# Topics in Cylindrical Algebraic Decomposition

**Corin Lee**

Department of Mathematical Sciences
University of Bath

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# Overview

- An introduction to Cylindrical Algebraic Decomposition:
  - Definitions, applications and construction algorithms.
- Highlight of current work enhancing CAD in computer algebra systems:
  - Comparing and analysing construction methods, and tailoring algorithms for specific cases.
  - Implementations and tools to spread and enhance understanding.

# Cylindrical Algebraic Decomposition

First presented in [Col75] as an algorithm to tackle real QE problems (taking a formula with quantifiers $\forall, \exists$ and obtaining an equivalent one without quantifiers).

Applications in various fields: robotics [Man+12], economics [MDE18] and biology [RS21].

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Examples of invariance include:

- *sign-invariant*: input polynomials have constant sign in each cell i.e. $+$, - or 0,
- *truth-invariant*: sign-invariant cells are marked true/false according to *Tarski formula* $\mathcal{T}_n$: boolean combinations of these $f_i$ with sign conditions $(<, >, =)$ and quantifiers $(\forall, \exists)$, e.g. $\exists x\ (f_1 > 0) \wedge (f_2 > 0)$.

# Cylindrical Algebraic Decomposition

- **Decomposition** of $\mathbb{R}^n$ into finitely many *cells* $C_i$ where $\bigcup C_i = \mathbb{R}^n$, $C_i \cap C_j = \emptyset$ if $i \neq j$,

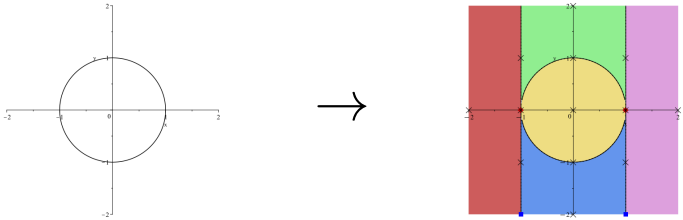

Figure: The graph of $f = x^2 + y^2 - 1$ and an associated sign-invariant CAD of $\mathbb{R}^2$.

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- **Decomposition** of $\mathbb{R}^n$ into finitely many *cells* $C_i$ where $\bigcup C_i = \mathbb{R}^n$, $C_i \cap C_j = \emptyset$ if $i \neq j$,
  - and for $0 \leq j \leq n$, a *j-cell* in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is homeomorphic to $\mathbb{R}^j$.

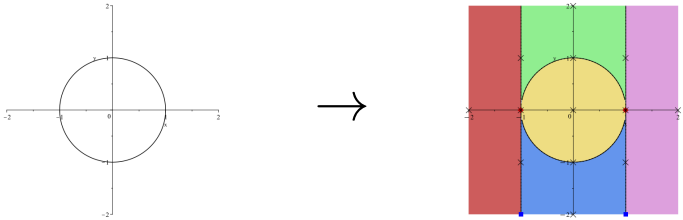

Figure: This CAD is made up of 13 cells: five 2-cells, six 1-cells and two 0-cells. The black crosses represent the sample points for each cell.

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- **Cylindrical:** The *projections* of any two cells into lower-dimensional space are either equal or disjoint (they 'stack up' over lower-dimensional cells).

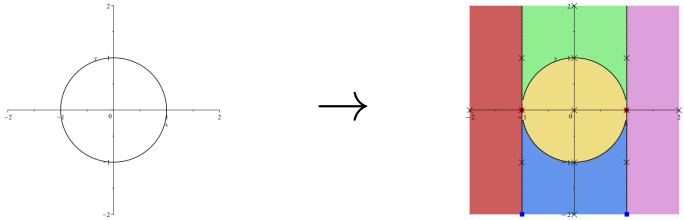

Figure: These cells stack in *cylinders* over the 5 cells of the CAD of $\mathbb{R}^1$ (the two points at the bottom and the regions between them).

# Cylindrical Algebraic Decomposition

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- **Cylindrical:** The *projections* of any two cells into lower-dimensional space are either equal or disjoint (they 'stack up' over lower-dimensional cells).
- **Algebraic:** Each cell can be described by a finite set of polynomial equations and inequalities.

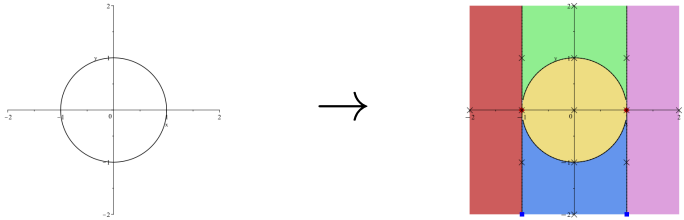

Figure: Each cell is described by constraints on $x$ and $y$.

# Projection and Lifting

**Projection Phase:** Apply a *projection operator* Proj to $\mathcal{F}_n$ to obtain a new set $\mathcal{F}_{n-1}$ in $n-1$ variables. Repeat this until one obtains $\mathcal{F}_1$.

$$
\begin{array}{c}
\mathcal{F}_n \\
\downarrow \text{Proj} \\
\mathcal{F}_{n-1} \\
\downarrow \text{Proj} \\
\vdots \\
\downarrow \text{Proj} \\
\mathcal{F}_2 \\
\downarrow \text{Proj} \\
\mathcal{F}_1
\end{array}
$$

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**Projection Phase:** Apply a *projection operator* Proj to $\mathcal{F}_n$ to obtain a new set $\mathcal{F}_{n-1}$ in $n-1$ variables. Repeat this until one obtains $\mathcal{F}_1$.

**Base Phase:** Decompose $\mathbb{R}^1$ into the roots of each polynomial in $\mathcal{F}_1$ and the open intervals either side of them to obtain CAD($\mathcal{F}_1$).

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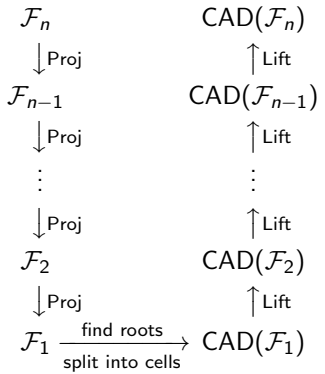
\mathcal{F}_n \\
\quad \downarrow \text{Proj} \\
\mathcal{F}_{n-1} \\
\quad \downarrow \text{Proj} \\
\quad \vdots \\
\quad \downarrow \text{Proj} \\
\mathcal{F}_2 \\
\quad \downarrow \text{Proj} \\
\mathcal{F}_1 \xrightarrow[\text{split into cells}]{\text{find roots}}
\end{array}
$$

# Projection and Lifting

**Projection Phase:** Apply a *projection operator* Proj to $\mathcal{F}_n$ to obtain a new set $\mathcal{F}_{n-1}$ in $n-1$ variables. Repeat this until one obtains $\mathcal{F}_1$.

**Base Phase:** Decompose $\mathbb{R}^1$ into the roots of each polynomial in $\mathcal{F}_1$ and the open intervals either side of them to obtain $\mathrm{CAD}(\mathcal{F}_1)$.

**Lifting Phase:** For each cell $C$ of $\mathrm{CAD}(\mathcal{F}_1)$, substitute a *sample point* into $\mathcal{F}_2$ to obtain a new set of univariate polynomials. Decompose these to obtain a set of cells that lie above $C$, and the full collection of these cells for every $C$ forms $\mathrm{CAD}(\mathcal{F}_2)$. Repeat this process until one obtains $\mathrm{CAD}(\mathcal{F}_n)$.



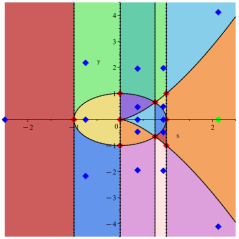
# Current work

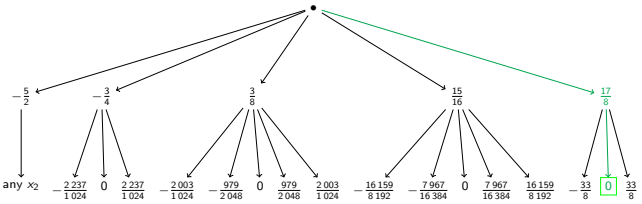
- Tools and implementations:
  - Open CAD in *Macaulay2*
  - CAD graphing tool in *Maple*
- Focus on a specific type of input polynomials (multilinearity)
  - Definition
  - Customised construction algorithm
- Competing construction algorithms (PL-CAD vs RC-CAD)
  - Timings of implementations
  - Complexity analysis of RC-CAD

# The CADecomposition Package for *Macaulay2*

- First implementation of CAD in *M2*.
- Produces tree of sample points in $\mathbb{Q}^n$ representing the open cells of a CAD of $\mathbb{R}^n$ for input polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$.
- Able to: return full tree or branches, check these for points where all polynomials are positive, step through the algorithm.
- Improves/fixes `RealRoots`' real root isolation.

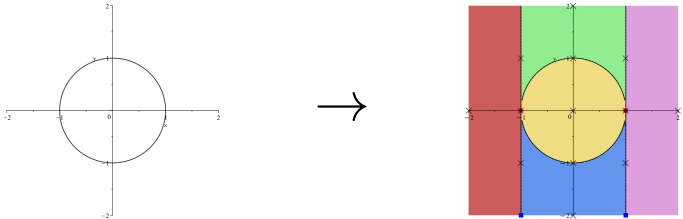
**Output:** $\mathcal{S}_n$, a tree of sample points in $n$ variables which represent each *open cell* of CAD($\mathcal{F}_n$).





# CAD Graphing in *Maple*

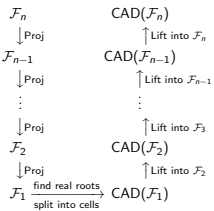
- Production of commands that graph a CAD of $\mathbb{R}^2$.
- Currently takes output of *QuantifierElimination*'s *CylindricalAlgebraicDecompose*, returning 2-cells as coloured regions, 1-cells as black lines and 0-cells as red points.
- Also returns sample points and 1d projection.
- Future aims are to produce a 1d/2d/3d projection of any higher-dimension CAD, take output from *RegularChains* package, and offer truth-invariant CAD colouring.



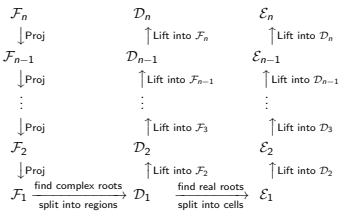
# Multilinearity

- Property where for $\mathcal{F}_n = \{f_1, \ldots, f_r\}$ in variables $x_1, \ldots, x_n$, we have some $1 \leq r \leq n$ variables of maximum degree 1 in any $f_i$.
- Such variables will be simpler to project, not requiring certain parts of the projection operator and producing a smaller number of cells - so 'projecting away' these variables first should produce a 'nicer' CAD,
- Such polynomials often arise in practical applications, so optimising for these is worthwhile.

# CAD Construction Algorithms




**PL-CAD**




**RC-CAD**

**Projection:** Apply a *projection operator* Proj to $\mathcal{F}_n$ to obtain a new set $\mathcal{F}_{n-1}$ in $n-1$ variables. Repeat this until one obtains $\mathcal{F}_1$.

**Construct CD $\mathcal{D}_n$:** Decompose $\mathbb{C}^1$ into $r+1$ regions, the sets of roots of the $r$ polynomials of $\mathcal{F}_1$ and the set of all other points. Substitute sample points of each region into $\mathcal{F}$ to obtain a new set of univariate polynomials and decompose again. The full collection is $\mathcal{D}_2$. Repeat to $\mathcal{D}_n$.

**Construct CAD $\mathcal{E}_n$:** Decompose $\mathbb{R}^1$ into the roots of each polynomial in $\mathcal{D}_1$ and the open intervals either side of them to obtain $\mathcal{E}_1$. Substitute sample points of each cell of $\mathcal{E}_1$ into every region of $\mathcal{D}_2$ above it to obtain a new set of univ. polynomials, decompose again. Full collection is $\mathcal{E}_2$. Repeat to $\mathcal{E}_n$.

# PL-CAD vs RC-CAD

- RC-CAD has been implemented in *Maple* via the *RegularChains* package.
- Papers on RC-CAD are promising, showing RC-CAD often performing better than implementations of PL-CAD, and producing fewer cells.
- Despite this, it has not been extensively compared, or run against the newest implementation of RC-CAD in *Maple*.
- Preliminary timings have been run, but aim is to find a fair way to compare both algorithms and attempt to understand when one algorithm may work better than another.
- Future aim is also to perform complexity analysis on RC-CAD in order to identify conditions on which it will run faster.

# References I


[Col75] George E. Collins. "Quantifier elimination for real closed fields by cylindrical algebraic decomposition". In: *Lecture Notes in Computer Science* (1975).

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[MDE18] Casey B. Mulligan, James H. Davenport, and Matthew England. "TheoryGuru: A Mathematica Package to Apply Quantifier Elimination Technology to Economics". In: *Mathematical Software – ICMS 2018*. Ed. by James H. Davenport et al. Springer International Publishing, 2018.

[RS21] Gergely Röst and AmirHosein Sadeghimanesh. "Exotic Bifurcations in Three Connected Populations with Allee Effect". In: *International Journal of Bifurcation and Chaos* 31.13 (2021).

# The End

Figure: https://github.com/cel34-bath/CAD_M2_ICMS_2024