These conditions hold at all points in S, if and only if $f(z)$ is holomorphic on S.*

(Why do you think we require the partial derivatives to be continuous in order for the Cauchy-Reimann (CR) equations to be sufficient conditions for complex differentiability? I.e. what happens if the CR equations are satisfied but the partial derivatives are ***not*** continuous?)

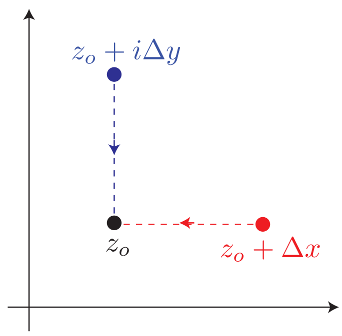


Figure 2.4: A holomorphic function has the same derivative no matter which direction the limit is taken in. Taking the limit vertically and horizontally allows us to derive the Cauchy-Reimann equations.

Exercise 2.1: CR equations and circular domains

BY EXPLICIT CALCULATION, show that functions (with continuous partial derivatives) that obey the Cauchy-Reimann equations must map locally circular domains to locally circular ranges, and that such maps must consist of only a scaling and rotation (with no reflections).

Example 2.3: The exponential function again

PROVE THAT THE FUNCTION $f(z) = e^z = e^x \cos y + ie^x \sin y$ is ***entire*** (i.e. is holomorphic on the entire complex plane) and find its derivative.
Solution: Since

$$\begin{aligned} \partial_x u &= e^x \cos y, \qquad \text{and} \qquad \partial_y v = e^x \cos y \\ \partial_y u &= -e^x \sin y, \qquad \text{and} \qquad \partial_x v = e^x \sin y \end{aligned}$$

the first partial derivatives are continuous and satisfy the Cauchy-Reimann equations at every point in the plane. Hence $f(z)$ is entire. Also we notice that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = f(z)$$

as we expect.

2.4 *Holomorphic and Harmonic Functions*

Laplace's equation in 2-dimensions is an extremely useful equation in mathematical physics. It can describe the behaviour of a variety of physical systems, such as 2-D electrostatic potentials, or the stream functions of irrotational incompressible fluid flow.

Laplace's Equation

Definition 2.3 (Harmonic function). *A real-valued function $\phi(x, y)$ is said to be* **harmonic** *on a (simply-connected open) set S if all its* **second-order partial derivatives exist** *and are* **continuous** *on S and ϕ satisfies* **Laplace's equation***:*

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (2.11)$$

at each point in S.

Other common notations for the Laplacian operator ∇^2 are Δ or $\nabla \cdot \nabla$.

Taking the partial derivatives of the Cauchy-Reimann equations gives us

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \qquad (2.12)$$

Similarly we have

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial y \partial x}. \tag{2.13}$$

The equality of mixed partials (the Clairaut-Schwartz theorem), then allows us to prove:

Harmonic Real and Imaginary Parts

Theorem 2.6. *If $f(z) = u(x,y) + iv(x,y)$ is* **holomorphic** *on a (simply-connected open) set S, then the real functions $u(x,y)$ and $v(x,y)$ are both* **harmonic** *on S.*

Definition 2.4 (Harmonic Conjugate)**.** *Conversely, if we are given a function $u(x,y)$ that is harmonic on some open* **simply-connected** *set, we can find some $v(x,y)$, also harmonic, such that $f(z) = u + iv$ is holomorphic on the same set. The function $v(x,y)$ is called the* **harmonic conjugate of** *u,*

Note will show later that if $f(z)$ is holomorphic then u and v have continuous partial derivatives of all orders.

Example 2.4: Finding a harmonic conjugate

Construct an analytic function whose real part is

$$u(x,y) = x^3 - 3xy^2 + y$$

Solution: First we verify that u is harmonic on the whole plane:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

Now we have to find the harmonic conjugate of u, $v(x,y)$ such that the Cauchy-Reimann equations are satisfied. Thus we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \tag{2.14}$$

$$\text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy - 1. \tag{2.15}$$

If we hold x constant and integrate Eq. 2.14 with respect to y we get

$$v(x,y) = 3x^2y - y^3 + \psi(x)$$

where $\psi(x)$ is the "constant" of integration (since we only integrate with respect to y). Taking the partial derivative of v with respect to x we find

$$\frac{\partial v}{\partial x} = 6xy + \psi'(x) = 6xy - 1$$

where the second equality comes from Eq. 2.15 above. This gives $\psi'(x) = -1$, so $\psi(x) = -x + a$ where a is some constant. Hence the harmonic conjugate of u is

$$v(x,y) = 3x^2y - y^3 - x + a$$

and the analytic function is

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + y + i(3x^2y - y^3 - x + a) \\ &= z^3 - i(z - a). \end{aligned}$$

The Harmonic Conjugate

Theorem 2.7. *If $u(x, y)$ is harmonic on an open simply-connected domain, the harmonic conjugate of u is given by the line integral*

$$v(\tilde{z}) = \int_{\tilde{z}_o}^{\tilde{z}} (\partial_x u)dy - (\partial_y u)dx + C \qquad (2.16)$$

where C is a constant of integration, and the line integral is performed over **any** *path from some arbitrary fixed point $\tilde{z}_o = \{x_o, y_o\}$ to the point $\tilde{z} = \{x, y\}$ (i.e. the result is path independent).*

Note that if the domain is ***not*** simply-connected (e.g. a punctured disc) the harmonic conjugate is not always guaranteed to exist. (In this case the line integral may be path dependent.)

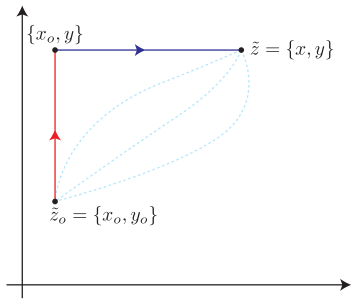


Figure 2.5: Above in Example 2.4 the path $\tilde{z}_o \to \tilde{z}$ we chose to integrate over was in two pieces, the vertical line segment $\{x_o, y_o\} \to \{x_o, y\}$ (along which $dx = 0$) followed by the horizontal line segment $\{x_o, y\} \to \{x, y\}$ (along which $dy = 0$). Since any path will do (e.g. the dotted lines), this one makes the integrals easy!

It may seem surprising that the condition that a complex function being complex differentiable leads to such strong restrictions on the functions u and v. This fact does have an extraordinary benefit though, often problems involving Laplace's equation on strange domains can be more easily solved by considering composition of holomorphic functions.

We know that if we have two functions

$$f : S_1 \to S_2, \qquad \text{and} \qquad g : S_2 \to S_3, \qquad (2.17)$$

that are both holomorphic with $S_1, S_2, S_3 \subseteq \mathbb{C}$ then the composition of the two functions,

$$g \circ f : S_1 \to S_3 \qquad (2.18)$$

is also holomorphic on S_1 (by the chain rule).

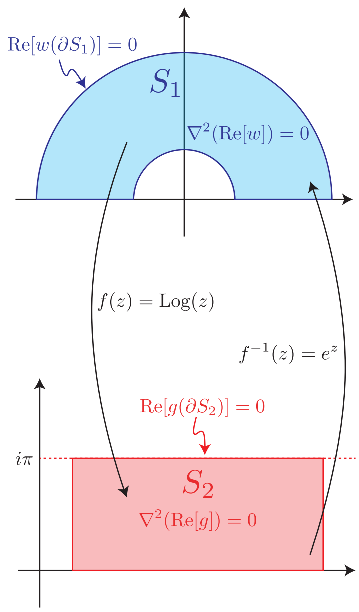


Figure 2.6: To solve Laplace's equation on the half-annular domain S_1 (here with a Dirichlet boundary condition), we could use the holomorphic map $f(z) = \mathrm{Log}\, z$ to map onto the rectangular domain S_2 where it is easier to find the harmonic function $\mathrm{Re}(g)$. The solution on S_1 is then given by $\mathrm{Re}[w(z)] = \mathrm{Re}[(g \circ f)(z)]$.

Application: Solving Laplace's Equation on Weird Domains (by Conformal Mapping)

Suppose we are trying to find a real function u satisfying

$$\nabla^2 u = 0 \qquad (2.19)$$

in S_1, with boundary condition $u = h(x, y)$ on the boundary ∂S_1.
This is of course equivalent to finding a holomorphic function, $w(z)$, whose real part satisfies the boundary condition on ∂S_1.
If S_1 is an awkward shape, but we can find a holomorphic function $f(z)$ that maps it to a more helpful domain S_2, then

we can define

$$w = g \circ f$$
$$\text{or} \qquad w(z) = g(f(z)) \tag{2.20}$$

We are now looking for a holomorphic function g on the nice domain S_2 such that the boundary condition

$$\text{Re}[(g \circ f)(z)] = h(x, y) \text{ on } \partial S_1 \tag{2.21}$$

is satisfied. As long as $f(z)$ is invertible we can write the equivalent boundary condition on ∂S_2 as

$$\text{Re}[g(z)] = h(f^{-1}(z)) \text{ on } \partial S_2 \tag{2.22}$$

Cultural sidenote: a function that is holomorphic and invertible is often called ***a holomorphism***.

2.5 *Angles under holomorphic maps*

Another property of holomorphic functions is that they are ***conformal***. Roughly, speaking this means that they preserve angles. In order to understand this fully, we need to figure out what we mean by angles on the complex plane.

Let's consider an open set $S \subseteq \mathbb{C}$ and define $\gamma : [a, b] \to S$ to be a ***curve*** in S, so we can write

$$\gamma(t) = x(t) + iy(t) \tag{2.23}$$

such that $t \in [a, b]$ is the parameter that traces the curve. We can assume that γ is differentiable, so that it's derivative is

$$\gamma'(t) = x'(t) + iy'(t) \in \mathbb{C}. \tag{2.24}$$

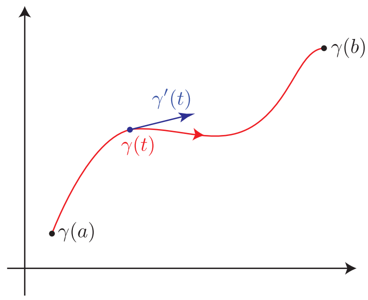


Figure 2.7: The derivative $\gamma'(t)$ is in the direction of the tangent to the curve γ at point $\gamma(t)$.

We can interpret the complex number $\gamma'(t)$ as a "vector" in the direction of a tangent at the point $\gamma(t)$. Thus the derivative, $\gamma'(t)$, if it is not 0, defines the direction of the curve at the point.

If two curves $\gamma(t)$ and $\eta(t)$ both pass through some point z_o, such that $z_o = \gamma(t_o) = \eta(t_1)$, then the tangent vectors $\gamma'(t_o)$ and $\eta'(t_1)$ determine an angle θ which is defined to be the ***angle between the curves***.

If we have two complex numbers $z = a + ib$ and $w = c + id$, then

$$z\overline{w} = ac + bd + i(bc - ad). \tag{2.25}$$

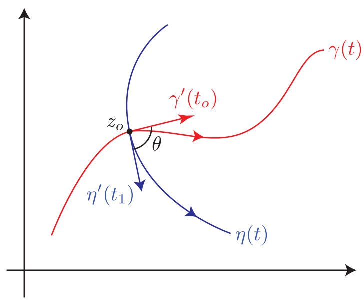


Figure 2.8: The angle between two curves is the angle between the tangent vectors.

We notice that $ac + bd$ is what we would expect from the dot or scalar product of the vectors $\{a, b\}$ and $\{c, d\}$. We can thus define an equivalent scalar product of the complex numbers to be

$$\langle z, w \rangle = \text{Re}(z\overline{w}). \tag{2.26}$$

The cosine and sin of the angle θ between z and w on the complex plane can be determined using this scalar product,

$$\cos\theta = \frac{\langle z, w \rangle}{|z||w|}, \qquad \text{and} \qquad \sin\theta = \frac{\langle z, -iw \rangle}{|z||w|}. \tag{2.27}$$

Conformal Maps

Definition 2.5 (A Conformal Map)**.** *A mapping that* **preserves angles and orientation** *(handedness) is called* **a conformal map**.

Theorem 2.8. *If $f\ : S \to \mathbb{C}$ is holomorphic and $f'(z_o) \neq 0$, at some $z_o \in S$, then the* **angle** *between the curves γ and η at z_o is the* **same** *as the* **angle** *between the curves $f \circ \gamma$ and $f \circ \eta$ at $f(z_o)$ with the* **same orientation***, and the mapping is* **conformal** *at z_o.*

Proof: We already have a sense, from discussing Figure 2.2, of how holomorphic functions only allow local scaling and rotations, and thus preserve angles and orientations, but here we will give a more formal approach. Let the curves γ and η be defined as above and $\gamma'(t_o) = z$, and $\eta'(t_1) = w$. We then have the angle between γ and η at z_o given by Equation 2.27. We must also have $(f \circ \gamma)'(t_o) = f'(z_o)\gamma'(t_o)$ and $(f \circ \eta)'(t_1) = f'(z_o)\eta'(t_1)$.
Let $f'(z_o) = \alpha \in \mathbb{C}$. Then the inner product of the tangent vectors of $f \circ \gamma$ and $f \circ \eta$ is given by,

$$\langle \alpha z, \alpha w \rangle = \mathrm{Re}(\alpha z \overline{\alpha w}) = \alpha\bar{\alpha}\langle z, w \rangle, \tag{2.28}$$

since $\alpha\bar{\alpha} = |\alpha|^2$ is real. The angle between $f \circ \gamma$ and $f \circ \eta$ at $f(z_o)$ is then given by

$$\cos\theta = \frac{\langle \alpha z, \alpha w \rangle}{|\alpha z||\alpha w|} = \frac{\langle z, w \rangle}{|z||w|}, \tag{2.29}$$

$$\text{and} \qquad \sin\theta = \frac{\langle \alpha z, -i\alpha w \rangle}{|\alpha z||\alpha w|} = \frac{\langle z, -iw \rangle}{|z||w|}, \tag{2.30}$$

This clearly fails when $\alpha = f'(z_o) = 0$.

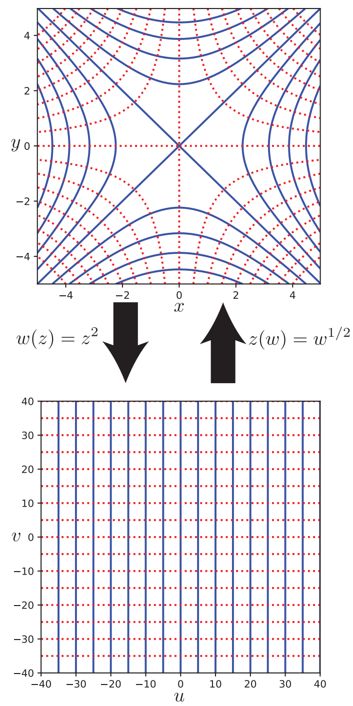


Figure 2.9: Level curves of $u(x,y) = x^2 - y^2$ (solid) and its harmonic conjugate $v(x,y) = 2xy$ (dotted). The holomorphic map $w(z) = z^2 = u + iv$ is conformal everywhere except $z = 0$ since $w'(0) = 0$, and will of course map these level curves to lines of constant u and v. Note that the level curves of u and v meet at right angles, since they map to the orthogonal grid, except at $z = 0$ (at this non-conformal point, angles between curves are actually doubled by $w(z)$).

Note that while maps like $z \mapsto \bar{z}$ also preserve angle, they involve a ***reflection*** such that the orientation of the angles is reversed. Such maps are ***not*** considered conformal because of this orientation change.

This conformal property of holomorphic maps, allows us to determine another interesting relationship between a harmonic function $u(x,y)$ and its harmonic conjugate $v(x,y)$. If we define the level curves of u and v (see Figure 2.9) to be where

$$u(x,y) = \text{constant} \tag{2.31}$$

$$v(x,y) = \text{constant.} \tag{2.32}$$

then it ***must be*** that on the z-plane the level curves of u are ***orthogonal*** to the level curves of v everywhere the function $w(z) = u + iv$ is holomorphic with non-zero derivative, since, on the w-plane, the lines of constant u are vertical lines, and lines of constant v are horizontal lines.

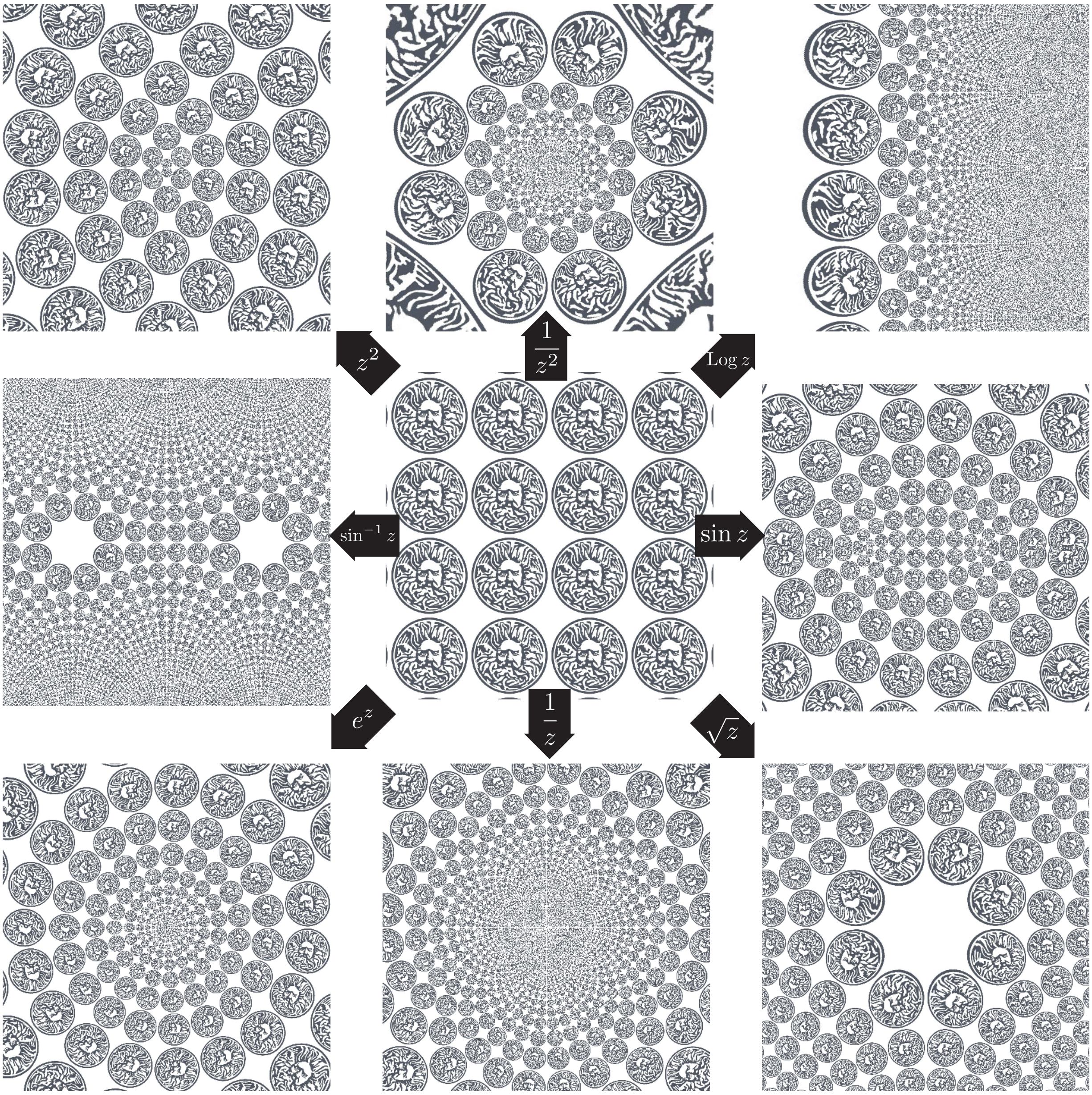


Figure 2.10: Conformally mapping a mosaic of Bath Gorgons using different holomorphic maps (up to constant scale factors).

2.6 *Poles and Branch Cuts*

As discussed above some complex functions are only holomorphic on part of the complex plane. In some locations they may encounter a pole, where the value of the function is infinite, or a branch cut, where the function experiences a jump. In this section we will examine a few examples of these in more detail to prepare us for contour integration

Types of Singular Points

A singular point is a point at which a mathematical "misbehaves", i.e. it is not well defined, or in our case, blows up or is no longer differentiable. There are few different classes of singularities which we will encounter in this unit. Let U be an open subset of $\mathbb{C}$, containing point $z_0 \in U$, and let f be a complex differentiable function in the domain U but excluding point z_0 (expressed as $U \setminus \{z_0\}$).

Definition 2.6 (Removable Singularities). *A singularity is called* **removable** *if there exists a holomorphic function g defined on all of U, such that $f(z) = g(z)$ for all $z \in U \setminus \{z_0\}$. $g(z)$ is a continuous replacement for the function f.*

A removable singularity is a point z_0 where the function $f(z_0)$ appears to be undefined, but if we define $g(z_0) = \lim_{z \to z_0} f(z)$ it stays well behaved. The standard example of a removable singularity is the Sinc function $f(z) = \sin(z)/z$ at point $z = 0$.

Definition 2.7 (Pole/Non-essential Singularity). *A singularity is called a* **pole** *or a* **non-essential singularity** *of f if there exists a holomorphic function g on U and natural number $n \in \mathbb{N}$ such that*

$$f(z) = \frac{g(z)}{(z - z_0)^n}, \qquad \forall z \in U \setminus \{z_0\} \tag{2.33}$$

The smallest number n for which this is true is called the **order of the pole** *of f at z_0.*

Poles play an important role in contour integration. A first order pole, i.e. a pole where $(z - z_0)f(z)$ is holomorphic on U, is often called ***a simple pole***.

Definition 2.8 (Essential Singularity). *A singularity at z_0 is called* **essential** *if $\forall n \in \mathbb{N}$, the function $g(z) = (z - z_0)^n f(z)$ is not holomorphic at z_0.*

It can be shown that f has an essential singularity at a point if and only if the (Laurent) power series expansion has infinitely many negative powers (more on this later).
An example of an ***essential singularity*** is point z_0 with the function $f(z) = \exp(1/(z - z_0))$.

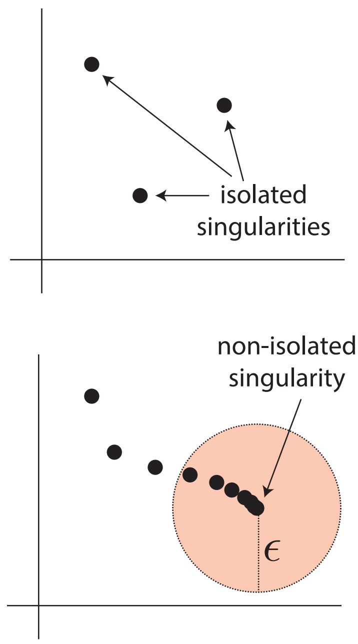


Figure 2.11: The singular points discussed in Definitions 2.6 - 2.8 are ***isolated*** singularities. That is one can always draw a small enough circle around each one such that no other singularity is inside that circle. A ***Non-isolated*** singularity, z_0, is a singularity that is also the limit point of a sequence of other singularities, such that $\forall$ real $\epsilon > 0$, there is always another singularity within the region $|z - z_0| < \epsilon$.

Branch Points

Another type of special point for functions in the complex plane are ***branch points***.

Definition 2.9 (Branch Point)**.** *The point z_0 is called a* **branch point** *for the complex (multiple) valued function $f(z)$ if the value of $f(z)$ does not return to its initial value as a closed curve around the point z_0 is traced, such that f varies continuously while the path is traced.*

What matters here is the ***local*** behaviour of the function f near z_0. What may happen on paths that are some distance away from z_0 is not relevant. More precisely, this behaviour must occur for all curves that enclose the point z_0 and are sufficiently close to it.

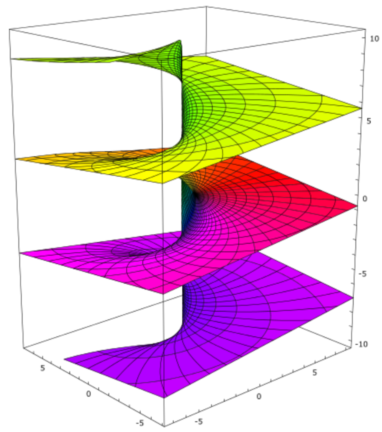


Figure 2.12: Surface plot of $\mathrm{Im}(\log z)$. The different branches of $\log z$ arise because it is multi-valued, with value continuously spiralling upwards when going around the branch point. Such a surface indicating the branches is called a ***Riemann surface***. Different branches are akin to different "floors" in the spiral parking garage. (Source: Wikimedia Commons user Leonid2.)

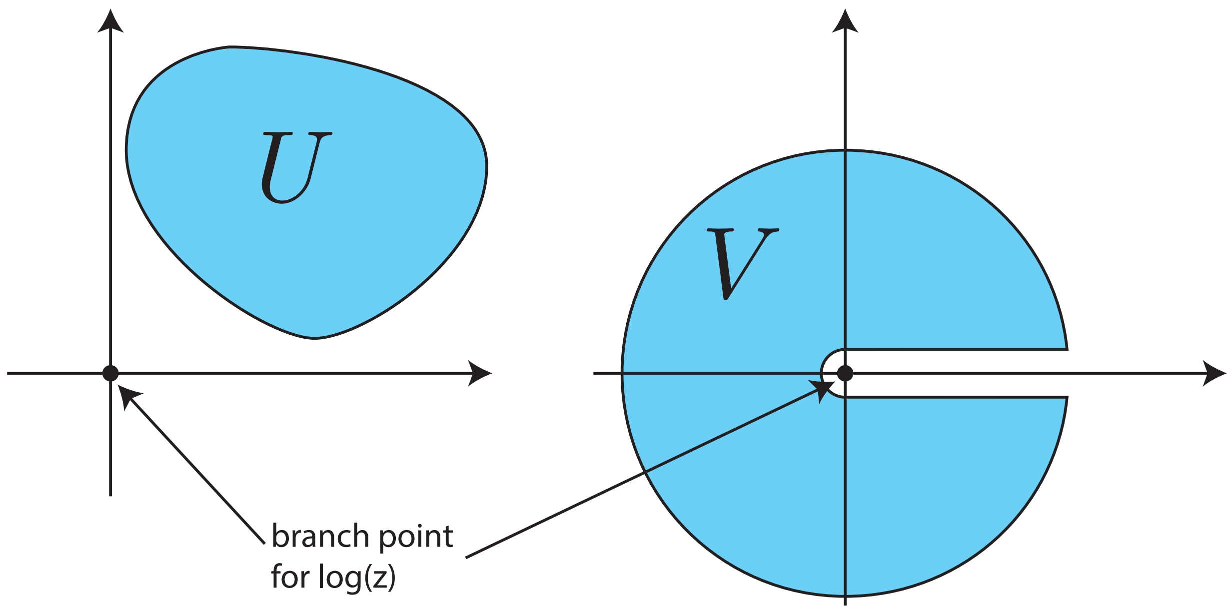


Figure 2.13: Two different regions on which we can choose a branch of $\log(z)$ to be holomorphic, since it does not enclose the branch point.

Consider the function $\log(z)$. If we restrict ourselves to the region U in the figure above to the left we may define $\log(z)$ uniquely, since there is no way to draw a loop that encloses the branch point $z = 0$ within the domain U. To define $\log(z)$ in U, we can, for example, simply choose the angle θ to be between zero and $\pi/2$ for any point $z = re^{i\theta} \in U$.Alternatively we could define the θ in region U to be between 2π and $3\pi/2$.

We can enlarge the region we are looking at to the domain V, and $\log(z)$ in V can still be defined uniquely, as long as we don't enclose a branch point to allow loops that will become discontinuous.

Branch Points at Infinity

You may encounter some texts discussing branch points (or singularities) at infinity, which may a bit difficult to understand. However, it is easy to understand by considering the inverse map:

$$w = 1/z \tag{2.34}$$

such that $w \to 0$ as $z \to \infty$. Then we can examine what happens at the point $w = 0$ in the w-plane. For example:

$$\log(z) = -\log(w) \tag{2.35}$$

and since $w = 0$ is a branch point of $-\log(w)$ we can conclude that $z = \infty$ is a branch point of $\log(z)$. Similar arguments can also be made about singularities at infinity.

Note that the choice of ***branch cut*** is completely arbitrary for a particular function. ***Branch cuts are usually, but not always, taken between pairs of branch points***. For $\log(z)$ we have branch points at $z = 0$ and $z = \infty$, which we can approach from any direction. The figure to the right shows multiple ways to choose the branch of $\log(z)$. All of them are equally valid, though only the first two are easy to write in terms of the principal value $\mathrm{Log}(z)$. Each of these choices of ***branch cut*** joins the ***branch point*** at $z = 0$ with the branch point at $z = \infty$.

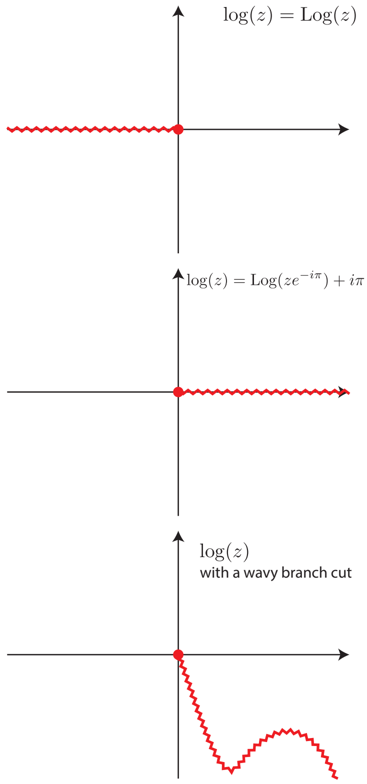


Figure 2.14: Possible different branch cuts for the same multi-valued function $\log(z)$. ***Top:*** The principal branch $\log(z) = \mathrm{Log}(z)$, with a branch cut along the negative real axis, so that $z = re^{i\theta}$ has $\theta \in (-\pi, \pi)$. ***Middle:*** A different branch where $\log(z) = \mathrm{Log}(ze^{-i\pi}) + i\pi$. Since Log has the standard branch cut along the negative real axis, $\mathrm{Log}(ze^{-i\pi})$ has a branch cut along the positive real axis, such that $\theta \in (0, 2\pi)$. ***Bottom:*** A wavy branch cut in the complex plane.

2.6.1 *Complex Powers and Inverse Trigonometric Functions*

One important use of the logarithmic function is to define complex powers of z. The definition is motivated by the identity

$$z^n = (e^{\log z})^n = e^{n \log z} \tag{2.36}$$

for any integer n.

Complex Powers

Definition 2.10 (Complex Powers). *If α is a complex constant, and $z \neq 0$, then we can define z^α by*

$$z^\alpha \equiv e^{\alpha \log(z)}. \tag{2.37}$$

Among other things, this means that each value of $\log(z)$ leads to a particular value of z^α.

Example 2.5: $(-2)^i$

Find all the values of $(-2)^i$:
Solution: Since $\log(-2) = \mathrm{Log}2 + i(\pi + 2k\pi)$ we have

$$(-2)^i = e^{i\log(-2)} = e^{i\mathrm{Log}2} e^{-\pi - 2k\pi}, \qquad k \in \mathbb{Z}. \tag{2.38}$$

It is clear that each branch of $\log z$ yields a branch of z^α. For example using the principal branch of $\log z$ we obtain the principal branch of z^α, namely $e^{\alpha \mathrm{Log} z}$.

Since e^z is entire, and $\mathrm{Log}(z)$ is analytic in the slit domain $\mathbb{C} \setminus (-\infty, 0]$, the chain rule implies that the principal branch of z^α is also analytic in the slit domain.

Other branches can be constructed using other branches of $\log(z)$, with different holomorphic domains.

"Branch chasing" — figuring out which branch to choose — for complicated functions is often a tedious task; fortunately, for simple applications this is not often necessary. We examine some of the subtleties involved in the examples below.

Figure 2.15: Bruce Banner consults with the Ancient One about his choice of branch cut.

Example 2.6: Clip the Branches!

Define a branch of $(z^2-1)^{1/2}$ that is holomorphic in the exterior of the unit circle, $|z| > 1$.

Solutions: Our task, restated, is to find a function $w = f(z)$ that is holomorphic outside the unit circle and satisfies

$$w^2 = z^2 - 1. \tag{2.39}$$

Note that the principal branch of $(z^2-1)^{1/2}$, namely,

$$e^{(1/2)\mathrm{Log}(z^2-1)}. \tag{2.40}$$

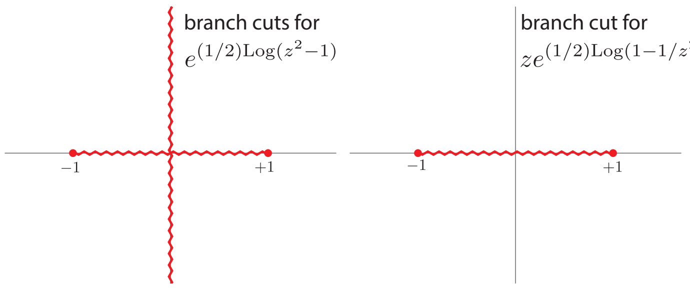
will not work, since it has branch cuts wherever $z^2 - 1$ is negative real, and this constitutes the whole y-axis as well as a portion of the x-axis. But if we experiment with some alternative expressions for w we are led to consider the solution

$$w = z\left(1 - \frac{1}{z^2}\right)^{1/2}, \tag{2.41}$$

such that the principal branch of $(1-1/z^2)^{1/2}$, i.e., $e^{(1/2)\mathrm{Log}(1-1/z^2)}$, has branch cuts where $1 - 1/z^2$ is negative real, and this only occurs when $1/z^2$ is real and greater than one, i.e. the cut is between the segment on the real line $[-1, 1]$. Thus,

$$w = f(z) = ze^{(1/2)\mathrm{Log}(1-1/z^2)} \tag{2.42}$$

satisfies the requirement of being holomorphic outside the unit circle.



Since we know that trigonometric functions can be expressed in terms of exponentials, and the inverses of exponentials are multi-valued logarithms, it should come as no surprise that the inverse of trig functions are logarithms.

The Inverse Sine Function

The inverse sine function $w = \sin^{-1} z$ is defined by the equation

$$z = \sin w, \tag{2.43}$$

and is a multi-valued function given by

$$w = \sin^{-1} z = -i \log[iz + (1 - z^2)^{1/2}]. \tag{2.44}$$

Proof: From the equation

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i} \tag{2.45}$$

we find that

$$e^{2iw} - 2ize^{iw} - 1 = 0. \tag{2.46}$$

Using the quadratic formula we can solve for e^{iw}:

$$e^{iw} = iz + (1 - z^2)^{1/2}, \tag{2.47}$$

where of course the square root has two possible values. The expression for w above thus follows.

Note that we can obtain a branch of the multi-valued function $\sin^{-1}(z)$ by first choosing a branch of the square root, and then selecting a suitable branch of the logarithm.

Example 2.7: Principal Branch of Sin^{-1}

Suppose z is real and lies in the interval $(-1, 1)$. If the principal values are used for the terms in (2.44), what is the range of $\sin^{-1} z$?

Solution: With principle values we have

$$\mathrm{Sin}^{-1} z \equiv -i\mathrm{Log}[iz + e^{(1/2)\mathrm{Log}(1-z^2)}]. \tag{2.48}$$

For $|z| = |x| < 1$, clearly $1 - z^2$ lies in the interval $(0, 1]$, and its Log is real. Hence the exponential term is positive real. Consequently the bracketed expression lies in the right half plane. In fact, it also lies on the unit circle, since

$$|iz + (1 - z^2)^{1/2}| = \sqrt{x^2 + (1 - x^2)} = 1. \tag{2.49}$$

Taking the Log and multiplying by i, we get

$$-\frac{\pi}{2} < \mathrm{Sin}^{-1} x < \frac{\pi}{2}, \tag{2.50}$$

which matches the normal convention for arcsin.

Exercise 2.2: Complex Inverse Trig

Find the inverse cosine and inverse tangent functions, and show they are given by

$$\cos^{-1} z = -i \log[z + (z^2 - 1)^{1/2}] \tag{2.51}$$

$$\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}, \qquad (z \neq \pm i). \tag{2.52}$$

Exercise 2.3: Derivatives of Inverse Trig

Use the chain rule to determine the derivatives of $\sin^{-1} z$, $\cos^{-1} z$ and $\tan^{-1} z$.

3
Complex Integration and Cauchy's Integral Theorem

In the previous chapter, we discussed how the derivative on the complex plane is modified by the extra degree of freedom with which a point can be approached. This two-dimensional property of the complex plane also modifies integration, since a general (line) integral between two points will require us to choose one of an infinite number of possible paths. In this chapter, we will find that if a function $f(z)$ is the complex derivative of an entire function $F(z)$, then we can use the standard fundamental theorem of calculus approach and find that the integral only depends on $F(z)$ evaluated at the end points. We will also find that if a function is holomorphic inside (and on) a closed loop, then its integral over that loop must be zero. This result is known as Cauchy's integral theorem.

3.1 Contours

Contour integrals are carried out by integrating along a series of ***curves*** in the complex plane. While we have an intuitive grasp for what a curve is, let's be a bit more explicit about the definitions of ***smooth curves***, ***contours***, and the ***interior and exterior*** of a domain.

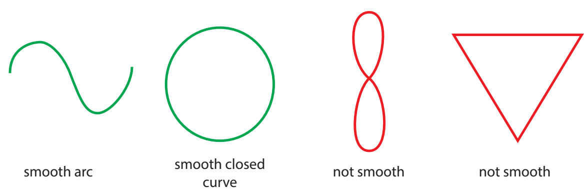
Smooth curves

Definition 3.1 (A smooth arc)**.** *A set of points γ in the complex plane is said to be a* **smooth arc** *if it is the range of some coninuous complex-valued function $z = z(t)$, where $t \in [a,b]$ is a real-valued parameterisation, and satisfies the following conditions*

(1) *$z(t)$ has continuous derivative on $[a,b]$,*

(2) *$z'(t)$ never vanishes on $[a,b]$,*

(3a) *$z(t)$ is one-to-one on $[a,b]$.*



Definition 3.2 (Smooth Closed Curves). *The set of points γ is called a* **smooth closed curve** *if it is the range of some continuous function $z = z(t)$ with $t \in [a, b]$, satisfying the conditions (1) and (2) above, as well as*

(3b) *z(t) is one-to one on the half-open interval $[a, b)$ but $z(b) = z(a)$ and $z'(b) = z'(a)$.*

Both smooth arcs and smooth closed curves are considered to be ***smooth curves***. Smooth arcs have distinct end points, while smooth closed curves have their endpoints joined.

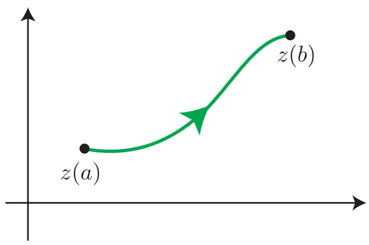


Figure 3.1: The directed smooth arc described by $z(t)$, taken from $z(a)$ to $z(b)$.

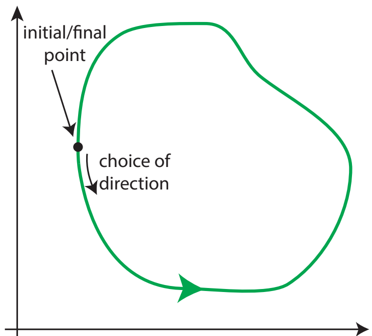


Figure 3.2: The directed smooth closed curve is given by a smooth closed curve with an initial/end point and a choice of direction (counterclockwise).

You can imagine that an artist with a pen on a paper, drawing a smooth curve. She is not allowed to lift the pen from the paper during the sketch; mathematically, we are requiring that $z(t)$ is continuous. Second we insist that the curves be drawn with an even, steady stroke, specifically, the pen point must move with a well-defined (finite) velocity that must also vary continuously. Finally, we require that no point on the curve be drawn twice, the artist cannot cross over her own points. The only exception to this is for smooth closed curves, where we allow the endpoints to coincide.

Suppose that the artist is able to draw a smooth arc between two points which obeys the rules we have laid out above. Then it should be clear that there are exactly two natural orderings over which we can traverse the smooth arc. We can start at one point and go to the other, or vice versa.

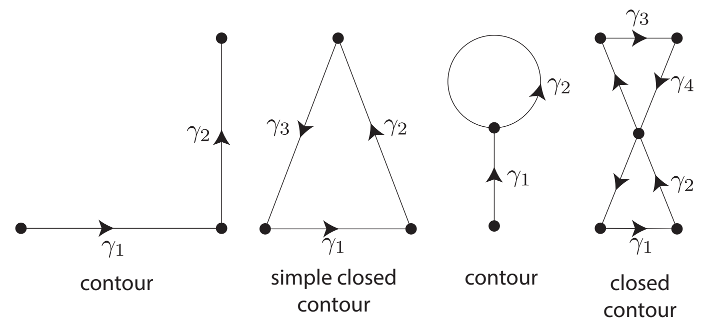
A smooth arc, together with a choice of the order over which to traverse it is called a ***directed smooth arc***. The ordering can be indicated by an arrow on the arc.

A directed smooth closed curve is slightly more complicated, and we must decide which direction to traverse the curve along from an initial point. We say say that a smooth closed curve is ***directed*** when we have (a) designated an initial point, and (b) chosen one of the two directions from this point.

Contours

Definition 3.3 (Contours). *A* **contour** *Γ is either a single point z_0 or a* **finite** *sequence of directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1}, for each $k = 1, 2, \ldots, n-1$.*

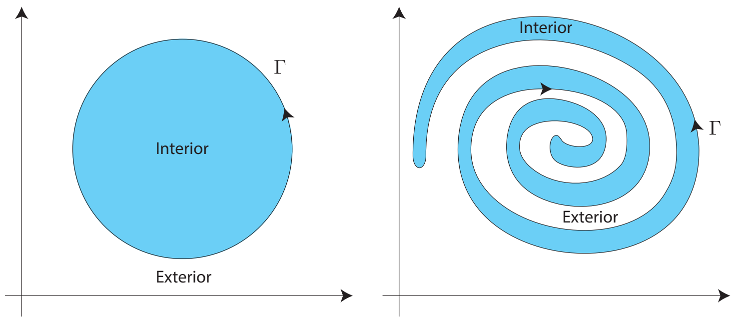
Definition 3.4 (Simple Closed Contours). *A contour is said to be* **closed** *or a* **loop** *if its initial and terminal points coincide. A* **simple closed contour** *is a closed contour with* **no repeated points** *other than the initial/final point.*



With any simple closed contour, it turns (via a theorem due to French mathematician Camille Jordan) out that we can always define an ***interior*** domain, and an ***exterior*** domain.

The Jordan Curve Theorem

Theorem 3.1 (The Jordan Curve Theorem). *A simple closed contour separates the plane into two domains, each having the curve as its voundary. One of these domains, called the* **interior**, *is bounded; the other, called the* **exterior**, *is unbounded.*



The formal proof of the Jordan Curve Theorem is quite involved, so we will not attempt it here.

The Orientation of a Contour

Imagine yourself walking along the complex plane, following the path of a contour. If the interior of the contour is ***on your left*** we would consider the contour to have a ***positive orientation***. If the interior is on your ***right***, we call it ***negative***.

Definition 3.5 (Orientation of Contours). *A contour is said to be* **positively oriented** *if the interior domain is on the* **left** *as the contour is traversed.*
A contour is said to be **negatively oriented** *if the interior domain is on the* **right** *as the contour is traversed.*

A circular contour is positively oriented if it is followed ***counterclockwise***, and negatively oriented if the it is followed ***clockwise***.
The contours in the figures above are both positive.

Alternate names for counterclockwise include "anticlockwise", "lefthandwise", and "widdershins" (meaning "counter the sun's direction"). The use of widdershins is often contrasted with "deisul" (clockwise - meaning "following the sun") in folklore.

With these definitions of the contours along which we can integrate, we are now ready to discuss contour integrals.

3.2 Contour Integrals

3.2.1 Complex-valued Integrals over the Real Line

Suppose we have a continuous function $f(t)$, that takes a value from the ***real interval*** $[a, b]$ and maps it to the complex plane.

$$f : [a, b] \to \mathbb{C}. \tag{3.1}$$

We can write f in terms of its real and imaginary parts

$$f(t) \equiv u(t) + iv(t), \tag{3.2}$$

and the indefinite integral of f can just be written in terms of normal integrals of real functions

$$\int f(t)dt = \int u(t)dt + i\int v(t)dt \tag{3.3}$$

Exercise 3.1: Integration By Parts

Verify that integration by parts is valid, i.e.,

$$\int f(t)g'(t)dt = f(t)g(t) - \int g(t)f'(t)dt, \tag{3.4}$$

assuming that f' and g' exist and are continuous. (Hint: The proof is the same as in ordinary real calculus, from the product rule)

Similarly, we can define the definite integral of F over $[a, b]$ to be

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt. \tag{3.5}$$

Thus the integral is defined in terms of the ordinary integrals of the real functions u and v. Consequently, by the fundamental theorem of calculus, the function

$$F(t) = \int_a^t f(s)ds \tag{3.6}$$

is ***differentiable***, and its derivative is $f(t)$.

Exercise 3.2: Absolute Value of an Integral vs Integral of an Absolute Value

Show that using simple properties of the integral of real-valued functions, one has the inequality,

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt \tag{3.7}$$

(Hint: Break it up into Riemann sums, and use the triangle inequality.)

Integral Along A Curve

Definition 3.6 (Integral Along a Curve). *Let smooth curve $\gamma : [a,b] \to U \subseteq \mathbb{C}$ be defined on a closed interval of real numbers $[a,b]$, and let f be a continuous function on the open set U. We can define the* **integral of f along γ** *to be*

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\frac{d\gamma}{dt}dt. \tag{3.8}$$

Note that it does not really matter what parameterisation we use, as can be easily seen defining the new parameterisation as $s(t)$, and applying the chain rule to convert an integral over s to one over t. The result is the same no matter what parameterisation is used.

Example 3.1: A Very Important Example

COMPUTE THE INTEGRAL $\oint_{C_r}(z-z_o)^n dz$, where n ***is an integer***, and C_r is the circle $|z-z_o| = r$ traversed once in the counterclockwise direction.

Solution: A suitable parameterisation for C_r is given by $z = \gamma(t) = z_o + re^{it}$, with $t \in [0, 2\pi]$. Setting $f(z) = (z-z_o)^n$, we have

$$f(\gamma(t)) = (z_o + re^{it} - z_o)^n = r^n e^{int}, \qquad \text{and} \qquad \gamma'(t) = ire^{it}$$

Hence, we must have

$$\begin{aligned} \oint_{C_r}(z-z_o)^n \mathrm{d}z &= \int_0^{2\pi}(r^n e^{int})(ire^{it})\mathrm{d}t \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt. \end{aligned} \tag{3.9}$$

If $n \neq -1$ we have

$$\begin{aligned} \oint_{C_r}(z-z_o)^n \mathrm{d}z &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt \\ &= ir^{n+1}\frac{e^{i(n+1)t}}{i(n+1)}\Bigg|_0^{2\pi} = 0. \end{aligned} \tag{3.10}$$

However, if $n = -1$ then we must have

$$\oint_{C_r}(z-z_o)^n \mathrm{d}z = ir^0\int_0^{2\pi} e^0 dt = i\int_0^{2\pi} dt = 2\pi i \tag{3.11}$$

Thus, regardless of the value of r,

$$\oint_{C_r}(z-z_o)^n \mathrm{d}z = \begin{cases} 0 & \text{for } n \neq -1, \\ 2\pi i & \text{for } n = -1. \end{cases} \tag{3.12}$$

We will see that this calculation plays an important role later.

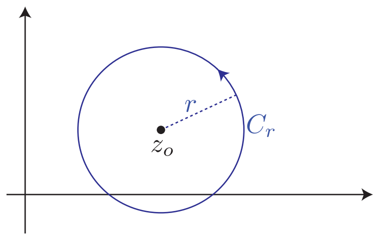


Figure 3.3: The contour C_r for Example 3.1.

3.2.2 *Bounding Contour Integrals*

Many times, in theory and in practice, it is not actually necessary to evaluate a contour integral. What may be required is simply a good upper bound on its magnitude. We therefore turn to the problem of estimating contour integrals.

Suppose that function f is continuous on the directed smooth curve γ and that $f(z)$ is bounded by the constant M on $\gamma : [a, b] \to \mathbb{C}$, i.e.,

$$|f(z)| \leq M, \qquad \forall z \text{ on } \gamma. \tag{3.13}$$

Let's divide the parameterisation of γ up into n partitions each of length $\Delta t = (b-a)/n$, The integral can be approximated by the ***Riemann sum***

$$\int_\gamma f(z)dz \approx \sum_{k=1}^{n} f(t_k)\Delta z_k \tag{3.14}$$

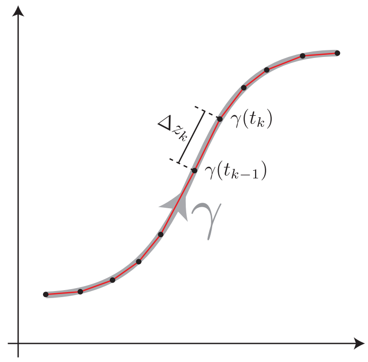


Figure 3.4: An integral along a directed smooth curve can be approximated using a Riemann sum if you partition the parameterisation of curve γ.

where $t_k \equiv a + k\Delta t$, and $\Delta z_k = \gamma(t_k) - \gamma(t_{k-1})$.

We have by the triangle identity

$$\left|\sum_{k=1}^{n} f(t_k)\Delta z_k\right| \leq \sum_{k=1}^{n} |f(t_k)||\Delta z_k| \leq M \sum_{k=1}^{n} \Delta z_k \tag{3.15}$$

Furthermore, we notice that the sum of the lengths Δz_k cannot be greater than the length $\ell(\gamma)$ of γ (since ***chords*** are always shorter than the ***arc length***), hence

$$\left|\sum_{k=1}^{n} f(t_k)\Delta z_k\right| \leq M\ell(\gamma). \tag{3.16}$$

Taking the limit as $n \to \infty$ gives us

$$\left|\int_\gamma f(z)dz\right| \leq M\ell(\gamma). \tag{3.17}$$

Applying this to a finite sum of different curves $\Gamma = \sum_m \gamma_m$, we have proved:

Bounding a Contour Integral

Theorem 3.2. *If f is continuous on the contour Γ and if $|f(z)| \leq M$ for all z on Γ, then*

$$\left|\int_\Gamma f(z)dz\right| \leq M\ell(\Gamma). \tag{3.18}$$

In particular, we have,

$$\left|\int_\Gamma f(z)dz\right| \leq \left[\max_{z \text{ on } \Gamma} |f(z)|\right] \times \ell(\Gamma). \tag{3.19}$$

This theorem is sometimes called ***Darboux's inequality***.

Integrands with Antiderivatives

Theorem 3.3. *Suppose that there exists*[†] *a holomorphic function $F(z)$ on U such that $F'(z) = f(z)$, and that curve γ has value $\alpha = \gamma(a)$, and $\beta = \gamma(b)$ at the end points. Then,*

$$\int_\gamma f(z)dz = F(\beta) - F(\alpha) \tag{3.20}$$

Proof: We know from the definition above

$$\int_\gamma f(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt.$$

By the chain rule, the expression under the integral sign is the derivative $\frac{d}{dt}F(\gamma(t))$. Then by ordinary calculus on the real and imaginary parts of F, the integral is equal to

$$\int_\gamma f(z)dz = F(\gamma(t))\Big|_a^b = F(\gamma(b)) - F(\gamma(a)). \quad \square \tag{3.21}$$

[†] Such a function F, where $F'(z) = f(z)$, is sometimes called a ***primative*** of f, or an ***antiderivative*** of f. Note that this is ***not*** guaranteed to exist for arbitrary f.

In general for such an f, if a path Γ consists of n connected curves $\gamma_1, \ldots, \gamma_n$, with z_j being the end point of the jth curve, then we can chain together the above to find

$$\begin{aligned}\int_\Gamma f(z)dz &= F(z_1) - F(z_0) + \ldots + F(z_n) - F(z_{n-1}) \\ &= F(z_n) - F(z_0).\end{aligned} \tag{3.22}$$

Thus, if the antiderivative of the integrand exists, the integral over a path Γ acts exactly as we expect from regular calculus.

Lemma 3.1. *If f is a continuous function on U that has a holomorphic antiderivative F, and Γ is any* **closed** *path, then*

$$\int_\Gamma f(z)dz = 0. \tag{3.23}$$

This follows from the theorem above. It is also straightforward to prove the following:

Theorem 3.4. *Let U be a connected open set, and let f be a continuous function on U. If the integral of f along* **any closed path** *in U is equal to 0, then f has an antiderivative F on U, that is, there exists a function F which is holomorphic such that $F'(z) = f(z)$.*

Theorems 3.3 and 3.4 are the complex analogues of the path independence of a potential function that is the antiderivative of a vector field.

It may seem that Theorem 3.4 is a little bit useless, since it seems that we'd have to test ***all*** possible closed paths and show the integrals are all zero. In the next section we will see that Cauchy's integral theorem will provide a simple condition for this property to hold.

For now we can simply summarise the two theorems above by saying that a given continuous function has an antiderivative in domain U ***if and only if*** its integral around every loop in U is zero.

3.3 *Cauchy's Integral Theorem*

In the last section we saw that if a continuous function f possesses a holomorphic anti-derivative in a domain U, its integral around any loop in U is zero and vice versa. Now we are going to show how this property ties in with whether f is itself holomorphic or not. Our first task will be to develop the necessary geometry.

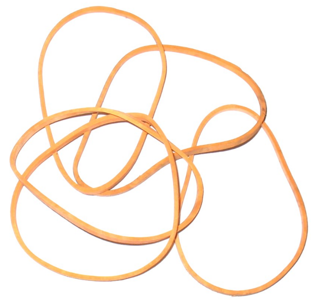
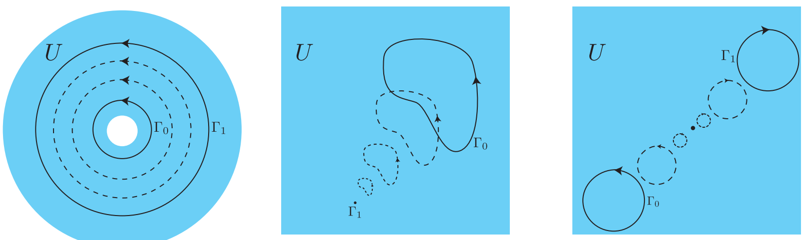


Figure 3.5: Deforming contours can be thought of as stretching and shrinking an infinitely elastic rubber band.

3.3.1 *Deformation of Contours*

The critical notion in this regard is the ***continuous deformation of one loop into another*** within the domain U. Deformations are quite easily visualised but somewhat harder to express in precise mathematical language. (We already discussed these briefly when reviewing the notion of simply connected domains.) In informal terms, we say that a loop Γ_0 can be continuously deformed into a loop Γ_1 in the domain U if Γ_0 (considered as an elastic string with indicated orientation) can be continuously moved, stretched, and shrunk about the complex plane ***without leaving*** U, in such a manner that it ultimately coincides with Γ_1.



Expanding circular contours — Shrinking a contour to a point — Reversing orientation of a contour by shrinking to a point

Figure 3.6: Examples of continuous deformations of contours within the domain U.

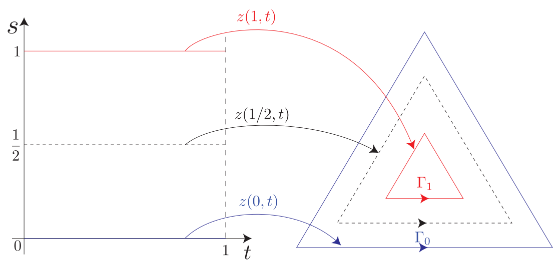
Continuous Deformation

Definition 3.7 (Continuously Deformable Loops). *The loop Γ_0 is said to be* **continuously deformable** *to the loop Γ_1 in the domain U, if there exists a function $z(s,t)$ continuous on the unit square $0 \le s \le 1, 0 \le t \le 1$, that satisfies the following conditions*

(i) For each fixed s in $[0,1]$, the function $z(s,t)$ parameterises a closed contour in U.

(ii) The function $z(0,t)$ parameterises the closed contour Γ_0.

(iii) The function $z(1,t)$ parameterises the closed contour Γ_1.



Loops that can be continuously deformed into one another are sometimes called ***homotopic***. The deformation between the two contours is called a ***homotopy***.

Example 3.2: Deformation in an Annulus

By finding a deformation function $z(s,t)$, prove that the loop $\Gamma_0 : z = e^{2\pi i t}$, for $0 \leq t \leq 1$, can be continuously deformed to the loop $\Gamma_1 : z = 2e^{2\pi i t}$, for $0 \leq t \leq 1$ in the domain U consisting of an annulus with $\frac{1}{2} < |z| < 3$.

Solution: The intermediate loops Γ_s, for $0 \leq s \leq 1$ are concetric circles with radii varying from 1 to 2. The function

$$z(s,t) = (1+s)e^{2\pi i t}, \qquad \text{for } 0 \leq s \leq 1, 0 \leq t \leq 1 \tag{3.24}$$

is a deformation function which takes Γ_0 to Γ_1.

A few elementary observations about continuous deformations are in order. First, notice that if $z(s,t)$ generates a deformation of loop Γ_0 into loop Γ_1, then $z(1-s,t)$ deforms Γ_1 into Γ_0. Furthermore, if in a given domain Γ_0 can be deformed to a single point and Γ_1 can be deformed to a point, then Γ_0 can be deformed into Γ_1.

It should be clear that ***simply connected domains***, i.e. connected domains without holes, are of particular interest to the study of deformed loops since we have,

Definition 3.8 (Simply Connected Domains)**.** *Any domain U possessing the property that* **every** *loop in U can be deformed (through U) to a point is called a* **simply connected domain**.

Deformation Invariance Theorem

Theorem 3.5 (Deformation Invariance Theorem)**.** *Let f be a function that is holomorphic on a domain U, containing closed contours Γ_0 and Γ_1. If these loops can be continuously deformed into on another in U then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz \tag{3.25}$$

A rigorous proof of the Deformation Invariance Theorem is beyond the scope of this course, however, we will instead attempt to prove a slightly weaker version of this theorem for the special case when the two contours are linked by a deformation function $z(s,t)$ whose second-order partial derivatives are continuous, and that $f'(z)$ is continuous.

—

(Weaker) Proof: Assume that the deformation function $z(s,t)$ has continuous partial derivatives up to second order for $0 \leq s \leq 1$, and $0 \leq t \leq 1$, and $f'(z)$ is continuous. Now, for each fixed s the equation $z = z(s,t)$, $0 \leq t \leq 1$, defines the loop Γ_s in U. Let $I(s)$ be the integral of f along this loop,

$$I(s) \equiv \int_{\Gamma_s} f(z)dz = \int_0^1 f(z(s,t))\frac{\partial z(s,t)}{\partial t}dt. \tag{3.26}$$

We wish to take the derivative of $I(s)$ with respect to s. The assumptions guarantee that the integrand above is continuously differentiable, so with constant limits of integration, we are allowed to bring the derivative inside the integral (this is called the Liebniz Integral Rule; see margin),

$$\frac{dI(s)}{ds} = \int_0^1 \left[f'(z(s,t)) \frac{\partial z}{\partial s}\frac{\partial z}{\partial t} + f(z(s,t)) \frac{\partial^2 f}{\partial t \partial s} \right] dt. \tag{3.27}$$

On the other hand observe that

$$\frac{\partial}{\partial t}\left[f(z(s,t)) \frac{\partial z}{\partial s} \right] = f'(z(s,t)) \frac{\partial z}{\partial t}\frac{\partial z}{\partial s} + f(z(s,t)) \frac{\partial^2 z}{\partial t \partial s}. \tag{3.28}$$

Because of the equality of mixed partials, the expression above is the same as the the integrand of dI/ds above. Thus, we have

$$\frac{dI(s)}{ds} = \int_0^1 \frac{\partial}{\partial t}\left[f(z(s,t)) \frac{\partial z}{\partial s} \right] dt \tag{3.29}$$

$$= f(z(s,1))\frac{\partial z}{\partial s}(s,1) - f(z(s,0))\frac{\partial z}{\partial s}(s,0). \tag{3.30}$$

But since each Γ_s is closed we have $z(s,1) = z(s,0)$ and $\partial_s z(s,1) = \partial_s z(s,0)$, so that dI/ds is zero, and consequently $I(s)$ must be a constant. In particular $I(0) = I(t)$ so that

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz. \quad \square \tag{3.31}$$

Theorem 3.6 (The Liebniz Integral Rule). *Let $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $\partial_x f(x,t)$ are continuous in t and x. Suppose that the boundary functions $a(x)$ and $b(x)$ are both continuous and have continuous derivatives. Then,*

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \partial_x f(x,t)dt.$$

This is straightforward to prove using the fundamental theorem of calculus.

A rather straightforward, but important, consequence of the deformation invariance theorem is known as ***Cauchy's integral theorem***:

Cauchy's Integral Theorem

Theorem 3.7 (Cauchy's Integral Theorem). *If f is holomorphic in a simply connected domain U, and Γ is any closed contour in U, then*

$$\int_\Gamma f(z)dz = 0 \tag{3.32}$$

Proof: In a simply connected domain, any loop can be shrunk to a point. The integral of a continuous function over a single point is zero, therefore by the deformation invariance theorem, the integral over any loop that can be continuously deformed to a point must be zero.
This leads us naturally to apply the results above to Theorems 3.3 and 3.4 from the previous section:

Theorem 3.8. *If a function is* **holomorphic on a simply connected domain**, *then in that domain it has an anti-derivative, its contour integrals are independent of path, and its loop integrals vanish.*

Example 3.3: Very Important Example Revisited

In the previous section we showed that if C_r is any positively oriented circle centred at z_o with any radius r, and n is an integer then

$$\oint_{C_r} (z - z_o)^n \mathrm{d}z = \begin{cases} 0 & \text{for } n \neq -1, \\ 2\pi i & \text{for } n = -1. \end{cases} \tag{3.33}$$

If n is a positive integer or zero, then Cauchy's integral theorem applies to the integral, since the integrand is holomorphic in the entire complex plane, which is obviously simply connected.
$(z - z_o)^n$ has an anti-derivative $(z - z_o)^{n+1}/(n+1)$, and the loop integral must be zero.
If n is negative, then $(z - z_o)^n$ is only analytic in the punctured plane, with the point z_o deleted. This domain is not simply connected, so the theorems of this section do not apply.
In fact for $n = -1$ the function $(z - z_o)^n$ does not even have an antiderivative in the punctured plane (since any branch of $\log(z)$ will have a discontinuity at a branch cut), and the loop integral fails to vanish.
For $n \leq -2$ $(z - z_o)^n$ still has an antiderivative, $(z - z_o)^{n+1}/(n+1)$, away from point z_o, and the loop integral is zero.
Thus we see that both cases can occur when the domain is not simply connected.

The main value of the deformation invariance theorem is that it allows us to replace complicated contours with more familiar ones, for the purpose of integration.

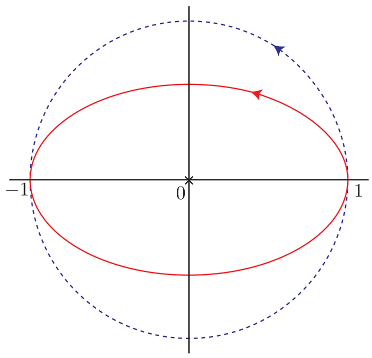


Figure 3.7: Deforming an elliptical contour to a circular one.

Example 3.4: Elliptical contour

Evaluate $\int_\Gamma \frac{1}{z} dz$ where Γ is the ellipse defined by $x^2 + 4y^2 = 1$ traversed once in the positive (counter-clockwise) sense.
Solution: The integrand $1/z$ is holomorphic everywhere on the complex plane except for the origin. We can continuously deform the this contour to the unit circle Γ_0 oriented positively. Thus we must have

$$\int_\Gamma \frac{1}{z} dz = \int_{\Gamma_0} \frac{1}{z} dz = 2\pi i \tag{3.34}$$

Example 3.5: Missing the poles

Evaluate

$$\oint_{|z|=2} \frac{e^z}{z^2 - 9} dz \tag{3.35}$$

where the integral is around the circle with $|z| = 2$ once in the positive (ccw) direction.

Solution: The integrand $e^z/(z^2 - 9)$ is holomorphic everywhere except at $z = \pm 3$, where the denominator vanishes. We can see that since the contour does not contain either of these poles, it can be shrunk to a point in the domain of analyticity, and thus the integral is zero. We can obtain the same result by applying Cauchy's theorem directly.

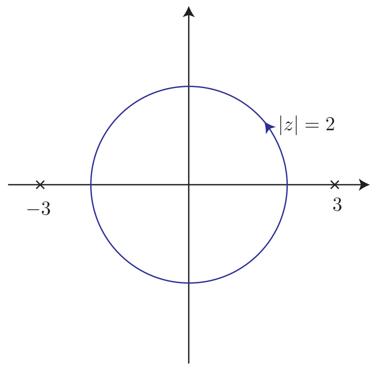


Figure 3.8: Contour and poles for Example 3.5.

Example 3.6: In or Out?

Determine the possible values for

$$\oint_\Gamma \frac{1}{z - a} dz, \tag{3.36}$$

where Γ is ***any*** circle not passing through $z = a$, traversed once in the counterclockwise direction.

Solution: The integrand is holomorhic in the domain $U = \mathbb{C} \setminus \{a\}$ consisting of the complex plane with the point $z = a$ removed. If this point lies exterior to Γ, then Γ can be continuously deformed to a point in U, and so the integral vanishes. If a lies in the interior of Γ, the contour can be continuously deformed in U to a positively oriented circle centred at $z = a$, and thus by Equation (3.12) in Example 3.1, we have

$$\oint_\Gamma \frac{dz}{z - a} = \begin{cases} 0 & \text{if } a \text{ lies outside } \Gamma, \\ 2\pi i & \text{if } a \text{ lies inside } \Gamma. \end{cases} \tag{3.37}$$

Example 3.7: Two Poles One Contour

Find

$$\int_\Gamma \frac{3z - 2}{z^2 - z} dz \tag{3.38}$$

where Γ is the simple closed contour shown in Figure 3.9.

—

Solution: We don't need an exact description of the contour Γ. Since the integrand $f(z) = (3z - 2)/(z^2 - z)$ is holomorphic in $U = \mathbb{C} \setminus \{0, 1\}$, i.e., the entire complex plane with the points $z = 0$ and $z = 1$ removed, we can deform Γ into the

barbell shaped contour shown in Figure 3.9 without changing the value of the integral (by the Deformation Invariance Theorem). This integral can be further simplified by noting that the integration along the line segment is to the left for one part of the contour, and to the right later in the contour, at the same place in the complex plane and therefore must cancel out.
Thus we must have

$$\int_\Gamma f(z)dz = \int_{C_0} f(z)dz + \int_{C_1} f(z)dz \tag{3.39}$$

where C_0 is the counterclockwise circular contour around the pole at $z=0$ and C_1 is the counterclockwise circular contour around the pole at $z=1$.
We will derive a much more powerful and flexible method to evaluate the integrals around these poles when we cover ***Cauchy's Residue Theorem***, however, for now we can use a partial fraction expansion to rewrite the integrand as

$$\begin{aligned}\frac{3z-2}{z^2-z} &= \frac{A}{z} + \frac{B}{z-1}, \\ &= \frac{Az - A + Bz}{z(z-1)}, \\ \Rightarrow A = 2, &\qquad B = 1.\end{aligned}$$

which gives us

$$\begin{aligned}\oint_\Gamma \frac{3z-2}{z(z-1)}dz &= \oint_{C_0}\left(\frac{2}{z}+\frac{1}{z-1}\right)dz + \oint_{C_1}\left(\frac{2}{z}+\frac{1}{z-1}\right)dz, \\ &= \underbrace{\oint_{C_0}\frac{2dz}{z}}_{2\times 2\pi i} + \underbrace{\oint_{C_0}\frac{dz}{z-1}}_{0} + \underbrace{\oint_{C_1}\frac{2dz}{z}}_{0} + \underbrace{\oint_{C_1}\frac{dz}{z-1}}_{2\pi i}, \\ &= 6\pi i. \quad \square\end{aligned}$$

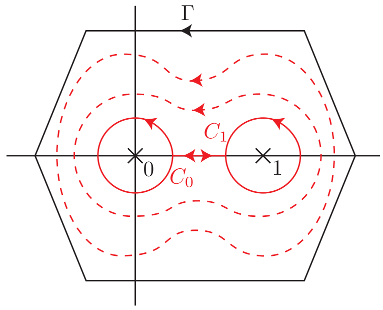


Figure 3.9: Contours and poles for Example 3.7.

3.4 *Cauchy's Integral Formula*

We now turn to the second important result of this chapter due to Cauchy, ***Cauchy's Integral Formula***. Given a function f that is holomorphic inside and on the simple closed contour Γ, we know from Cauchy's Integral Theorem that $\oint_\Gamma f(z)dz = 0$. However, if we consider the integral

$$\oint_\Gamma \frac{f(z)}{z-z_o}dz,$$

where z_o is an ***interior point*** to the contour Γ, we should not expect that this integral vanishes, since the integrand is no longer holomorphic inside the contour, and Cauchy's theorem no longer applies.

In fact, rather remarkably, we will show that for all z_o inside Γ, the value of this integral is proportional to $f(z_o)$.

Cauchy's Integral Formula

Theorem 3.9 (Cauchy's Integral Formula). *Let Γ be a simple closed positively oriented contour. If f is holomorphic in some simply connected domain U containing Γ and z_o is any point in the interior of Γ, then*

$$f(z_o) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_o} dz. \tag{3.40}$$

Proof: The function $f(z)/(z - z_o)$ is holomorphic everywhere in U except for point z_o. Hence by the Deformation Invariance Theorem, we can deform the contour freely to the circle C_r centred at z_o (see Figure 3.10) which is also positively oriented.
We can then write,

$$\oint_\Gamma \frac{f(z)}{z - z_o} dz = \oint_{C_r} \frac{f(z)}{z - z_o} dz, \tag{3.41}$$

$$= \underbrace{\oint_{C_r} \frac{f(z_o)}{z - z_o} dz}_{f(z_o) \times 2\pi i} + \oint_{C_r} \frac{f(z) - f(z_o)}{z - z_o} dz. \tag{3.42}$$

where we've just split the denominator into two parts since $f(z) = f(z_o) + (f(z) - f(z_o))$. For the second integral we can shrink the circle C_r to arbitrarily small radius r. Since f is holomorphic and thus does not blow up anywhere on C_r, we can define the maximum magnitude

$$M_r \equiv \max_{z \text{ on } C_r} [|f(z) - f(z_o)|]. \tag{3.43}$$

Then for z on C_r we have, by Theorem 3.2,

$$\left| \oint_{C_r} \frac{f(z) - f(z_o)}{z - z_o} dz \right| \leq \int_{C_r} \frac{|f(z) - f(z_o)|}{r} dz \tag{3.44}$$

$$\leq \frac{M_r}{r} \times \ell(C_r) = \frac{M_r}{r} \times 2\pi r, \tag{3.45}$$

so that we find

$$\left| \oint_{C_r} \frac{f(z) - f(z_o)}{z - z_o} dz \right| \leq 2\pi M_r. \tag{3.46}$$

But since $f(z)$ is holomorphic, and thus a ***continuous*** function, we must have $M_r \to 0$ as $r \to 0$, so this term must vanish as we shrink the contour. Therefore we are left with

$$\oint_\Gamma \frac{f(z)}{z - z_o} dz = 2\pi i f(z_o), \tag{3.47}$$

which proves the theorem. □

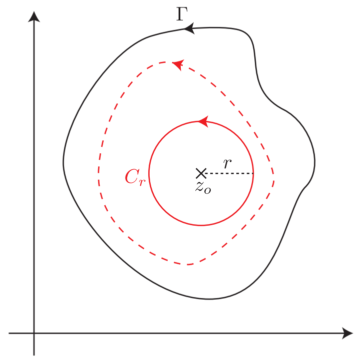


Figure 3.10: Contours and pole for the proof of Cauchy's Integral Formula.

One remarkable consequence of Cauchy's formula is that by merely knowing the values of the holomorphic function f on Γ we can compute the integral $\oint_\Gamma [f(z)/(z-z_0)]dz$ for any z_0, and hence all values of f inside Γ. In other words, ***the behaviour of a function holomorphic in a region is completely determined by its behaviour on the boundary!***

Example 3.8: Using Cauchy's Integral Formula

Compute the integral

$$\oint_\Gamma \frac{e^z + \sin z}{z} dz \tag{3.48}$$

where Γ is the circle $|z-2| = 3$ traversed once in the counter-clockwise direction.

Solution: The function $f(z) = e^z + \sin z$ is holomorphic inside and on Γ, and the point $z_0 = 0$ lies inside this circle. Hence we can use Cauchy's Integral Formula to get

$$\oint_\Gamma \frac{e^z + \sin z}{z} dz = 2\pi i f(0) = 2\pi i (e^0 + \sin 0) = 2\pi i. \quad \square \tag{3.49}$$

Example 3.9: Using Cauchy's Int. Formula Again

Evaluate the integral

$$\int_\Gamma \frac{\cos z}{z^2-4} dz \tag{3.50}$$

along the contour shown in Figure 3.11.

Solution: The integrand fails to be holomorphic at points $z = \pm 2$. However, only one of these ($z = 2$) is inside the contour Γ. Thus we can recast the integrand as

$$\begin{aligned}
\int_\Gamma \frac{\cos z}{z^2-4} dz &= \int_\Gamma \frac{(\cos z)/(z+2)}{z-2} dz \\
&= 2\pi i \left.\frac{\cos z}{z+2}\right|_{z=2} \\
&= 2\pi i \frac{\cos 2}{4} \\
&= \frac{i\pi \cos 2}{2}. \quad \square
\end{aligned}$$

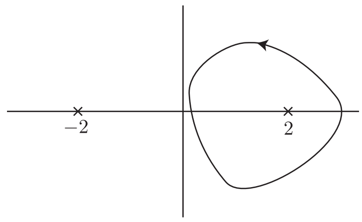


Figure 3.11: Contour and poles for Example 3.9.

We will return to Cauchy's Integral Formula shortly, but first we need to consider the general properties of complex power series and introduce the concept of analyticity.

4
Series Representations of Complex Functions

4.1 Convergence of Series

In Chapter 1, we briefly reviewed the concept of sequences. Here we will develop a few more tools to deal with series, which, as you should already know from 1st year Maths, can form a sequence made of partial sums.

Formal Definition of a Series

Definition 4.1 (Series). *A* **series** *is an expression of the form* $c_0 + c_1 + c_2 + \ldots$, *or,* $\sum_{k=0}^{\infty} c_k$ *where the terms* c_k *are complex numbers. The nth* **partial sum** *of a series, usually denoted* S_n*, is defined to be the sum of the first* $n+1$ *terms, i.e.* $S_n \equiv \sum_{k=0}^{n} c_k$*. If the* **sequence** *of partial sums* $\{S_n\}$ *has a limit* S*, the series is said to* **converge** *to* S*, and we write* $S = \sum_{k=0}^{\infty} c_k$*. A series that does not converge is said to* **diverge**.

Notice that the notion of a convergence of a series has been defined in terms of convergence for a sequence.

Clearly one way to demonstrate that a series converges to S is to show that the remainder after summing the first $n+1$ terms, i.e.,

$$S - \sum_{k=0}^{n} c_k \to 0 \tag{4.1}$$

as $n \to \infty$. Here we will use this to prove that the Geometric Series converges only inside the unit circle.

The Geometric Series

We will use this extensively in the proofs to come!

Theorem 4.1 (Geometric Series). *The geometric series* $1 + z^2 + z^3 + z^4 + \ldots$ *converges to the value* $1/(1-z)$ *for any complex number* z *with* $|z| < 1$.

Proof: Let $S_n(z) = 1 + z + \ldots + z^n$. Multiplying by z we find

$$zS_n = S_n + z^{n+1} - 1$$

Solving for S_n we get

$$S_n = \frac{1 - z^{n+1}}{1 - z}.$$

The series converges to $S = 1/(1-z)$ iff $S - S_n \to 0$ as $n \to 0$,

$$S - S_n = \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} \tag{4.2}$$

$$= \frac{z^{n+1}}{1-z} \tag{4.3}$$

Taking the limit $n \to \infty$ we must then have

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad \text{iff } |z| < 1, \tag{4.4}$$

since $z^{n+1} \to 0$ only for z in the interior of the unit circle. □

Another important way to establish the convergence of a series involves comparing it with another series whose convergence is known.

The Comparison Test

Theorem 4.2 (The Comparison Test)**.** *Suppose that the terms c_k satisfy the inequality*

$$|c_k| \leq M_k \tag{4.5}$$

where $0 \leq M_k \in \mathbb{R}$, for all integers k larger than some number K. Then, if the series $\sum_{k=0}^{\infty} M_k$ converges, so does $\sum_{k=0}^{\infty} c_k$.

Proof: This theorem seems very obvious, however, we will prove it just to be pedantic. This is most easily done using Theorem 1.6, the Cauchy Criterion for the convergence of sequences.

For any real $\varepsilon > 0$, since the series $\sum_k M_k$ converges, $\exists N \in \mathbb{N}$ such that the partial sums must obey,

$$\left| \left(\sum_{k=0}^{n+m} M_k \right) - \left(\sum_{k=0}^{n} M_k \right) \right| < \varepsilon, \qquad \forall n \geq N, \forall m > 0.$$

And hence we must have that

$$|M_{n+1} + M_{n+2} + \ldots + M_{n+m}| < \varepsilon$$
$$\Rightarrow M_{n+1} + M_{n+2} \ldots + M_{n+m} < \varepsilon$$

(since each M_k is real and positive). Choosing $n > \max(N, K)$, we therefore must have

$$|c_{n+1}| + |c_{n+2}| + \ldots + |c_{n+m}| < \varepsilon$$

Applying the triangle identity, $|c_n + \ldots + c_{n+m}| \leq |c_n| + \ldots + |c_{n+m}|$, we find that

$$|c_{n+1} + c_{n+2} + \ldots + c_{n+m}| < \varepsilon,$$
$$\Rightarrow |S_{n+m} - S_n| < \varepsilon, \qquad \forall n > \max(N, K), \forall m > 0,$$

where $\{S_n\} = \{\sum_k^n c_k\}$ is the sequence of partial sums, which we have just shown satisfies the Cauchy Criterion, and thus must also converge. □

Example 4.1: Comparing To the Geom. Series

Show that the series

$$\sum_{k=0}^{\infty} \frac{3+2i}{(k+1)^k} \tag{4.6}$$

converges.

Solution: We can compare the series above with the convergent ***geometric series***, with $z = 1/2$,

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1-\frac{1}{2}} = 2. \tag{4.7}$$

Since $|3+2i| = \sqrt{13} < 4$, we see that for $k \geq 3$,

$$\left| \frac{3+2i}{(k+1)^k} \right| < \frac{4}{(k+1)^k} \leq \frac{1}{2^k}, \tag{4.8}$$

hence, the series must converge. $\square$

Example 4.2: Absolute Convergence

Definition 4.2 (Absolutely Convergent). *A series $\sum_{k=0}^{\infty} c_k$ is said to be* **absolutely convergent** *if the series $\sum_{k=0}^{\infty} |c_k|$ is convergent.*

Prove the following:

Theorem 4.3. *Any absolutely convergent series must also be convergent.*

Proof: Let $M_k = |c_k|$, and apply the comparison test. $\square$

The Ratio Test

Theorem 4.4 (Ratio Test). *Suppose that the terms of the series $\sum_{k=0}^{\infty} c_k$ have the property that the ratios $|c_{k+1}/c_k|$ approach a limit L as $k \to \infty$. Then the series converges if $L < 1$ and diverges if $L > 1$.*

You will prove this on a worksheet.

Example 4.3: Applying the Ratio Test

Show that the series $\sum_{k=0}^{\infty} 4^k/k!$ converges.

Solution:

$$\left| \frac{c_{k+1}}{c_k} \right| = \frac{4^{k+1}}{(k+1)!} \frac{k!}{4^k} = \frac{4}{k+1}. \tag{4.9}$$

This ratio approaches zero as $k \to \infty$, therefore by the Ratio Test, the series converges. $\square$

The kinds of sequences and series that often arise in complex analysis are those where the terms are functions of a complex z. Thus if we have a sequence of functions $f_1(z), f_2(z), \ldots$, we must consider the possibility that for some values of z the sequence converges, while for others it diverges, as we saw for the geometric series.

In applying this theory to holomorphic functions, we need a somewhat stronger notion of convergence. Consider the sequence of real functions

$$\{f_n(x)\} = \{x^n\} \tag{4.10}$$

on the half open interval $x \in [0, 1)$. Clearly for any given x in the interval $x^n \to 0$ for sufficiently large n. We call this kind of convergence ***pointwise convergence***.

However, the curve $y = x^n$ for large n is a poor approximation to the curve $y = 0$ since as $x \to 1$ the difference between the functions remains large. What we want instead is the property of ***uniform convergence***.

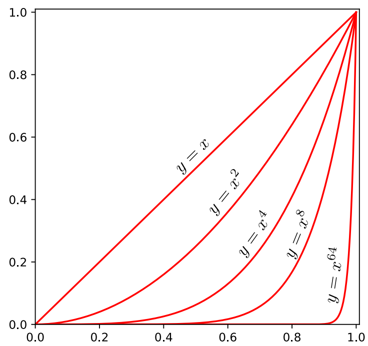


Figure 4.1: The functional sequence $\{f_n(x)\} = \{x^n\}$ is pointwise convergent, but not uniformly convergent on the interval $0 \le x < 1$.

Uniform Convergence

Definition 4.3. *The functional sequence $\{f_n(z)\}_n$ is said to* **converge uniformly** *to $f(z)$ on the set U if for any real $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$,*

$$|f(z) - f_n(z)| < \varepsilon, \qquad \forall z \in U. \tag{4.11}$$

We say that the series $\sum_{k=0}^{\infty} f_k(z)$ **converges uniformly** *to $S(z)$ on U if the sequence of partial sums $\{S_n(z)\}$ converges uniformly to $S(z)$ on U.*

The essential feature of uniform convergence is that for a given $\varepsilon > 0$, one must be able to find an integer N that is ***independent of*** $z \in U$ such that the error $|f(z) - f_n(z)|$ is less than ε for all possible $n > N$. In contrast, for pointwise convergence, N can depend upon z. Note, uniform convergence on U implies pointwise convergence on U, i.e. uniform convergence is a stronger property.

4.2 *Power Series*

Power Series

Definition 4.4 (Power Series)**.** *A* **power series** *is defined as a series of the form*

$$\sum_{n=0}^{\infty} a_n(z - z_o)^n = a_0 + a_1(z - z_o) + a_2(z - z_o)^2 + \ldots \tag{4.12}$$

where a_n is the complex coefficient of the nth term, and z_o is some fixed point on the complex plane.

By using the comparison test, it is easy to prove the following:

Theorem 4.5. *Let $\{a_n\}$ be a sequence of complex numbers, and let $r > 0$ be a real number such that the series $\sum_{n=0}^{\infty} |a_n| r^n$ converges. Then the series*

$$\sum_{n=0}^{\infty} a_n z^n \tag{4.13}$$

must converge absolutely and uniformly for $|z| \leq r$.

Example 4.4: $\sum z^n/n!$

For any $r > 0$ show that the series

$$\sum z^n/n! \tag{4.14}$$

converges absolutely and uniformly for $|z| \leq r$.
Solution: Let us define the series $\sum_n c_n$ where $c_n = r^n/n!$, then we have

$$\frac{c_{n+1}}{c_n} = \frac{r^{n+1}}{(n+1)!} \frac{n!}{r^n} = \frac{r}{n+1}. \tag{4.15}$$

Take $n \geq 2r$. Then the right hand side is $\leq 1/2$ Hence for all n sufficiently large, we have

$$c_{n+1}/c_n \leq 1/2 \tag{4.16}$$

Therefore there exists some positive integer N such that

$$c_n \leq c_N/2^{n-N} = (c_N 2^N)/2^n \tag{4.17}$$

for all $n > N$. We may therefore apply the comparison test with the geometric series, with $z = 1/2$, to get absolute and uniform convergence.

A given power series about a particular point can be shown to converge within a particular radius r, called the radius of convergence. Outside this radius, the series diverges.

Radius of Convergence

Theorem 4.6 (Radius of Convergence)**.** *Let $\sum a_n z^n$ be a power series. If it does not converge absolutely for all z, then there exists a number r (that may be zero) called* **the radius of convergence***, such that the series converges absolutely for $|z| < r$, and does not converge absolutely for $|z| > r$*

Proof: Suppose that the series does not converge absolutely for all z. Let r be the least upper bound of those numbers $s \geq 0$ such that $\sum |a_n| s^n$ converges. Then $\sum |a_n||z|^n$ diverges if $|z| > r$, and converges if $|z| < r$ by the comparison test.

If a power series has non-zero radius of convergence, then it is called a ***convergent*** power series. If D is a disc centred at the origin and contained in the disc $D(0,r)$, where r is the radius of convergence, then we say that the power series ***converges on*** D.

Example 4.5: Radius of Convergence of $\sum nz^n$

Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} nz^n. \tag{4.18}$$

Solution: We can apply the ratio test and see that this series converges if $\lim_{n\to\infty} |c_{n+1}/c_n| < 1$ and diverges if this limit is larger than 1.

$$\lim_{n\to\infty} \left| \frac{(n+1)z^{n+1}}{nz^n} \right| = \lim_{n\to\infty} \frac{n+1}{n}|z| = |z| \tag{4.19}$$

Thus $\sum nz^n$ converges if $|z| < 1$ and diverges if $|z| > 1$.

4.3 *Analytic Functions*

It is convenient for us to have a name for the type of functions that can be expressed as a convergent power series. We call such functions ***analytic*** and with some clever use of Cauchy's Integral Formula, we will see that a function is analytic if and only if it is holomorphic!

Analytic Functions

Definition 4.5 (Analytic Function)**.** *A function $f(z)$ is called* **analytic** *about a point z_o if there exists a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n \tag{4.20}$$

and some $r > 0$ such that the series converges absolutely for $|z - z_o| < r$, and that for such z we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n. \tag{4.21}$$

It should be obvious that polynomial functions can be expressed as power series (just set $a_n = 0$ for n greater than the degree of the polynomial), but a much wider class of functions turn out to be expressible as convergent power series as well.

Suppose f is a function on the open set U. We say that f is ***analytic on*** U if f is analytic at every point of U.

The next theorem, is easy to prove, but it gives us practice in a way of finding power series expansions for a function at a point.

Power Series at $z_o = 0$

Theorem 4.7. *Let $f(z) = \sum a_n z^n$ be a power series (centred at $z = 0$) whose radius of convergence is r. Then f is analytic in the open disc $D(0, r) \equiv \{z : |z| < r\}$.*

Proof: We have to show that f has a power series expansion at an arbitrary point z_o of the disc so that $|z_o| < r$. Let $s > 0$ be such that $|z_o| + s < r$. We shall see that f can be represented by a convergent power series at z_o converging on a disc $D(z_o, s)$ with radius s centred at z_o.

We can write z as

$$z = z_o + (z - z_o) \tag{4.22}$$

so that $z^n = (z_o + (z - z_o))^n$, so that by the binomial theorem we have

$$f(z) = \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} z_o^{n-k} (z - z_o)^k \right). \tag{4.23}$$

If $|z - z_o| < s$, then $|z_o| + |z - z_o| < r$, and hence the series

$$\sum_{n=0}^{\infty} |a_n| (|z_o| + |z - z_o|)^n = \sum_{n=0}^{\infty} |a_n| \left[\sum_{k=0}^{n} \binom{n}{k} |z_o|^{n-k} |z - z_o|^k \right] \tag{4.24}$$

converges. Then we can interchange the order of the summations† to get

$$f(z) = \sum_{k=0}^{\infty} \underbrace{\left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_o^{n-k} \right]}_{b_n} (z - z_o)^k \tag{4.25}$$

which converges absolutely also. Thus, the function f is analytic at z_o. $\square$

Generalising slightly, we can say that the above theorem shows that if f has a power series expansion on a disc $D(z_o, r)$, that is

$$f(z) = \sum a_n (z - z_o)^n \tag{4.26}$$

for $|z - z_o| < r$, then f is analytic on this disc.

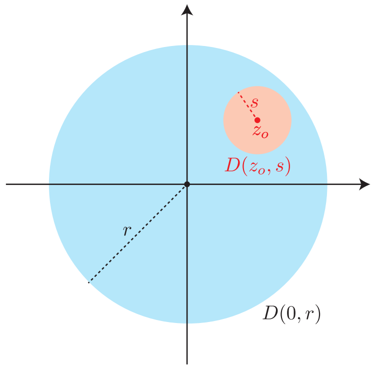


Figure 4.2: If f is a power series centred at $z = 0$, with a radius of convergence r, we can choose a point z_o within $D(0, r)$ and a disc $D(z_o, s)$ where $|z| + s < r$ such that f is analytic anywhere in $D(z_o, s)$.

† We are allowed to exchange the order of summation for any finite sum, by the normal commutativity of addition. If an infinite sum is ***absolutely convergent***, then for any $\epsilon > 0$ we can always find an N large enough so that the sum of of the the terms for $n > N$ are smaller than ϵ. Thus we can exchange the sums for the first (finite) N terms in the series to get the same value, whereas the difference the infinite tails can be made arbitrarily small.

4.4 *Differentiation of Power Series*

Let $D(0, r)$ be a disc of radius $r > 0$. A function f on the disc for which there exists a power series $\sum a_n z^n$ having a radius of convergence $\geq r$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4.27}$$

for all z in the disc is said to ***admit a power series expansion on this disc***. We shall now see that such a function is complex differentiable and thus is holomorphic on this domain, and the derivative is indeed given by the "obvious" power series,

$$\sum_{n=0}^{\infty} na_n z^{n-1} = a_1 + 2a_2 z^2 + 3a_3 z^3 + \ldots. \tag{4.28}$$

The Derivative of a Power Series

Theorem 4.8. *If $f(z) = \sum a_n z^n$ has radius of convergence $r > 0$, then:*

(i) The series $\sum na_n z^{n-1}$ has the same radius of convergence,

(ii) The function f is holomorphic on $D(0, r)$ and its derivative is equal to $\sum na_n z^{n-1}$.

Proof:

(i) For any $w \in D(0, r)$, except $w = 0$, choose $z \in D(0, r) \setminus \{0\}$ such that $|z| < |w|$. Then by Example 4.5 the series

$$\sum_{n=0}^{\infty} n \left| \frac{z}{w} \right|^n \tag{4.29}$$

converges (since $z/w < 1$) and hence each term $n|z/w|^n$ must be bounded by some $M \in \mathbb{R}$, so that

$$n|z|^n < M|w|^n, \qquad \forall n \in \mathbb{N}. \tag{4.30}$$

Thus we see that

$$n|a_n|z^{n-1} < \frac{M}{|z|}|a_n||w|^n, \qquad \forall n \in \mathbb{N} \tag{4.31}$$

and the series must be convergent by the comparison test since $\frac{M}{|z|} \sum a_n w^n$ is convergent for $|w| < r$. Therefore the radii of convergence must be the same.

(ii) Let $|z| < r$, and choose some small enough real $\delta > 0$ such that $|z| + \delta < r$. We consider complex numbers h such that $|h| < \delta$. We have

$$\begin{aligned} f(z+h) &= \sum a_n (z+h)^n \\ &= \sum a_n (z^n + nz^{n-1}h + h^2 P_n(z, h)) \end{aligned} \tag{4.32}$$

where $P_n(z, h)$ is a polynomial in z and h given by

$$P_n(z, h) = \sum_{k=2}^{n} \binom{n}{k} h^{k-2} z^{n-k}. \tag{4.33}$$

We can then use the estimate

$$|P_n(z, h)| \leq \sum_{k=2}^{n} \binom{n}{k} \delta^{k-2} |z|^{n-k} = P_n(|z|, \delta). \tag{4.34}$$

Rearranging we see

$$f(z+h) - \underbrace{\sum a_n z^n}_{f(z)} - \sum na_n z^{n-1} h = \sum h^2 P_n(z, h) \tag{4.35}$$

Since we showed above that both series on the left are absolutely convergent for any z within the radius of convergence of f, the series on the right hand side must also be absolutely convergent.

We divide by h to obtain the finite difference

$$\frac{f(z+h)-f(z)}{h} - \sum n a_n z^{n-1} = h \sum a_n P_n(z,h) \tag{4.36}$$
$$\leq |h| P_n(|z|,\delta) \tag{4.37}$$

where the right hand side goes to zero as $h \to 0$ (since δ is independent of h). Therefore we must have that

$$f'(z) \equiv \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum n a_n z^{n-1}. \quad \square \tag{4.38}$$

Thus we see that the following lemma must be true:

Analytic Functions are Holomorphic

Lemma 4.1. *Any analytic function is also holomorphic within its radius of convergence.*

The kth Derivative of a Power Series

From Theorem 4.8 we see also see that the kth derivative of $f = \sum a_n z^n$ is given by the series

$$f^{(k)}(z) = k! a_k + h_k(z) \tag{4.39}$$

where $h_k(z)$ is a power series that has no constant term, so that $h_k(z) \to 0$ as $z \to 0$. Therefore we obtain the standard expression for the coefficients of the power series in terms of the derivatives

$$a_n = \frac{f^{(n)}(0)}{n!} \tag{4.40}$$

If instead we are expanding about a point z_o, then we find

$$a_n = \frac{f^{(n)}(z_o)}{n!}. \tag{4.41}$$

To show that the definitions of analytic and holomorphic are interchangeable, it only remains to be shown that the implication also goes the other way, by circling back to Cauchy's Integral Formula.

4.5 *Holomorphic Functions are Analytic*

Let's consider a function $f(z)$ that is holomorphic on and inside a circle $C_R : |z - z_o| = R$, centred on point z_o.

Using Cauchy's Integral Formula on C_R we can swap the names of symbols around replace z by w and z_o by z, we find

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w-z} dw \tag{4.42}$$

for any z inside C_R.

Note that the denominator of the integrand above can be replaced by a geometric series in ***the interior*** of the C_R

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{w-z_o-(z-z_o)} \\ &= \frac{1}{w-z_o}\left(\frac{1}{1-\zeta}\right) \\ &= \frac{1}{w-z_o}\left(1+\zeta+\zeta^2+\dots\right) \end{aligned} \tag{4.43}$$

where $\zeta \equiv \frac{z-z_o}{w-z_o}$, has $|\zeta| < 1$ allowing the geometric series to converge since $|w-z_o| = R$ and z is in the ***(open) interior*** of C_R, so we must have $|z-z_o| < R$.

Since $f(z)$ is holomorphic in and on the circle, it is also bounded, and the series is uniformly convergent in this domain, thus we are allowed† to integrate term by term, swapping the integral and the infinite sum,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{(w-z_o)} \zeta^n dw \tag{4.44}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{(w-z_o)} \left(\frac{z-z_o}{w-z_o}\right)^n dw \tag{4.45}$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{2\pi i} \left[\oint_{C_R} \frac{f(w)}{(w-z_o)^{n+1}} dw\right]}_{a_n} (z-z_o)^n \tag{4.46}$$

This proves the following theorem:

Holomorphic Functions are Analytic

Theorem 4.11. *In the circle C_R where $f(z)$ is holomorphic, $f(z)$ can be expressed in terms of a convergent power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_o)^n \tag{4.47}$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{(w-z_o)^{n+1}} dw. \tag{4.48}$$

Therefore combining the above with Theorem 4.8 we now see that ***a function is analytic if and only if it is holomorphic***. We may now use the terms interchangeably, as we previously hinted.

Theorem 4.9 (Integrals of Convergent Sequences of Functions). *Let $\{g_n\}$ be a sequence of continuous functions on a domain U, converging uniformly to a function g. Then*

$$\lim_{n\to\infty} \int_\gamma g_n(z) dz = \int_\gamma g(z) dz.$$

Proof: Uniform convergence on a domain U means that for any real number $\epsilon > 0$ one can find an $N \in \mathbb{N}$ such that $|g_n(z) - g(z)| < \epsilon$ for any $n \geq N$ and $z \in U$, therefore,

$$\begin{aligned} \left| \int_\gamma g_n dz - \int_\gamma g dz \right| &\leq \int_\gamma |g_n - g| dz \\ &\leq \epsilon \ell(\gamma), \end{aligned}$$

where $\ell(\gamma)$ is the length of the contour. Hence, the integral of the sequence also uniformly converges to the integral of the limit. □

—

Theorem 4.10 (†Integrals of Convergent Series). *If $\sum f_n$ is a series of continuous functions converging uniformly on U, then*

$$\int_\gamma \sum_{n=0}^{\infty} f_n(z) dz = \sum_{n=0}^{\infty} \int_\gamma f_n(z) dz.$$

Proof: Let $\{g_k\} = \{\sum_{n=0}^{k} f_n\}$. Then by the Theorem 4.9 above, the integral of this sequence of partial sums also converges to the infinite sum of the integrals of each term. □

Thus we see that any differentiable function can be expanded as a power series — a very remarkable fact that is characteristic of ***complex differentiability***.

4.5.1 Holomorphic/Analytic Functions are Infinitely Differentiable

Using very similar arguments we can prove another somewhat weird looking, but surprisingly useful theorem:

Theorem 4.12. *Let γ be a directed curve in an open set U and let $g(z)$ be a continuous function for all z on γ. For any $z \in U$ that is not on the curve γ we can define the function*

$$G(z) \equiv \int_\gamma \frac{g(w)}{w-z} dw. \tag{4.49}$$

This function is analytic (and holomorphic) on $U \setminus \gamma$, and its nth derivative is given by

$$G^{(n)}(z) = n! \int_\gamma \frac{g(w)}{(w-z)^{n+1}} dw. \tag{4.50}$$

Note that in contrast to Eq. (4.42), here γ is not necessarily a closed loop and $g(z)$ is a completely arbitrary function (not necessarily holomorphic) that is only required to be continuous along γ.

Proof: Let $z_o \in U$, and z_o not on γ. Since U is an open set, we can choose a circle C_R of some radius $R > 0$ centred at z_o such that it does not overlap the curve γ (see Figure 4.3).

For z in the interior of C_R we can expand the denominator of the integrand using a convergent geometric series as before, and we find

$$G(z) = \sum_{n=0}^{\infty} \int_\gamma \frac{g(w)}{(w-z_o)^{n+1}} dw (z-z_o)^n \tag{4.51}$$

since we know $|w - z_o| > |z - z_o|$ for all w on γ, and z in the interior of C_R. Thus the function $G(z)$ is analytic (and holomorphic by Theorem 4.8) since we can write it as a power series about any point z_o in $U \setminus \gamma$,

$$G(z) = \sum_{n=0}^{\infty} a_n (z-z_o)^n \tag{4.52}$$

$$a_n \equiv \int_\gamma \frac{g(w)}{(w-z_o)^{n+1}} dw,$$

which converges within the radius R.

From Equation (4.41) and Theorem 4.8 we see that we can write a_n as,

$$a_n = \frac{G^{(n)}(z_o)}{n!} \tag{4.53}$$

Therefore we must have that

$$G^{(n)}(z_o) = n! a_n = n! \int_\gamma \frac{g(w)}{(w-z_o)^{n+1}} dw. \quad \square \tag{4.54}$$

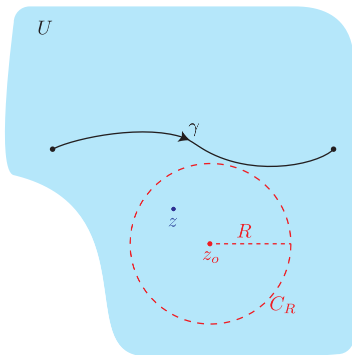


Figure 4.3: For a given point z_o, we can choose a radius $R < \min(|\gamma(t) - z_o|)$ so that C_R does not overlap with the curve $\gamma(t)$.

Why do we care about functions and derivatives of the strange integral form given by Equation (4.49)? Well, consider Cauchy's Integral Formula written in the form of Equation (4.42), for any function $f(z)$ holomorphic in the interior of C_R.

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w-z} dw.$$

for any z interior to C_R. It follows from Theorem 4.12 above, that such a function must be have a derivative given by

$$f'(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{(w-z)^2} dw. \tag{4.55}$$

But it if we set $G(z) = f'(z)$ and $g(w) = f(w)/(w-z)$ (which is continuous and well defined for w along C_R, since z is not on C_R), then we have that $f'(z)$ is also expressible as a power series by Equations (4.52), and hence is ***also analytic***.

This can be carried on to show that any analytic (holomorphic) function can be differentiated any number of times, and the nth derivative will also be analytic (holomorphic),

Holomorphic Functions are Infinitely Differentiable

Theorem 4.13. *If f is holomorphic (analytic) in a domain U, then all its derivatives $f', f'', \ldots f^{(n)}, \ldots$ exist and are also holomorphic (analytic) in U. Hence, holomorphic functions are* **infinitely differentiable**.

We also get a general expression for the ***derivatives*** of f, in terms of an ***integral***!

Generalised Cauchy Integral Formula

Theorem 4.14 (The Generalised Cauchy Integral Formula)**.** *If f is holomorphic inside and on the simple closed positively oriented contour Γ and if z is any point inside Γ, then its nth derivative is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(w)}{(w-z)^{n+1}} dw. \tag{4.56}$$

Example 4.6: Derivatives of e^{5z}

Compute $\int_\Gamma e^{5z}/z^3 dz$ where Γ is the circle $|z| = 1$ traversed once counterclockwise.

Solution: Observe that $f(z) = e^{5z}$ is holomorphic inside an on Γ. Therefore, from Theorem 4.14 above, with $z = 0$ and $n = 2$, we have

$$\int_\Gamma \frac{e^{5z}}{z^3} dz = \frac{2\pi i}{2!} f''(0) = 25\pi i. \quad \square \tag{4.57}$$

4.6 Taylor and Maclaurin Series

Taylor and Maclaurin Series

Definition 4.6 (Taylor Series). *If f is analytic at z_o, then the series*

$$f(z_o) + f'(z_o)(z - z_o) + \frac{f''(z_o)}{2!}(z - z_o)^2 + \ldots$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(z_o)}{k!}(z - z_o)^k, \tag{4.58}$$

is called the **Taylor Series** *for f around z_o. When $z_o = 0$, it is also known as the* **Maclaurin Series** *for f.*

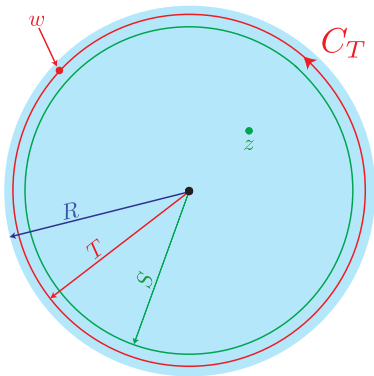


Figure 4.4: Given $S < R$, to prove that the Taylor Series converges uniformly for any $|z - z_o| \leq S$, we make use of the circular contour C_T, where $S < T < R$.

Theorem 4.15. *If f is analytic in the open disc $|z - z_o| < R$, then the Taylor series converges to $f(z)$ for all z in this disc. Futhermore, the convergence of the series is uniform for any closed subdisc $|z - z_o| \leq S < R$ (which implies pointwise convergence in the open disc of radius R).*

Proof: Let C_T be a (positive) circular contour with $S < T < R$. Then by Theorems 4.11 and 4.14, and we have for $|z| \leq S$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n \tag{4.59}$$

$$a_n = \frac{1}{2\pi i} \oint_{C_T} \frac{f(w)}{(w - z_o)^{n+1}} dw = \frac{f^{(n)}(z_o)}{n!}. \tag{4.60}$$

Thus the Taylor Series converges to $f(z)$ converges for $|z| < S$. We just need to show that this converges ***uniformly*** in this domain as well.

Let us define the partial sum of the Taylor series

$$f_N(z) \equiv \sum_{n=0}^{N} a_n (z - z_o)^n \tag{4.61}$$

From Equation (4.3) in our derivation of the Geometric Series, we know that for $|\zeta| < 1$, we have

$$\frac{1}{1 - \zeta} - (1 + \zeta + \zeta^2 + \ldots + \zeta^n) = \frac{\zeta^{n+1}}{1 - \zeta} \tag{4.62}$$

Taking $\zeta = (z - z_o)/(w - z_o)$ the error in the partial sum is

$$f(z) - f_N(z) = \oint_{C_T} \frac{f(w)}{w - z} dw - \sum_{n=0}^{N} a_n (z - z_o)^n \tag{4.63}$$

$$= \frac{1}{2\pi i} \oint_{C_T} \frac{f(w)}{w - z} \left(\frac{z - z_o}{w - z_o} \right)^{n+1} dw \tag{4.64}$$

since $|\zeta| < 1$ for $|z - z_o| < S < |w - z_o| = T$.

Since f is holomorphic and continuous on C_T, there must exist a maximum for $f(w)$ in the integrand. Setting

$$M \equiv \max_{w \text{ on } C_T} |f(w)|,$$

the error is then bounded by

$$|f(z) - f_N(z)| \leq \frac{1}{2\pi i} \frac{M 2\pi T}{(S-T)} \left(\frac{S}{T}\right)^{N+1} \tag{4.65}$$

The right hand side is independent of z, and since $S/T < 1$, given any $\varepsilon > 0$, we can always choose an N large enough such that the error is less than ε for any $n > N$. Therefore the convergence is uniform for $|z| \leq S$. $\square$

The above theorem immediately implies the following lemma:

Lemma 4.2. *The Taylor Series will converge to $f(z)$* **everywhere** *inside the largest* **open** *disc, centred at z_0, over which f is analytic.*

Example 4.7: Tinker, Something, Soldier, Sailor

Compute and state the convergence properties of the Taylor series for ***(a)*** $\operatorname{Log}(z)$ around $z_0 = 1$, ***(b)*** $1/(1-z)$ around $z_0 = 0$ and ***(c)*** e^z around $z_0 = 0$.

Solution:

(a) The derivatives of $\operatorname{Log} z$ in general are

$$\frac{d^k \operatorname{Log} z}{dz^k} = (-1)^{k+1}(k-1)! z^{-k} \qquad \text{for } k = 1, 2, \ldots,$$

so that evaulating at $z = 1$ gives us

$$\operatorname{Log} z = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(z-1)^k}{k}. \tag{4.66}$$

This converges for $|z-1| < 1$, since the largest open disk centred at $+1$ we can have before we run into the branch point at $z = 0$ has a radius of 1. $\square$

(b) The derivatives of $\frac{1}{1-z}$ are given by

$$\frac{d^k}{dz^k}(1-z)^{-1} = k!(1-z)^{-j-1}.$$

Evaluating these at $z = 0$ gives us the Taylor series,

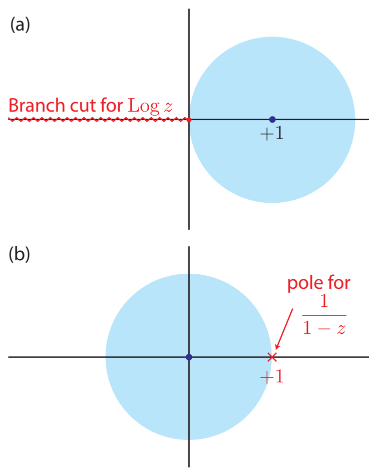
$$\frac{1}{1-z} = 1 + z + \frac{2! z^2}{2!} + \ldots = \sum_{k=0}^{\infty} z^k \tag{4.67}$$

(which of course is just the geometric series), which converges for $|z| < 1$, since there is a pole where the function is no longer holomorphic at $z = 1$. $\square$

(c) Since the derivative of e^z is just e^z, we can evaluate this at $z = 0$ to get

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \tag{4.68}$$

as expected. This converges for all z, since e^z is analytic on the entire complex plane. $\square$



4.7 Laurent Series

We are now ready to introduce a very important (for Complex Analysis anyways) generalisation of the power series, the ***Laurent Series***.

Laurent Series

Definition 4.7 (Laurent Series). *A* **Laurent Series** *is a series that can be written as*

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n. \tag{4.69}$$

Obviously, it is easily to generalise the following for a ***Laurent Series centred around a point*** z_o, where

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n.$$

Let U be a set of complex numbers. We say that the Laurent series ***converges absolutely*** on U if the two series,

$$f^+(z) \equiv \sum_{n\geq 0} a_n z^n \qquad \text{and} \qquad f^-(z) \equiv \sum_{n<0} a_n z^n$$

converge absolutely on U. If this is the case then $f(z)$ can be regarded as the sum,

$$f(z) = f^+(z) + f^-(z). \tag{4.70}$$

Similarly, $f(z)$ is said to ***converge uniformly*** on U if $f^+(z)$ and $f^-(z)$ both converge uniformly on U.

Laurent Series of a Function

Let $r, R \in \mathbb{R}$ be positive numbers with $0 \leq r < R$. Consider the ***closed***[†] ***annular domain*** A consisting of all complex numbers z such that $r \leq |z| \leq R$, which is bounded by the positively oriented contours C_R and C_r.

Theorem 4.16. *Let A be the closed annulus above, and let f be a holomorphic function everywhere in A. Let some s and S define the radii of the closed sub-annulus such that $r < s \leq |z| \leq S < R$. Then f has a Laurent expansion,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \tag{4.71}$$

which converges absolutely and uniformly on $s \leq |z| \leq S$. The coefficients a_n are obtained by the formula:

$$a_n = \begin{cases} \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w^{n+1}} dw, & \text{if } n \geq 0, \\ \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w^{n+1}} dw, & \text{if } n < 0. \end{cases} \tag{4.72}$$

[†]It is straightforward to instead prove a related theorem on the ***open*** annulus following a similar procedure to the proof of Theorem 4.15 (on the convergence properties of Taylor expansions) by introducing contours that sit in between the boundary and an interior closed annulus[††]. This would show uniform convergence of the Laurent Series for any closed annular domain ***interior*** to the open annulus, but only pointwise convergence in the whole open annulus.

[††]Under no circumstances should you refer to a closed annulus as a "clenched annulus".

Proof: For z such that $s \leq |z| \leq S$, we can use Cauchy's Integral Formula over the closed contour Γ shown in Figure 4.5,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z} dw \end{aligned} \tag{4.73}$$

where we note that the two radial paths (chosen such that they do not intersect z) cancel out, and C_r is traversed clockwise, acquiring a minus sign.

The first integral is handled in the same way as the proof of Theorem 4.11, by expanding using the geometric series, and interchanging the integral and the sum,

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}dw = \sum_{n=0}^{\infty}\left[\frac{f(w)}{w^{n+1}}dw\right]z^n. \tag{4.74}$$

The second integral can be expanded in a similar fashion, by writing

$$w - z = -z\left(1-\frac{w}{z}\right). \tag{4.75}$$

Then, since $|w/z| < 1$ for w on C_r, the geometric series converges, allowing us to expand,

$$\frac{1}{z}\frac{1}{1-w/z} = \frac{1}{z}\left(1+\frac{w}{z}+\left(\frac{w}{z}\right)^2+\dots\right) = \sum_{k=0}^{\infty}\frac{w^k}{z^{k+1}}. \tag{4.76}$$

So that the second integral becomes

$$-\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}dw = -\frac{1}{2\pi i}\int_{C_r}\sum_{k=0}^{\infty}f(w)\left(\frac{w^k}{z^{k+1}}\right)dw, \tag{4.77}$$

$$= \sum_{n=-1}^{-\infty}\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}dw\right)z^n, \tag{4.78}$$

where $k = -(n+1)$ and we can interchange the sum and the integral because the series converges uniformly. This completes the proof of Theorem 4.16. $\square$

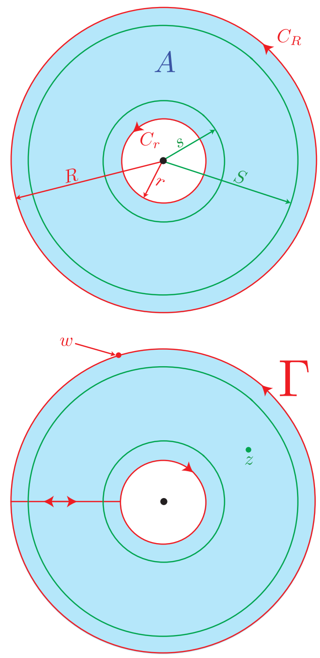


Figure 4.5: ***Above:*** The closed annulus A centred at $z = 0$ and bounded by the circular contours C_r and C_R. For any z such that $r < s \leq |z| \leq< R$, the Laurent Series for a function holomorphic on A converges. ***Below:*** For f holomorphic in the ***open interior*** of annulus A, i.e. $\{z : r < |z| < R\}$, we can use the contour Γ to prove Theorem 4.16.

Example 4.8: Laurent Series of $1/[z(z-1)]$

Find the the Laurent Series (centred at $z = 0$) for the function

$$f(z) = \frac{1}{z(z-1)}, \qquad \text{for } 0 < |z| < 1. \tag{4.79}$$

Solution: We can write f using partial fractions

$$f(z) = \frac{1}{z-1} - \frac{1}{z}. \tag{4.80}$$

Then for one term we get the geometric series,

$$\frac{1}{z-1} = -\frac{1}{1-z} = -1 - z - z^2 - \dots \tag{4.81}$$

$$f(z) = -\frac{1}{z} - 1 - z - z^2 + \dots \quad \square. \tag{4.82}$$

What happens if we want the Laurent series for $|z| > 1$? Then we can write

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-1/z}\right) = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right), \tag{4.83}$$

so that we find the Laurent Series,

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots. \tag{4.84}$$

Example 4.9: Different Regions, Different Series

For the function

$$\frac{1}{(z-1)(z-2)} \tag{4.85}$$

find the Laurent series (centred at $z_0 = 0$) in

(a) the region $|z| < 1$,

(b) the region $1 < |z| < 2$,

(c) the region $|z| > 2$.

Solution: Using partial fractions we can write

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \tag{4.86}$$

Now in each region we have to proceed slightly differently to keep the series convergent:

(a) For $|z| < 1$ we have

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k \tag{4.87}$$

$$\frac{1}{z-1} = -\sum_{k=0}^{\infty} z^k \tag{4.88}$$

This gives us the series expansion

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty}\left(1 - \frac{1}{2^{k+1}}\right) z^k \tag{4.89}$$

which converges in the region $|z| < 1$.

(b) For $1 < |z| < 2$, the first expansion in part (a) is valid, but we need to replace the second.

$$\frac{1}{z} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k} \tag{4.90}$$

So that the series can be written as

$$\frac{1}{(z-1)(z-2)} = -\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}} - \sum_{k=0}^{\infty}\frac{1}{z^{k+1}} \tag{4.91}$$

$$= \ldots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \ldots \tag{4.92}$$

for $1 < |z| < 2$.

(c) For $|z| > 2$ we can use Equation (4.90) as well as

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{k=0}^{\infty}\frac{2^k}{z^{k+1}}, \tag{4.93}$$

which gives us,

$$\frac{1}{(z-1)(z-2)} = \sum_{k=0}^{\infty}\frac{2^k - 1}{z^{k+1}} = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \ldots. \tag{4.94}$$

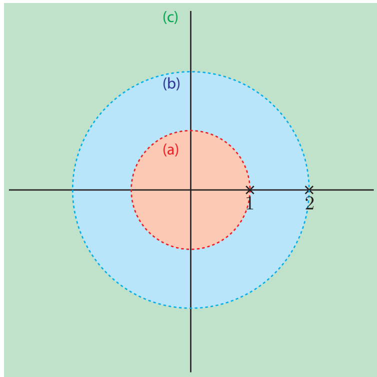


Figure 4.6: The different domains for Example 4.9. The function $1/[(z-1)(z-2)]$ has poles at $z = 1$ and $z = 2$.

4.8 Singularities and Zeros Revisited

With Laurent Series properly defined and well understood, we are now ready to revisit our discussion of singularities that were given by Definitions 2.6 - 2.8. In terms of Laurent Series, these definitions now become:

Isolated Singularities

Definition 4.8 (Singularities in terms of Laurent Series coefficients). *Let f have an isolated singularity at z_0, and let us have a Laurent Series expansion of f convergent in $0 < |z - z_0| < R$ given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{4.95}$$

Then,

(i) If $a_n = 0$ for all $n < 0$ we say that z_0 is a **removable singularity** *of f;*

(ii) If $a_{-m} = 0$ for some positive integer m, but $a_n = 0$ for all $n < -m$, we say that z_0 is a **pole of order** *m for f;*

(iii) If $a_n \neq 0$ for an infinite number of negative values of n we say that z_0 is an **essential singularity** *of f.*

We can also use the Laurent Series expansion to classify the zeros of an analytic function f, i.e., point z_0 where f is analytic and $f(z_0) = 0$.

Zeros of order m

Definition 4.9. *A point z_0 is called a* **zero of order** *m for the function f if f is analytic at z_0, and f and its first $m - 1$ derivatives vanish at z_0 but $f^{(m)}(z_0) \neq 0$.*

Using the Laurent series expansion, it is then easy to show

Theorem 4.17. *Let f be analytic on z_0. Then f has a zero of order m at z_0 if and only if f can be written as,*

$$f(z) = (z - z_0)^m g(z) \tag{4.96}$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

Just as a pole of order 1 is called a ***simple pole***, a zero of order 1 is often called a ***simple zero***. For instance the zeros of the function $\sin z$, which occur at integer multiples of π are all simple zeros, since the first derivative at these points is nonzero.

Cultural Sidenote: If f is holomorphic on an open set U ***except at a discrete set of points which are poles***, then f is sometimes called ***meromorphic*** on U. This derives from the Greek "***meros***" meaning ***part***, and "***morphe***" meaning ***shape*** or ***form*** and should not to be confused with the term Mer-Morphic.

5
The Calculus of Residues

We have already seen how the theory of contour integration provides insight into the properties of holomorphic functions. We have developed all the tools we need to compute integrals of analytic functions in terms of their power series expansions, and use these to evaluate certain real integrals.

5.1 Cauchy's Residue Theorem

Let us consider the problem of evaluating the integral

$$\oint_\Gamma f(z)dz, \tag{5.1}$$

where Γ is a simple closed positively oriented contour and $f(z)$ is holomorphic on and inside Γ ***except*** for a single isolated singularity at z_o lying ***interior*** to Γ.

As we showed in the previous chapter, the function $f(z)$ has a Laurent Series expansion,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n \tag{5.2}$$

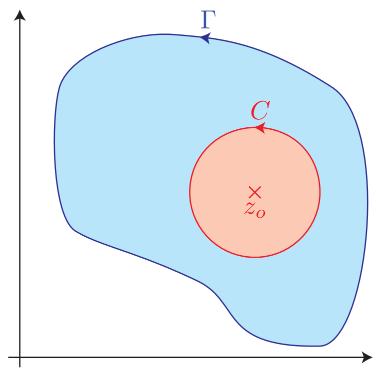


Figure 5.1: If $f(z)$ is holomorphic inside and on Γ except at a simple pole at z_o, we can deform the contour Γ to the circle C, where the Laurent Series for $f(z)$ centred at z_o converges in the punctured disc bounded by C.

converging in some punctured neighbourhood of z_o, such as the circle C (see Figure 5.1). Via the Deformation Invariance Theorem (Theorem 3.5), the integral over Γ can be deformed to integration over C without modifying the result of the integral,

$$\oint_\Gamma f(z)dz = \oint_C f(z)dz. \tag{5.3}$$

Since the Laurent Series (uniformly) converges everywhere on the contour C, the integral on the right hand side can be computed by term-wise integration of the Laurent Series along C. For all $n \neq -1$ the integral is zero [see Example 3.1], and for $n = -1$ we obtain the value of $2\pi i a_{-1}$. The coefficient a_{-1} of this Laurent expansion about the pole z_o is the key to the calculus of residues.

Cauchy's Residue Theorem

Definition 5.1 (Residues of a Pole). *If f has an isolated singularity at the point z_0, then the coefficient a_{-1} of $1/(z - z_0)$ of the Laurent series expansion for f around z_0 is called the* **residue of f at** z_0 *and is denoted by*

$$\text{Res}(f; z_0), \qquad \text{Res}_{z_0} f, \qquad \textit{or} \qquad \text{Res}(z_0),$$

where the latter can be used when there is no ambiguity as to which function we mean.

Theorem 5.1 (Cauchy's Residue Theorem). *Let U be an open set and Γ a closed positively oriented contour in U that is traversed once. Let f be analytic on U except at a finite number of points $z_1, \ldots, z_m$. Then,*

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{m} \text{Res}(f; z_k). \tag{5.4}$$

Proof: We can deform the contour Γ to the contour consisting of circles C_k surrounding each of the m poles, which are well separated, since the singularities are isolated, and connected by linear paths that are traversed once in each direction. Since the integrals along the straight paths cancel out, the contour integral becomes,

$$\begin{aligned} \oint_\Gamma f(z)dz &= \sum_{k=1}^{m} \oint_{C_k} f(z)dz & (5.5)\\ &= 2\pi i \sum_{k=1}^{m} \text{Res}(z_k), & (5.6) \end{aligned}$$

by applying the results of Example 3.1 for each circular contour.
□

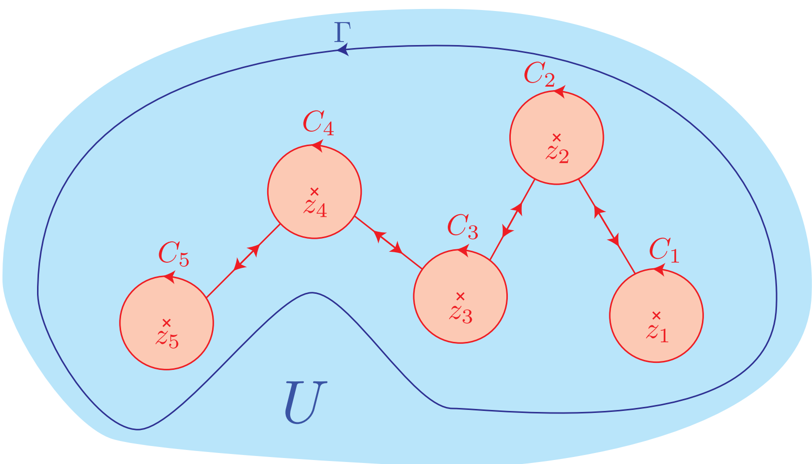


Figure 5.2: Contours for the proof of Cauchy's Residue Theorem. Such a finite sum of closed contours C_n is often called a ***chain*** (sometimes shown with the connecting cancelling paths omitted).

Example 5.1: Residue of $ze^{3/z}$

Find the residue at $z = 0$ of the function $f(z) = ze^3/z$ and compute

$$\oint_{|z|=4} ze^{3/z} dz, \tag{5.7}$$

where the contour is traversed once in the counterclockwise direction.

Solution: Since e^w has a Taylor expansion,

$$e^w = \sum_{k=0}^{\infty} \frac{w^k}{k!}, \tag{5.8}$$

for all $w \in \mathbb{C}$. The Laurent expansion for $f(z)$ around $z = 0$ is then given by,

$$ze^{3/z} = z \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{3}{z}\right)^k = z + 3 + \frac{3^2}{2!z} + \frac{3^3}{3!z^2} + \ldots \tag{5.9}$$

Hence we find,

$$\mathrm{Res}(0) = a_{-1} = \frac{3^2}{2!} = \frac{9}{2}. \tag{5.10}$$

Since $z = 0$ is the only singularity inside $|z| = 4$, we have

$$\oint_{|z|=4} ze^{3/z} dz = 2\pi i \cdot \frac{9}{2} = 9\pi i. \tag{5.11}$$

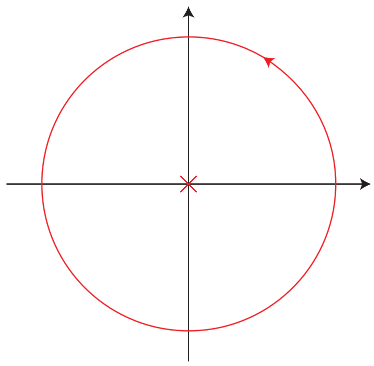


Figure 5.3: Contour and pole for the function $f(z) = ze^{3/z}$ for Example 5.1. Note that the pole at $z = 0$ is an ***essential singularity***.

5.2 *Calculating Residues*

Cauchy's Residue theorem allows us to calculate the contour integrals around isolated singularities in terms of their residues, which are given by the value of the Laurent series coefficient a_{-1}. In particular, for the three different types of isolated singularities, we see:

- If f has a ***removable*** singularity at z_0, then all the coefficients of the negative powers of $(z - z_0)$ in the Laurent expansion are zero, and so, ***the residue of a function at a removable singularity is zero***.

- If f has an ***essential singularity*** at z_0 (e.g. in Example 5.1) to determine the residue we must ***determine the Laurent series expansion for f about z_0, and the coefficient a_{-1}***.

- If f has a ***pole*** at z_0, it turns out that we can find a ***simple formula*** to evaluate the residue.

  If the singularity is a ***simple pole*** (i.e. a pole of order 1) then for z

near z_o where the Laurent Series is convergent, we have

$$f(z) = \frac{a_{-1}}{z - z_o} + a_0 + a_1(z - z_o) + a_2(z - z_o)^2 + \dots \quad (5.12)$$

$$\Rightarrow (z - z_o)f(z) = a_{-1} + (z - z_o)\left[a_0 + a_1(z - z_o) + \dots\right], \quad (5.13)$$

giving us the following theorem:

Residue of a Simple Pole

Theorem 5.2 (Residue at a Simple Pole). *Let f be a function that has an isolated* **simple pole** *at z_o. Then the residue of f at z_o can be given by,*

$$\operatorname{Res}(f; z_o) = \lim_{z \to z_o} (z - z_o) f(z). \quad (5.14)$$

Example 5.2: Simple Poles

Find the residues at the simple poles of

$$f(z) = \frac{e^z}{z(z+1)}. \quad (5.15)$$

Solution: $f(z)$ has simple poles at $z = 0$ and $z = -1$. Therefore we have

$$\operatorname{Res}(f; 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^z}{z+1} = 1, \quad (5.16)$$

$$\operatorname{Res}(f; -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{e^z}{z} = -e^{-1}. \quad (5.17)$$

Example 5.3: Denominators with Simple Zeros

Prove the following theorem:

Theorem 5.3. *Let $f(z) = P(z)/Q(z)$, where the functions $P(z)$ and $Q(z)$ are both analytic at z_o, and $Q(z)$ has a* **simple zero** *at z_o, while $P(z_o) \neq 0$. Then, the residue of f at z_o is given by,*

$$\operatorname{Res}(f; z_o) = \frac{P(z_o)}{Q'(z_o)}. \quad (5.18)$$

Proof: Obviously, f has a simple pole at z_o, so we can apply Theorem 5.2. Using the fact that $Q(z_o) = 0$, we see that

$$\begin{aligned} \operatorname{Res}(f; z_o) &= \lim_{z \to z_o} (z - z_o) \frac{P(z)}{Q(z)}, \\ &= \lim_{z \to z_o} \frac{P(z)}{\left[\frac{Q(z) - Q(z_o)}{z - z_o}\right]}, \\ &= \frac{P(z)}{Q'(z)}. \quad \square \end{aligned}$$

To obtain the general formula for the residue at a pole of order m we need some method of picking out the coefficient a_{-1} from the Laurent expansion.

Residue of an mth Order Pole

Theorem 5.4 (Residue of an mth Order Pole). *If f has a pole of order m at z_0, then*

$$\operatorname{Res}(f;z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]. \tag{5.19}$$

Proof: The Laurent expansion for f around z_0 is

$$\begin{aligned} f(z) &= \frac{a_{-m}}{(z-z_0)^m} + \ldots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots, \\ (z-z_0)^m f(z) &= a_{-m} + \ldots + a_{-1}(z-z_0)^{m-1} \\ &\quad + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \ldots. \end{aligned}$$

We can differentiate the above $m-1$ times to find

$$\begin{aligned} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] &= (m-1)!\,a_{-1} + m! a_0(z-z_0) \\ &\quad + \frac{(m+1)!}{2} a_1(z-z_0)^2 + \ldots, \\ \Rightarrow \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right] &= (m-1)! a_{-1}. \quad \square \end{aligned}$$

Example 5.4: Higher Order Poles

Compute the residues at the singularities of

$$f(z) = \frac{\cos z}{z^2(z-\pi)^3}. \tag{5.20}$$

Solution: $f(z)$ has a pole of order 2 at $z=0$ and a pole of order 3 at $z=\pi$. Applying Theorem 5.4 we find

$$\begin{aligned} \operatorname{Res}(f;0) &= \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left[z^2 f(z)\right] \\ &= \lim_{z\to 0} \frac{d}{dz}\left[\frac{\cos z}{(z-\pi)^3}\right] \\ &= \lim_{z\to 0}\left[\frac{-(z-\pi)\sin z - 3\cos z}{(z-\pi)^4}\right] \\ &= -\frac{3}{\pi^4}. \\ \operatorname{Res}(f;\pi) &= \lim_{z\to \pi} \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-\pi)^3 f(z)\right] \\ &= \lim_{z\to \pi} \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{\cos z}{z^2}\right] \\ &= \lim_{z\to \pi} \frac{1}{2}\left[\frac{(6-z^2)\cos z + 4z\sin z}{z^4}\right] \\ &= -\frac{6-\pi^2}{2\pi^4}. \end{aligned}$$

Exercise 5.1: Residue Practice

Compute the residue at each singularity of the following functions

(a) $\cot z$;

(b) $\frac{e^{3z}}{z-2}$;

(c) $\frac{z+1}{z^2-3z+2}$;

(d) $\frac{\cos z}{z^2}$;

(e) $\frac{e^{1/z}}{z-2}$;

(f) $\frac{z-1}{\sin z}$.

Example 5.5: Missing a Pole

Evaluate

$$\oint_{|z|=2} \frac{1-2z}{z(z-1)(z-3)} dz. \tag{5.21}$$

where the contour is traversed once in the counterclockwise direction.

Solution: The integrand $f(z) = (1-2z)/[z(z-1)(z-3)]$ has simple poles at $z = 0$, $z = 1$, and $z = 3$. However, only the first two of these points lie inside the contour. Thus by the residue theorem

$$\oint_{|z|=2} f(z)dz = 2\pi i[\text{Res}(0) + \text{Res}(1)]. \tag{5.22}$$

Evaluating the residues we find,

$$\text{Res}(0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{1-2z}{(z-1)(z-3)} = \frac{1}{3}, \tag{5.23}$$

$$\text{Res}(1) = \lim_{z\to 1} (z-1)f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-3)} = \frac{1}{2}. \tag{5.24}$$

This gives us

$$\oint_{|z|=2} f(z)dz = 2\pi i\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5\pi i}{3}. \quad \square \tag{5.25}$$

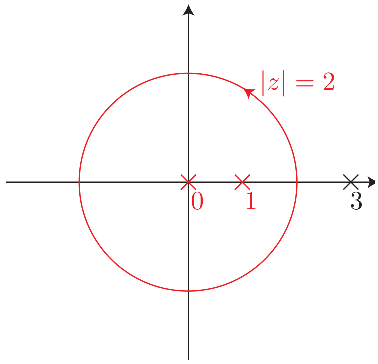


Figure 5.4: Contour and poles for $f(z) = (1-2z)/[z(z-1)(z-3)]$ in Example 5.5. Note that the contour does not enclose the pole at $z = 3$.

Exercise 5.2: Contour Integral Practice

Evaluate each of the following integrals by means of Cauchy's Residue Theorem. You may assume that all contours are positively oriented.

(a) $\oint_{|z|=5} \frac{\sin z}{z^2-4}\, dz$

(b) $\oint_{|z|=3} \frac{e^z}{z(z-2)}\, dz$

(c) $\oint_{|z|=2\pi} \tan z\, dz$

(d) $\oint_{|z|=3} \frac{e^{iz}}{z^2(z-2)(z+5i)}\, dz$

(e) $\oint_{|z|=1} \frac{1}{z^2 \sin z}\, dz$

(f) $\oint_{|z|=3} \frac{3z-2}{z^4+1}\, dz$

(g) $\oint_{|z|=8} \frac{1}{z^2+z+1}\, dz$

(h) $\oint_{|z|=1} e^{1/z} \sin(1/z)\, dz$

6
Applications of Cauchy's Residue Theorem

Having established Cauchy's residue theorem and the basics of the calculus of residues, we now turn to their applications in evaluating certain useful ***real*** integrals.

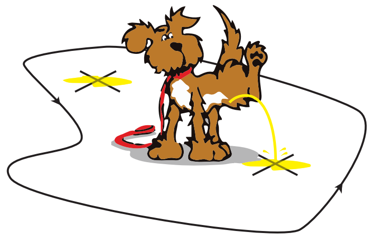


Figure 6.1: ***Q:*** Why did the Mathematician name his dog Cauchy?
A: Because his favourite long-dead uncle was named Cauchy and the dog was the only family he had left.

6.1 *Trigonometric Integrals Over* $[0, 2\pi]$

We would like to use Cauchy's residue theorem to evaluate real integrals of the form

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta \tag{6.1}$$

where $Q(\cos\theta, \sin\theta)$ is a ***rational function*** (with real coefficients) of $\cos\theta$ and $\sin\theta$ which is finite over $[0, 2\pi]$.

We want to recast this integral into a parameterised form of a contour integral

$$\int_{C_1} F(z) dz \tag{6.2}$$

where C_1 is the positively oriented unit circle $|z| = 1$. We start by parameterising C_1 by

$$z = e^{i\theta}, \qquad (0 \leq \theta \leq 2\pi). \tag{6.3}$$

For such z we have $1/z = e^{-\theta}$. We can use these substitutions in the expressions for the trigonometric functions

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right); \tag{6.4}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right). \tag{6.5}$$

When integrating along C_1, we also have

$$dz = ie^{i\theta} d\theta \qquad \Rightarrow \qquad d\theta = \frac{dz}{iz}. \tag{6.6}$$

With these substitutions we can rewrite the trigonometric integral as

$$\int_0^{2\pi} Q(\cos\theta, \sin\theta) d\theta = \oint_{C_1} F(z) dz \tag{6.7}$$

where the new integrand F is

$$F(z) \equiv Q\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right] \cdot \frac{1}{iz}. \tag{6.8}$$

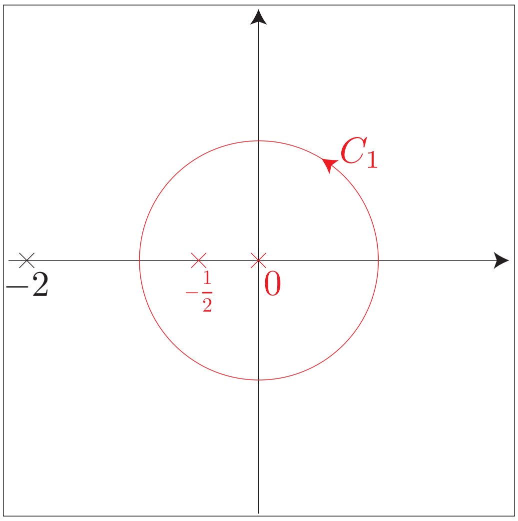
By Cauchy's residue theorem our trigonometric integral must equal $2\pi i$ times the sum of the residues of those poles of F which lie inside C_1.

Example 6.1: Trig Integral

Evaluate

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta. \tag{6.9}$$

Solution: We note that the denominator, $5+4\cos\theta$ is never zero, so the integrand is finite over $[0, 2\pi]$. We can then perform the substitutions above for $\cos\theta$, $\sin\theta$, and $d\theta$ to find,



$$I = \oint_{C_1} \frac{\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^2}{5+4\left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]} \frac{dz}{iz}. \tag{6.10}$$

Some algebra reduces this to

$$I = -\frac{1}{4i} \oint_{C_1} \frac{(z^2-1)^2}{z^2(2z^2+5z+2)} dz. \tag{6.11}$$

The integrand,

$$F(z) = -\frac{1}{4i} \frac{(z^2-1)^2}{z^2(2z^2+5z+2)} \tag{6.12}$$

has simple poles at $z = -1/2$ and $z = -2$, and a pole of order 2 at $z = 0$. However, only $-1/2$ and 0 lie inside the unit circle, so we have

$$I = 2\pi i \left[\mathrm{Res}\left(F; -\frac{1}{2}\right) + \mathrm{Res}(F; 0)\right]. \tag{6.13}$$

Using the techniques of the preceding section, we have

$$\begin{aligned}
\mathrm{Res}\left(F, -\frac{1}{2}\right) &= \lim_{z\to-\frac{1}{2}} (z+1/2)F(z) \\
&= -\frac{1}{4i} \lim_{z\to-\frac{1}{2}} \frac{(z^2-1)^2}{2z^2(z+2)} \\
&= -\frac{3}{16i};
\end{aligned}$$

$$\begin{aligned} \operatorname{Res}(F,0) &= \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}[z^2F(z)] \\ &= -\frac{1}{4i}\lim_{z\to 0}\frac{d}{dz}\left[\frac{(z^2-1)^2}{2z^2+5z+2}\right] \\ &= -\frac{1}{4i}\frac{(2z^2+5z+2)\cdot 2(z^2-1)2z-(z^2-1)^2(4z+5)}{(2z^2+5z+2)^2}\bigg|_{z=0} \\ &= \frac{5}{16i}. \end{aligned}$$

Thus, the contour integral is given by

$$I = 2\pi i\left[\frac{-3}{16i}+\frac{5}{16}\right] = \frac{\pi}{4}. \quad \square \tag{6.14}$$

Example 6.2: Trig Integrals Cont'd

Evaluate

$$I = \int_0^\pi \frac{d\theta}{2-\cos\theta}. \tag{6.15}$$

Solution: The trick here is that the integral is taken over $[0,\pi]$ instead of $[0,2\pi]$. However, it is easy to see that since $\cos\theta = \cos(2\pi-\theta)$,

$$\int_0^\pi \frac{d\theta}{2-\cos\theta} = \int_\pi^{2\pi}\frac{d\theta}{2-\cos\theta},$$

and, therefore,

$$\int_0^{2\pi}\frac{d\theta}{2-\cos\theta} = 2I.$$

Substituting for $\cos\theta$ and $d\theta$, we have

$$\begin{aligned} 2I &= \oint_{C_1}\frac{1}{2-\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz} \\ &= -\frac{2}{i}\oint_{C_1}\frac{dz}{z^2-4z+1}. \end{aligned}$$

By the quadratic formula the zeros of the denominator are

$$z_- \equiv 2-\sqrt{3}, \qquad z_+ \equiv 2+\sqrt{3},$$

so that the integrand

$$F(z) \equiv \frac{2i}{z^2-4z+1} = \frac{2i}{(z-z_-)(z-z_+)}$$

has simple poles at these points. But only z_- lies inside C_1, which has residue given by

$$\begin{aligned} \operatorname{Res}(F;z_-) &= \lim_{z\to z_-}(z-z_-)F(z) \\ &= \lim_{z\to z_-}\frac{2i}{z-z_+} \\ &= \frac{2i}{z_- - z_+} = -\frac{i}{\sqrt{3}}. \end{aligned}$$

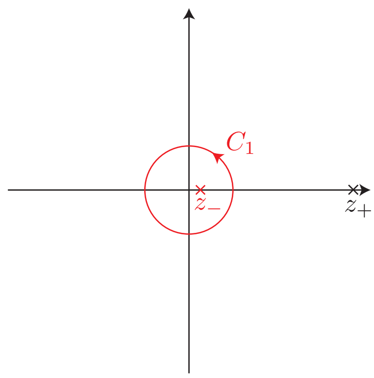


Figure 6.2: Contour and poles for Example 6.2.

This gives us

$$2I = -2\pi i \frac{i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$
$$I = \frac{\pi}{\sqrt{3}}. \quad \square$$

Exercise 6.1: Practice on Trig Integrals on the Unit Circle

Using Cauchy's Residue Theorem, verify each of the following:

(a) $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta} = \frac{2\pi}{\sqrt{3}}$;

(b) $\int_0^{\pi} \frac{8d\theta}{5+2\cos\theta} = \frac{8\pi}{\sqrt{21}}$;

(c) $\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$;

(d) $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \pi\sqrt{2}$;

(e) $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$, for $a^2 < 1$;

(f) $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta = \frac{2\pi}{b^2}(a - \sqrt{a^2-b^2})$, for $a > |b| > 0$.

6.2 *Improper Integrals and the Cauchy Principal Value*

If $f(x)$ is a function continuous on the nonnegative real axis $a \leq x < \infty$, then the improper integral of f over $[a, \infty)$ is defined by

$$\int_a^\infty f(x)dx \equiv \lim_{b\to\infty} \int_a^b f(x)dx, \qquad (6.16)$$

provided this limit exists.

Example 6.3: An Improper Integral

Find

$$\int_0^\infty e^{-2x} dx. \qquad (6.17)$$

Solution:

$$\begin{aligned}\int_0^\infty e^{-2x} dx &= \lim_{b\to\infty} \int_0^b e^{-2x} dx \\ &= \lim_{b\to\infty} \left.\frac{-e^{-2x}}{2}\right|_0^b \\ &= \lim_{b\to\infty} \left[\frac{-e^{-2b}}{2} + \frac{1}{2}\right] = \frac{1}{2}. \quad \square\end{aligned}$$

Similarly when $f(x)$ is continuous on $(-\infty, a]$, we set

$$\int_{-\infty}^a f(x)dx \equiv \lim_{c\to-\infty} \int_c^a f(x)dx. \qquad (6.18)$$

If it turns out that both of the above limits exist for a function f continuous on the whole real line, then f is said to be integrable over $(-\infty, \infty)$, and we can define

$$\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{c\to-\infty}\int_{c}^{0} f(x)dx + \lim_{b\to\infty}\int_{0}^{b} f(x)dx \tag{6.19}$$

$$= \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx. \tag{6.20}$$

Thus the improper integral over $(-\infty, \infty)$ can be computed by taking a single limit,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx. \tag{6.21}$$

However, this last limit may exist even for certain ***non-integrable functions*** f. Consider $f(x) = x$. This function is non integrable over $(-\infty, \infty)$ since the limit

$$\lim_{b\to\infty}\int_{0}^{b} xdx = \lim_{b\to\infty}\frac{b^2}{2} \tag{6.22}$$

is not finite. However, we if we instead take the dual limit

$$\lim_{R\to\infty}\int_{-R}^{R} xdx = \lim_{R\to\infty}\left.\frac{x^2}{2}\right|_{-R}^{R} = \lim_{R\to\infty} 0 = 0. \tag{6.23}$$

For this reason we introduce the following definition:

Cauchy Principal Value of an Improper Integral

Definition 6.1 (The Cauchy Principal Value). *Given any function f continuous on $(-\infty, \infty)$, the limit*

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{6.24}$$

(if it exists) is called the **Cauchy Principal Value** *of the integral of f over $(-\infty, \infty)$. We typically write*

$$\mathcal{P}\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{R\to\infty}\int_{-R}^{R} f(x)dx. \tag{6.25}$$

This is not to be confused with the principal value we discussed in Chapter 1, that chooses a branch of a multi-valued complex function.

Several different notations are used for the Cauchy Principal Value of the integral of a function f including

- p.v. $\int f(x)dx$;
- PV $\int f(x)dx$;
- V.P. $\int f(x)dx$;
- $\int_L^* f(z)dz$.

In our previous example, $\mathcal{P}\int_{-\infty}^{\infty} f(x)dx = 0$. Whenever the improper integral ***exists***, it must equal its principal value. Contour integration and the theory of residues can be used to compute principal value integrals for certain functions.

Cauchy Principal Value and Semicircular Contours

Let $f(z)$ be a holomorphic except at a ***finite number of poles***. Suppose that $f(x)$ is the restriction to the real line of the function f, and is a continuous function of real variable x that is well defined (finite) for the entire real line. Let γ_R be the closed positively oriented path, denoted in Figure 6.3, consisting of the real interval between $[-R, R]$ and the semi-circle of radius R, S_R, in the upper half plane.

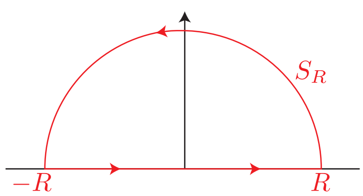


Figure 6.3: The contour consisting of the real line between $[-R, R]$, and the semi-circle S_R.

Theorem 6.1 (Evaluating Improper Integrals). *If there exists real numbers $B > 0$ and $\alpha > 0$ such that for all $|z|$ sufficiently large, we have*

$$|f(z)| \leq B/|z|^{1+\alpha} \tag{6.26}$$

Then

$$\lim_{R\to\infty} \int_{S_R} f(z)dz = 0, \tag{6.27}$$

and

$$\mathcal{P} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{residues of } f \text{ in the upper half plane.} \tag{6.28}$$

Proof: The integral over S_R has f bounded by $B/R^{1+\alpha}$ by assumption, and thus is itself bounded by $(B/R^{1+\alpha}) \times \pi R$. Since $\pi B/R^{\alpha}$ tends to 0 as $R \to \infty$ for $\alpha > 0$, the integral over S_R must be zero. Then we must have

$$\mathcal{P} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz \tag{6.29}$$

so that application of Cauchy's residue theorem gives us the desired result. □

Example 6.4: Cauchy Principal Value

Evaluate

$$I = \mathcal{P} \int_{-\infty}^{\infty} \frac{dx}{x^4+4}. \tag{6.30}$$

Solution: We can define the integral

$$I_R \equiv \int_{-R}^{R} \frac{dx}{x^4+4} \tag{6.31}$$

which can be interpreted as a contour integral of an analytic function,

$$I_R = \int_{\gamma_R} \frac{dz}{z^4+4} \tag{6.32}$$

where γ_R. This complex function is holomorphic except at 4 different poles given by

$$z^4 = -4 \qquad \Rightarrow z_k = 2^{1/2} e^{i\pi k/4}, \qquad k = 1, -1, 3, -3 \tag{6.33}$$

The poles within the contour γ_R as $R \to \infty$ are those in the ***upper*** half plane, which are the poles $z_1 = \sqrt{2}e^{i\pi/4} = 1+i$ and $z_3 = \sqrt{2}e^{i3\pi/4} = -1+i$. Meanwhile the poles $z_{-1} = 1-i$ and $z_{-3} = -1-i$ are in the exterior of the contour γ_R.
We can ignore the integral along the semicircular path S_R as $R \to \infty$ as long as the integrand goes to zero faster than $1/R$. The inequality

$$\left|\frac{1}{z^4+4}\right| \leq B/R^4 \tag{6.34}$$

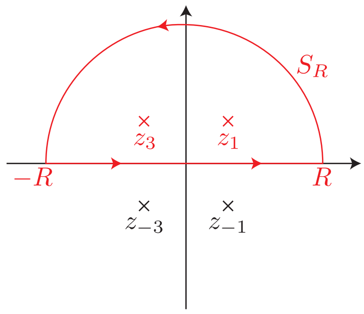


Figure 6.4: Contour and Poles for Example 6.4.

is indeed satisfied for some constant B when $|z| = R$, hence the theorem above applies, and we have

$$\begin{aligned}
\mathcal{P}\int_{-\infty}^{\infty}\frac{dx}{x^4+4} &= 2\pi i[\mathrm{Res}(z_1)+\mathrm{Res}(z_3)] \\
&= 2\pi i\left(\lim_{z\to z_1}\frac{z-z_1}{z^4+4}+\lim_{z\to z_3}\frac{z-z_1}{z^4+4}\right) \\
&= 2\pi i\left[\frac{1}{(z_1-z_{-1})(z_1-z_3)(z_1-z_{-3})}\right. \\
&\qquad\left.+\frac{1}{(z_3-z_1)(z_3-z_{-1})(z_3-z_{-3})}\right] \\
&= 2\pi i\left[\frac{1}{4i(2+2i)}+\frac{1}{-4i(-2+2i)}\right] \\
&= 2\pi i\left[\frac{-1-i}{16}+\frac{1-i}{16}\right] \\
&= \frac{\pi}{4}. \quad \square
\end{aligned}$$

Exercise 6.2: Practice on Integrals over $(-\infty, \infty)$

Using the calculus of residues verify each of the following,

(a) $\mathcal{P}\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2} = \pi$;

(b) $\mathcal{P}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+9)^2}dx = \frac{\pi}{6}$;

(c) $\int_0^{\infty}\frac{x^2+1}{x^4+1}dx = \frac{\pi}{\sqrt{2}}$;

(d) $\mathcal{P}\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$;

(e) $\mathcal{P}\int_{-\infty}^{\infty}\frac{x}{(x^2+4x+13)^2}dx = -\frac{\pi}{27}$;

(f) $\int_0^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{6}$.

6.3 *Improper Integrals Involving Trig Functions*

In this section we develop methods to evaluate integrals of the general forms

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos mx dx, \qquad \text{or} \qquad \mathcal{P}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin mx dx, \tag{6.35}$$

where m is real and $P(x)/Q(x)$ is a rational function continuous on the real line. We can often use the semicircular contour approach from above, but some modifications are necessary.

Example 6.5: Real Trig, Complex Problems

Compute

$$I = \mathcal{P}\int_{-\infty}^{\infty}\frac{\cos 3x}{x^2+4}dx \tag{6.36}$$

Solution: In utilising semicircular contours our first inclination is to deal with the complex function

$$f(z) = \frac{\cos 3z}{z^2+4}. \tag{6.37}$$

However, with this choice for $f(z)$ the modulus does not go to zero in either the upper or lower half-plane. We see that when $z = \pm\rho i$ we have

$$\lim_{\rho\to\infty}\left|\frac{\cos 3z}{z^2+4}\right| = \lim_{\rho\to\infty}\frac{e^{-3\rho}+e^{3\rho}}{2|-\rho^2+4|} \to \infty. \tag{6.38}$$

To get around this problem, we notice that since $\cos 3x$ ***is the real part of*** e^{3ix}, we have

$$I = \mathrm{Re}(I_o), \qquad \text{where } I_o \equiv \int_{-\infty}^{\infty}\frac{e^{3ix}}{x^2+4}dx. \tag{6.39}$$

To evaluate this new integral we can deal with the function

$$f(z) \equiv \frac{e^{3iz}}{z^2+4} \tag{6.40}$$

This function has singularities at $z = \pm 2i$. We can evaluate the absolute value of $f(z)$

$$|f(z)| = |f(x+iy)| = \frac{|e^{3ix}e^{-3y}|}{|z^2+4|} = \frac{e^{-3y}}{|z^2+4|}, \tag{6.41}$$

and see that in the upper half plane where $y \geq 0$,

$$|f(z)| \leq \frac{1}{|z^2+4|}, \tag{6.42}$$

so that for any $R > 2$, the integral over the upper semi-circle S_R is bounded by

$$\left|\int_{S_R} f(z)dz\right| \leq \frac{\pi R}{R^2-4} \tag{6.43}$$

which goes to zero as $R \to \infty$.
We then have

$$I_o = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = 2\pi i\,\mathrm{Res}(f;2i) \tag{6.44}$$

$$\mathrm{Res}(f;2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{e^{3iz}}{z+2i} = \frac{e^{-6}}{4i} \tag{6.45}$$

$$\Rightarrow I_o = \frac{2\pi i}{4ie^6} = \frac{\pi}{2e^6}. \tag{6.46}$$

Thus we have

$$I = \mathrm{Re}(I_o) = \frac{\pi}{2e^6}. \quad \square \tag{6.47}$$

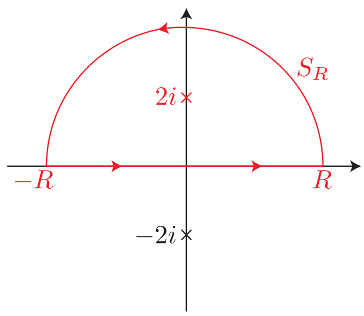


Figure 6.5: Contour and poles for Example 6.5.

A useful result for evaluating indefinite integrals of trigonometric functions is known as Jordan's Lemma.

Jordan's Lemma[†]

Lemma 6.1 (Jordan's Lemma). *Let R be a positive real number, and let f be a continuous complex valued function defined everywhere on the (positively oriented) semicircle S_R in the upper half plane. Suppose that for a given R there exists a non-negative real number $N(R)$ such that $|f(z)| \le N(R), \forall z \in S_R$. Then for all real $s > 0$,*

$$\left| \int_{S_R} f(z) e^{isz} dz \right| \le \frac{\pi N(R)}{s}. \tag{6.48}$$

[†] Jordan's Lemma is not to be confused with Jordan's Lambda.

Proof: The contour S_R can be parameterised by $z = Re^{i\theta}$ where $\theta \in [0, \pi]$

$$\int_{S_R} f(z) e^{isz} dz = \int_0^\pi f(Re^{i\theta}) e^{iRs\cos\theta - Rs\sin\theta} (iRe^{i\theta} d\theta).$$

Note that an equivalent lemma holds for the semicircular contour in the ***lower*** half plane if $s < 0$.

Then we have

$$\begin{aligned} \left| \int_{S_R} f(z) e^{isz} dz \right| &\le R \int_0^\pi |f(Re^{i\theta})| |e^{iRs\cos\theta - Rs\sin\theta}| d\theta \\ &\le RN(R) \int_0^\pi e^{-Rs\sin\theta} d\theta. \end{aligned}$$

The integrand in the last line above is symmetric about $\theta = \pi/2$, so that we see

$$\int_0^\pi e^{-Rs\sin\theta} d\theta = 2 \int_0^{\pi/2} e^{-Rs\sin\theta} d\theta$$

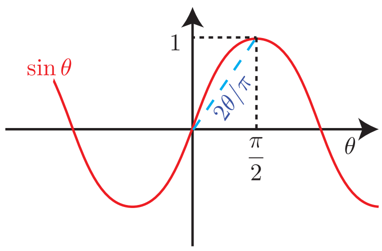


Figure 6.6: From the plot of $\sin\theta$ we can clearly see that $\sin\theta \ge 2\theta/\pi$ for $\theta \in [0, \pi/2]$.

From Figure 6.6, we see that $\sin\theta \ge 2\theta/\pi$ for $\theta \in [0, \pi/2]$, so that integral is bounded by

$$2 \int_0^{\pi/2} e^{-Rs\sin\theta} d\theta \le 2 \int_0^{\pi/2} e^{-2Rs\theta/\pi} d\theta = \left(\frac{\pi}{Rs}\right) [1 - e^{-Rs}] \le \frac{\pi}{Rs}.$$

Thus we must have

$$\left| \int_{S_R} f(z) e^{isz} dz \right| \le \frac{\pi N(R)}{s},$$

which proves Jordan's Lemma. □

Let's take a look at an example where Jordan's Lemma proves useful.

Example 6.6: Using Jordan's Lemma

Compute

$$I = \mathcal{P} \int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^2} dx \tag{6.49}$$

We can evaluate the integral

$$I_o \equiv \mathcal{P} \int_{-\infty}^{\infty} \frac{x e^{ix}}{1 + x^2} dx \tag{6.50}$$

and then take the imaginary part to evaluate I. To apply

Jordan's lemma, we see

$$\left|\frac{z}{1+z^2}\right| \leq M/R \tag{6.51}$$

for $z \in S_R$ for some constant M given large enough R. Thus Jordan's lemma (with $s = 1$) tells us

$$\lim_{R\to\infty}\left|\int_{S_R}\frac{ze^{iz}}{1+z^2}dz\right| \leq \lim_{R\to\infty}\frac{\pi M}{1\times R} = 0 \tag{6.52}$$

so that by Cauchy's residue theorem we have

$$I_o = 2\pi i \operatorname{Res}\left(\frac{ze^{iz}}{(z+i)(z-i)}; +i\right) = \frac{2\pi i}{2e}. \tag{6.53}$$

Taking the imaginary part gives us

$$I = \operatorname{Im}(I_o) = \frac{\pi}{e}. \quad \square \tag{6.54}$$

Converting the trigonometric function into the real or imaginary part of an integral doesn't always work, consider the following example.

Example 6.7: Using Jordan's Lemma Redux

Evaluate the integral

$$I = \mathcal{P}\int_{-\infty}^{\infty}\frac{\sin x}{x+i}dx. \tag{6.55}$$

Solution: We note that we ***cannot*** say that this integral is given by the imaginary part of the integral

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x+i}dx \tag{6.56}$$

since the complex valued denominator spoils this approach. Moreover we can't use $\sin z/(z+i)$ either since it is unbounded in both upper and lower half-planes. Instead we will try the substitution $\sin x = (e^{ix} - e^{-ix})/(2i)$, which splits the integral into two parts

$$I = \frac{1}{2i}\left(\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x+i}dx - \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{-ix}}{x+i}dx\right). \tag{6.57}$$

For the first integral we have

$$I_1 \equiv \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x+i}dx = 0 \tag{6.58}$$

since by Jordan's lemma we must have

$$\lim_{R\to\infty}\int_{S_R^+}\frac{e^{iz}}{z+i}dz = 0 \tag{6.59}$$

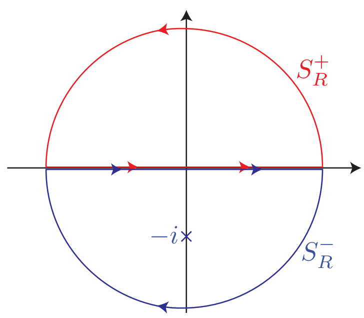


Figure 6.7: Pole and Contours for Example 6.7.

where S_R^+ is the semicircular contour in the upper half plane, since $1/(z+i) \to 0$ along S_R^+ as $R \to \infty$, and there are no singularities in the upper half plane.
For the second integral we have

$$I_2 \equiv \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x+i}dx \tag{6.60}$$

involves the function e^{-iz} which is unbounded in the upper half-plane, so we can close the contour in the lower half-plane with the semicircle S_R^-. Applying Jordan's Lemma for $s < 0$ in the lower half plane we have

$$\lim_{R\to\infty} \int_{S_R^-} \frac{e^{-iz}}{z+i}dz = 0. \tag{6.61}$$

Noting that the closed contour for the lower half plane is negatively oriented, we can obtain

$$I_2 = -2\pi i \operatorname{Res}\left(\frac{e^-iz}{z+i}; -i\right) = -2\pi i \lim_{z\to -i} e^{-iz} = -\frac{2\pi i}{e}, \tag{6.62}$$

giving the solution

$$I = \frac{1}{2i}(I_1 - I_2) = \frac{\pi}{e}. \quad \square \tag{6.63}$$

Exercise 6.3: Practice on Improper Trig Integrals

Using the calculus of residues verify each of the following,

(a) $\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx = \frac{\pi}{e^2}$;

(b) $\mathcal{P}\int_{-\infty}^{\infty} \frac{x\sin x}{x^2-2x+10}dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$;

(c) $\int_0^{\infty} \frac{\cos x}{(x^2+1)^2} = \frac{\pi}{2e}$.

Exercise 6.4: More Improper Trig Integrals

Compute the following integrals,

(a) $\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{3ix}}{x-2i}dx$;

(b) $\mathcal{P}\int_{-\infty}^{\infty} \frac{x\sin(3x)}{x^4+4}dx$;

(c) $\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+4}dx$.

6.4 *Indented Contours: Singularities on the Real Line*

In the previous sections the integrands $f(x)$ were assumed to be defined and continuous over the whole interval of integration. In this section, we will explore definite real integrals where $|f(x)| \to$

∞ at particular (finite) points in that interval. We will see that these ***improper integrals*** can be evaluated using the Cauchy's principal value approach, and by choosing the appropriate contours to be able to apply the residue theorem.

Let $f(x)$ be continuous and finite on $[a, b]$ except at the point $c \in (a, b)$. Then the (improper) integral of f over $[a, b]$ is given by

$$\int_a^b f(x)dx \equiv \lim_{r\to 0} \int_a^{c-r} f(x)dx + \lim_{s\to 0} \int_{c+s}^b f(x)dx, \tag{6.64}$$

for $r > 0$ and $s > 0$, provided these limits exist.

Example 6.8: An Improper Integral on One Side

Determine the value of the improper integral,

$$\int_0^1 \frac{1}{\sqrt{x}}dx. \tag{6.65}$$

Solution: The integrand goes to infinity at $x = 0$, so our improper integral is one sided:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{s\to 0} \int_s^1 \frac{1}{\sqrt{x}}dx = \lim_{s\to 0} 2\sqrt{x}\Big|_s^1 = 2. \tag{6.66}$$

Let's consider an example where the integral of a singular integrand does not converge when taken from each side.

Example 6.9: An Undefined Indefinite Integral

Determine the limits of both,

$$\lim_{r\to 0} \int_1^{2-r} \frac{dx}{x-2} \qquad \text{and} \qquad \lim_{s\to 0} \int_{2+s}^4 \frac{dx}{x-2}. \tag{6.67}$$

Solution: Both limits are infinite, since

$$\begin{aligned}
\lim_{r\to 0} \int_1^{2-r} \frac{dx}{x-2} &= \lim_{r\to 0} \operatorname{Log}|x-2|\Big|_1^{2-r} = \lim_{r\to 0} \operatorname{Log} r - \operatorname{Log} 1 \to -\infty \\
\lim_{r\to 0} \int_{2+s}^4 \frac{dx}{x-2} &= \lim_{s\to 0} \operatorname{Log}|x-2|\Big|_{2+s}^{4} = \lim_{s\to 0} \operatorname{Log} 2 - \operatorname{Log} s \to \infty.
\end{aligned}$$

The value of the improper integral

$$\int_1^4 \frac{dx}{x-2} = \lim_{r\to 0} \int_1^{2-r} \frac{dx}{x-2} + \lim_{s\to 0} \int_{2+s}^4 \frac{dx}{x-2} \tag{6.68}$$

does not exist because it depends on how each of r and s approach 0. We can define the Cauchy Principal Value of this integral by approaching the singularity ***symmetrically***.

The Cauchy Principal Value of an Integral Over a Singularity

Definition 6.2 (The Cauchy Principal Value of an Integral Over a Singularity). *Let $f(x)$ be continuous and finite on $[a,b]$ except at the point $c \in (a,b)$. Then the symmetric limit*

$$\mathcal{P}\int_a^b f(x)dx \equiv \lim_{r\to 0^+}\left[\int_a^{c-r} f(x)dx + \int_{c+r}^b f(x)dx\right] \tag{6.69}$$

is called the **Cauchy Principal Value** *of the integral $\int_a^b f(x)dx$. When the $f(x)$ is continuous on the whole real line except at c, then the integral over $(-\infty,\infty)$ has Cauchy Principal Value*

$$\mathcal{P}\int_{-\infty}^{\infty} f(x)dx \equiv \lim_{\substack{R\to\infty \\ r\to 0^+}}\left[\int_{-R}^{c-r} f(x)dx + \int_{c+r}^R f(x)dx\right]. \tag{6.70}$$

If the integral is over an interval with a finite number of isolated singularities we extend the definition of the principal value in the natural way.

Thus we saw that while the improper integral $\int_1^4 \frac{1}{x-2}dx$ does not exist, its Cauchy Principal Value is

$$\begin{aligned}\mathcal{P}\int_1^4 \frac{dx}{x-2} &= \lim_{r\to 0^+} \operatorname{Log} 2 - \operatorname{Log} r + \operatorname{Log} r - \operatorname{Log} 1 \\ &= \operatorname{Log} 2.\end{aligned}$$

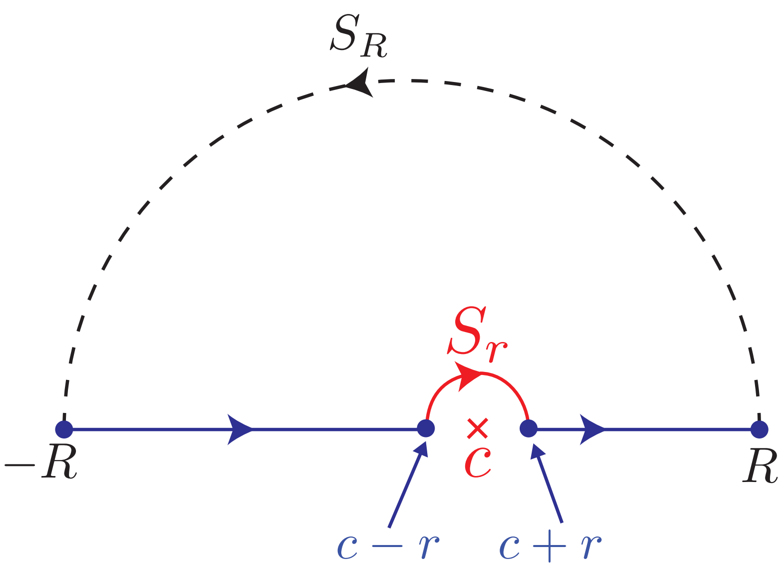


Figure 6.8: To apply Cauchy's Residue theorem to find the integral of a function which is singular at $z = c$, we use the semi-circular contour S_r to close the contour as we approach the singularity at c by taking $z \to 0$.

CONSIDER THE PRINCIPAL VALUE of an integral from $(-\infty,\infty)$, given by Equation (6.70). In order to use a closed contour and Cauchy's Residue Theorem to evaluate the principal values of such integrals we need to ***connect*** the two sides of the principal value at the singularity $x = c$. If we continued along the x-axis, we'd pass through c and the integrand would become undefined. We instead want to detour ***around*** c in a way that depends on r, so that we can take the limit as $r \to 0$. It is often convenient to choose the semi-circle S_r, as shown in Figure 6.10, so that we need to find the value

$$\lim_{r\to 0^+} \int_{S_r} f(z)dz. \tag{6.71}$$

If c is a ***first order pole*** we can determine the value of this integral with the following lemma.

Integral of an Arc Around a Simple Pole

Lemma 6.2 (An Arc Around a Simple Pole)**.** *If function f has a* **simple pole** *at $z = c$ and T_r is the counterclockwise directed circular arc defined by*

$$T_r = \{z = c + re^{i\theta} \,|\theta_1 \leq \theta \leq \theta_2\} \tag{6.72}$$

then

$$\lim_{r\to 0^+} \int_{T_r} f(z)dz = i(\theta_2 - \theta_1)\mathrm{Res}(f; c). \tag{6.73}$$

The value of (6.71) for the ***clockwise directed*** half-circle S_r in Figure 6.10 is given by

$$\lim_{r\to 0^+} \int_{S_r} f(z)dz = -i\pi\,\mathrm{Res}(f; c). \tag{6.74}$$

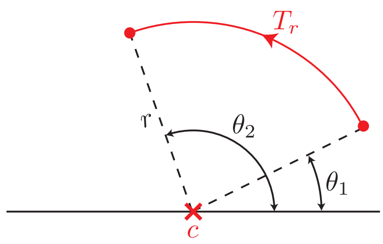


Figure 6.9: Contour of the circular arc for Lemma 6.2.

Proof: Since f has a simple pole at c, its Laurent expansion has the form

$$f(z) = \frac{a_{-1}}{z-c} + \sum_{k=0}^{\infty} a_k(z-c)^k \tag{6.75}$$

valid in some punctured neighbourhood of c, $0 < |z-c| < \rho_1$ where the Laurent expansion converges. For radius r such that $0 < r < \rho_1$, we can then write

$$\int_{T_r} f(z)dz = a_{-1}\int_{T_r} \frac{dz}{z-c} + \int_{T_r} g(z)dz \tag{6.76}$$

where $g(z) \equiv \sum_{k=0}^{\infty} a_k(z-c)^k$ is analytic at c and hence is bounded in some neighbourhood of this point,

$$|g(z)| \leq M, \qquad \text{for } |z-c| < \rho_2. \tag{6.77}$$

Therefore, for $0 < r < \rho_2$, we must have

$$\left|\int_{T_r} g(z)dz\right| \leq M\ell(T_r) = M(\theta_2 - \theta_1)r \tag{6.78}$$

which goes to zero as r goes to zero. This gives us

$$\lim_{r\to 0^+} \int_{T_r} g(z)dz = 0. \tag{6.79}$$

For the remaining term we can parameterise the arc as we did in Example 3.1 to obtain

$$\int_{T_r} \frac{dz}{z-c} = \int_{\theta_1}^{\theta_2} \frac{1}{re^{i\theta}} rie^{i\theta} d\theta = i \int_{\theta_1}^{\theta_2} d\theta = i(\theta_2 - \theta_1). \tag{6.80}$$

which is independent of r. Thus the limit of the integral around the circular arc T_r is given by

$$\lim_{r\to 0^+} \int_{T_r} f(z)dz = a_{-1} i(\theta_2 - \theta_1) = i(\theta_2 - \theta_1)\,\mathrm{Res}(f;c). \quad \square \tag{6.81}$$

Example 6.10: An Indented Contour

Evaluate

$$I = \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx. \tag{6.82}$$

Solution: First we note that the integrand is continuous except at $x = 0$. Hence

$$I = \lim_{\substack{R\to\infty \\ r\to 0^+}} \left(\int_{-R}^{-r} \frac{e^{ix}}{x} + \int_{r}^{R} \frac{e^{ix}}{x} \right). \tag{6.83}$$

We can introduce the complex function $f(z) \equiv e^{iz}/z$ which has a simple pole at the origin, but is elsewhere analytic. To apply Cauchy's Residue Theorem we must form a closed contour containing the segments $[-R, -r]$ and $[r, R]$. Observing that Jordan's Lemma applies to $f(z)$ we can close the contour on the "outside" using the semi-circular contour of radius R, S_R^+, in the upper half plane joining $z = -R$ and $z = R$. To join $z = -r$ to $z = r$ we indent around the origin using the half circle S_r going clockwise around the origin. Since there are no singularities ***inside*** this closed contour, the residue theorem gives us

$$\int_{-R}^{-r} \frac{e^{ix}dx}{x} + \int_{S_r} \frac{e^{iz}dz}{z} + \int_{r}^{R} \frac{e^{ix}dx}{x} + \int_{S_R^+} \frac{e^{iz}dz}{z} = 0 \tag{6.84}$$

Applying Jordan's Lemma, we have

$$\lim_{R\to\infty} \int_{S_R^+} \frac{e^{iz}dz}{z} = 0. \tag{6.85}$$

While Lemma 6.2 provides

$$\lim_{r\to 0^+} \int_{S_r} \frac{e^{iz}dz}{z} = -i\pi\,\mathrm{Res}(0) = -i\pi. \tag{6.86}$$

This gives us the solution

$$I = \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{ix}dx}{x} = i\pi. \quad \square \tag{6.87}$$

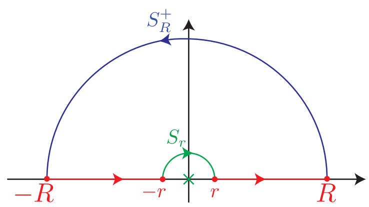


Figure 6.10: Contour for Example 6.10.

Extra: What About Higher Order Poles on the Real Line?

You are probably wondering what happens if instead of a simple pole of the integrand along the real line, there is a higher order pole or essential singularity. This case is more complicated than that of the simple pole since Lemma 6.2 no longer applies. One may be tempted to conclude that such an integral is always undefined, however, this is not always true.
If a function has a Laurent expansion with a term given by

$$a_{-n}(z-c)^{-n}, \tag{6.88}$$

with integer $n \geq 2$ and $a_{-n} \neq 0$, then this term has an anti-derivative

$$\frac{a_{-n}}{1-n}(z-c)^{1-n}. \tag{6.89}$$

The integral of this term along T_r of Figure 6.9 is then given by

$$\begin{aligned} I_n(r) \equiv \int_{T_r} a_{-n}(z-c)^{-n} dz &= \frac{a_{-n}}{1-n} r^{1-n} e^{i(1-n)\theta}\Big|_{\theta_1}^{\theta_2} && (6.90)\\ &= -\frac{a_{-n}}{n-1} r^{-(n-1)}\left[e^{-i(n-1)\theta_2} - e^{-i(n-1)\theta_1}\right] && (6.91)\\ &= -\frac{a_{-n}}{n-1} r^{-(n-1)} e^{-i(n-1)\theta_1}\left[e^{-i(n-1)(\theta_2-\theta_1)} - 1\right] && (6.92) \end{aligned}$$

Here clearly $I_r \to \pm\infty$ as $r \to \infty$ as long as the exponential terms do not cancel out. That is,

$$\lim_{r\to 0^+} I_n(r) = \begin{cases} 0 & \text{if } n \leq 0 \text{ (since this term is analytic and goes to zero by continuity);} \\ i(\theta_2-\theta_1)a_1 & \text{if } n = 1 \text{ (by Lemma 6.9);} \\ 0 & \text{if } n \geq 2 \text{ and } \exists k \in \mathbb{Z} \text{ such that } \theta_2 - \theta_1 = \frac{2k\pi}{n-1}; \\ \pm\infty & \text{otherwise.} \end{cases} \tag{6.93}$$

So we see that for higher order poles we ***usually*** get undefined integrals over these shrinking arcs ***unless*** the angle traversed by the arc happens to be just the right value for the order of the pole. For the clockwise semi-circular arc S_r like we saw in Figure 6.10, where $\theta_2 = 0$ and $\theta_1 = \pi$, we have

$$\lim_{r\to 0^+} I_n(r) = \begin{cases} 0 & \text{if } n \leq 0; \\ -i\pi a_{-1} & \text{if } n = 1; \\ 0 & \text{if } n \text{ is odd and } n > 2; \\ \pm\infty & \text{otherwise.} \end{cases} \tag{6.94}$$

Exercise 6.5: Practice on Indented Contours

Using the calculus of residues, verify each of the following,

(a) $\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x+1} dx = \pi i e^{-2i}$;

(b) $\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x-2)} dx = \pi i(e^{2i} - e^{i})$;

(c) $\int_0^{\infty} \frac{\sin(2x)}{x(x^2+1)^2} dx = \pi\left(\frac{1}{2} - \frac{1}{e^2}\right)$;

(d) $\int_0^{\infty} \frac{\cos x - 1}{x^2} dx = -\frac{\pi}{2}$;

(e) $\mathcal{P}\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)} dx = \frac{\pi}{5}[\cos(1) - e^{-2}]$;

(f) $\mathcal{P}\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-3x+2} dx = \pi[\sin(1) - 2\sin(2)]$.

6.5 Integrals Involving Branch Cuts

While trying to apply residue theory to compute an integral of $f(x)$ it may turn out that the complex function $f(z)$ is ***multi-valued***. If this happens we need to modify our approach by taking into account not only the isolated singularities of $f(z)$, but also the branch points and branch cuts. Since the functions are not analytic (holomorphic) at these branch points and branch cuts, we must carefully select the contours we use to apply Cauchy's Integral and Residue Theorems. Indeed, we will often find it necessary to integrate ***along*** the branch cuts in the following instructive examples.

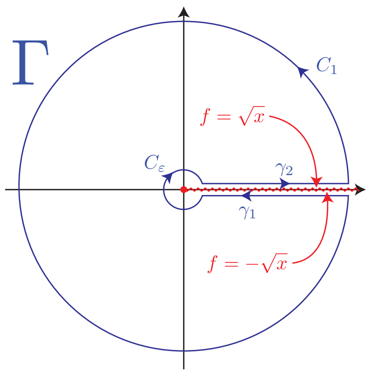


Figure 6.11: To calculate the integral of $\sqrt{z}$ around the unit circle, using the Cauchy Integral Theorem, we need to bypass the branch cut by taking the closed contour Γ. The straight lines γ_1 and γ_2 are drawn separated from the branch cut for clarity, but are to represent evaluations using the lower and upper values of the branch respectively along the real axis.

Example 6.11: Integrating $\sqrt{z}$ on the Unit Circle

Consider the square root function

$$f(z) = \sqrt{z} = \sqrt{r}e^{i\theta/2}, \qquad 0 \le \theta < 2\pi. \tag{6.95}$$

such that it has a branch cut along the positive real axis. Determine the integral of $f(z)$ around the unit circle.

Solution: While there are no poles inside the unit circle, we can't just apply the Cauchy Integral Theorem directly to the unit circle $|z| = 1$ since there is a discontinuity on this contour as we cross the positive real line at $z = 1$. Instead we will construct the contour Γ, such that

$$\oint_\Gamma f(z)dz = \int_{C_1} f(z)dz + \int_{\gamma_1} f(z)dz + \int_{C_\varepsilon} f(z)dz + \int_{\gamma_2} f(z)dz.$$

Note that here the integrals over the straight lines γ_1 and γ_2 ***do not*** cancel out, since along γ_1 the integrand has the value $\sqrt{x}e^{i\pi} = -\sqrt{x}$, while the integrand along γ_2 has the value $\sqrt{x}$.

We see that the integral around the branch point at the origin must be bound by

$$\left|\int_{C_\varepsilon} \sqrt{z}dz\right| \le 2\pi\varepsilon\sqrt{\varepsilon} \tag{6.96}$$

so that if we shrink the radius $\varepsilon \to 0^+$ we find

$$\lim_{r\to 0^+} \int_{C_\varepsilon} \sqrt{z}dz \to 0. \tag{6.97}$$

Since there are no poles inside the contour Γ, Cauchy's Integral Theorem says

$$\oint_\Gamma f(z)dz = 0 \tag{6.98}$$

So that we find

$$\begin{aligned}\int_{C_1} \sqrt{z}dz &= -\int_1^0 (-\sqrt{x})dx - \int_0^1 \sqrt{x}dx \\ &= -2\int_0^1 \sqrt{x}dx = -4/3 \end{aligned} \tag{6.99}$$

Example 6.12: Integrating Along a Branch Cut

Calculate the integral

$$I \equiv \int_0^\infty \frac{dx}{\sqrt{x}(x+4)} \tag{6.100}$$

where $\sqrt{x}$ denotes the principal (positive) value for $x > 0$.
Solution: Including the explicit limits, what we are looking for is

$$I = \lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \int_\varepsilon^R \frac{dx}{\sqrt{x}(x+4)}.$$

We start by considering the branch of $\sqrt{z}$ defined by

$$\sqrt{z} = e^{(\operatorname{Log} r + i\theta)/2} = \sqrt{r}e^{i\theta/2}, \qquad 0 \le 0 < 2\pi.$$

which has a branch cut along the positive real axis. With this choice of $\sqrt{z}$ we consider the complex function

$$f(z) \equiv \frac{1}{\sqrt{z}(z+4)}.$$

Then according to our choice of branch, on the upper side of the branch cut, we have $f(x) = 1/[\sqrt{x}(x+r)]$ as desired. Now, we would like to be able to use Cauchy's Residue Theorem to evaluate this, so we need to find a ***closed*** contour containing the positive real line, taking into account the branch point (and pole) at the origin as well as the pole at $z = -4$. We consider the closed contour of Figure 6.12, where ε is small enough and R large enough so that the pole at -4 lies inside the contour. Then we have

$$\int_\Gamma = \left(\int_{C_R} + \int_{\gamma_1} + \int_{C_\varepsilon} + \int_{\gamma_2}\right) f(z)dz = 2\pi i \operatorname{Res}(f; -4)$$

As we let the radius $R \to \infty$, we see

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{C_R} f(z)dz\right| &\le \lim_{R\to\infty}\int_{C_R}\frac{dz}{|\sqrt{z}||z+4|} \\
&\le \lim_{R\to\infty}\int_{C_R}\frac{dz}{\sqrt{R}(R-4)} \\
&\le \lim_{R\to\infty}\frac{2\pi R}{\sqrt{R}(R-4)} = 0.
\end{aligned}$$

As we let the inner radius $\varepsilon \to 0^+$ we find,

$$\lim_{\varepsilon\to 0^+}\left|\int_{C_\varepsilon} f(z)dz\right| \le \lim_{\varepsilon\to 0^+}\frac{2\pi\varepsilon}{\sqrt{\varepsilon}(4-\varepsilon)} = 0.$$

Thus all that remains are the integrals along the top and bottom of the branch cut. On the top of the branch cut the integrand takes the value

$$f(x) = \frac{1}{\sqrt{x}(x+4)}$$

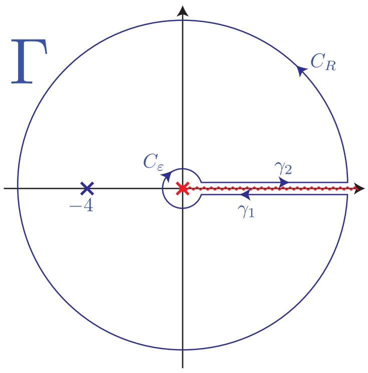


Figure 6.12: Contour and poles for Example 6.12.

while on the bottom of the branch cut, the integrand takes the value

$$f(x) = -\frac{1}{\sqrt{x}(x+4)}$$

Thus we find

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} \left[\int_R^{\varepsilon} \frac{-dx}{\sqrt{x}(x+4)} + \int_{\varepsilon}^{R} \frac{dx}{\sqrt{x}(x+4)}\right] = 2\pi i \operatorname{Res}(f;-4)$$

$$\lim_{\substack{R\to\infty \\ \varepsilon\to 0^+}} 2\int_{\varepsilon}^{R} \frac{dx}{\sqrt{x}(x+4)} = 2\pi i \operatorname{Res}(f;-4)$$

$$I = \pi i \operatorname{Res}(f;-4) = \frac{\pi}{2}. \quad \square$$

Example 6.13: A Pole Along the Branch Cut

Determine the Cauchy Principal Value

$$I \equiv \mathcal{P}\int_0^{\infty} \frac{dx}{x^{\lambda}(x-4)}, \qquad \text{where } 0 < \lambda < 1. \tag{6.101}$$

Solution: Here we have a singularity at $x = +4$, which lies ***on the interval of integration***, and we should also take the positive real branch of x^{λ} in the denominator. The integral can then be recast as the limit

$$I = \lim_{\substack{R\to\infty \\ \varepsilon,\delta\to 0^+}} \left(\int_{\varepsilon}^{4-\delta} + \int_{4+\delta}^{R}\right) \frac{dx}{x^{\lambda}(x-4)}. \tag{6.102}$$

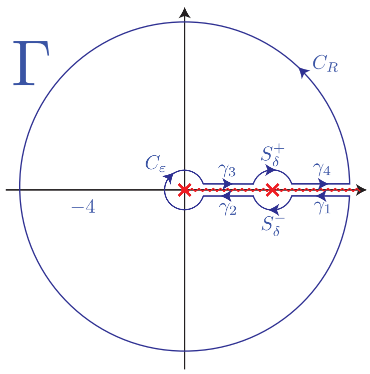


Figure 6.13: Poles and Contour for Example 6.13.

To evaluate this we must draw the contour as we did in the previous example around the branch, but also ***indent*** around the singularity at $z = 4$. We choose the branch

$$f(z) = \frac{1}{e^{\lambda(\operatorname{Log} r + i\theta)}(re^{i\theta}-4)}, \qquad \text{for } z = re^{i\theta}, \qquad 0 \le \theta < 2\pi.$$

and form the contour Γ shown in Figure 6.13. From Cauchy's Integral Theorem we then have,

$$\begin{aligned} 0 &= \oint_{\Gamma} f(z)dz \\ &= \left[\int_{C_R} + \int_{\gamma_1} + \int_{S_\delta^-} + \int_{\gamma_2} \right. \\ &\qquad \left. + \int_{C_\varepsilon} + \int_{\gamma_3} + \int_{S_\delta^+} + \int_{\gamma_4}\right] f(z)dz \end{aligned} \tag{6.103}$$

since $f(z)$ has no singularities in the interior of the closed contour. We can gather the contour integrals of γ_1 with γ_4 and γ_2 with γ_3, by noting that the integrands differ by a factor of $e^{-2\pi i\lambda}$ between the top and bottom of the branch cut yielding,

$$0 = (1 - e^{-2\pi i \lambda}) \left[\int_{\varepsilon}^{4-\delta} + \int_{4+\delta}^{R} \right] \frac{dx}{x^{\lambda}(x-4)} + \left[\int_{C_R} + \int_{S_\delta^-} + \int_{C_\varepsilon} + \int_{S_\delta^+} \right] f(z)dz \tag{6.104}$$

So that the solution is given by

$$\begin{aligned} I &= \mathcal{P} \int_0^{\infty} \frac{dx}{x^{\lambda}(x-4)} \\ &= \lim_{\substack{R\to\infty \\ \varepsilon,\delta\to 0^+}} \frac{-1}{1-e^{-2\pi i\lambda}} \left[\int_{C_R} + \int_{S_\delta^-} + \int_{C_\varepsilon} + \int_{S_\delta^+} \right] f(z)dz \end{aligned} \tag{6.105}$$

For $0 < \lambda < 1$ it is straightforward to repeat the estimates from Example 6.12, to find

$$\lim_{\varepsilon\to 0^+} \int_{C_\varepsilon} f(z)dz = 0, \qquad \text{and} \qquad \lim_{R\to\infty} \int_{C_R} f(z)dz = 0. \tag{6.106}$$

Along the upper half circle S_δ^+, function f agrees with the principal value

$$f_+(z) \equiv \frac{1}{e^{\lambda \operatorname{Log} z}(z-4)} \tag{6.107}$$

while along the lower half circle S_δ^-, the function f takes the value

$$f_-(z) = e^{-2\pi i\lambda} f_+(z) \tag{6.108}$$

The residues of interest are

$$\operatorname{Res}(f_+;4) = \lim_{z\to 4} e^{-\lambda \operatorname{Log} z} = 4^{-\lambda} \tag{6.109}$$

$$\operatorname{Res}(f_-;4) = 4^{-\lambda} e^{-2\pi i\lambda} \tag{6.110}$$

Since $z = 4$ is a simple pole of f_+ and f_-, Lemma 6.2 tells us

$$\int_{S_\delta^+} f(z)dz = -\pi i \operatorname{Res}(f_+;4) = -i\pi 4^{-\lambda} \tag{6.111}$$

$$\int_{S_\delta^-} f(z)dz = -\pi i \operatorname{Res}(f_-;4) = -i\pi 4^{-\lambda} e^{-2\pi i\lambda} \tag{6.112}$$

which are independent of δ.
Thus we have

$$\begin{aligned} I &= i\pi 4^{-\lambda} \frac{1+e^{-2\pi i\lambda}}{1-e^{-2\pi i\lambda}} \\ &= \pi 4^{-\lambda} i \frac{e^{i\pi\lambda}+e^{-i\pi\lambda}}{e^{i\pi\lambda}-e^{-i\pi\lambda}} \\ &= \pi 4^{-\lambda} \cot(\pi\lambda). \quad \square \end{aligned} \tag{6.113}$$

Exercise 6.6: Practice on Integrals with Branch Cuts

Using the calculus of residues, verify each of the following,

(a) $\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}}$;

(b) $\int_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx = \frac{\pi}{\sin(\pi\alpha)}$,
for $0 < \alpha < 1$;

(c) $\int_0^\infty \frac{x^{\alpha}}{(x+9)^2}\,dx = \frac{9^{\alpha-1}\pi\alpha}{\sin(\pi\alpha)}$,
for $-1 < \alpha < 1$, $\alpha \neq 0$;

(d) $\int_0^\infty \frac{x^{\alpha}}{(x+1)^2}\,dx = \frac{\pi(1-\alpha)}{4\cos(\alpha\pi/2)}$,
for $-1 < \alpha < 3$, $\alpha \neq 1$;

(e) $\int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\cos\left(\frac{2\alpha\pi+\pi}{6}\right)\csc(\alpha\pi)$,
for $0 < \alpha < 2, \alpha \neq 1$;

(f) $\mathcal{P}\int_0^\infty \frac{x^{\alpha}}{x^2-1}\,dx = \frac{\pi}{2\sin(\pi\alpha)}[1-\cos(\pi\alpha)]$,
for $-1 < \alpha < 1$, $\alpha \neq 0$.