December 9th 2015

ALEA in Europe Young Researchers' Workshop

A STROLL AROUND RANDOM INFINITE QUADRANGULATIONS OF THE PLANE

Alessandra Caraceni

“

What does a large random **planar map** look like?

G. Miermont L. Addario Berry J. Bouttier

J. Bettinelli O. Schramm

J.-F. Le Gall

W. T. Tutte B. Haas

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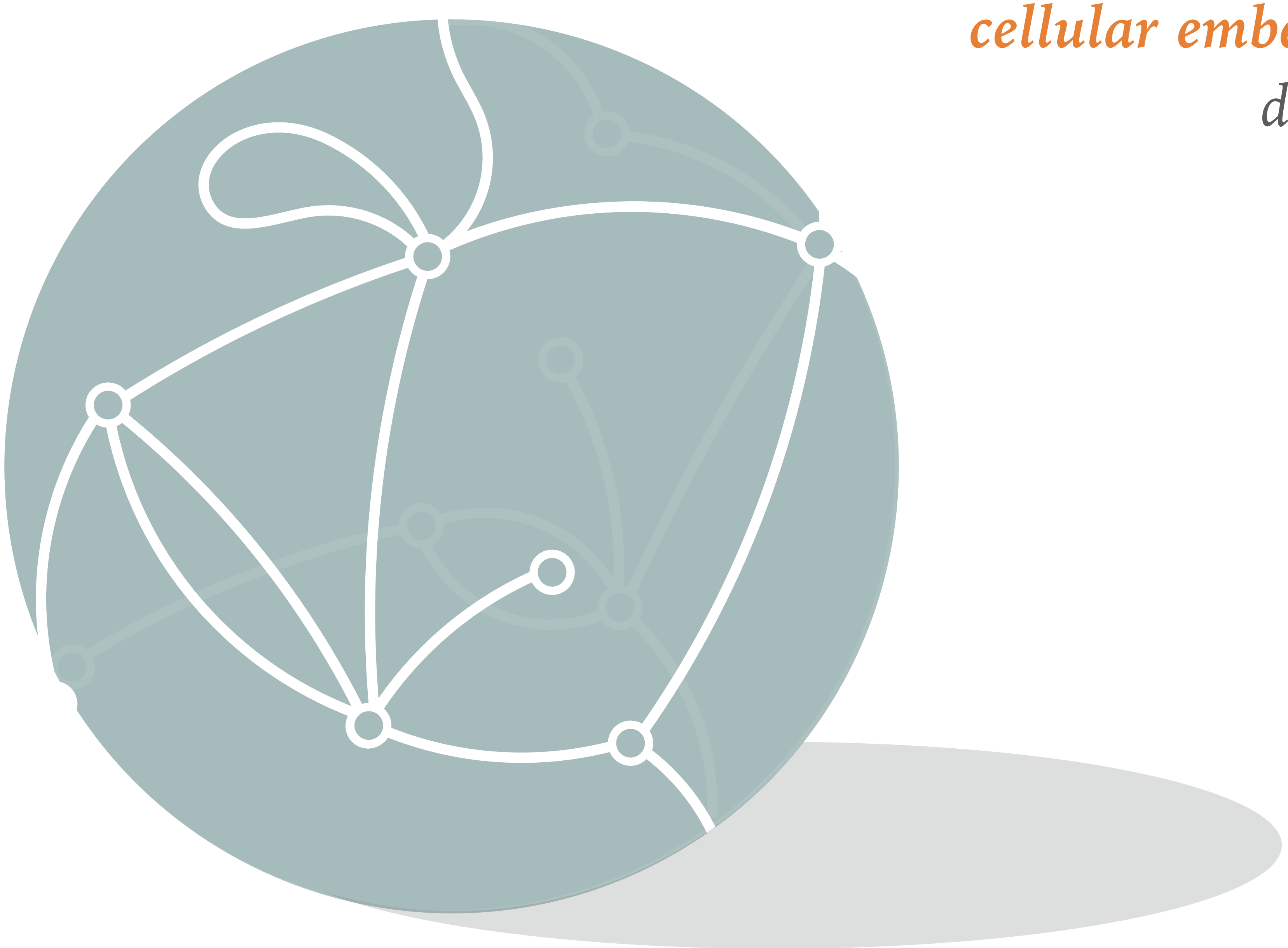

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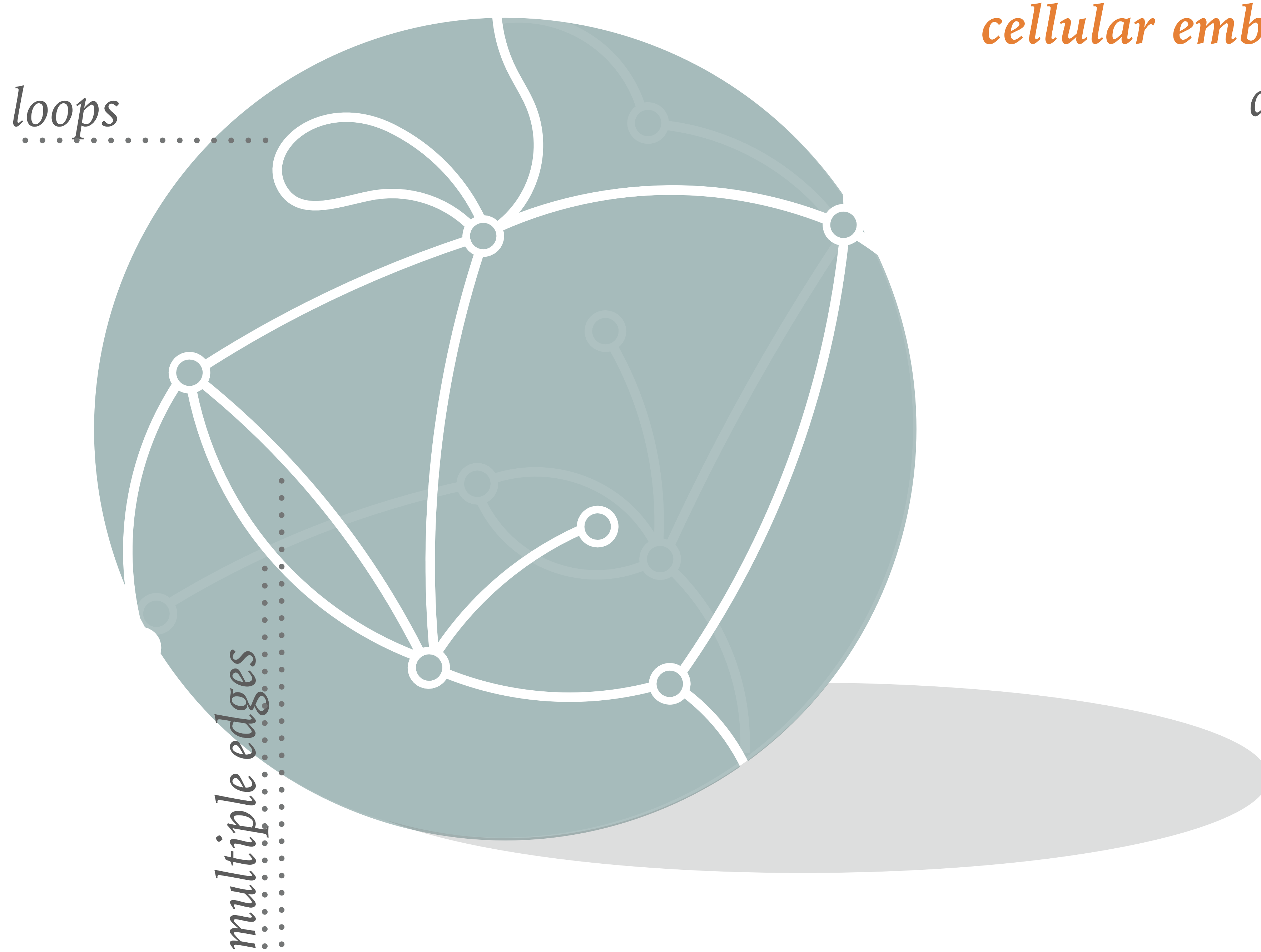

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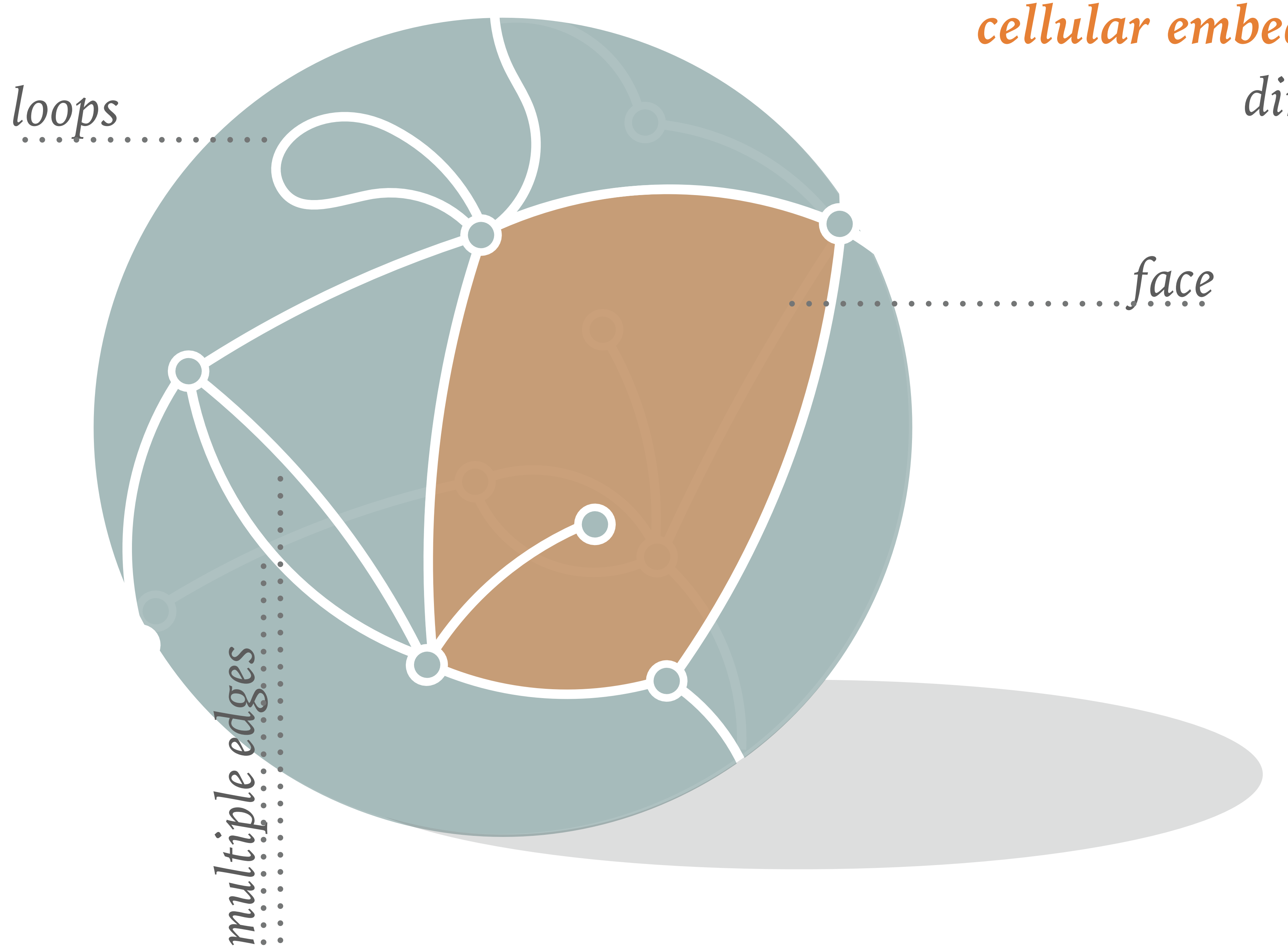

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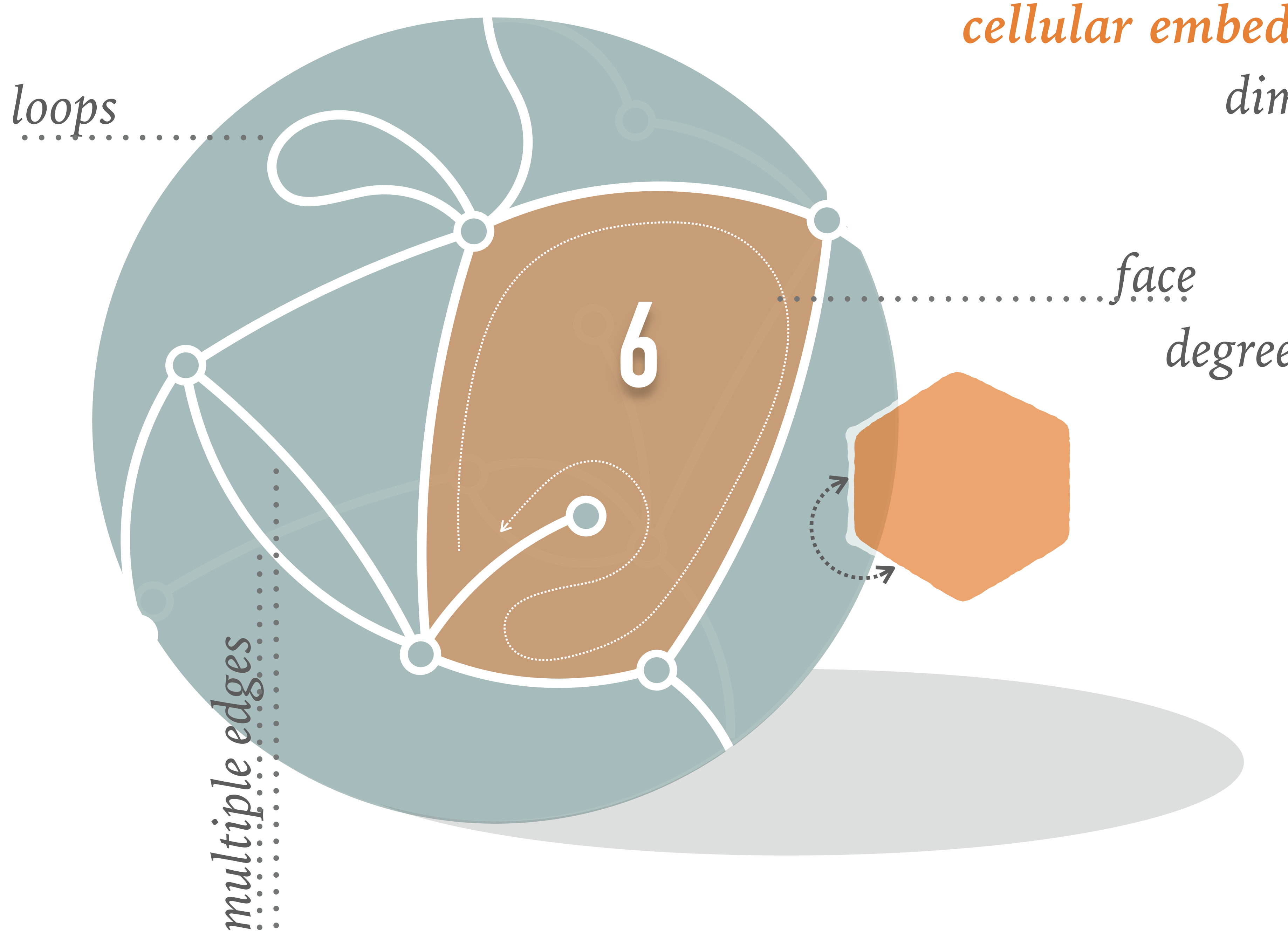

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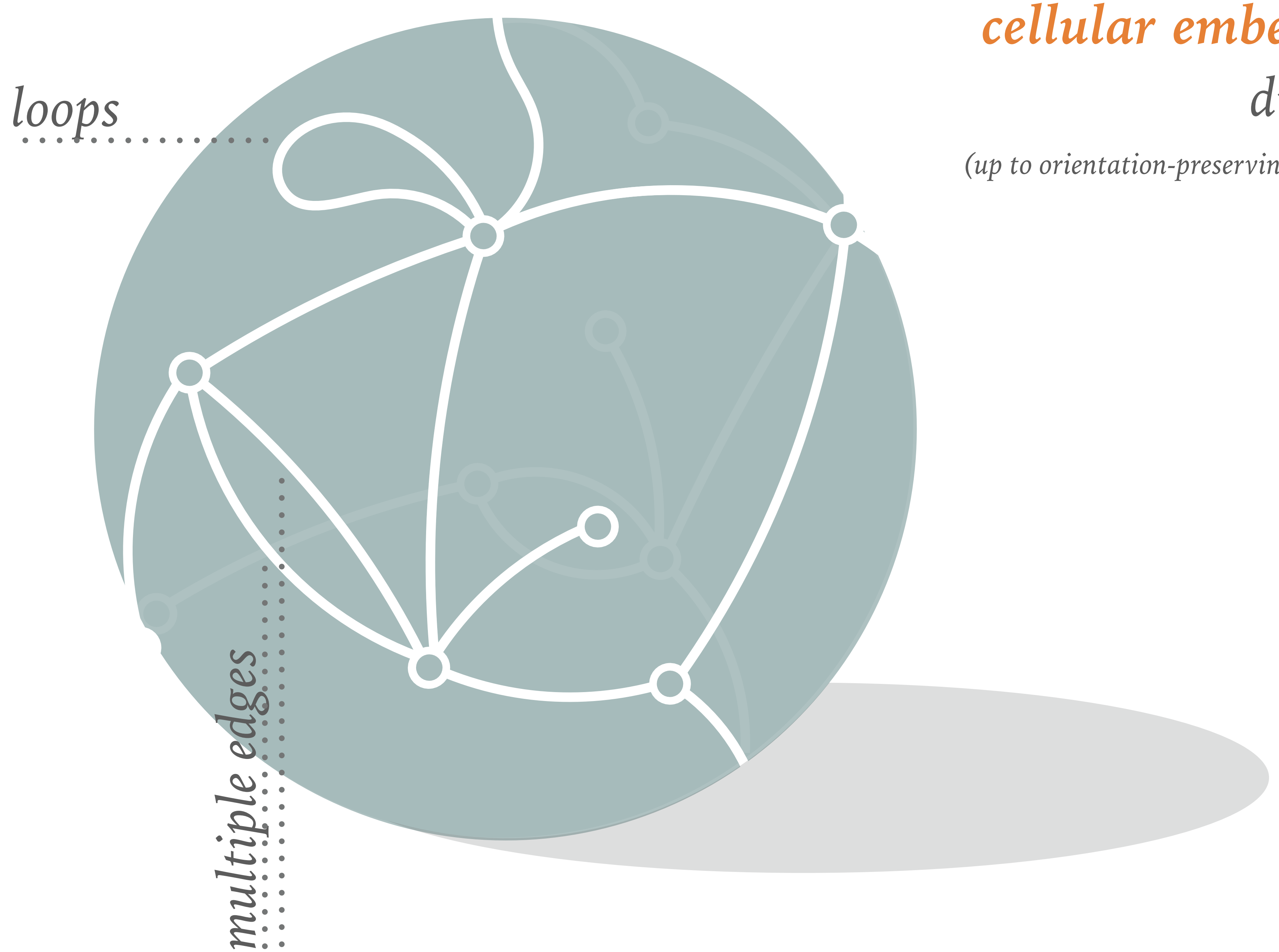

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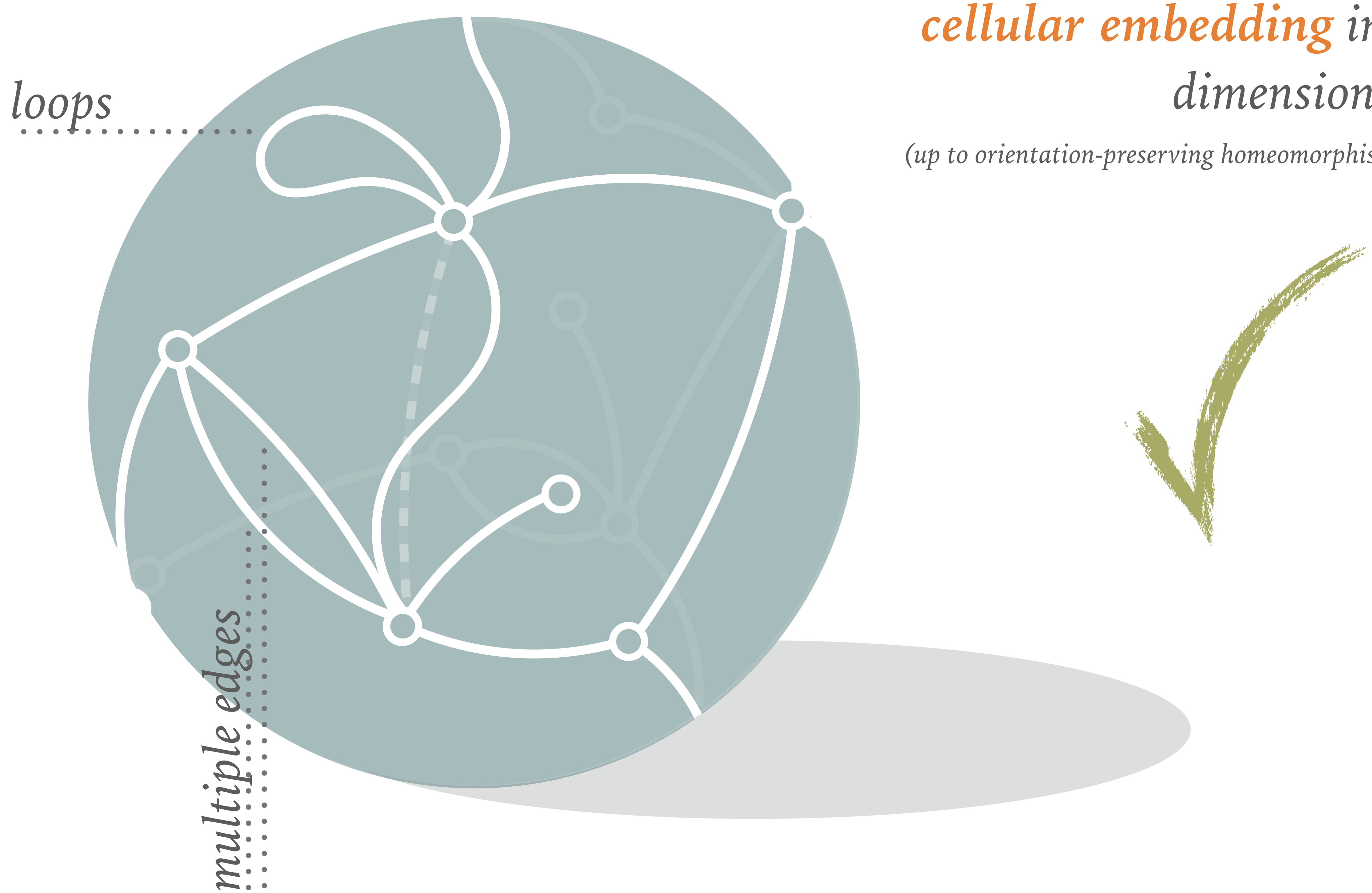

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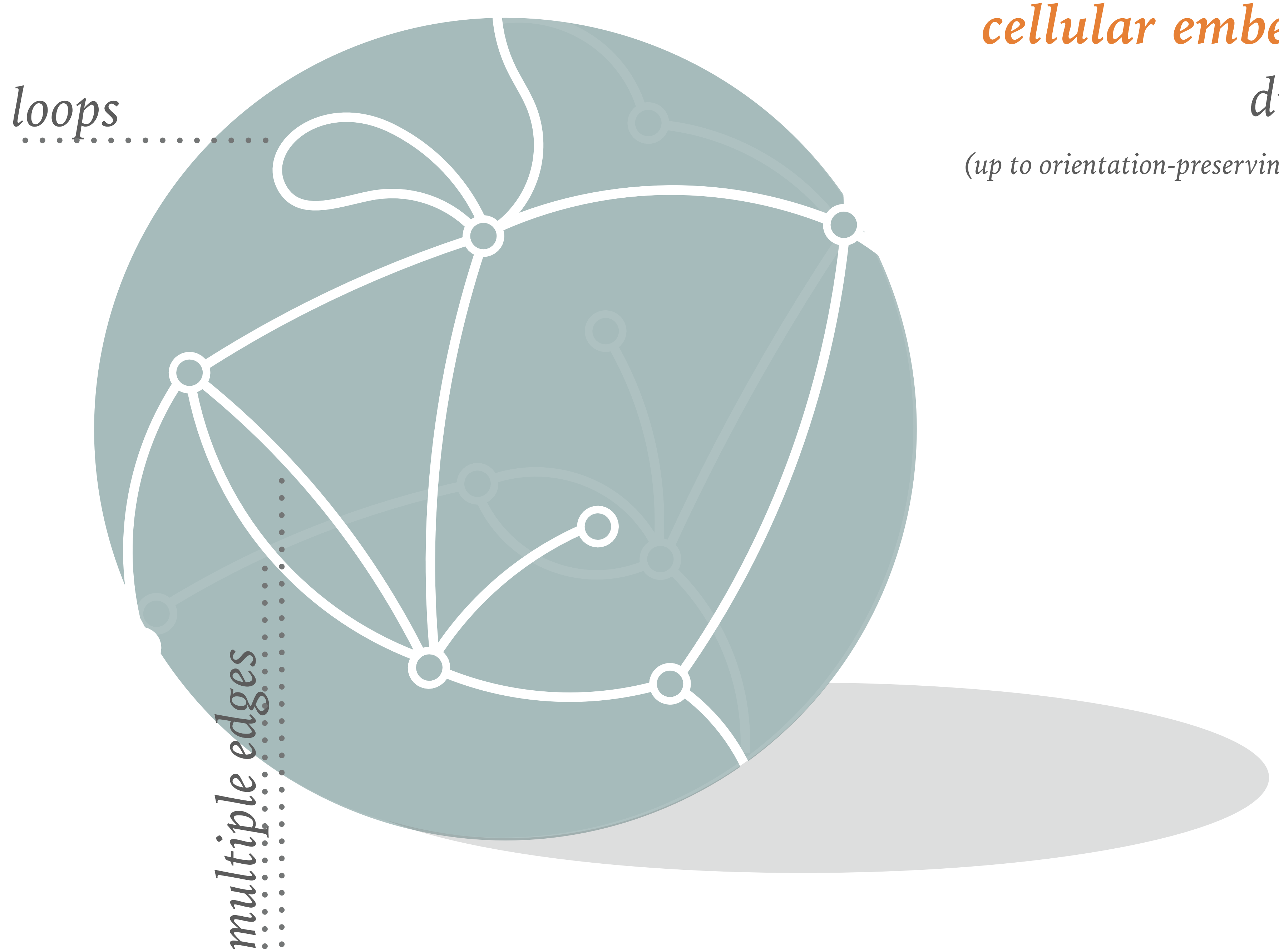

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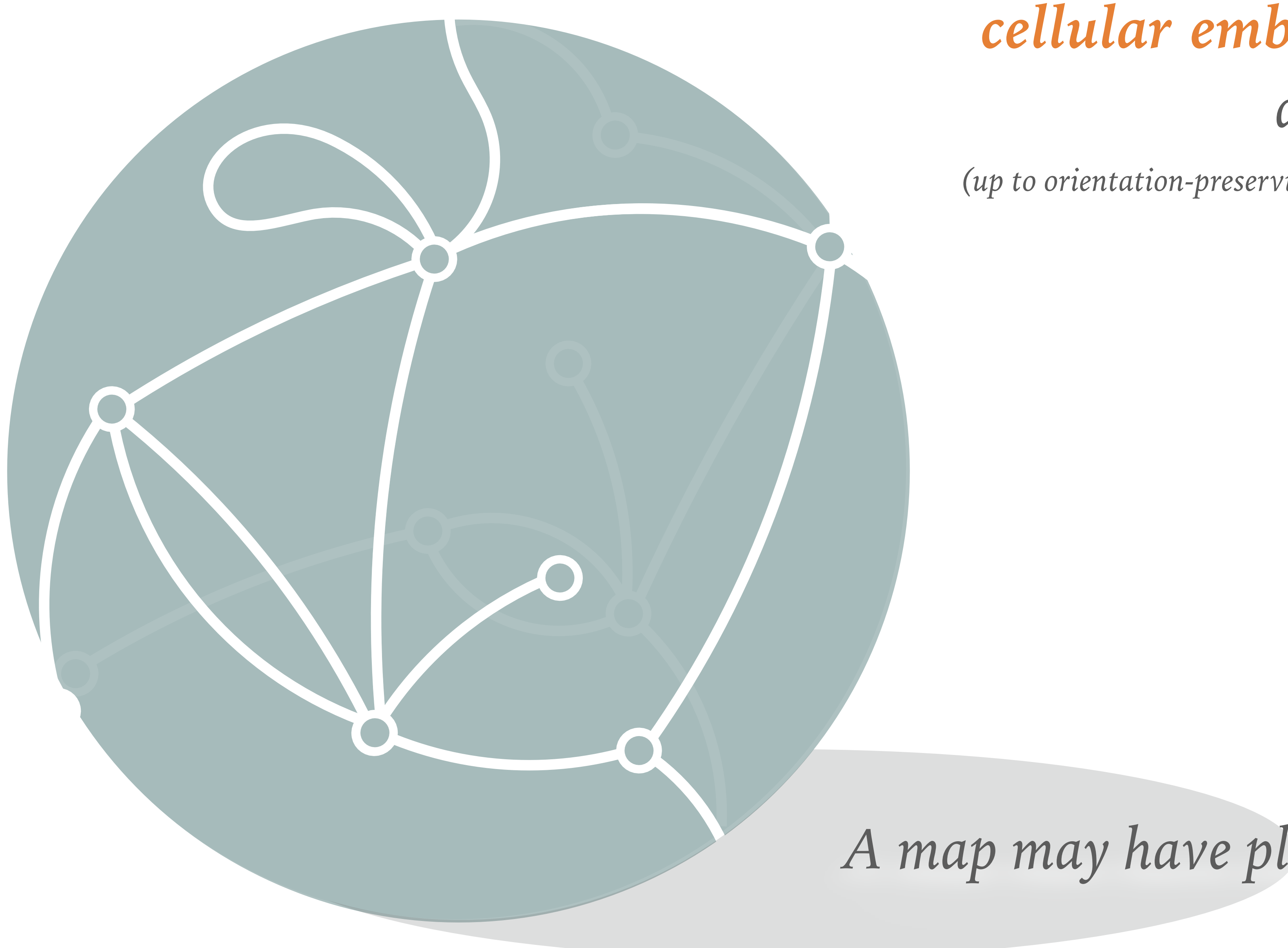

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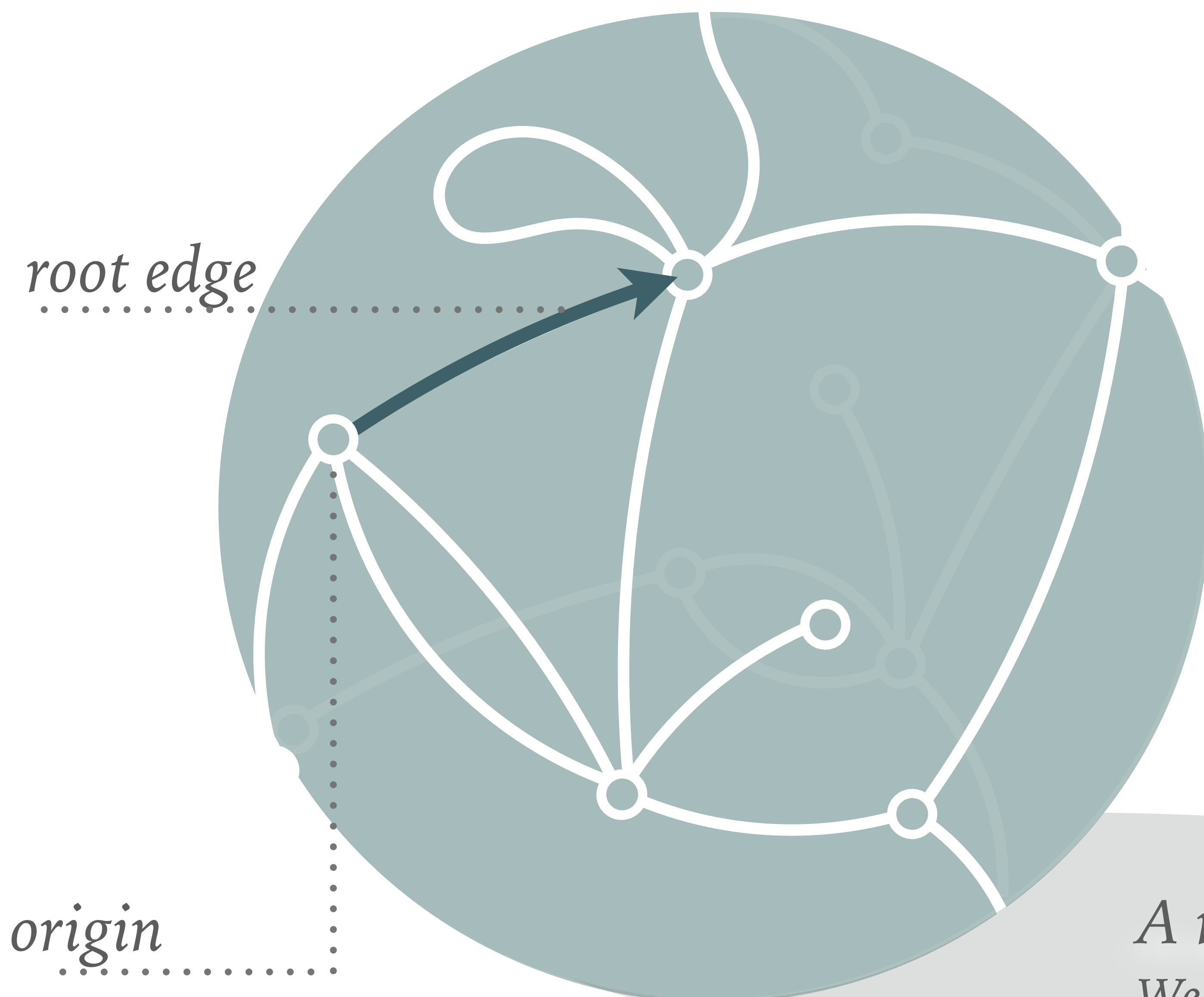

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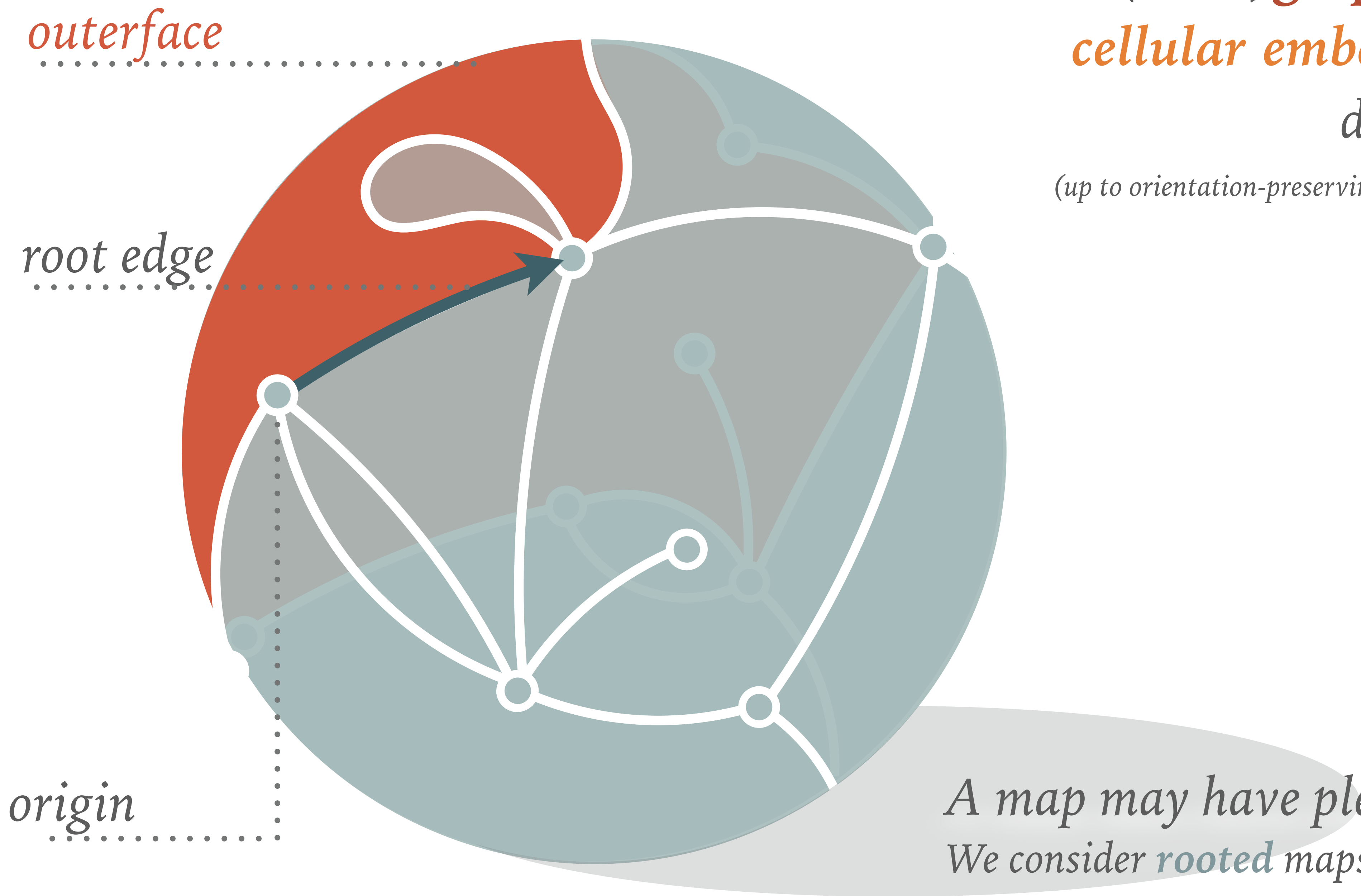

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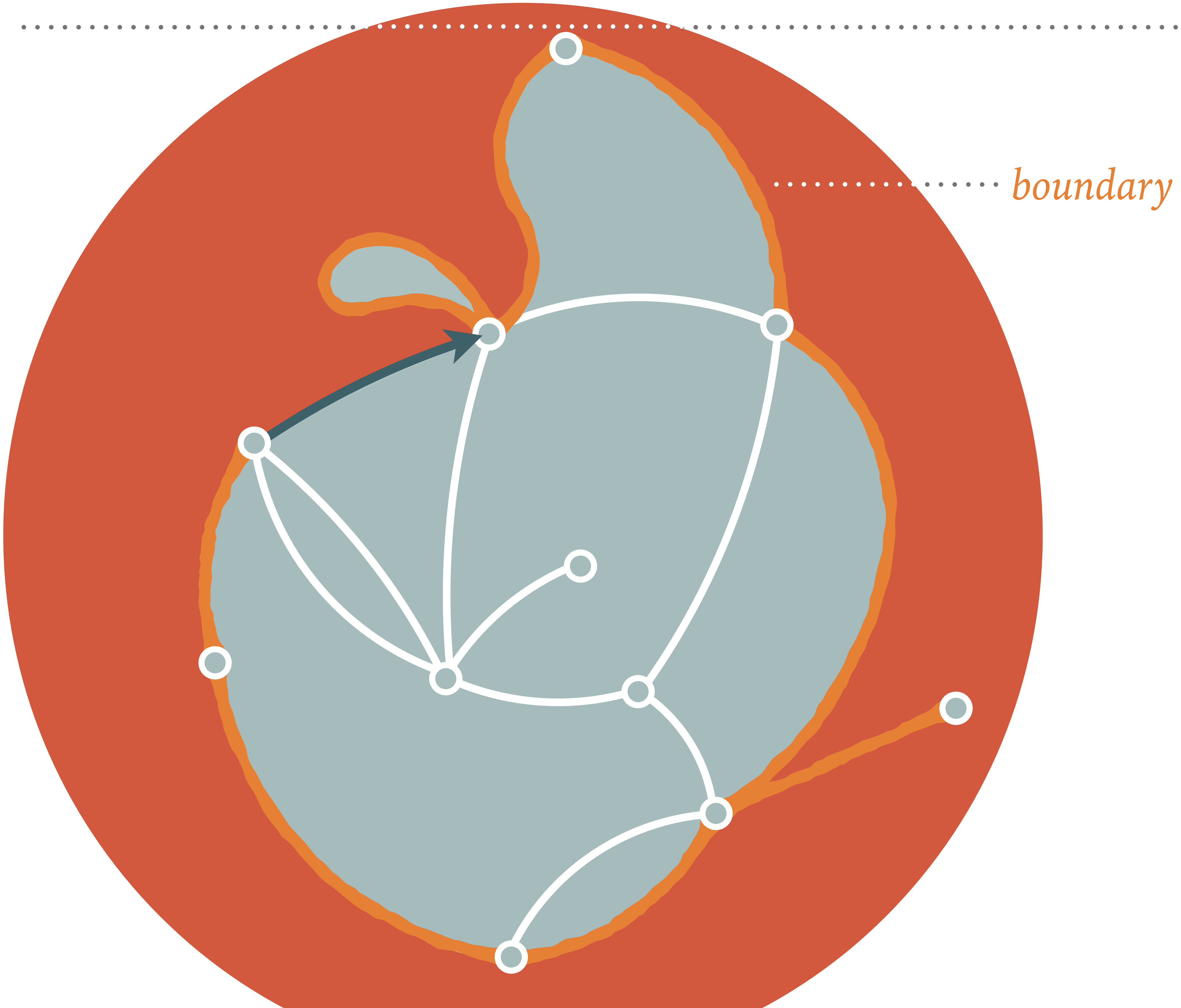

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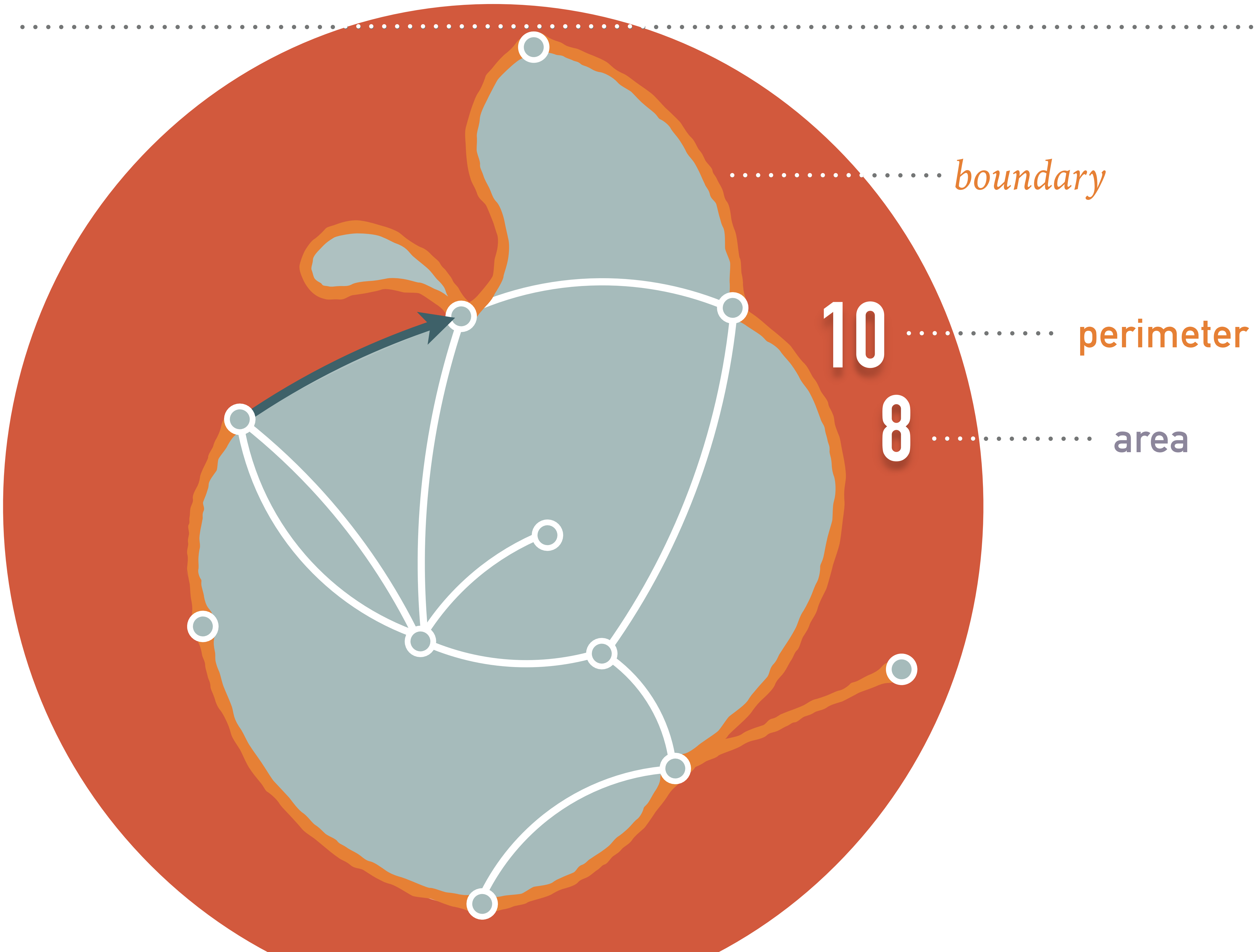

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a quadrangulation

only faces of degree p

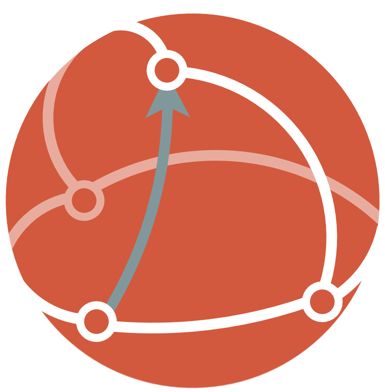
BIPARTITE

simple

unrooted

ROOTED

pointed



a rooted simple triangulation

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simple boundary

only faces of degree p

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fixed perimeter

BIPARTITE

simple

unrooted

dissection

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pointed

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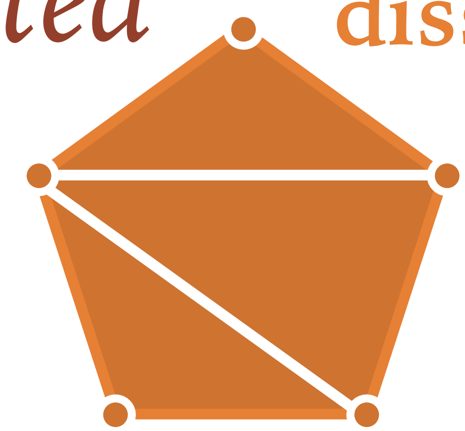

a triangulation of the pentagon

WHAT DOES A LARGE RANDOM PLANAR MAP LOOK LIKE?

RANDOM = *Usually uniformly sampled within an "interesting" class of planar maps.*

SIZE # faces # edges # vertices

LARGE = *of* **size** *approaching infinity*

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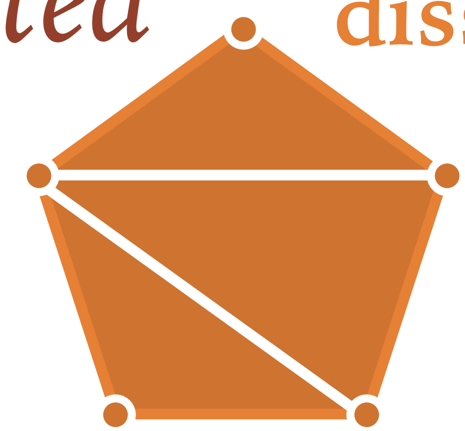

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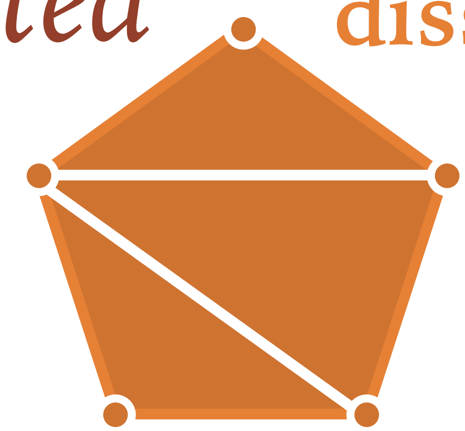

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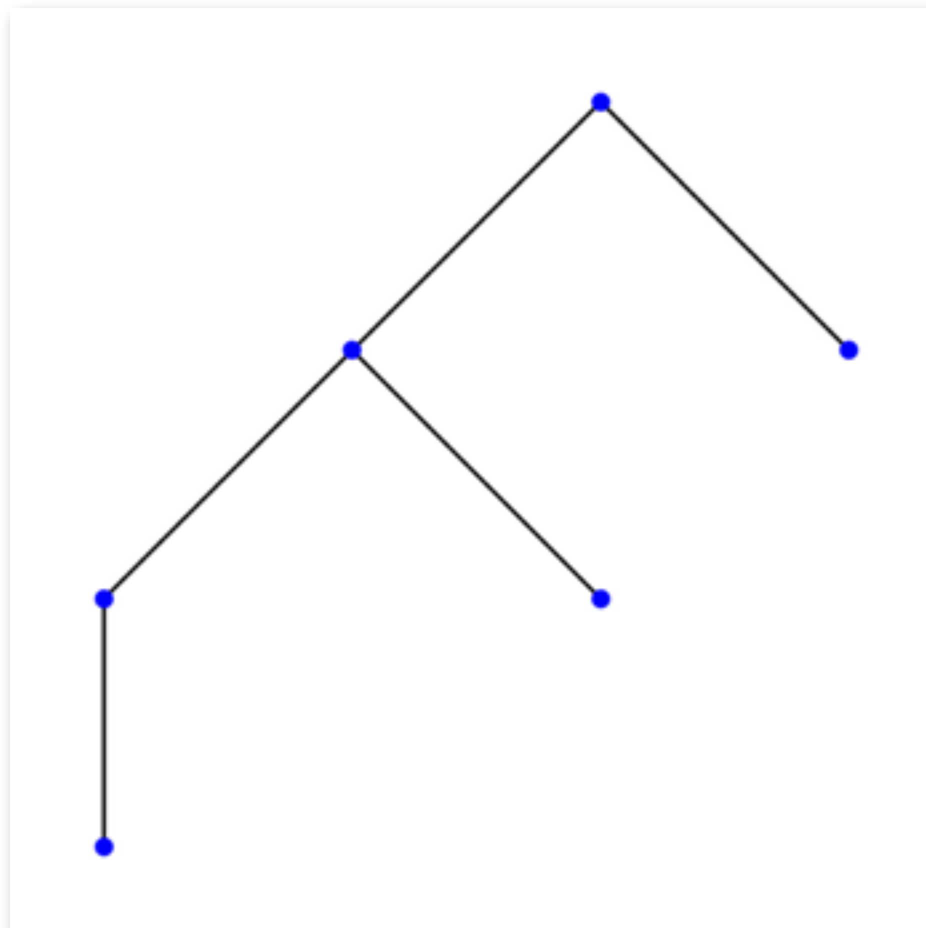
When do we consider two maps to be similar? When are they different?

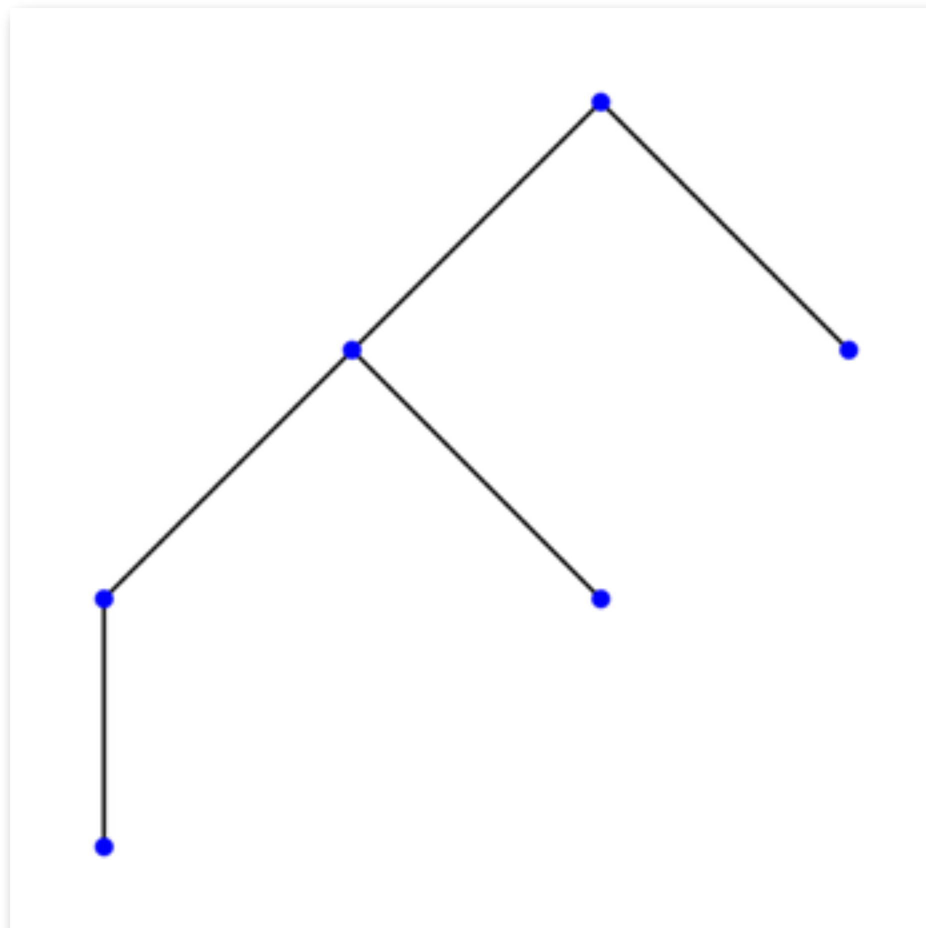
We need to consider DISTANCES on sets of maps.

THE SCALING LIMIT OF PLANE TREES



THE SCALING LIMIT OF PLANE TREES

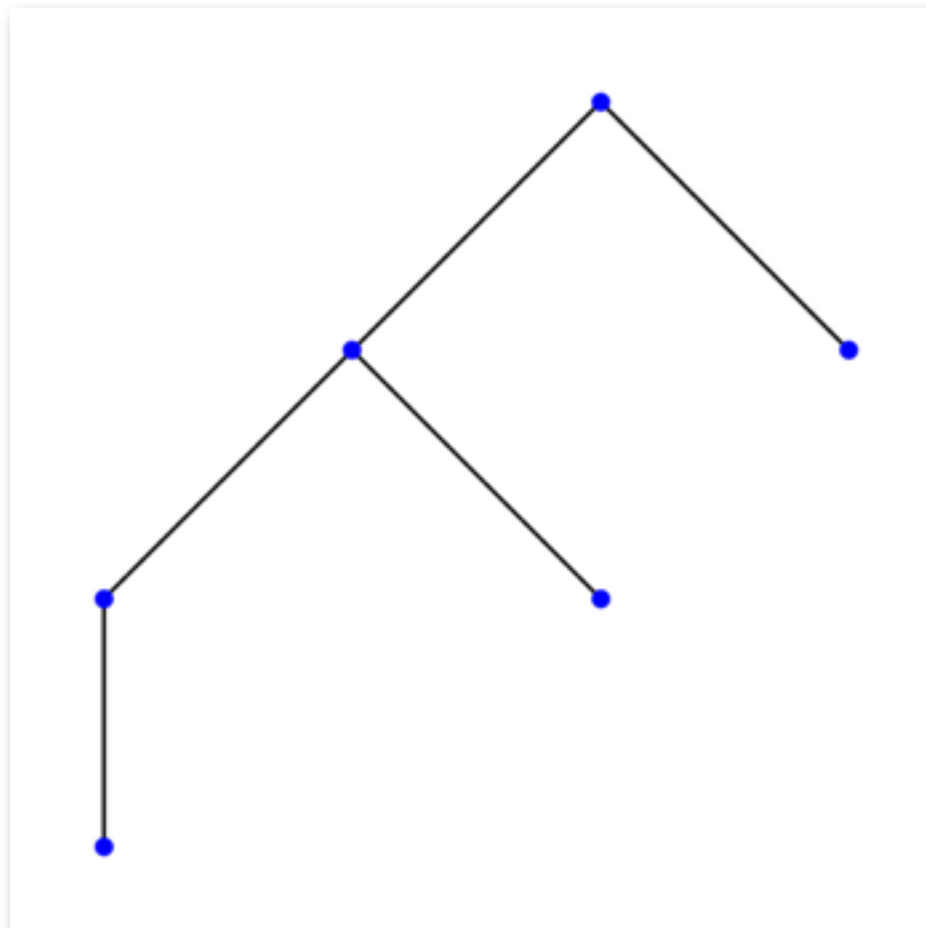
A **plane tree** is a (rooted) map with only one face.



6 vertices
5 edges

THE SCALING LIMIT OF PLANE TREES

t_6



Let t_n *be a (uniform) random* ***plane tree*** *with* n *vertices.*

THE SCALING LIMIT OF PLANE TREES

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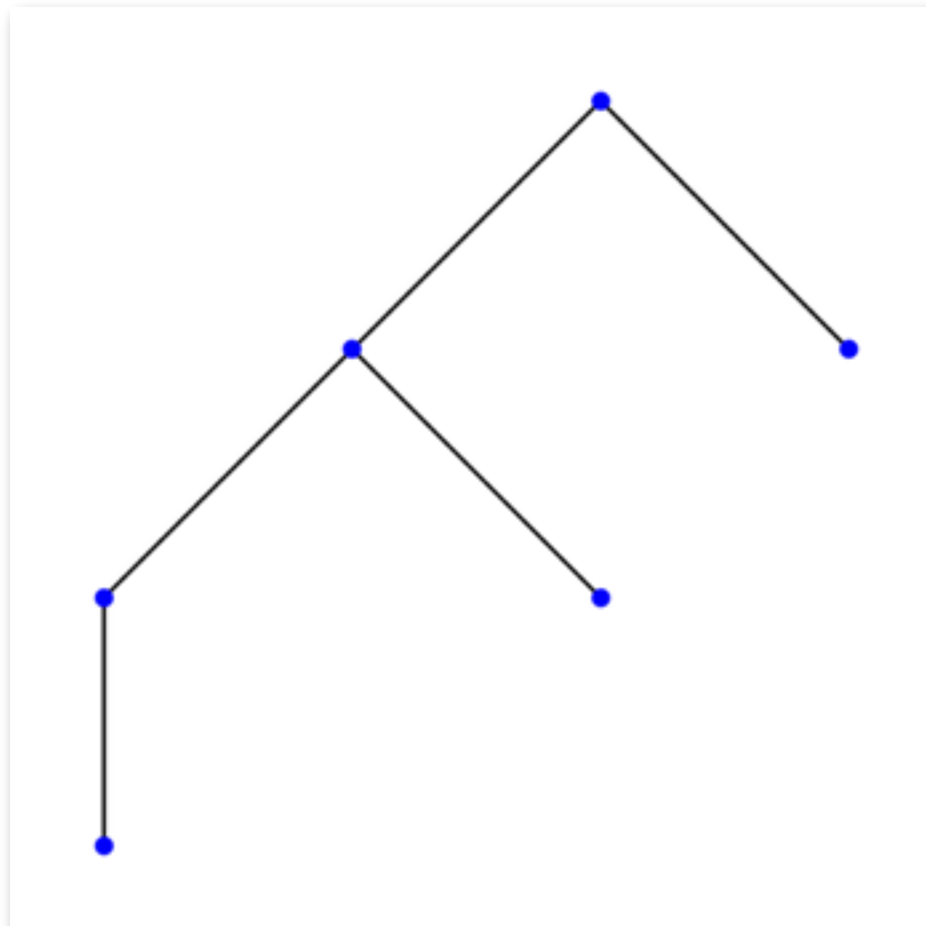
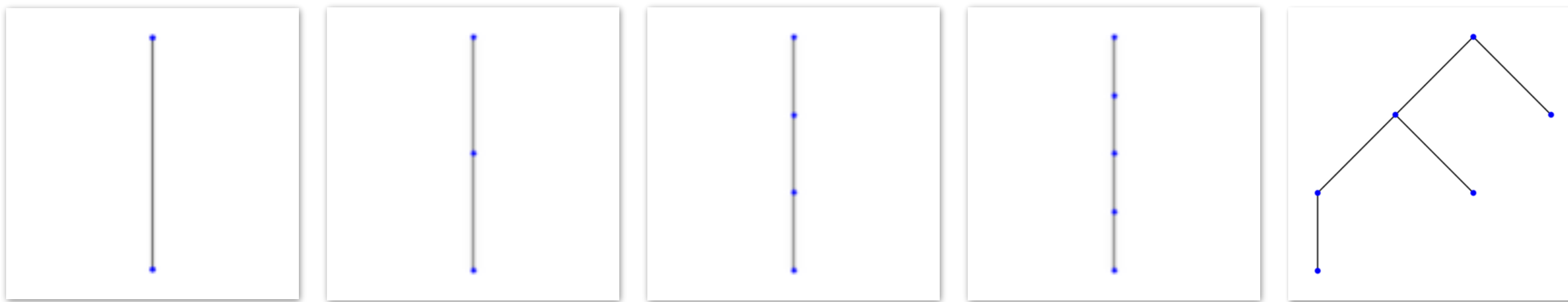


Let t_n *be a (uniform) random* ***plane tree*** *with* n *vertices.*

We can see $(t_n, d/n^{1/2})$ *as a random* ***metric space*** *(a measure on the space of compact m. s.* (X, d_{GH})*).*

THE SCALING LIMIT OF PLANE TREES



t_2 t_3 t_4 t_5 t_6 . . .

We can see $(t_n, d/n^{1/2})$ *as a random* ***metric space*** *(a measure on the space of compact m. s.* (X, d_{GH})*).*

What is the weak limit of the sequence $(t_n, d/n^{1/2})$ *if we let* n *go to infinity?*

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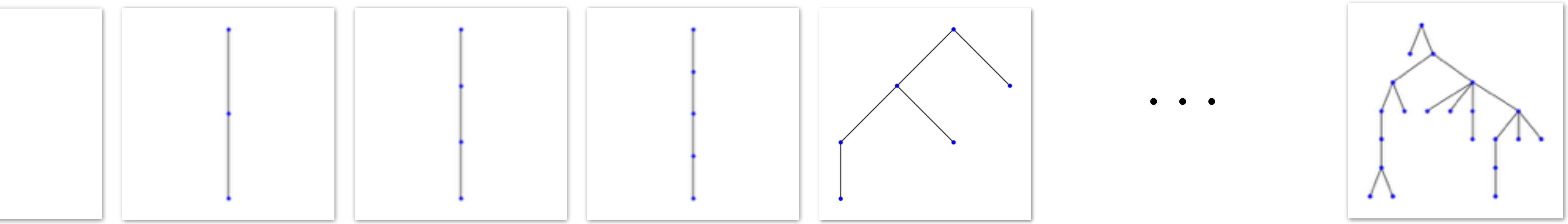

t_3 t_4 t_5 t_6 … t_{20}

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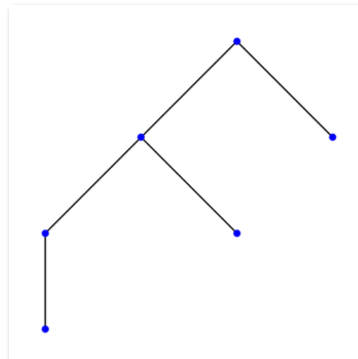
THE SCALING LIMIT OF PLANE TREES

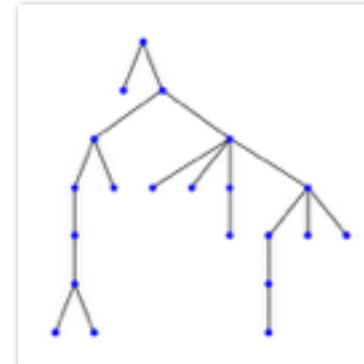
t_4

t_5



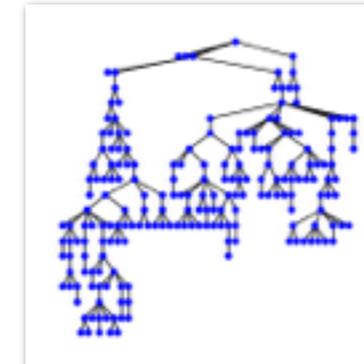
t_6

. . .



t_{20}

. . .



t_{200}

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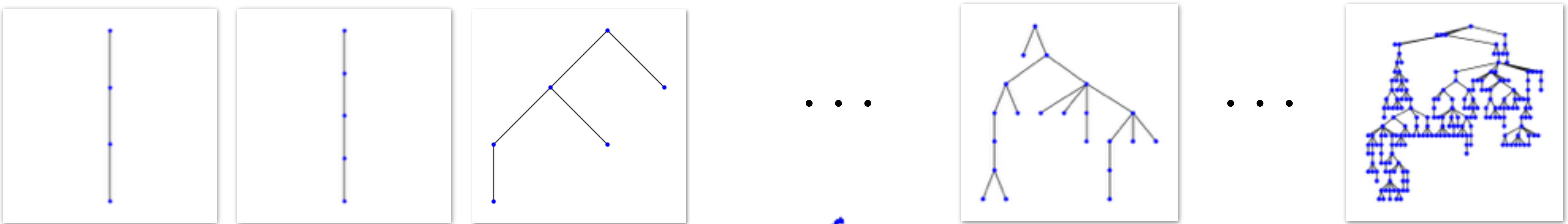
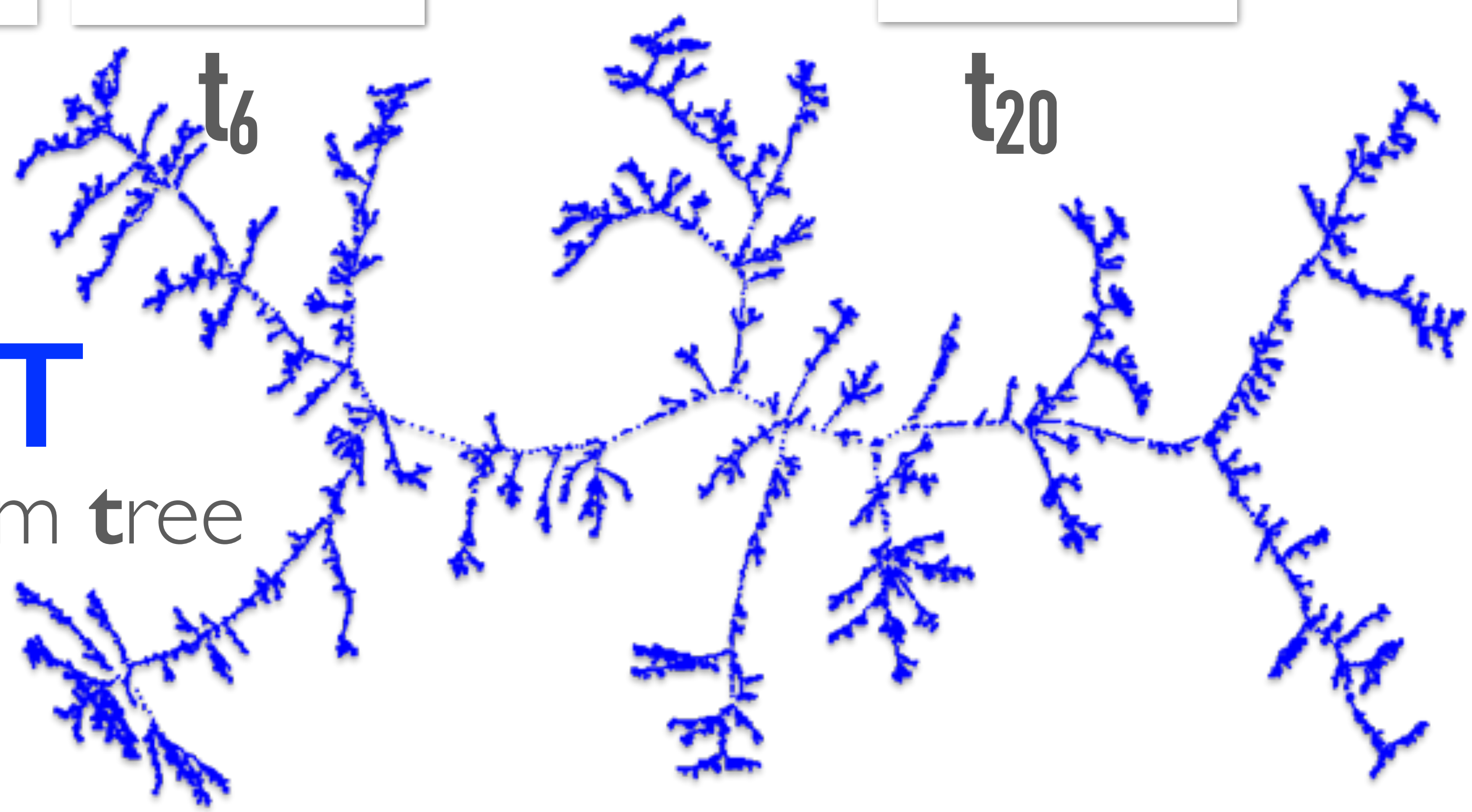


t_4 t_5 t_6 . . . t_{20} . . . t_{200}



THE **CRT**

continuum **r**andom **t**ree

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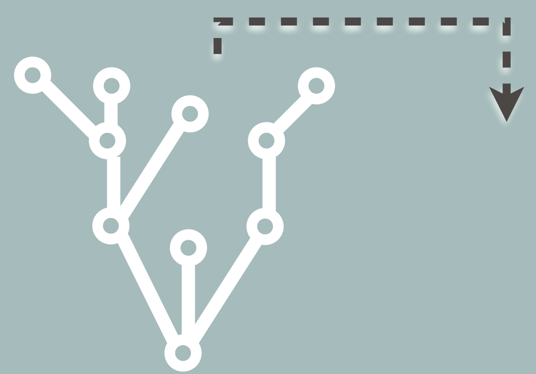
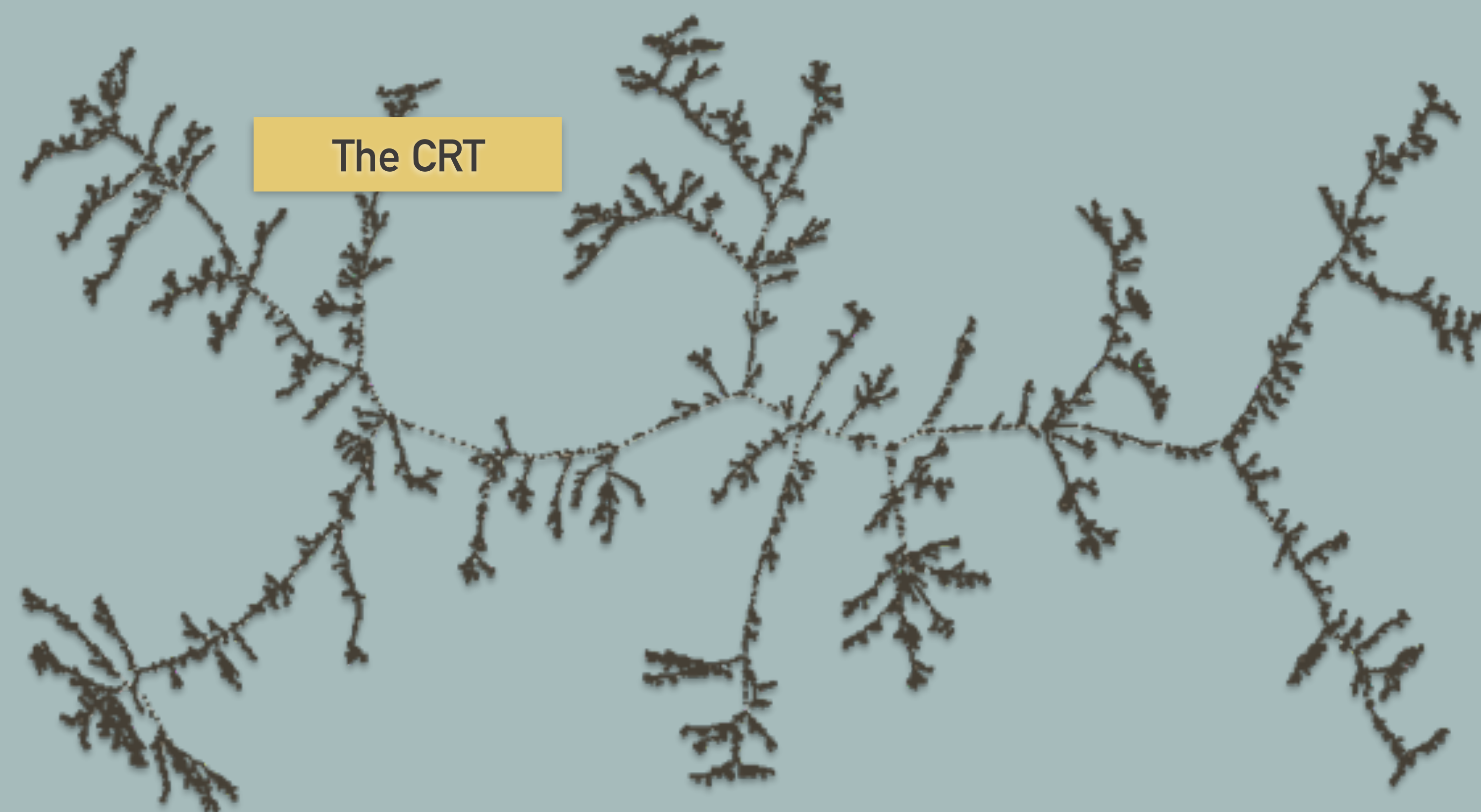


random plane trees with n vertices

scale by $n^{1/2}$



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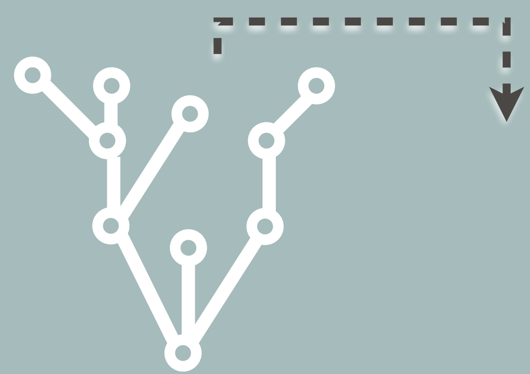
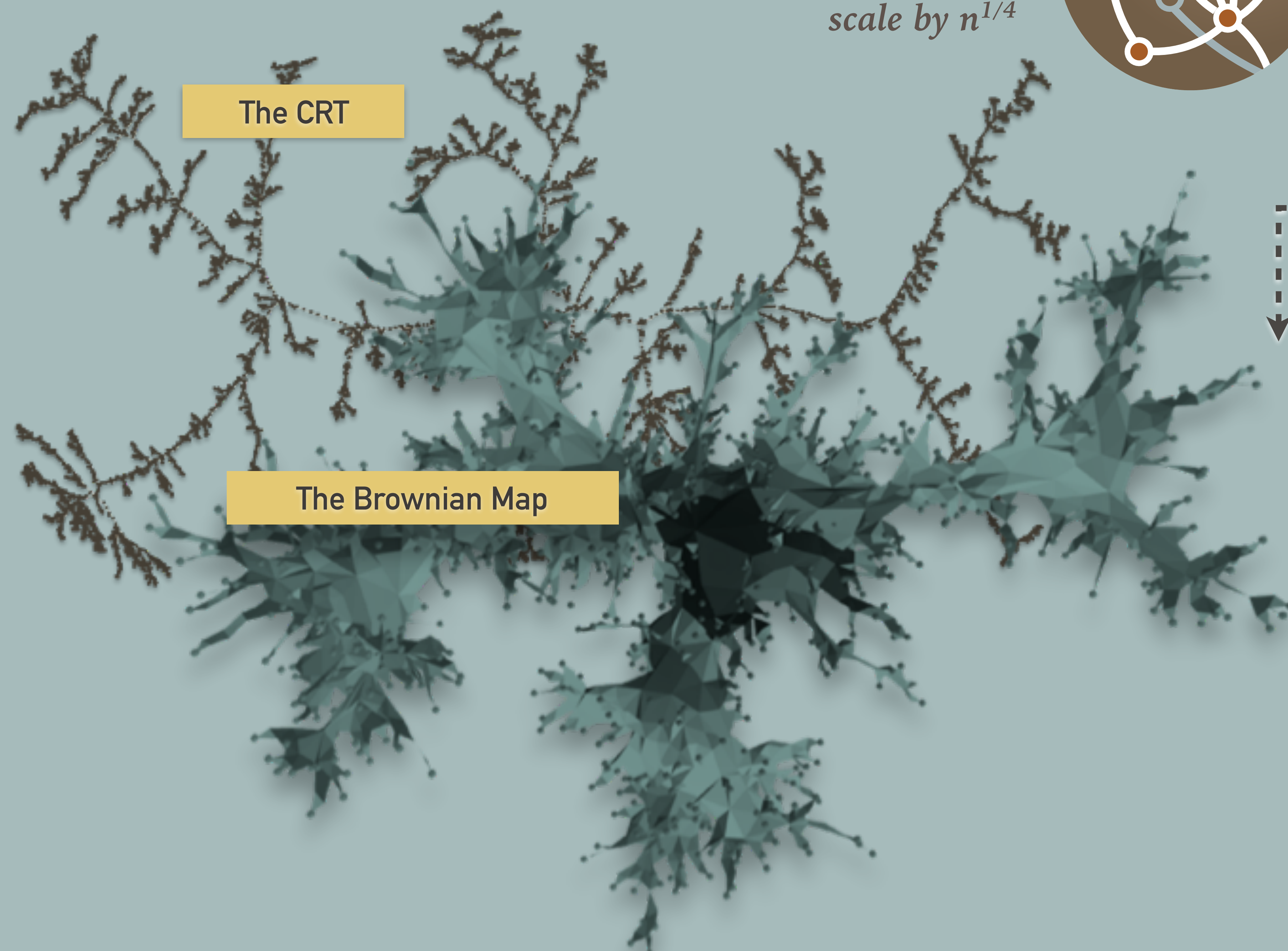


random plane trees with
n vertices

scale by $n^{1/2}$

random rooted
quadrangulations
maps with n faces

scale by $n^{1/4}$



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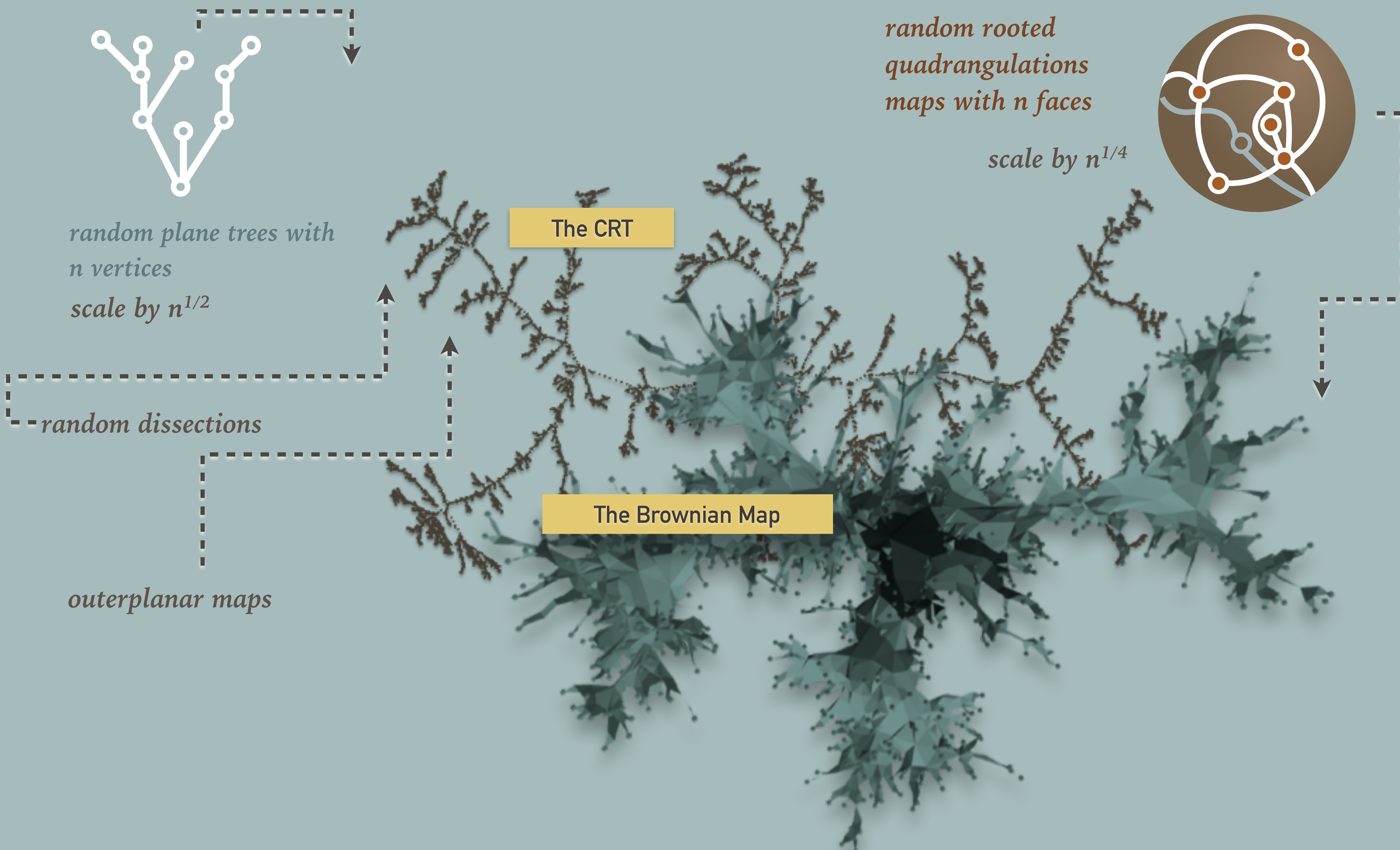

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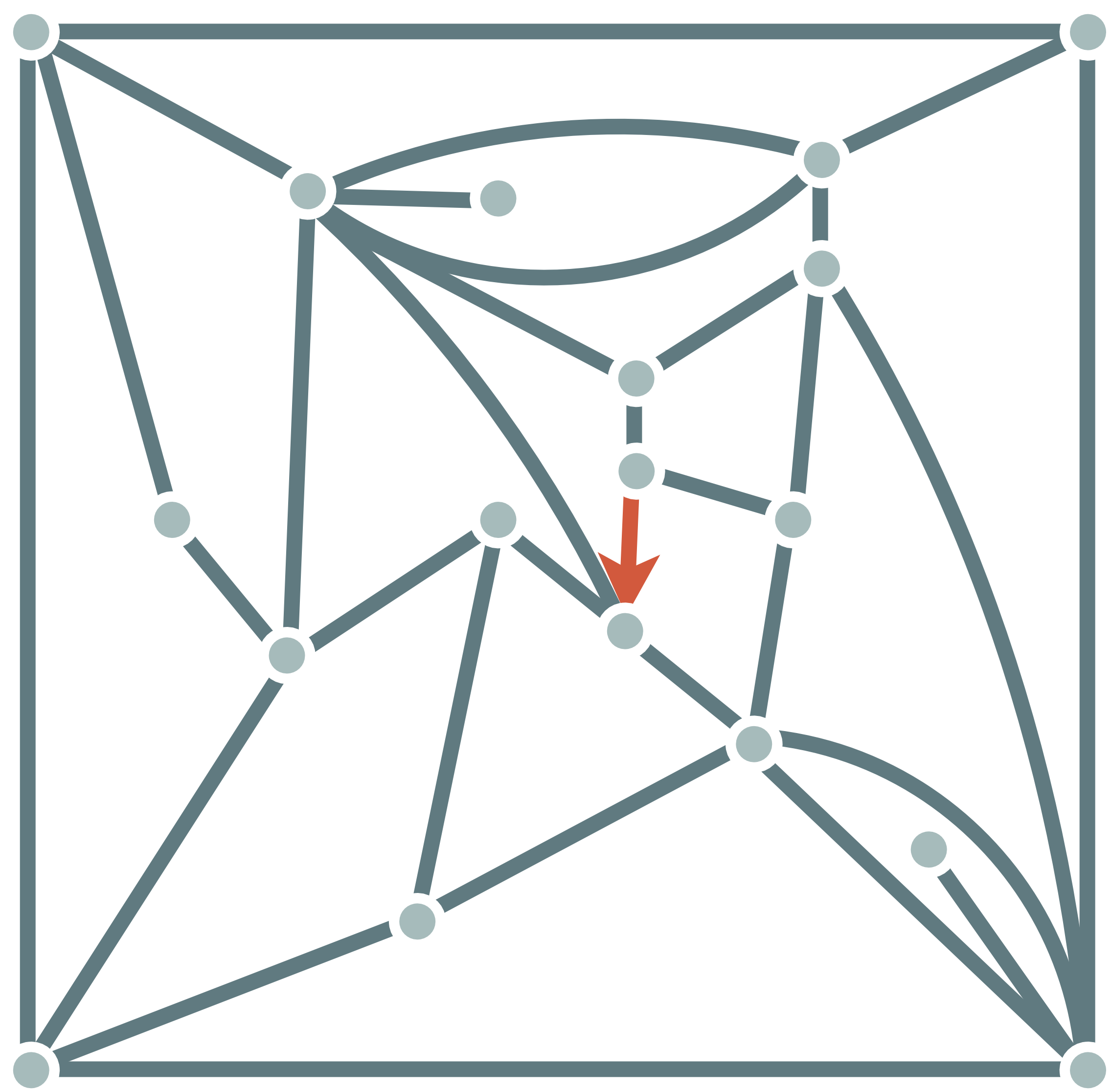

size **16** *random quadrangulation*

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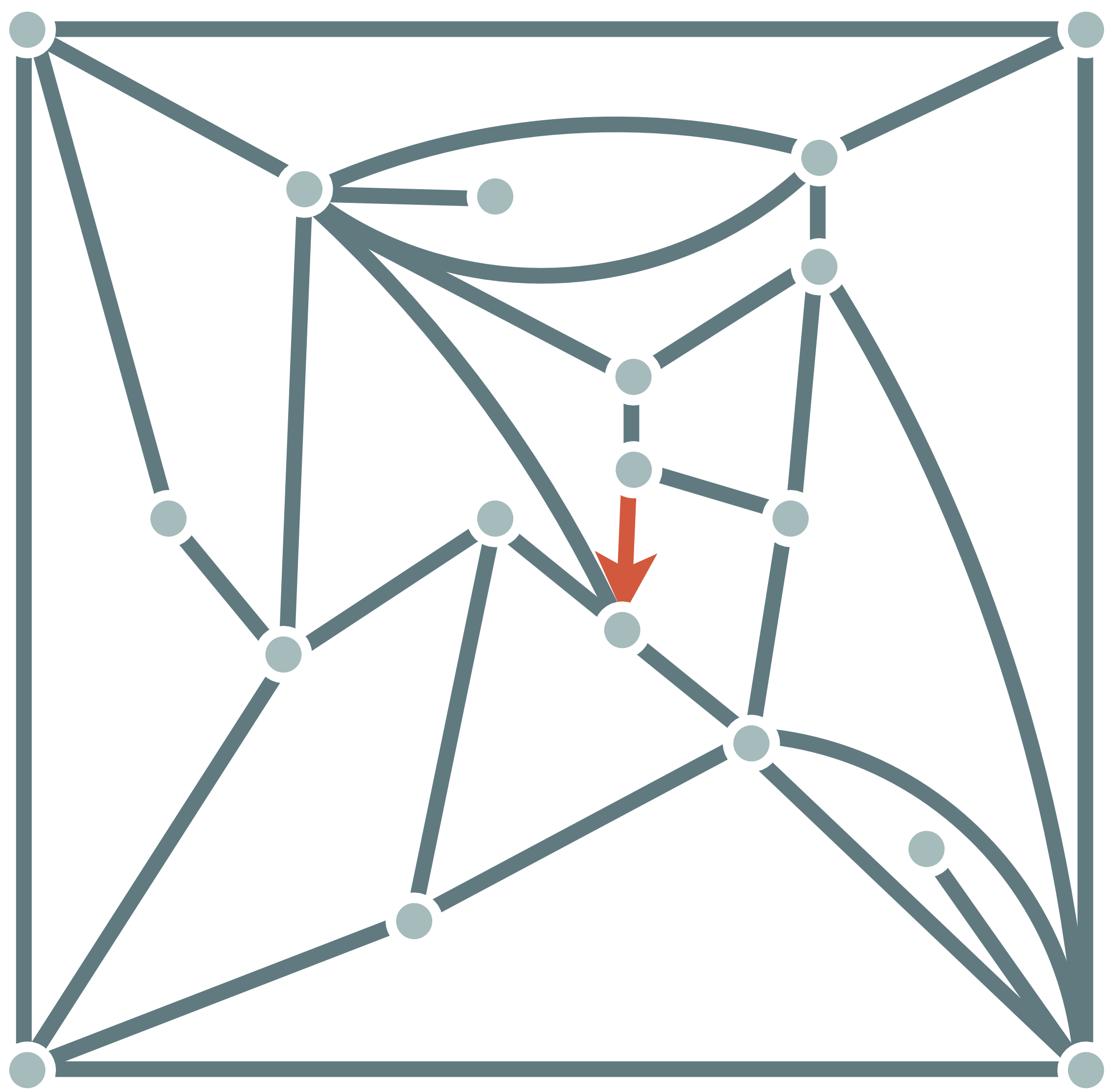

size 16 random quadrangulation

Fix a positive integer r *and consider the (rooted) map induced by vertices within graph distance* r *from the root vertex.*

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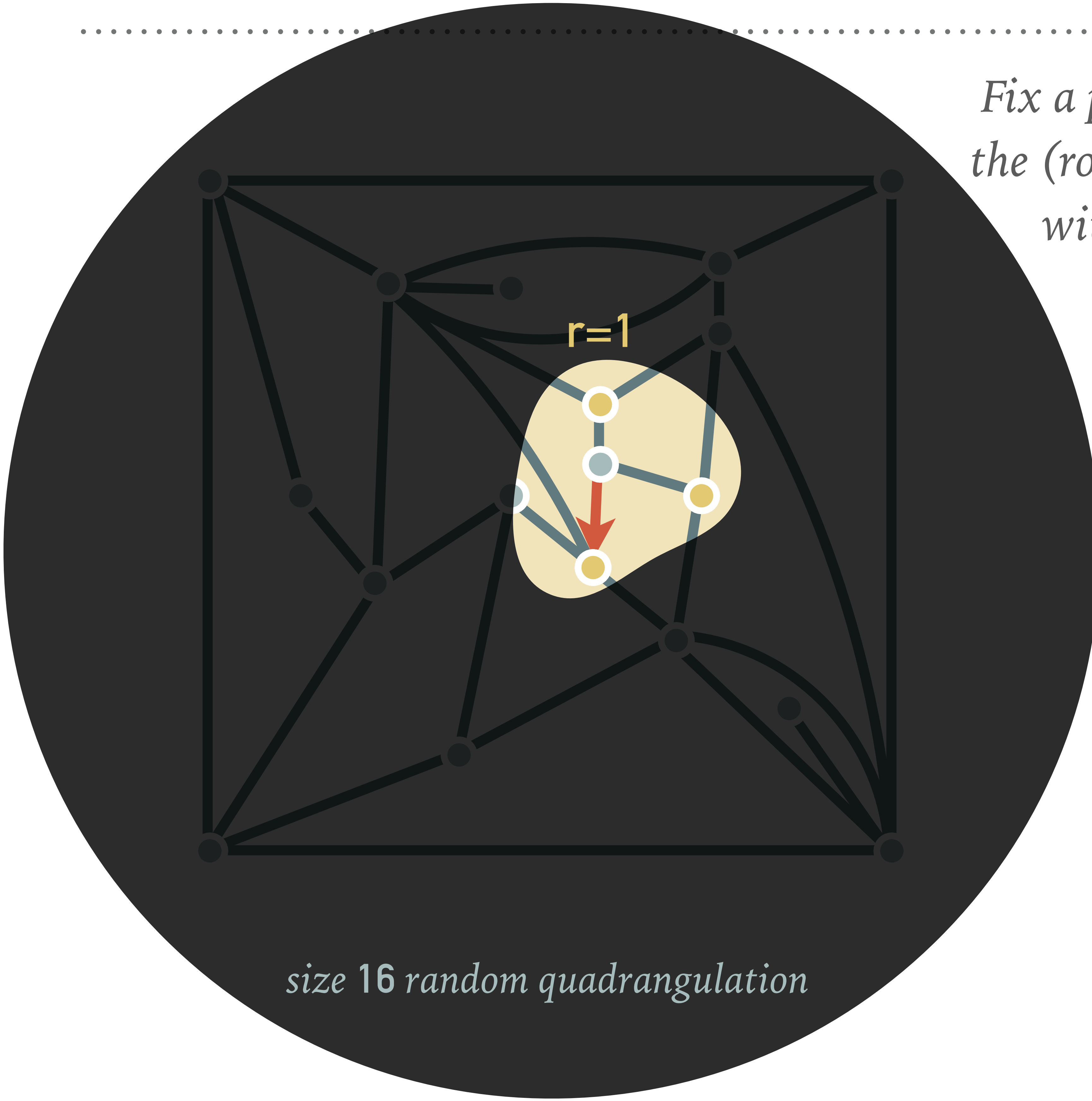

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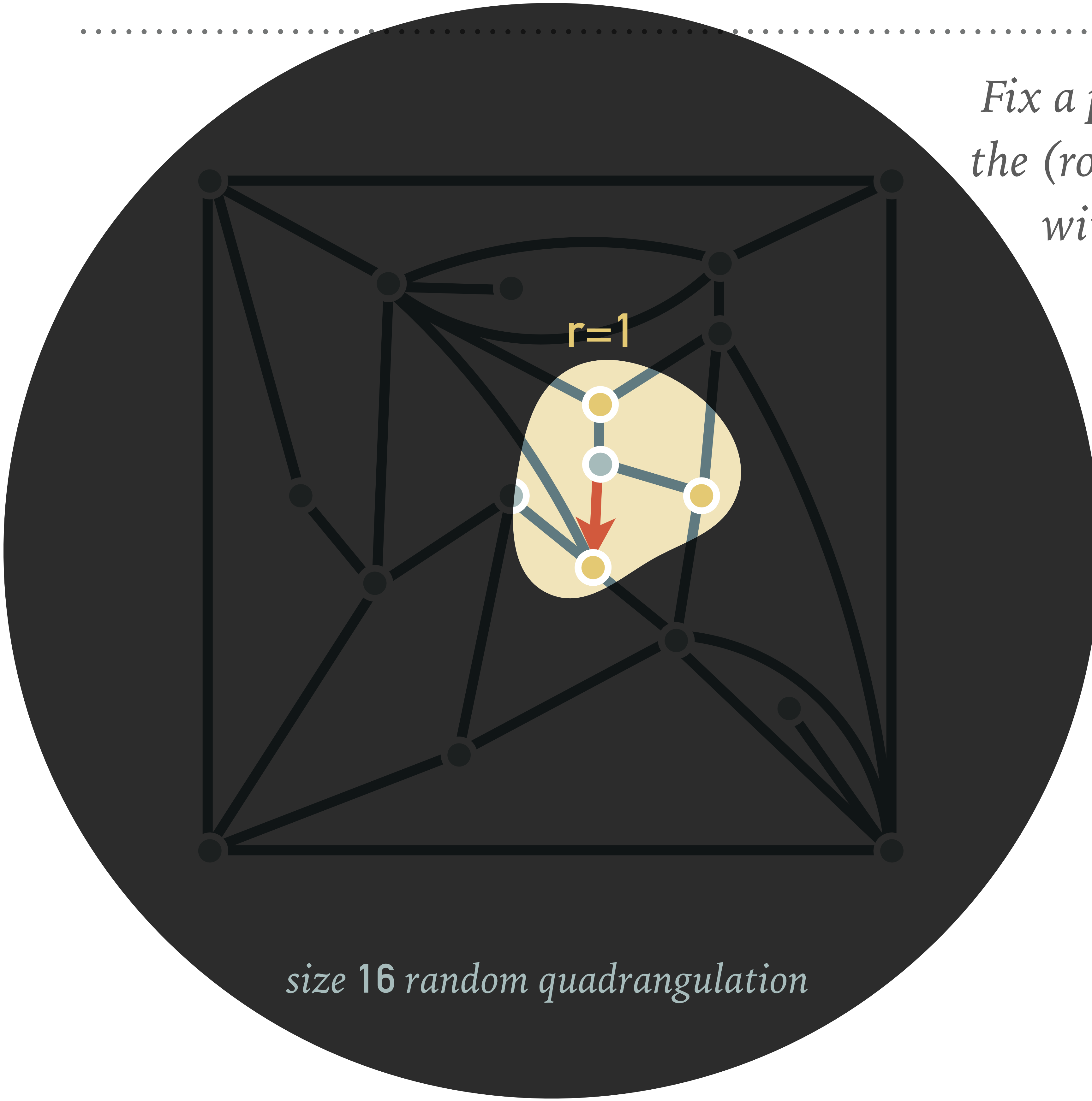

size **16** *random quadrangulation*

Fix a positive integer r *and consider the (rooted) map induced by vertices within graph distance* r *from the root vertex.*

Keep r *fixed and sample maps of increasing size.*

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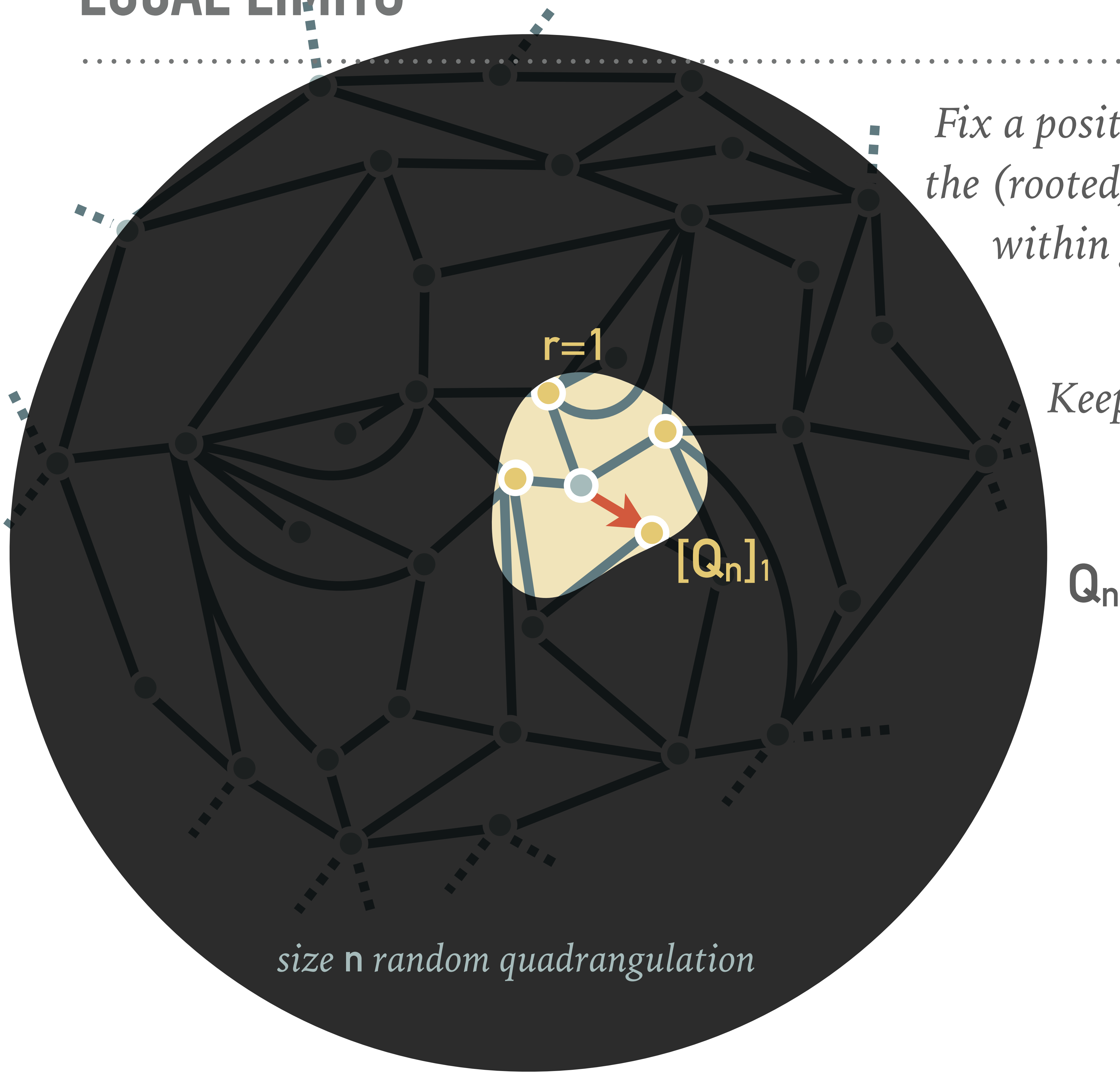

Fix a positive integer r *and consider the (rooted) map induced by vertices within graph distance* r *from the root vertex.*

Keep r *fixed and sample maps of increasing size.*

Send n *to infinity.*

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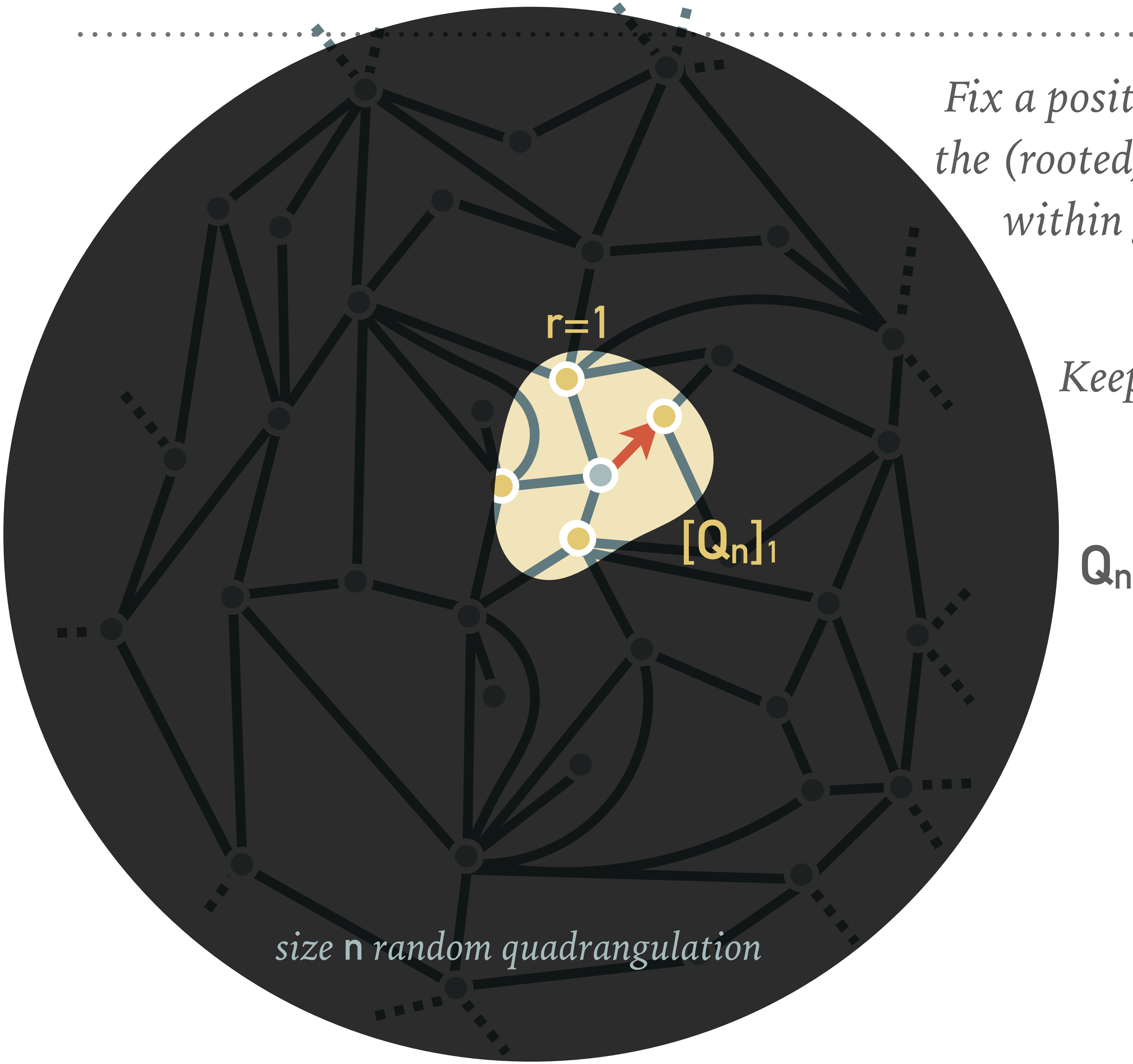

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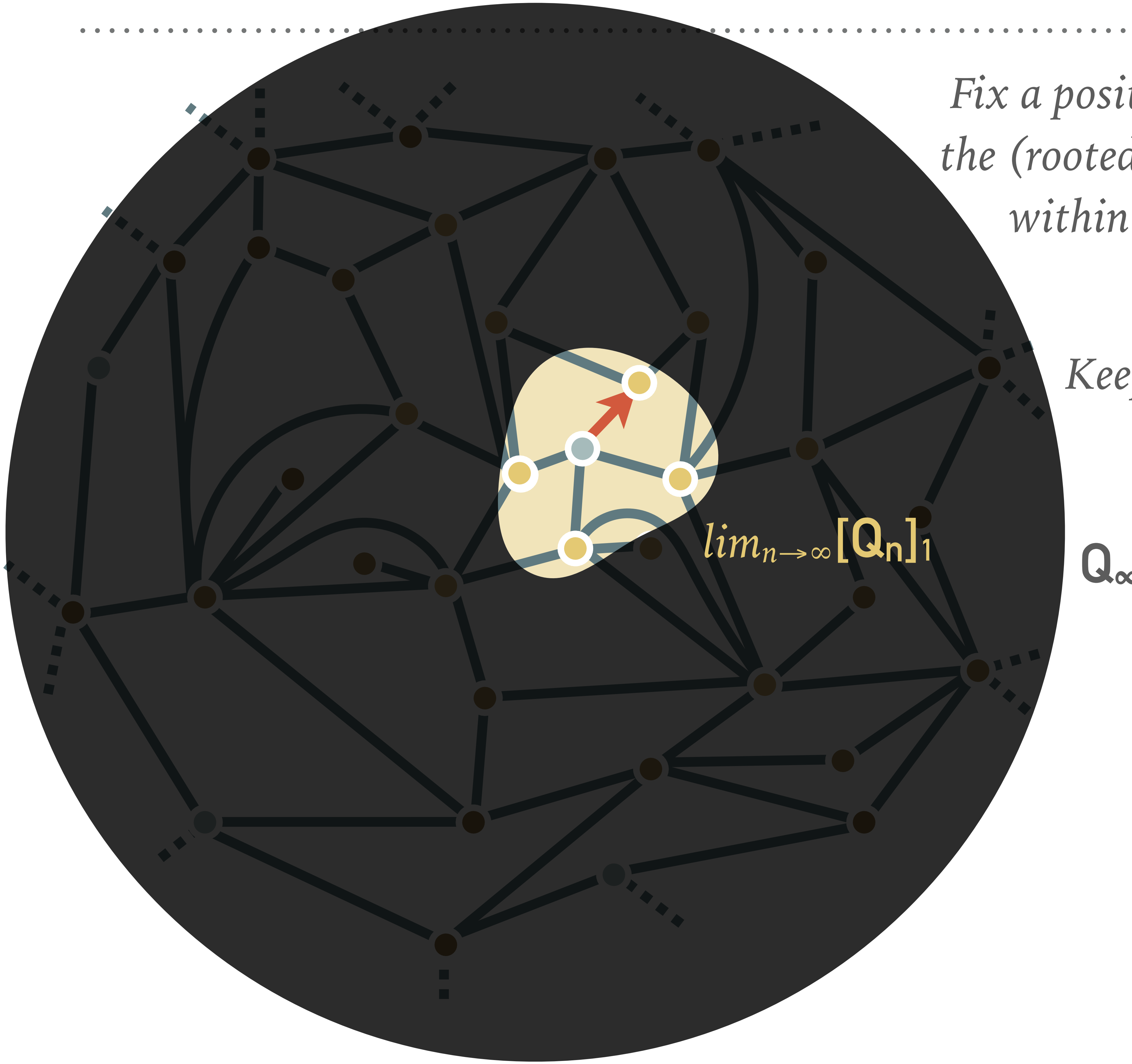

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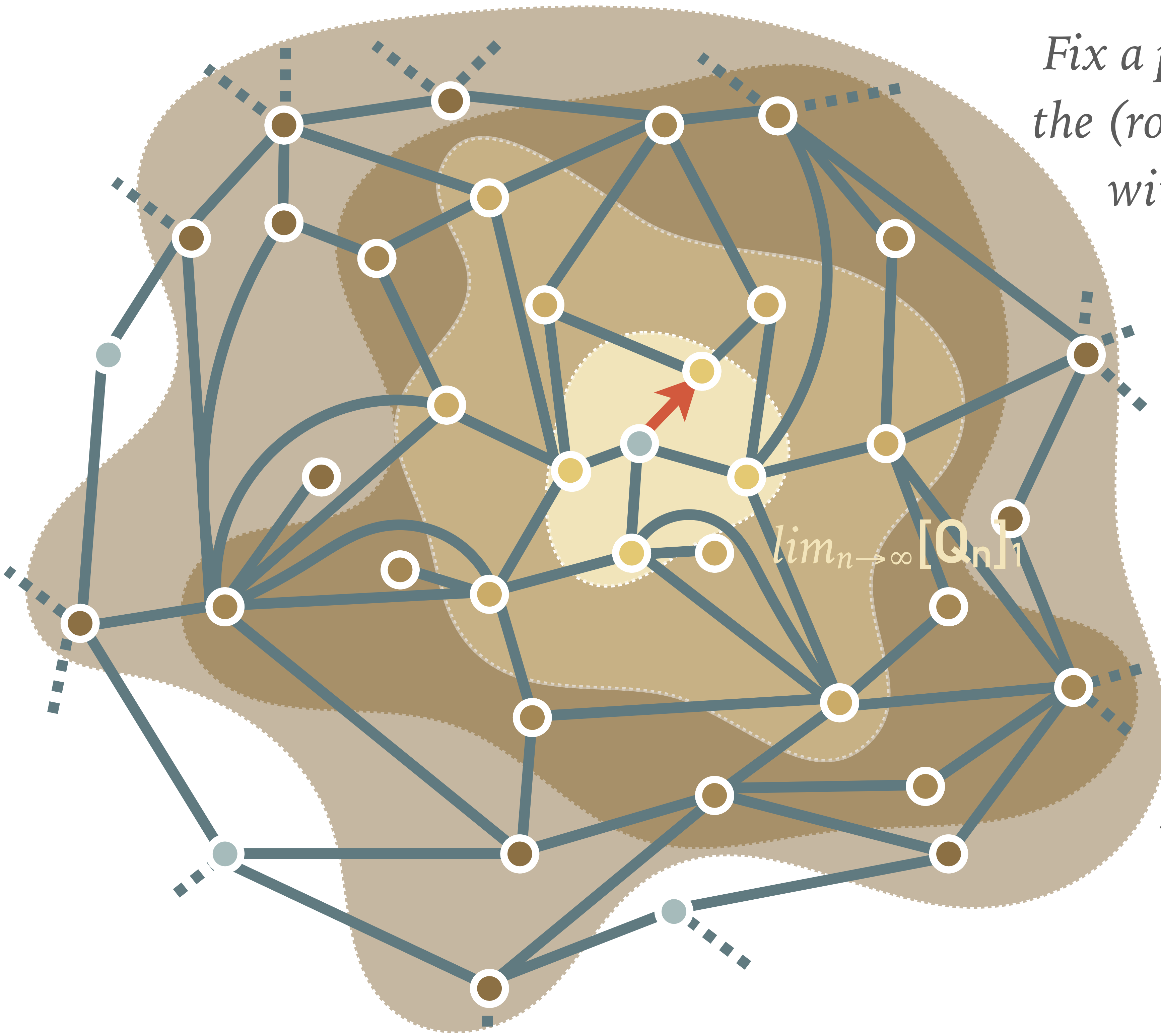

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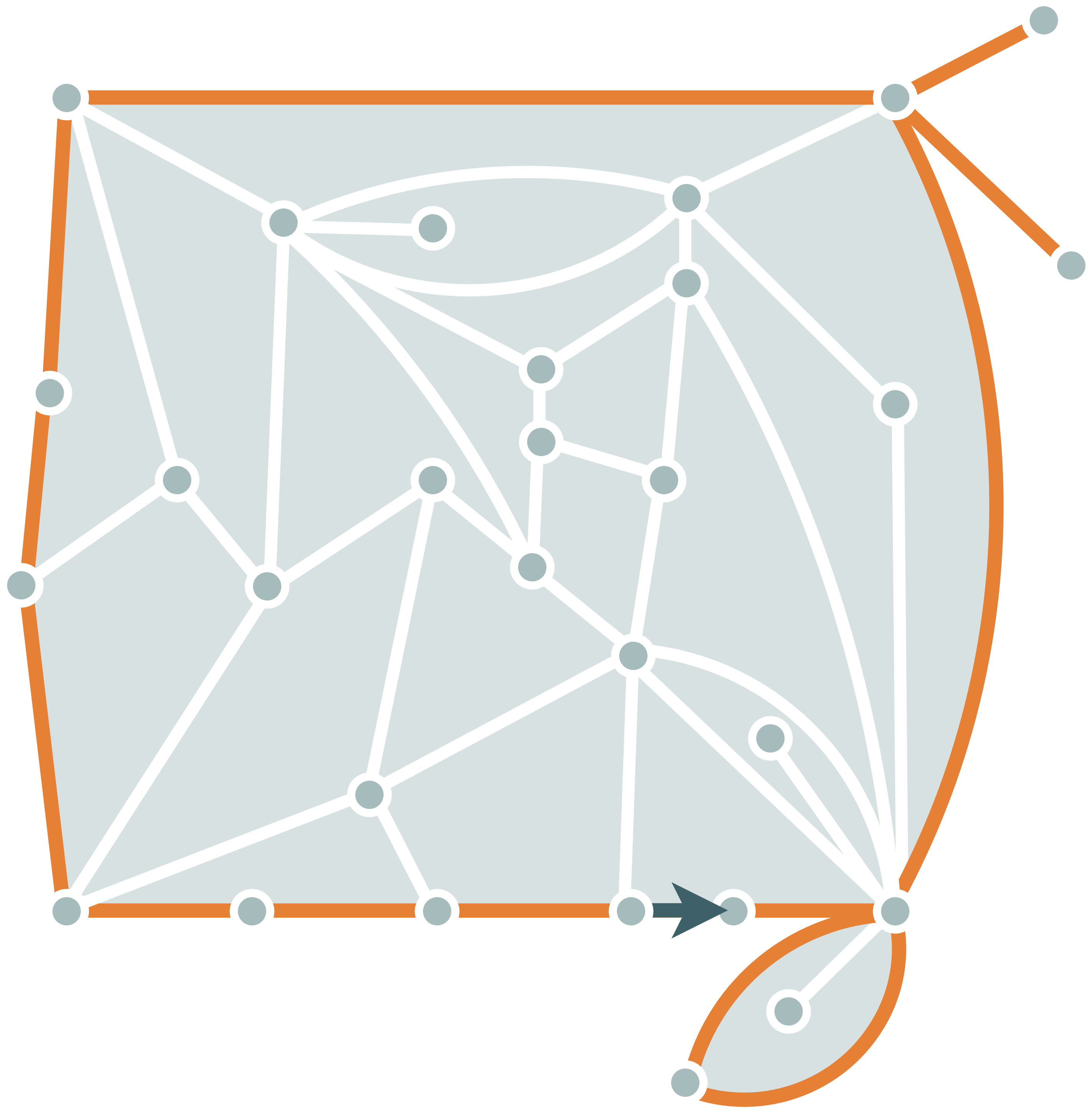
$\mathbf{Q}_\infty$

Send n *to infinity.*

The UIPQ $\mathbf{Q}_\infty$ *is an infinite random quadrangulation such that for each* r *we have* $[\mathbf{Q}_\infty]_r \sim lim_{n \to \infty}[\mathbf{Q}_n]_r$.

THE UIHPQ



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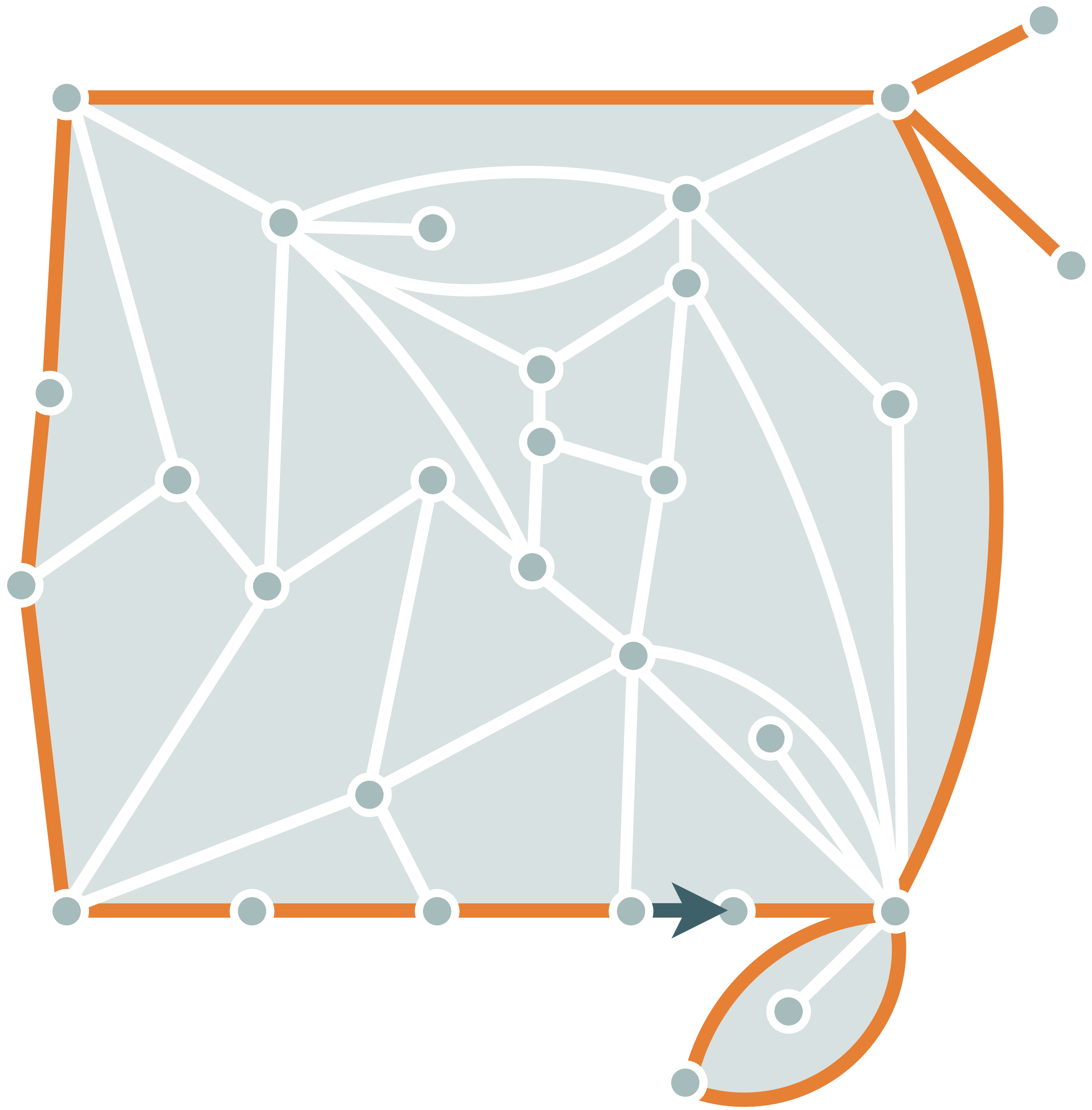


$Q_{n,p}$

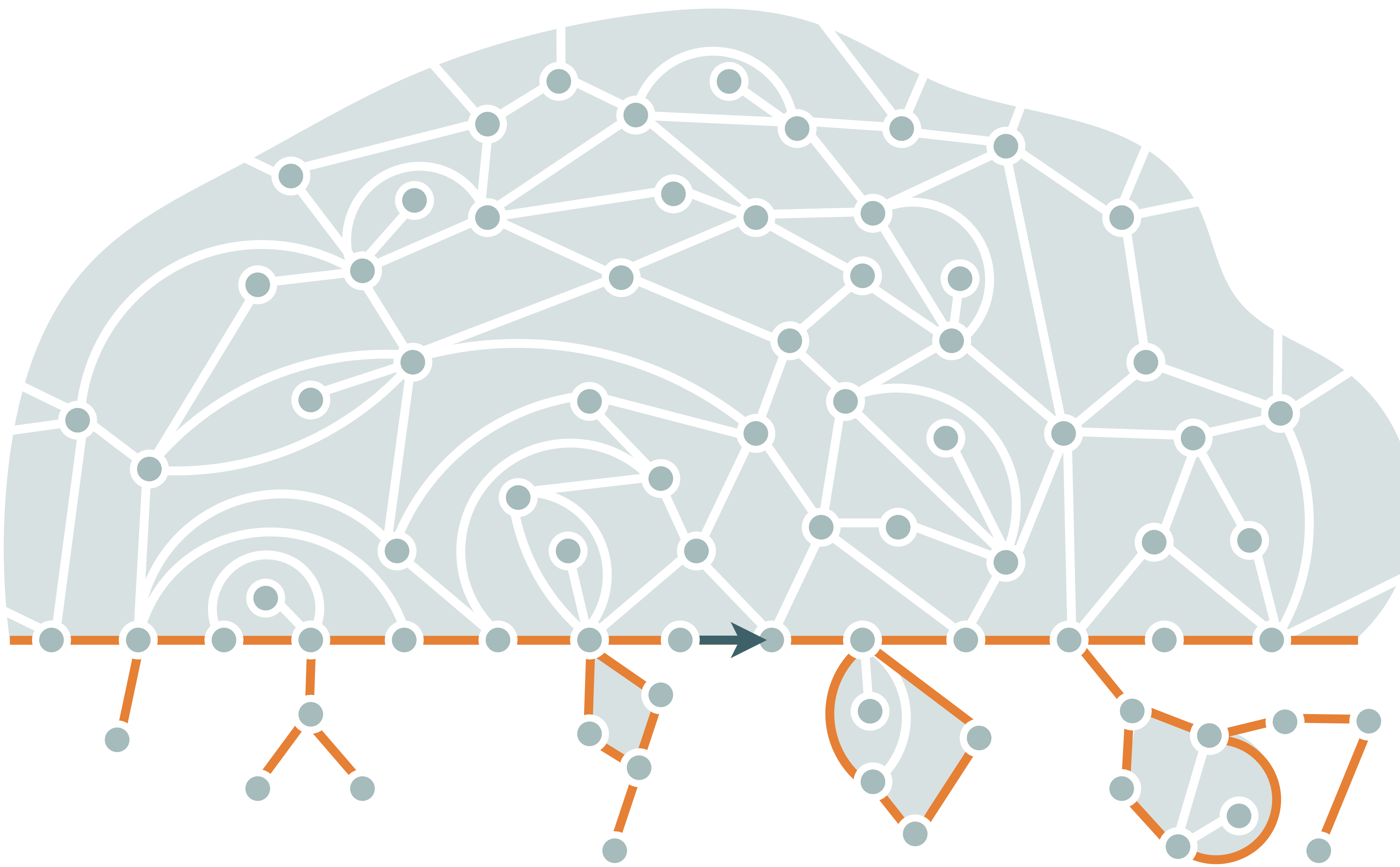
THE UIHPQ

$$Q_{n,p} \longrightarrow Q_{\infty,p}$$

THE UIHPQ

$$\begin{tikzcd} Q_{n,p} \arrow[r] & Q_{\infty,p} \arrow[d] \\ & H_\infty \end{tikzcd}$$

THE UIHPQ



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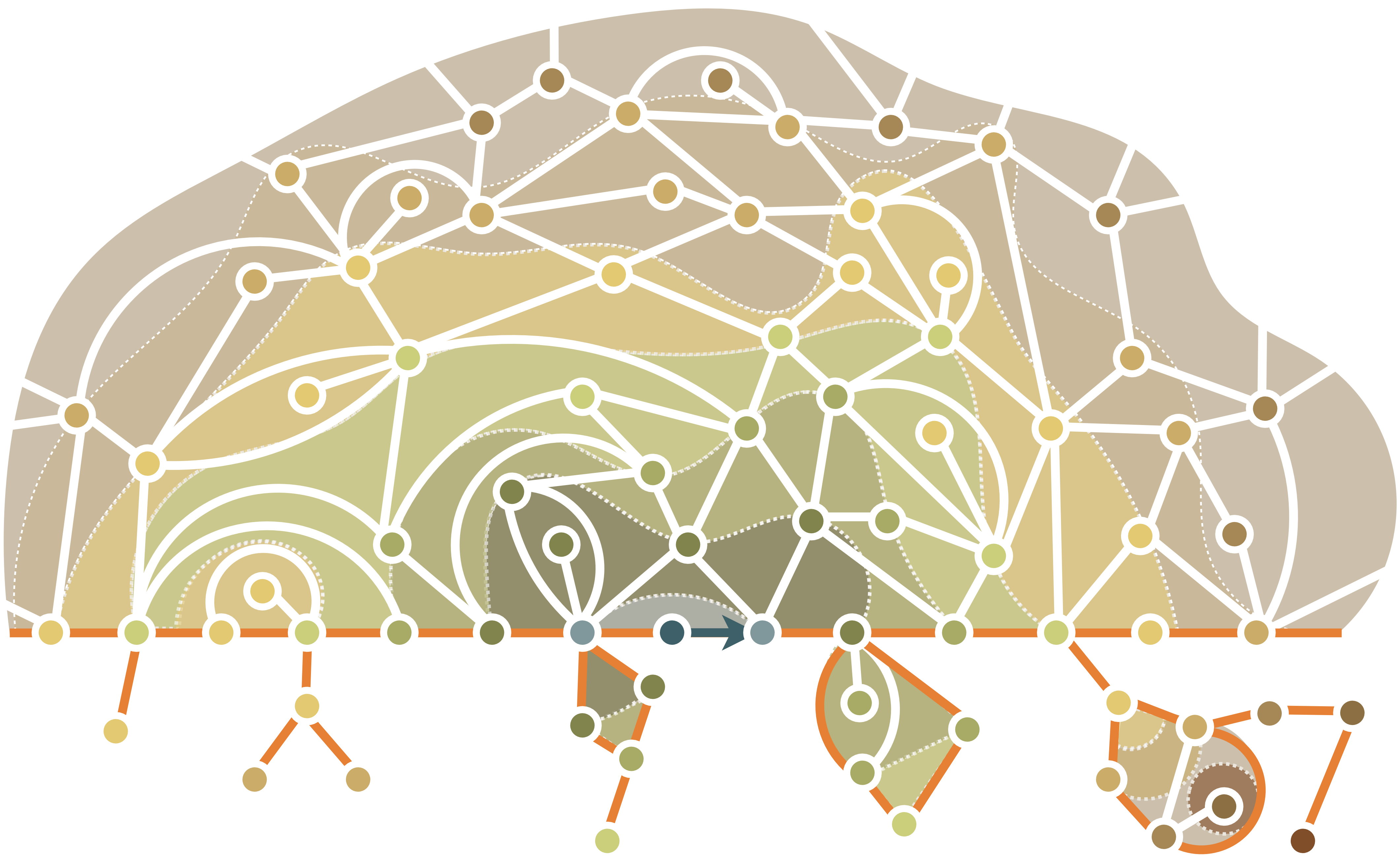
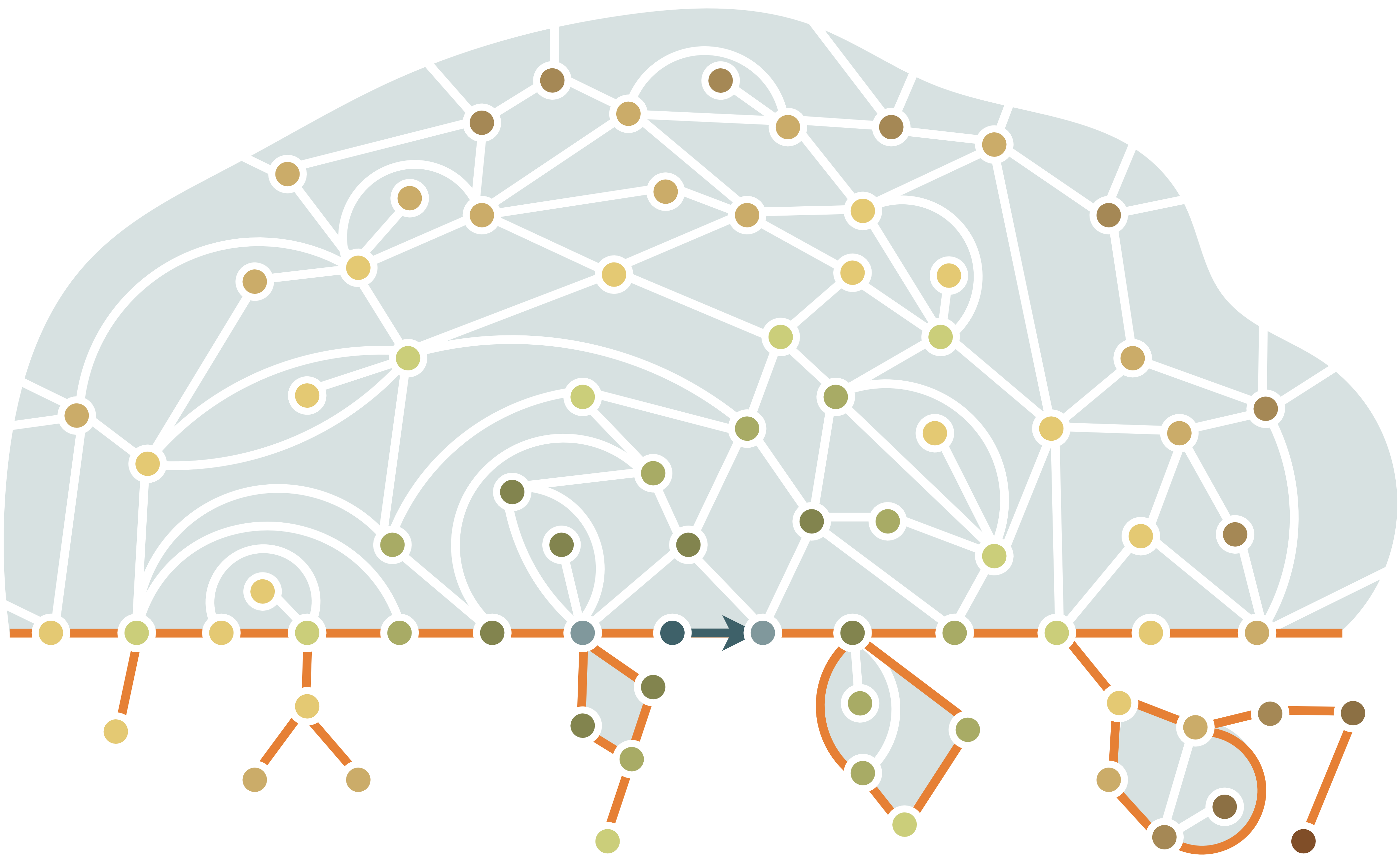


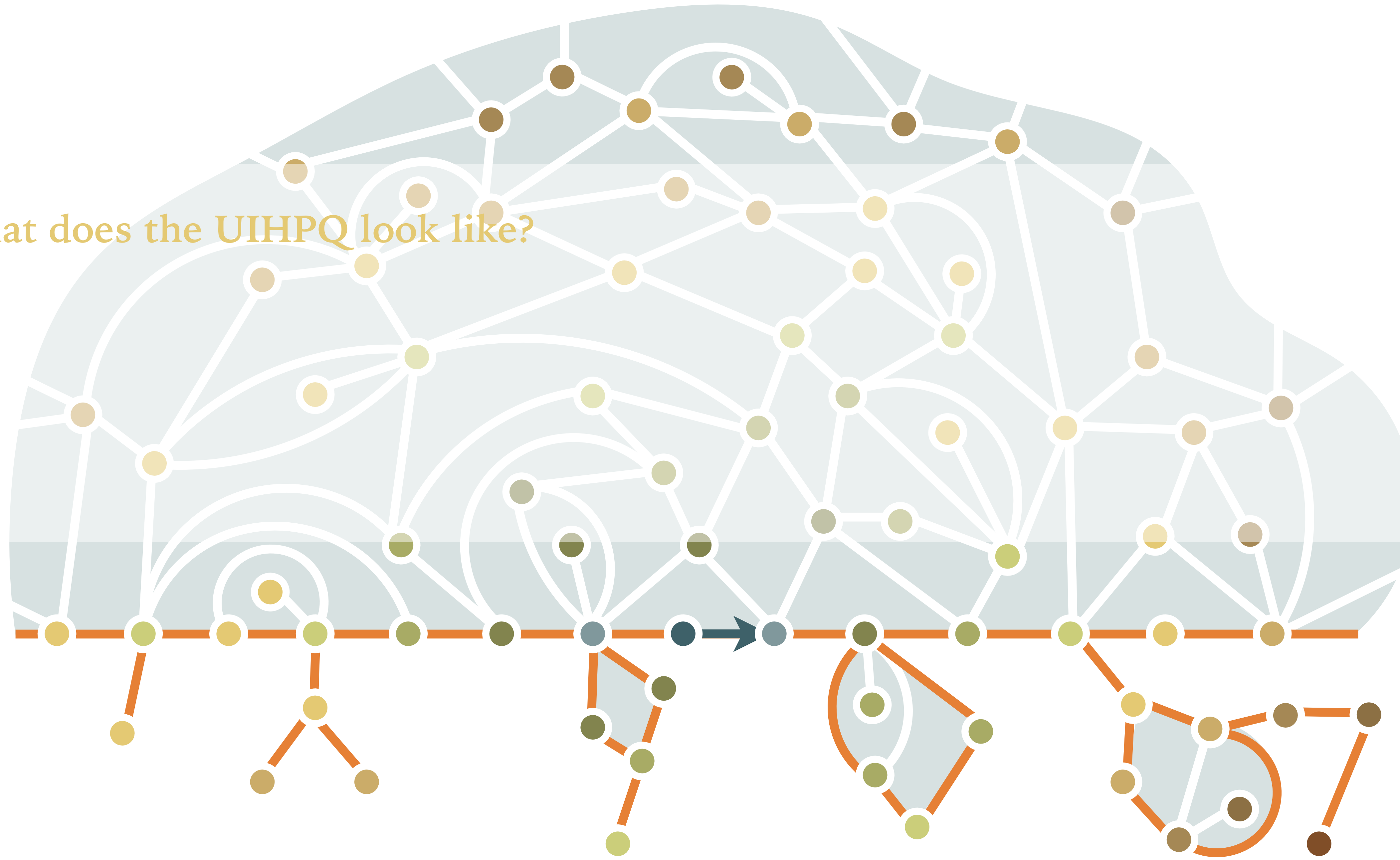
$$[H_\infty]_k = lim_{p \to \infty} [Q_{\infty,p}]_k$$

THE UIHPQ



THE UIHPQ

- What does the UIHPQ look like?



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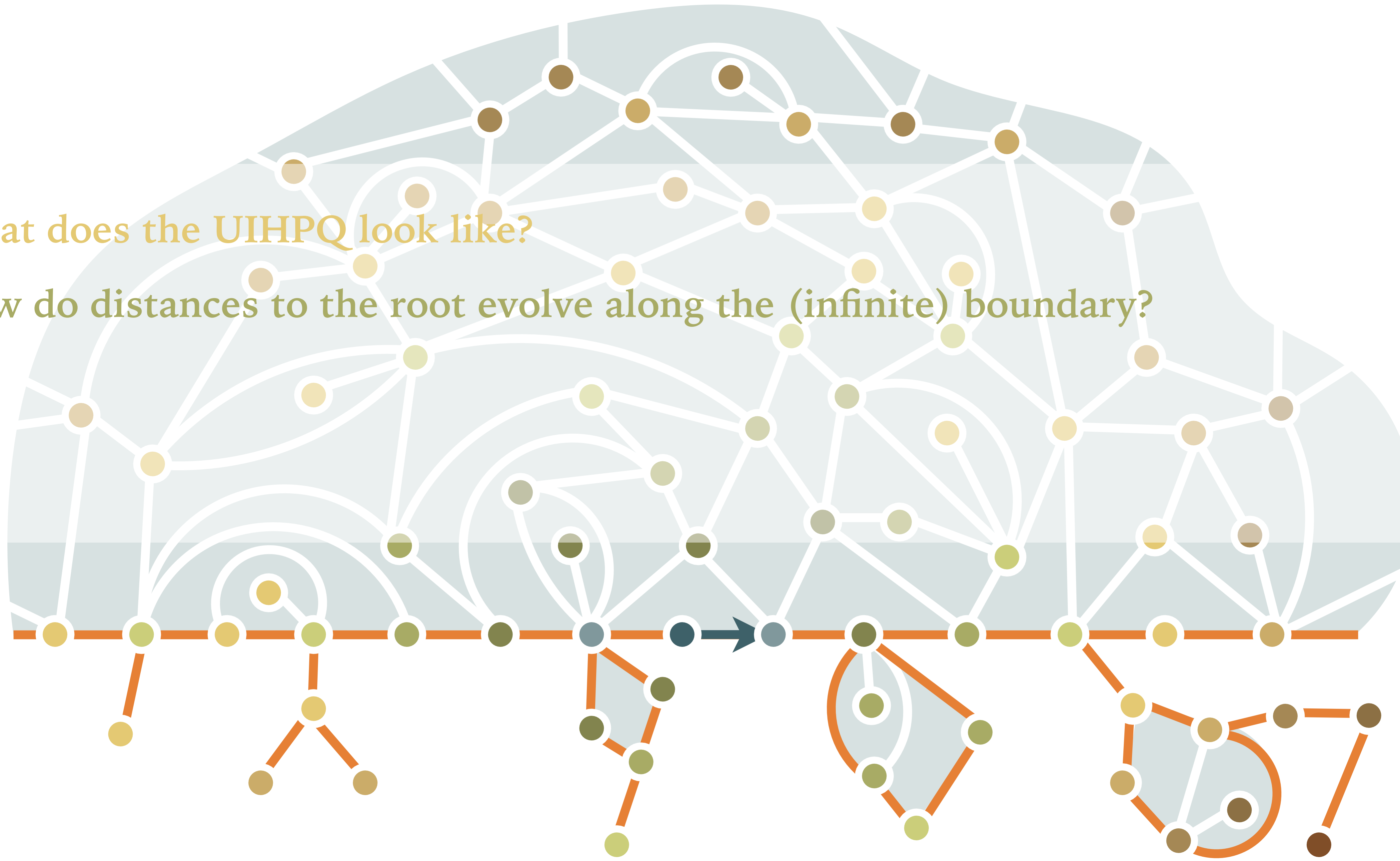
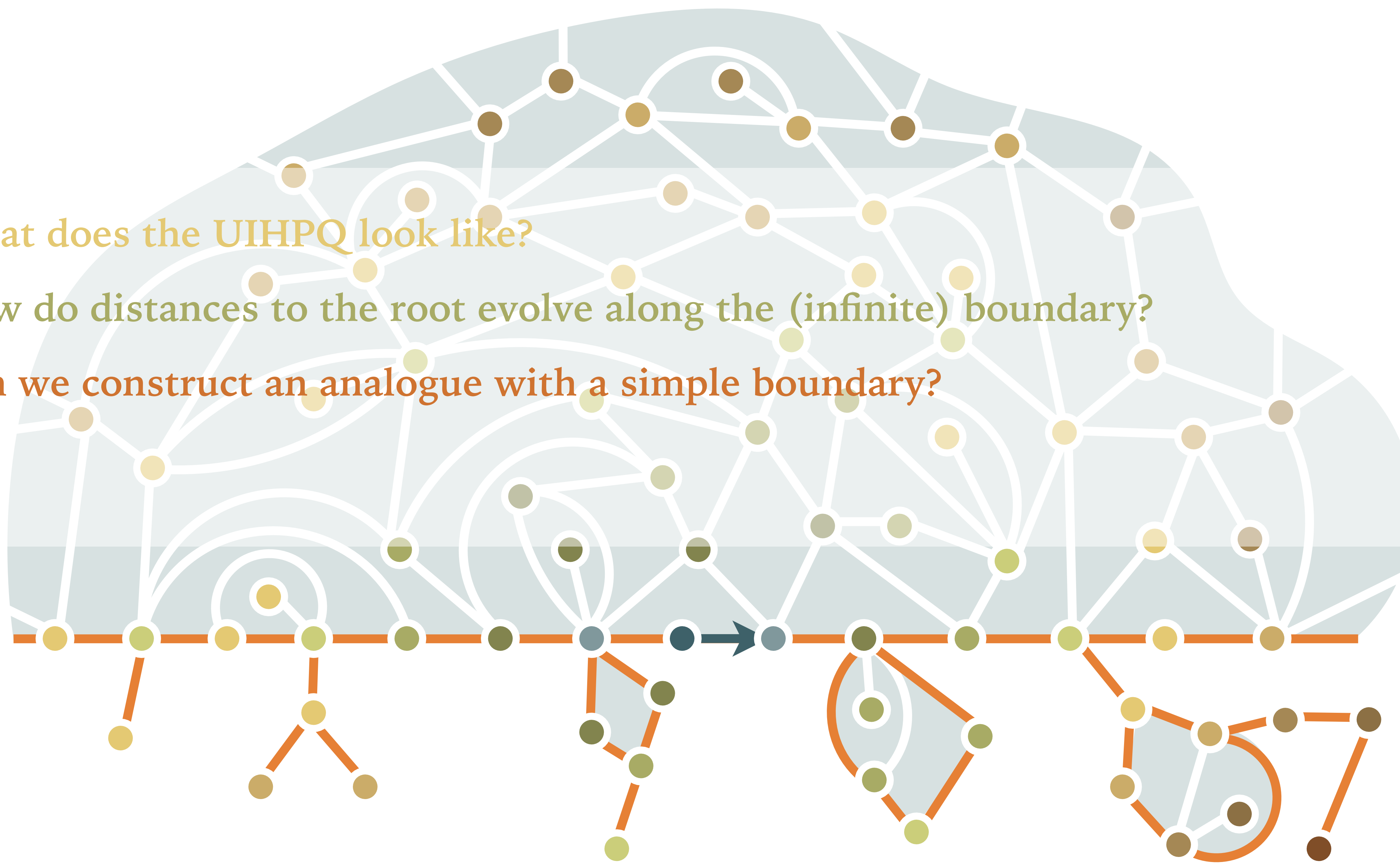


- What does the UIHPQ look like?
- How do distances to the root evolve along the (infinite) boundary?

THE UIHPQ



- What does the UIHPQ look like?
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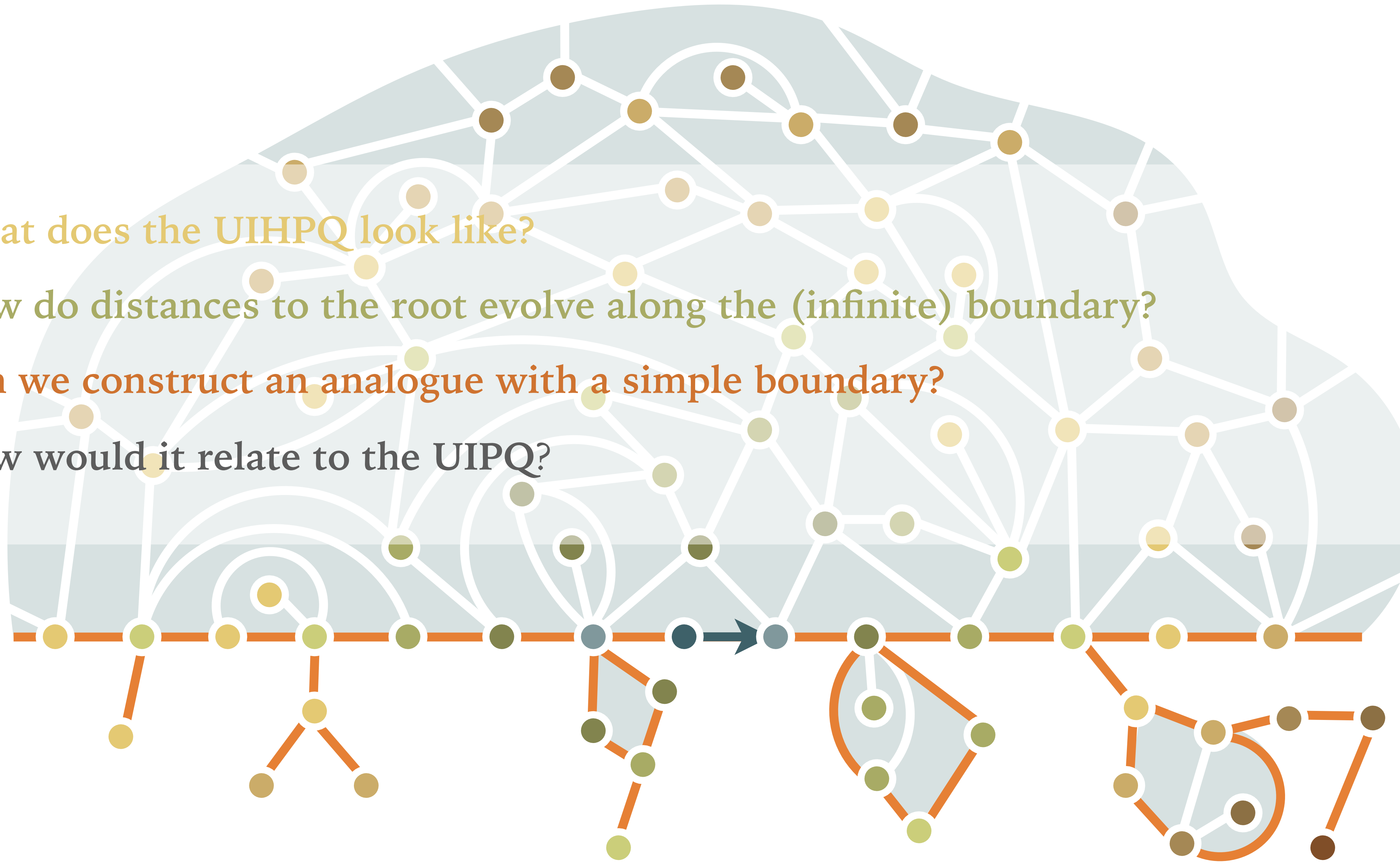
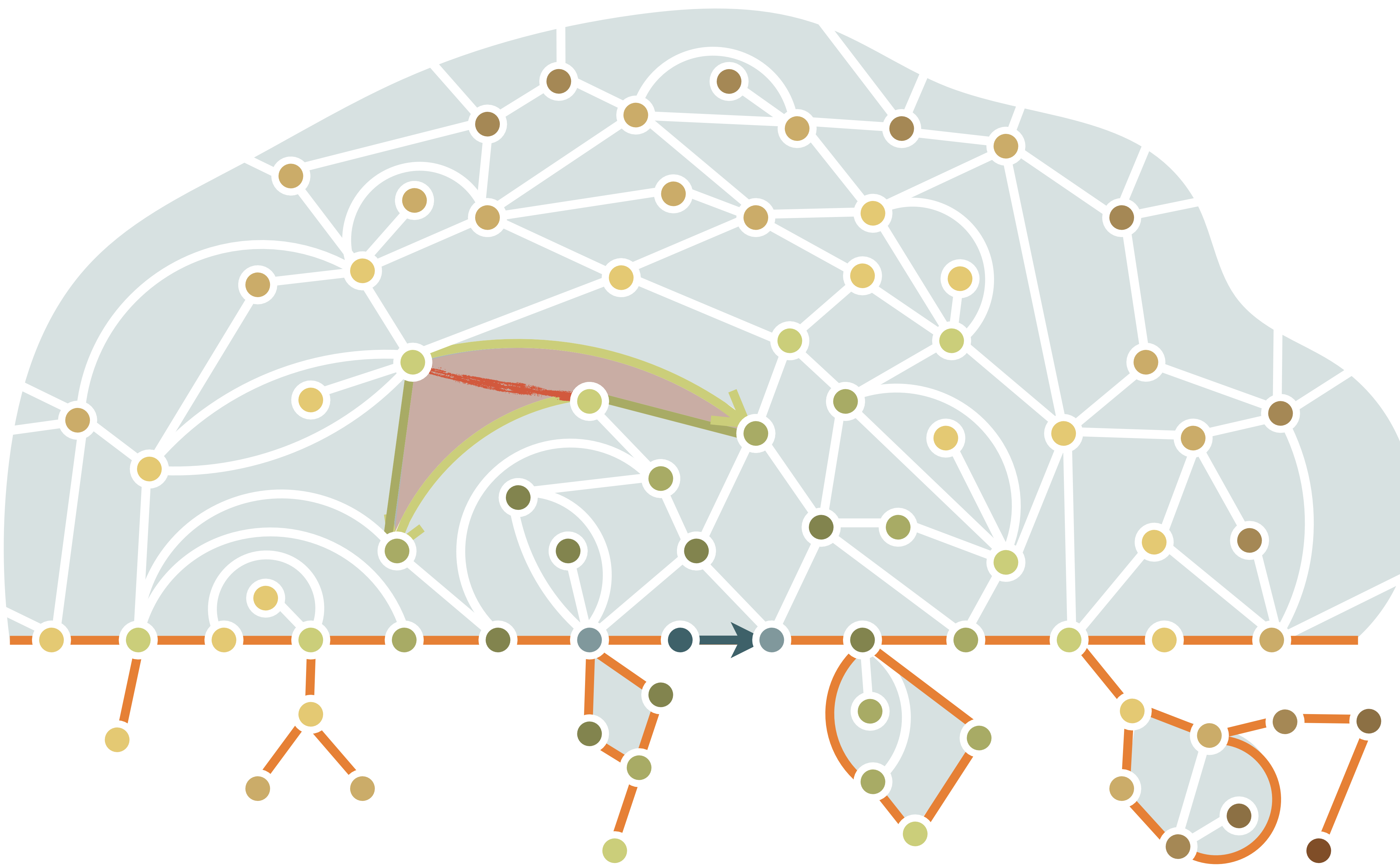
- Can we construct an analogue with a simple boundary?

THE UIHPQ

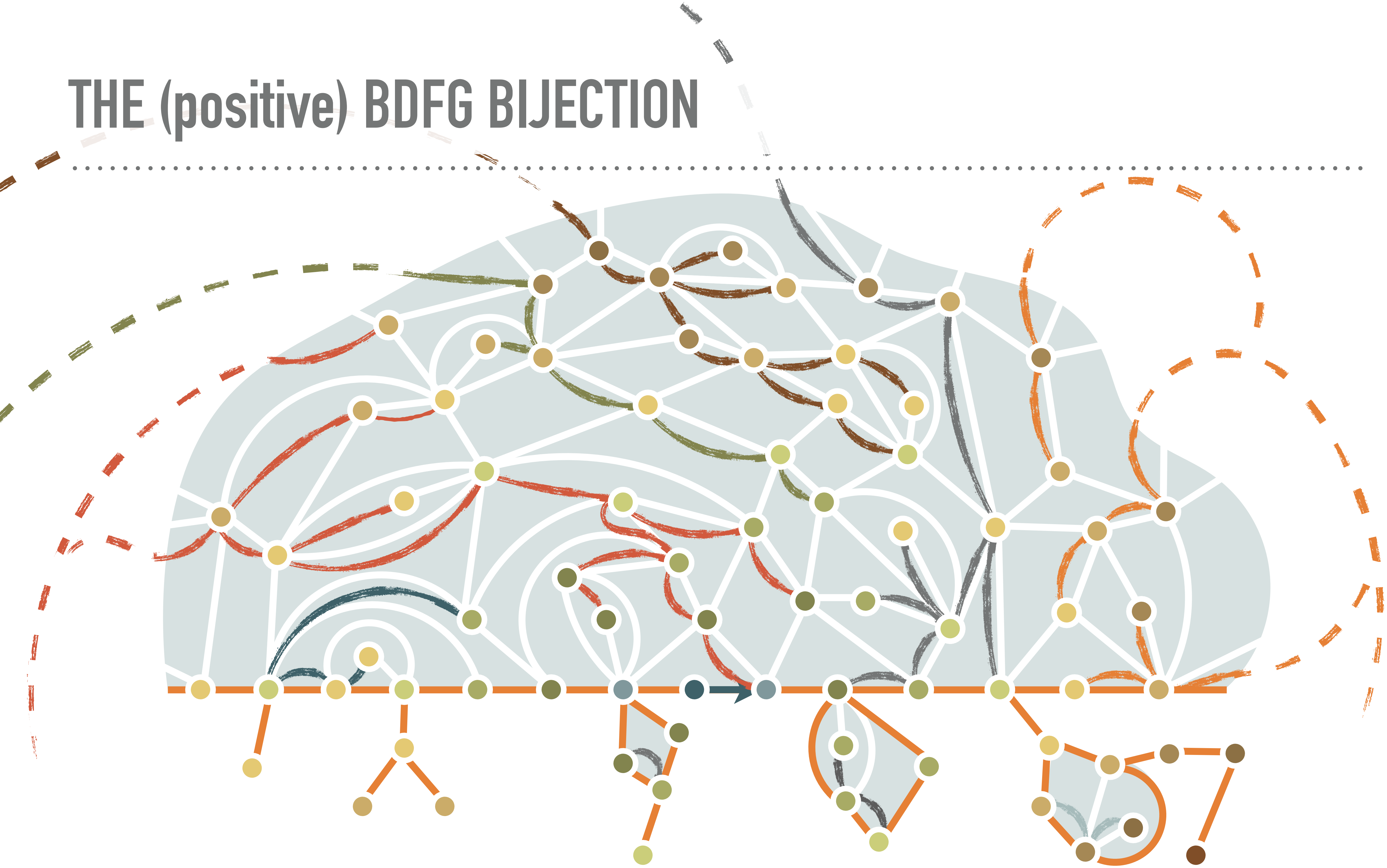
- What does the UIHPQ look like?
- How do distances to the root evolve along the (infinite) boundary?
- Can we construct an analogue with a simple boundary?
- How would it relate to the UIPQ?



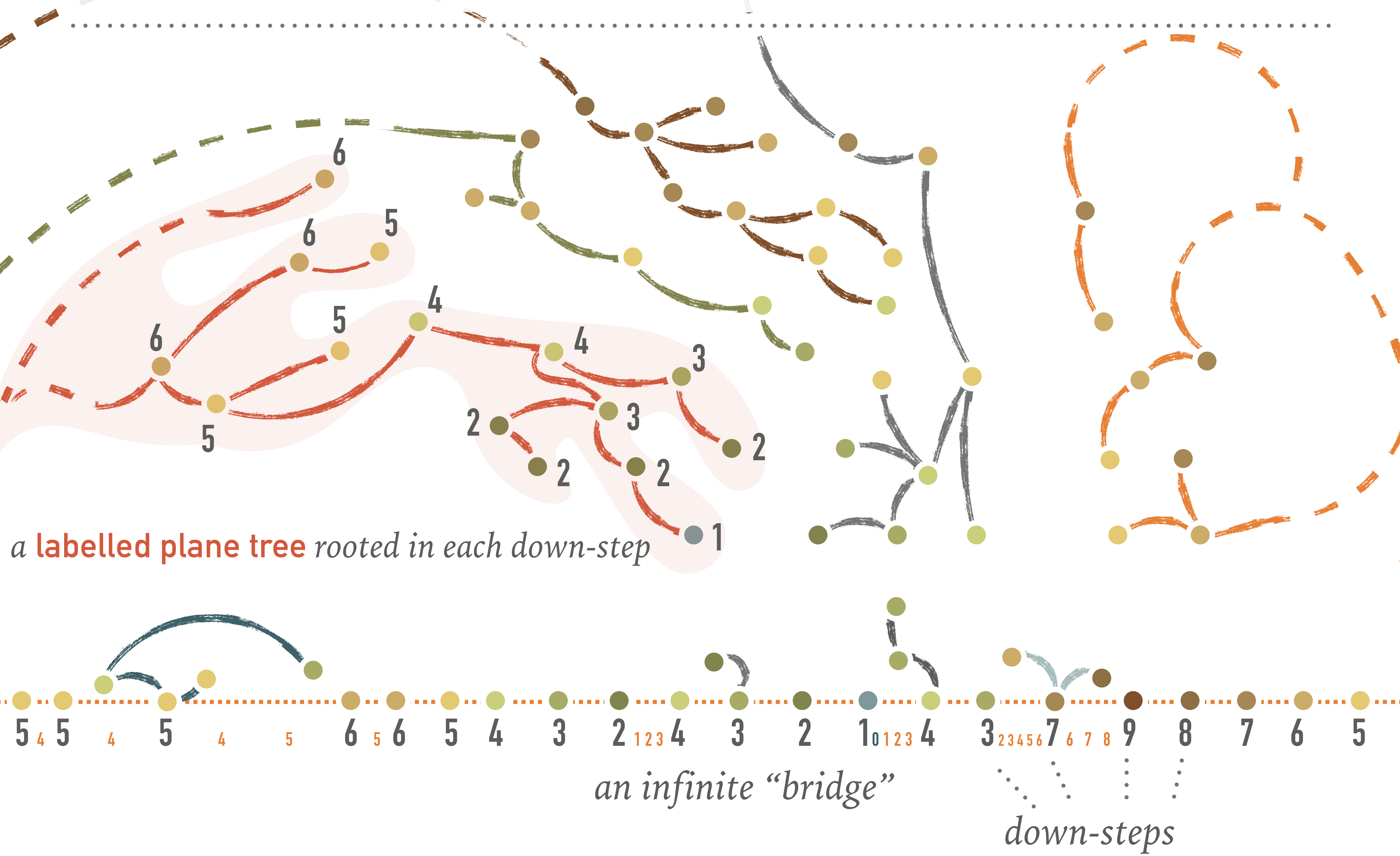
THE (positive) BDFG BIJECTION



THE (positive) BDFG BIJECTION



THE (positive) BDFG BIJECTION



THE UNIFORM INFINITE POSITIVE TREED BRIDGE

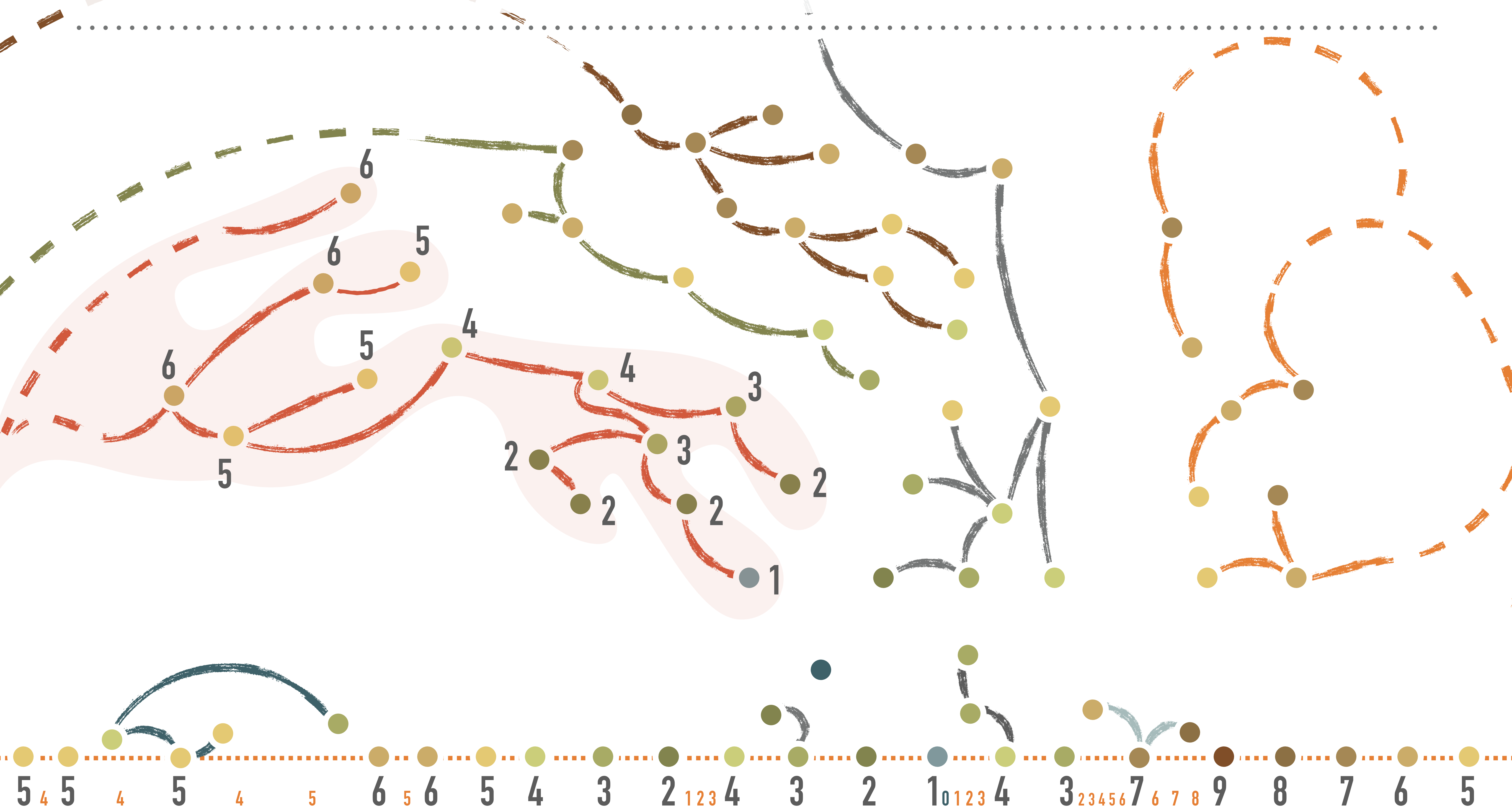
- The UIHPQ can be constructed as encoded by a random infinite treed bridge B_∞,which comprises
  - a random bridge $b=(X_i)_{i\in Z}$ (representing distances from the root vertex as read along the boundary of the UIHPQ)
  - a sequence of random positive labelled trees $(T(i))_{i\in DS(b)}$, where $T(i)$ has root label X_i, and the trees are conditionally independent given the bridge
- The two halves of the bridge $(X_i)_{i\geq 0}$ and $(X_i)_{i\leq 0}$ have the same law up to time-reversal, i.e. that of a Markov chain issued from 0, with transition probabilities given by

$$P(n,n-1)= \frac{n}{2(n+2)} \qquad P(n,n+1)= \frac{n+4}{2(n+2)}$$

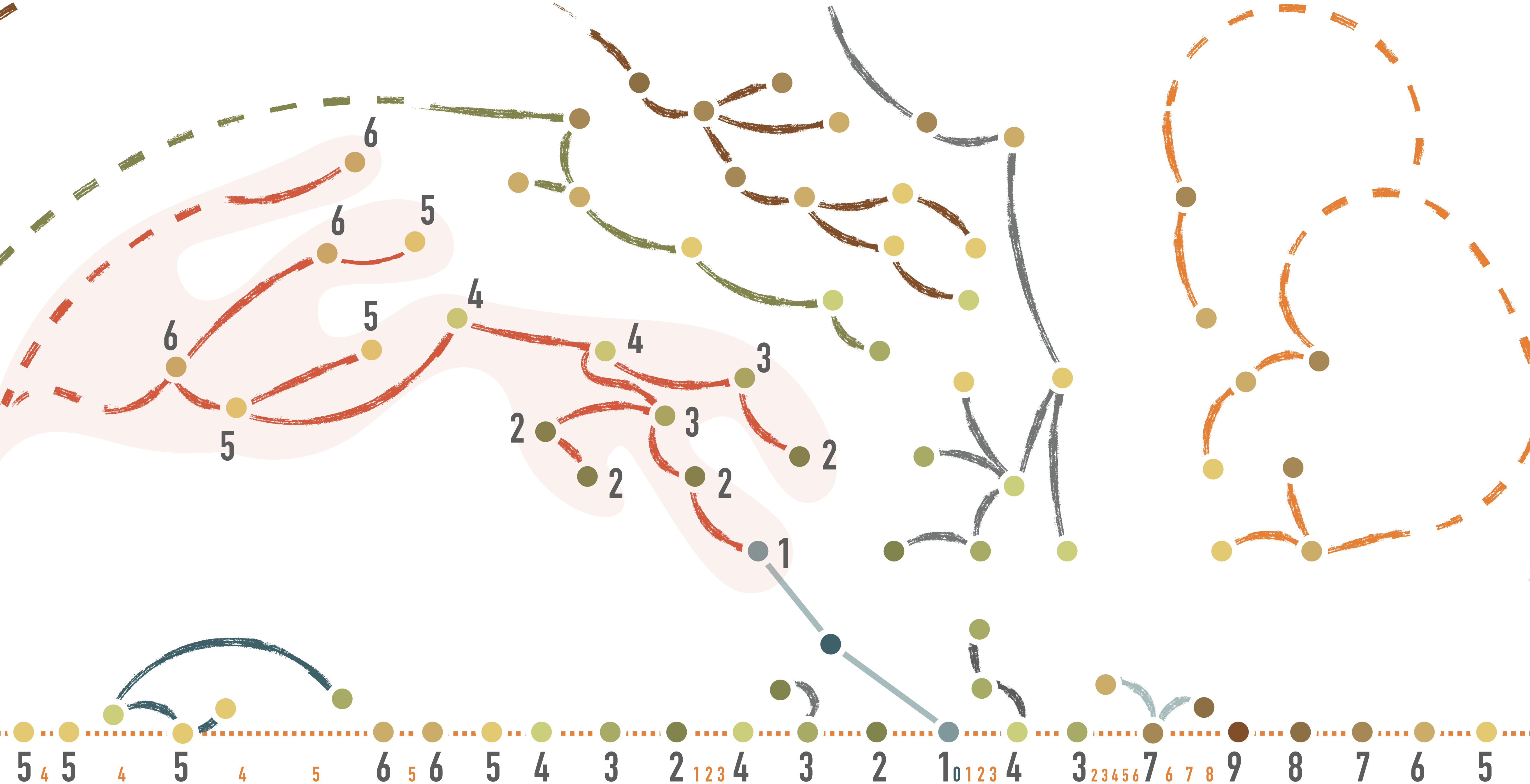
- The scaling limit of the process $(X_i)_{i\geq 0}$ is a Bessel process of dimension 5 issued from 0.

$$(n^{-1/2} X_{[nt]})_{t\in R} \xrightarrow[n\to\infty]{} (Z_t)_{t\in R} \qquad (Z_t)_{i\geq 0} \quad (Z_{-t})_{i\geq 0} \quad \text{Bessel-5}$$

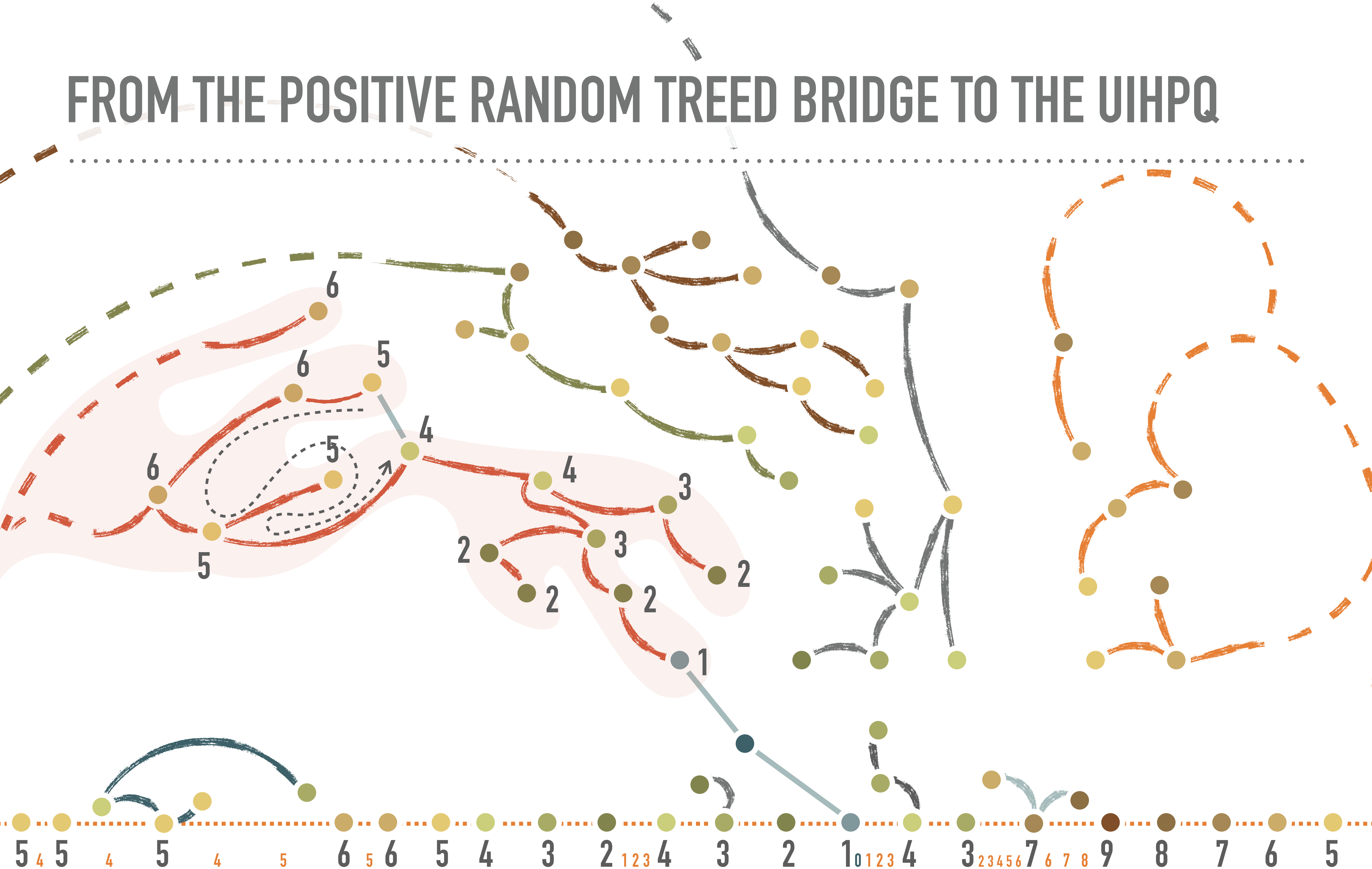
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ



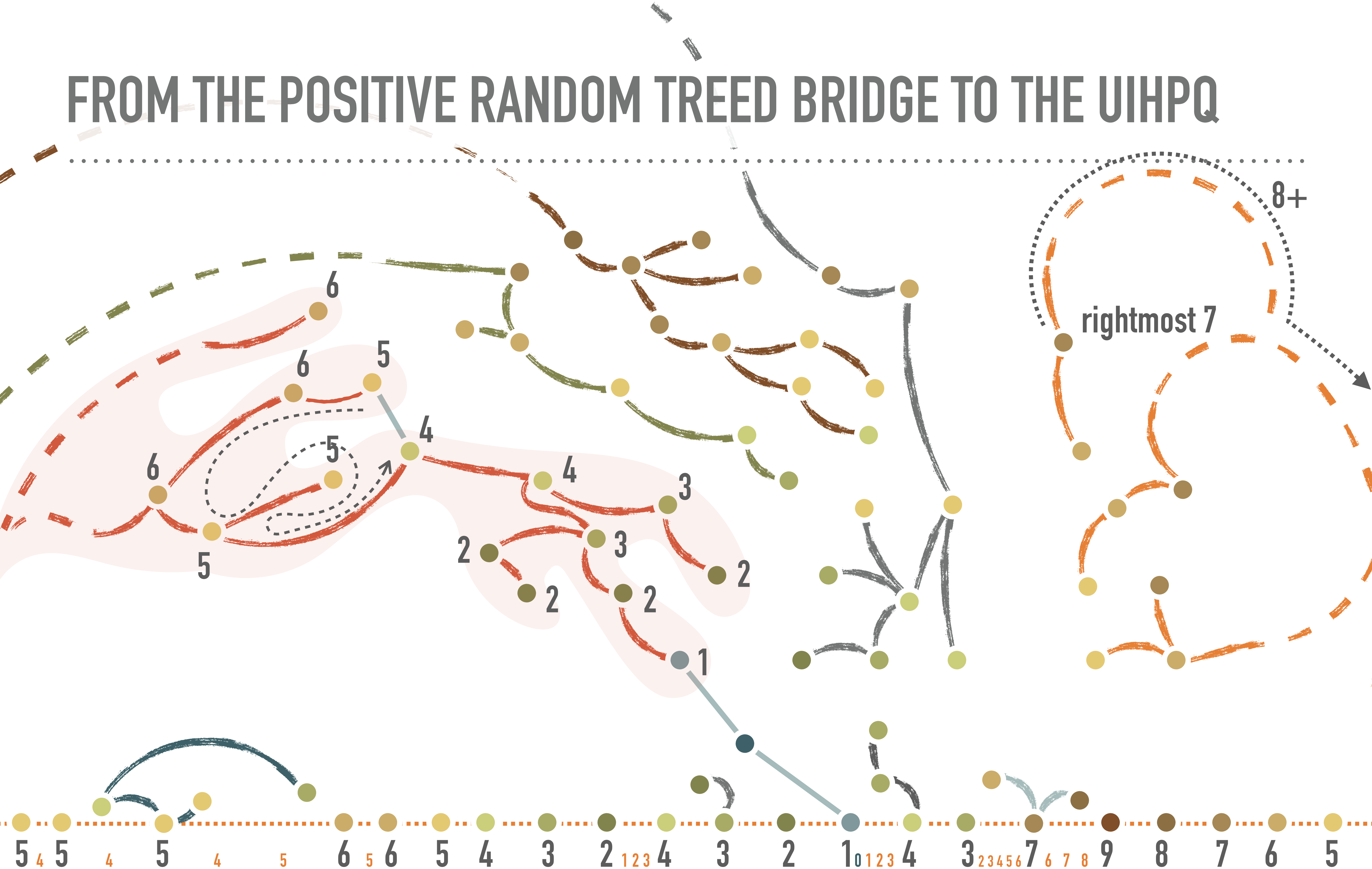
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ



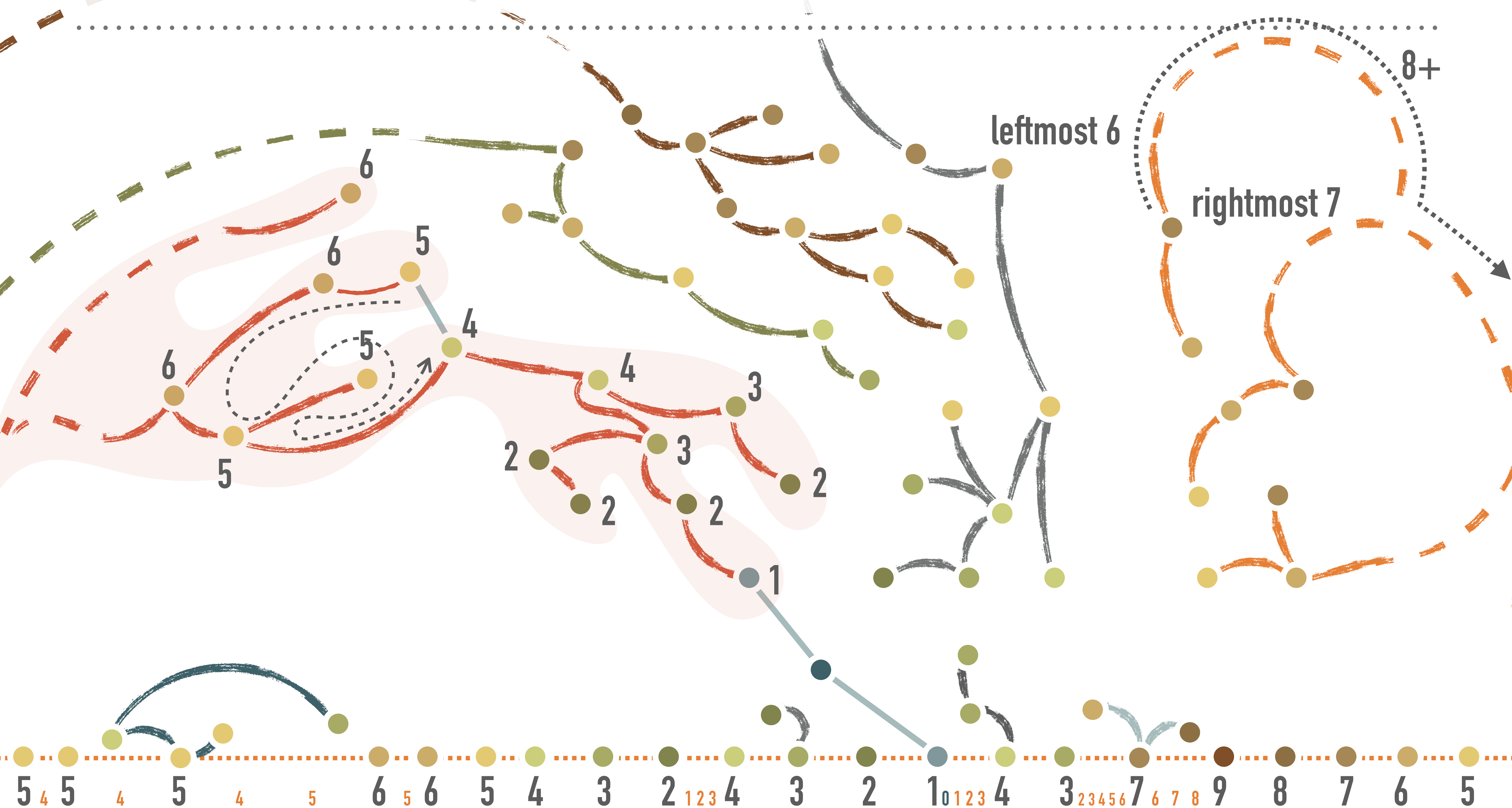
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ



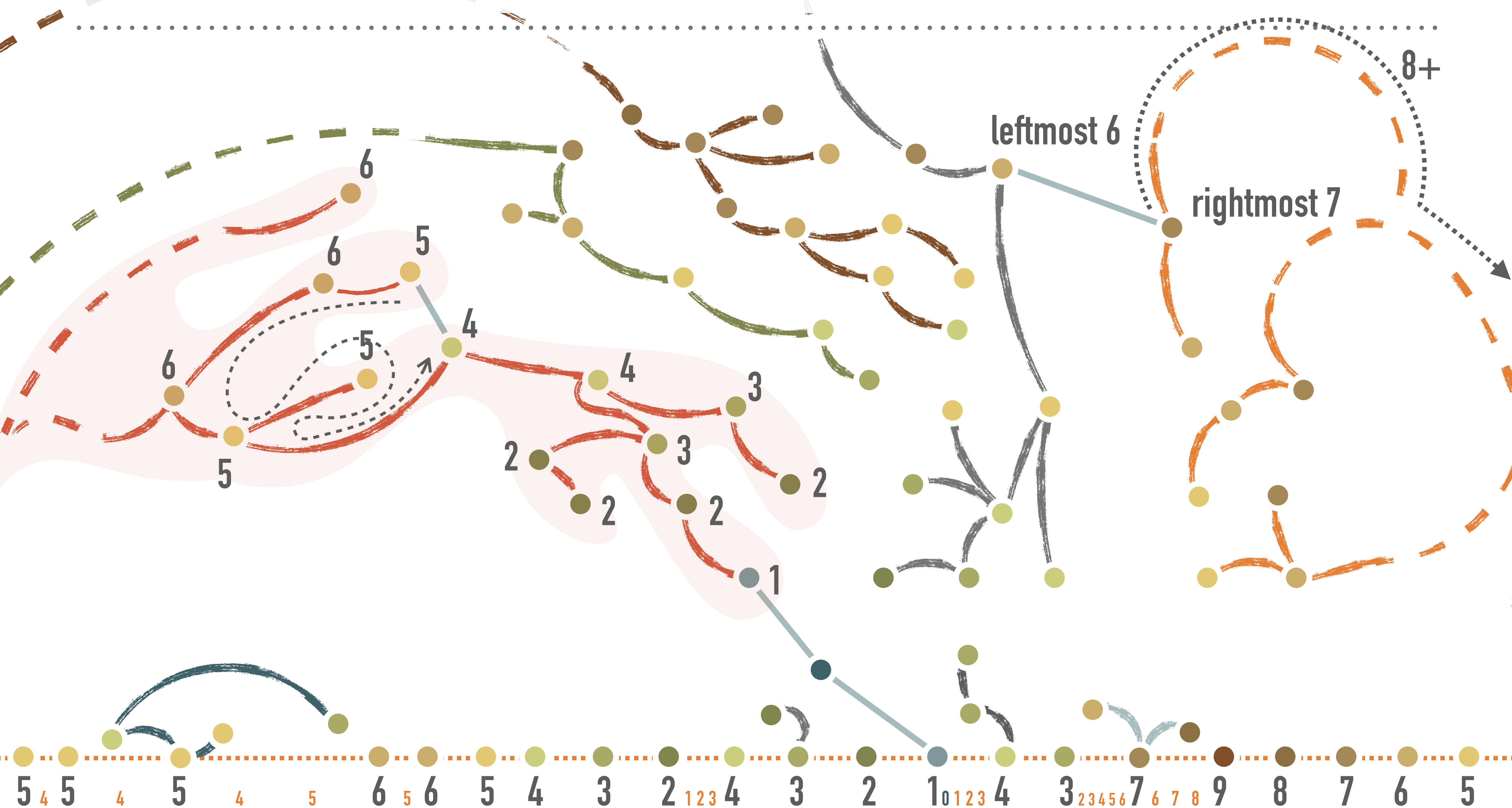
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ



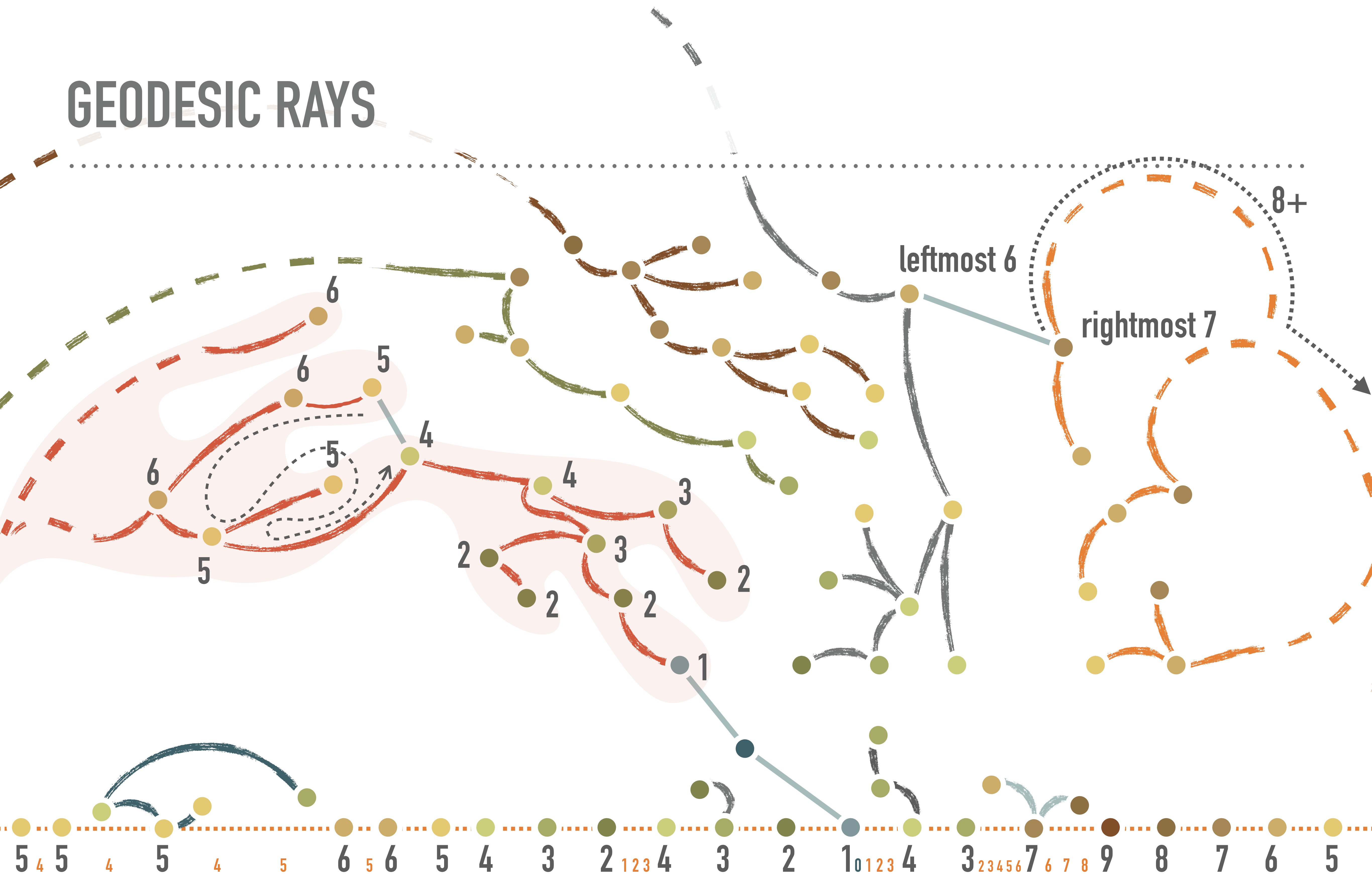
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ



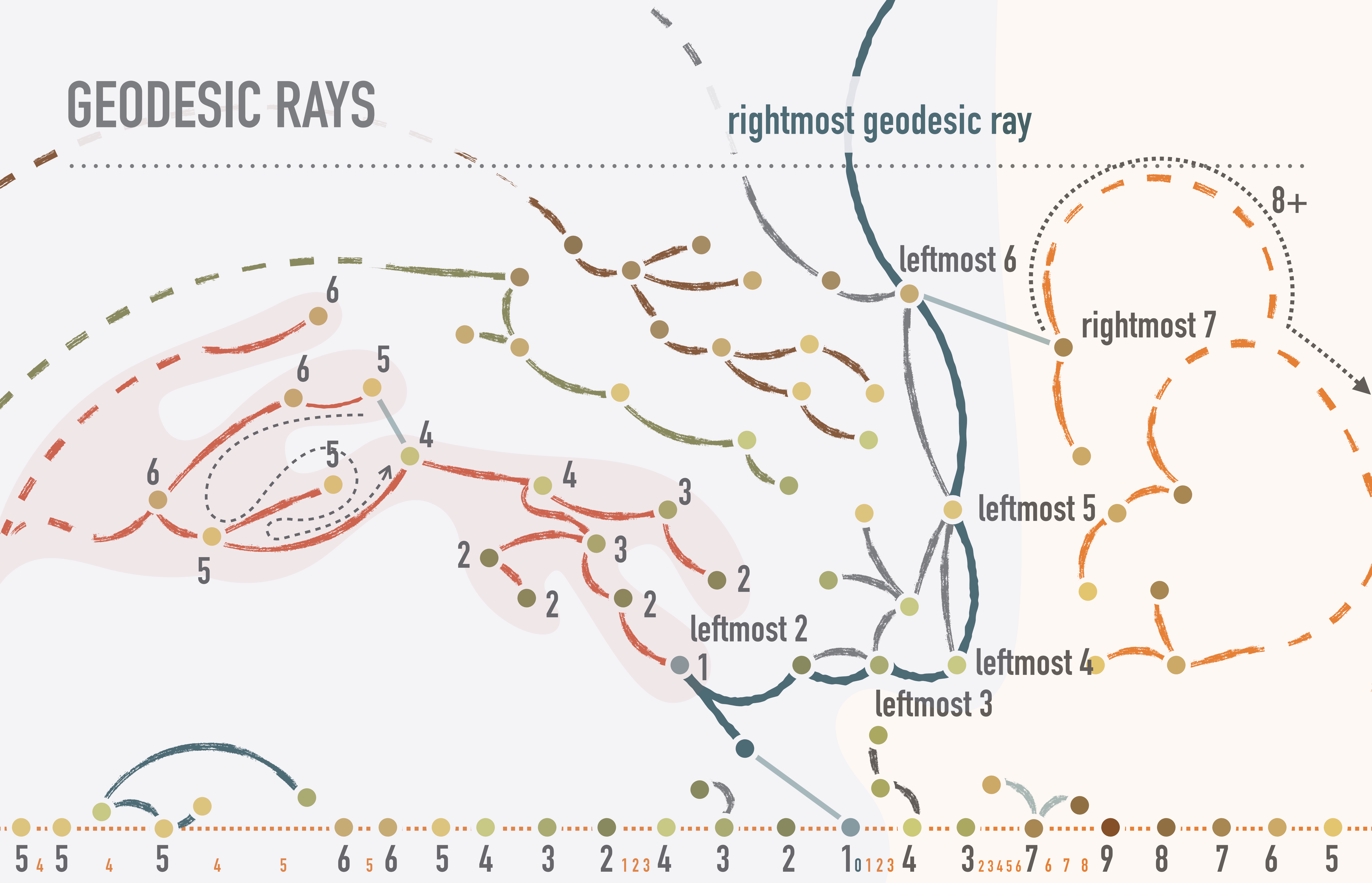
FROM THE POSITIVE RANDOM TREED BRIDGE TO THE UIHPQ

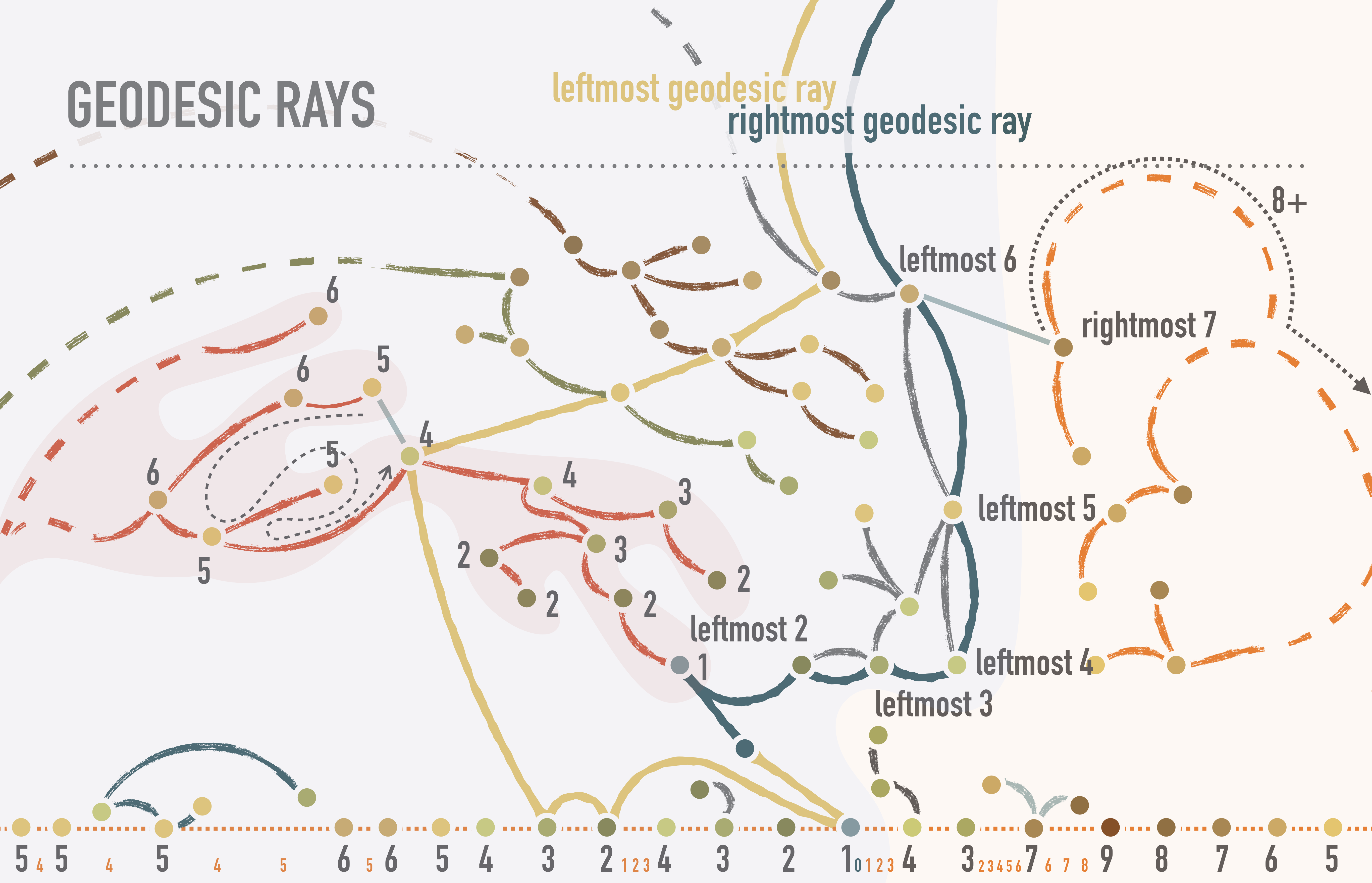


GEODESIC RAYS



GEODESIC RAYS



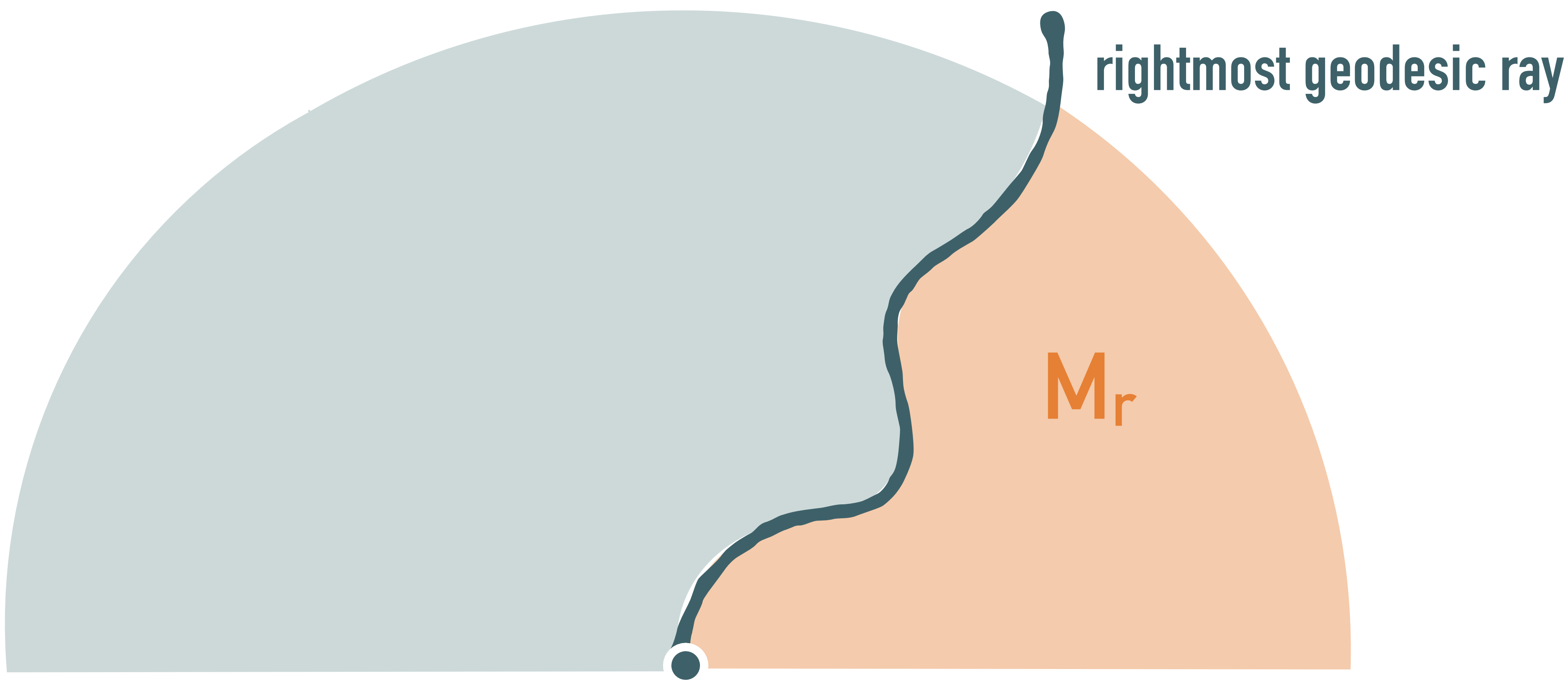


GEODESIC RAYS
leftmost geodesic ray
rightmost geodesic ray
leftmost 6
rightmost 7
8+
leftmost 5
leftmost 2
leftmost 4
leftmost 3
5 4 5 4 5 4 5 6 5 6 5 4 3 2 1 2 3 4 3 2 1 0 1 2 3 4 3 2 3 4 5 6 7 6 7 8 9 8 7 6 5

THE PENCIL DECOMPOSITION

- The UIHPQ has a leftmost and a rightmost geodesic rays, which induce a decomposition into 3 (random) submaps M_l, M_c and M_r. M_l and M_r contain no geodesic rays except for their "right" and "left" boundaries.
- The three random variables M_l, M_c and M_r are independent, and each can be constructed as the image of a certain random treed bridge via the BDFG bijection.

THE PENCIL DECOMPOSITION



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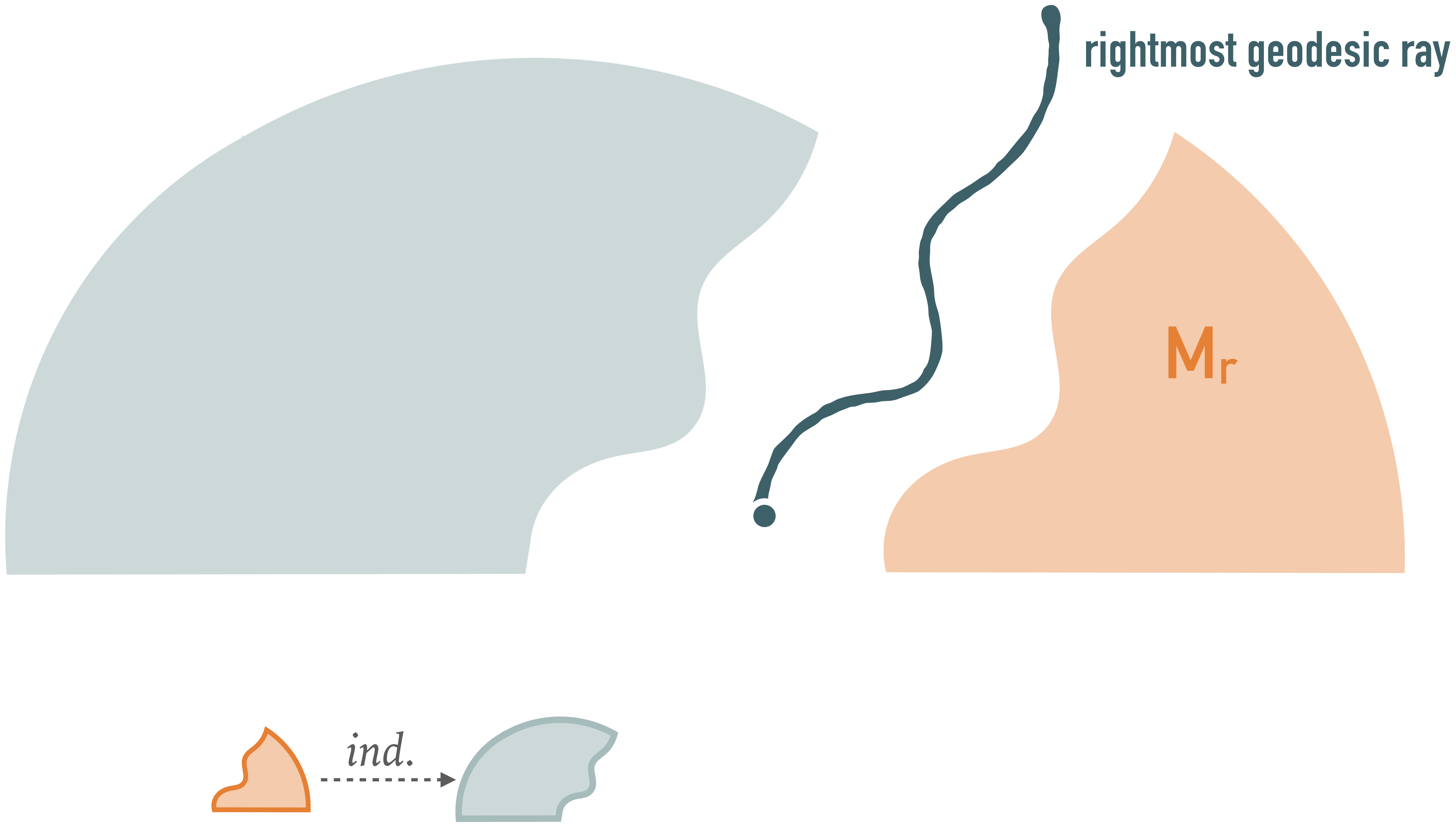

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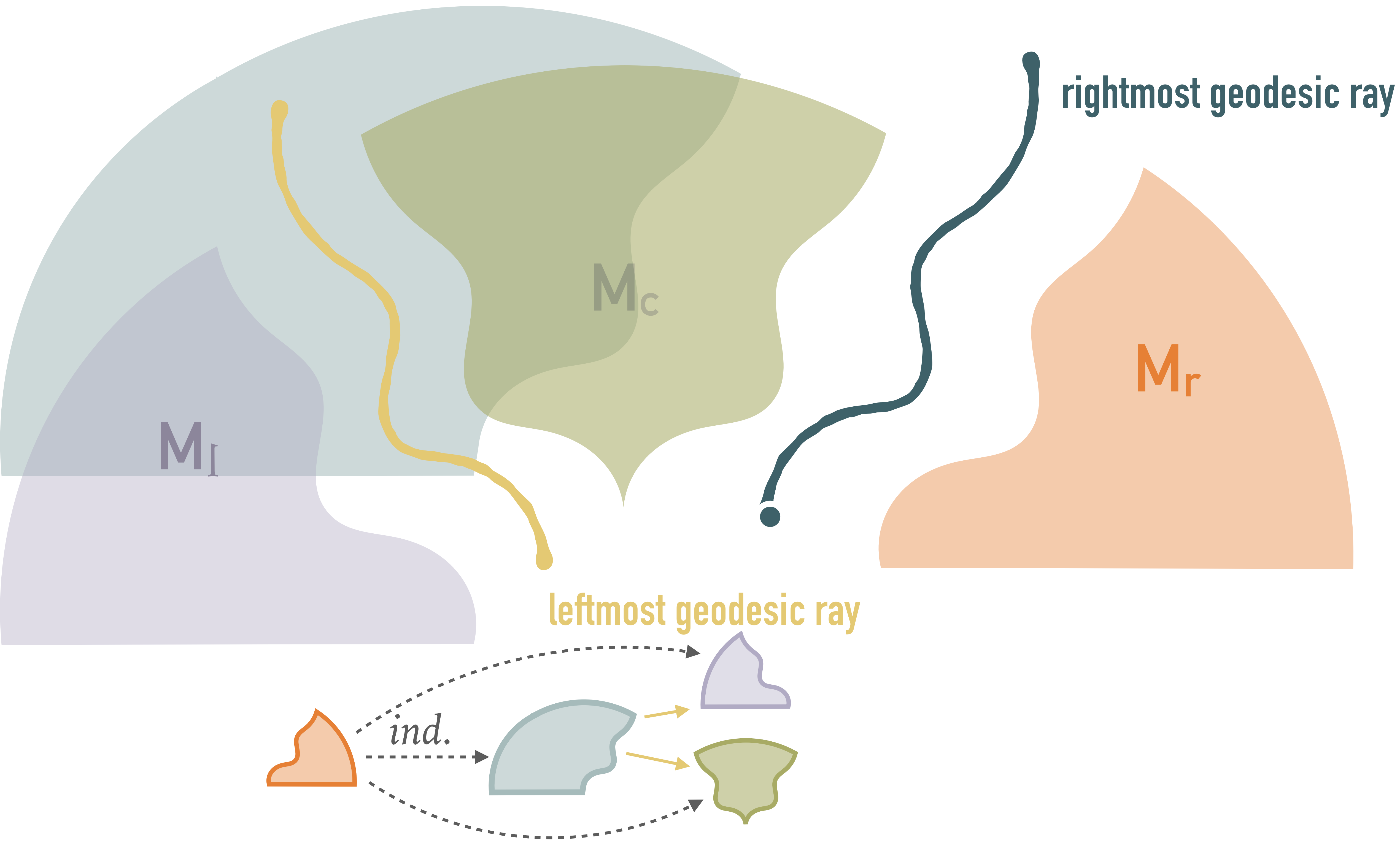

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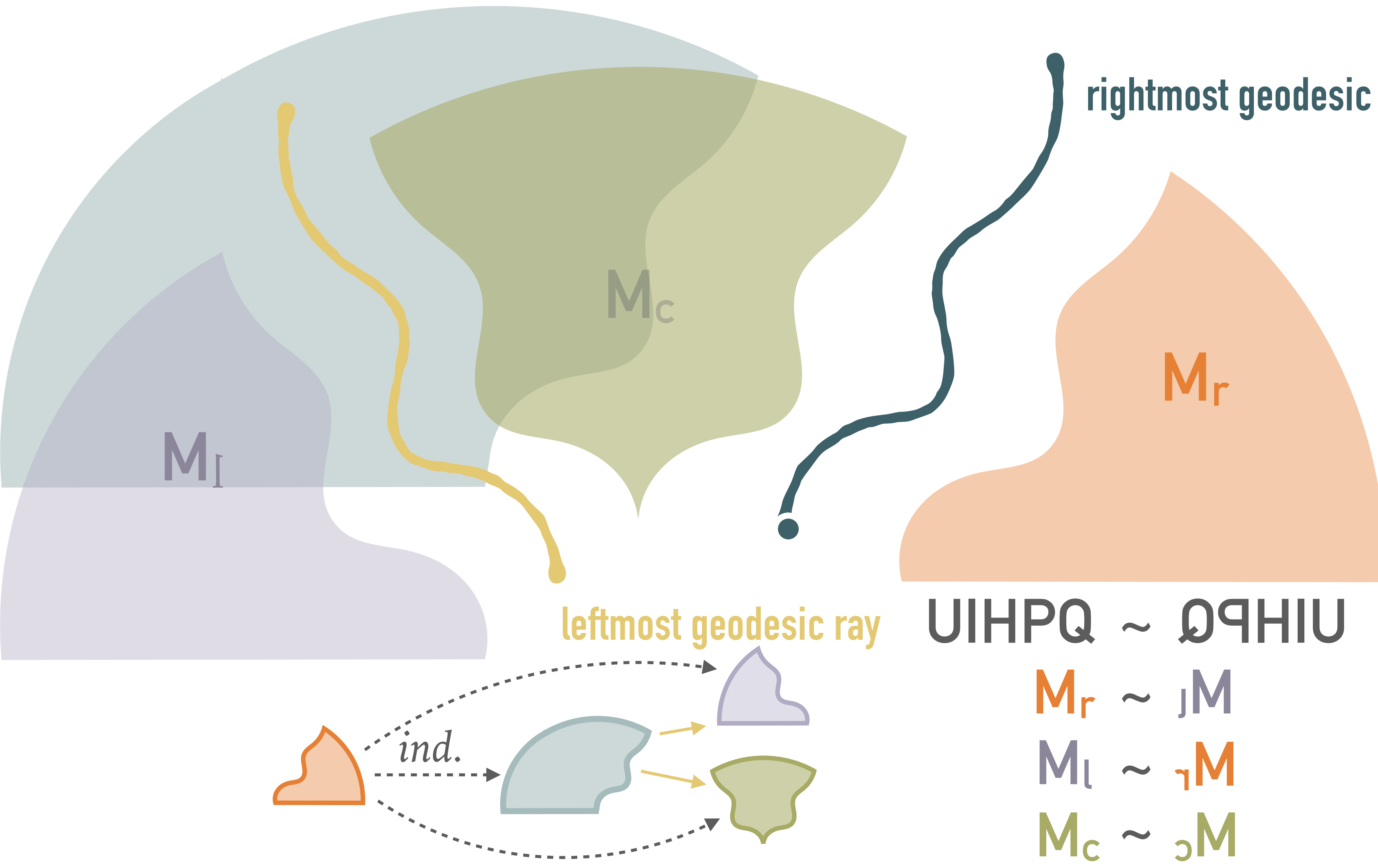
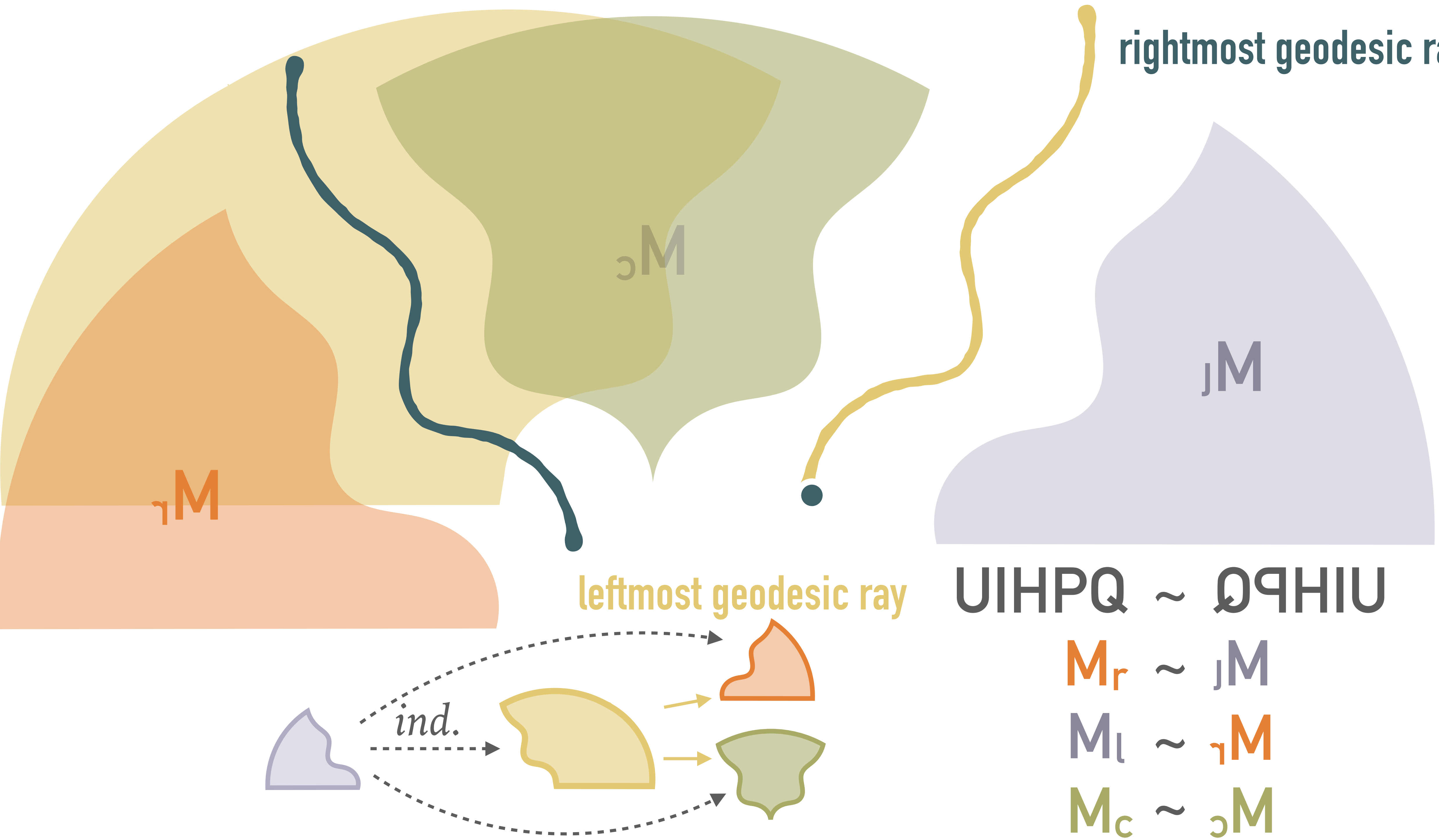
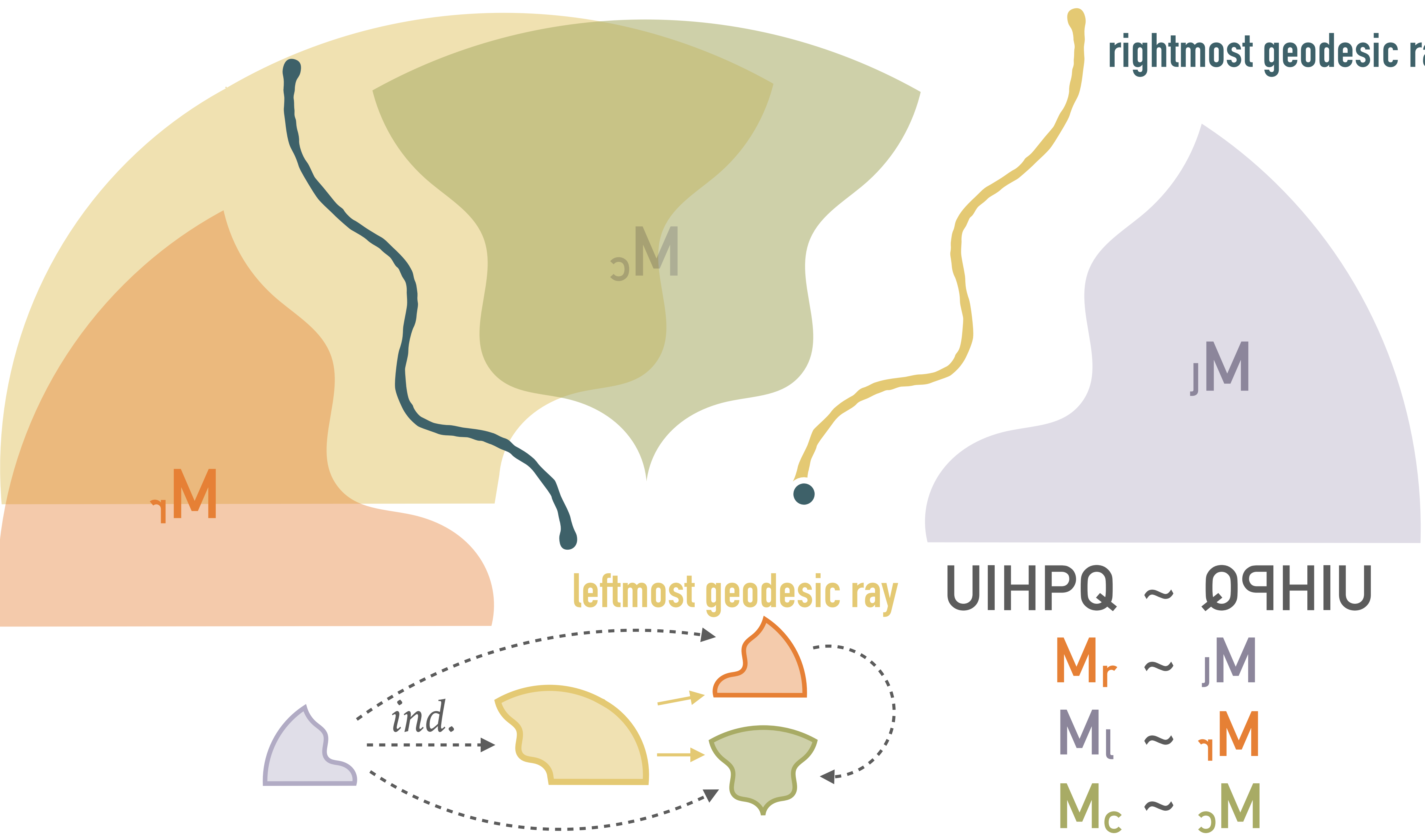


THE PENCIL DECOMPOSITION



THE PENCIL DECOMPOSITION



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- The three random variables M_l, M_c and M_r are **independent**, and each can be constructed as the image of a certain random treed bridge via the BDFG bijection.

- **Do the leftmost and rightmost geodesic rays meet?**

THE PENCIL DECOMPOSITION

- The UIHPQ has a leftmost and a rightmost geodesic rays, which induce a decomposition into 3 (random) submaps M_l, M_c and M_r. M_l and M_r contain no geodesic rays except for their “right” and “left” boundaries.
- The three random variables M_l, M_c and M_r are **independent**, and each can be constructed as the image of a certain random treed bridge via the BDFG bijection.
- The leftmost and rightmost geodesic rays (almost surely) meet an infinite number of times.

THE PENCIL DECOMPOSITION

- The UIHPQ has a leftmost and a rightmost geodesic rays, which induce a decomposition into 3 (random) submaps M_l, M_c and M_r. M_l and M_r contain no geodesic rays except for their “right” and “left” boundaries.
- The three random variables M_l, M_c and M_r are **independent**, and each can be constructed as the image of a certain random treed bridge via the BDFG bijection.
- The leftmost and rightmost geodesic rays (almost surely) meet an infinite number of times.
- **Do they also meet the boundary an infinite number of times, or do they eventually leave it?**

“

Thank You.