Spatial random permutations

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TU Darmstadt

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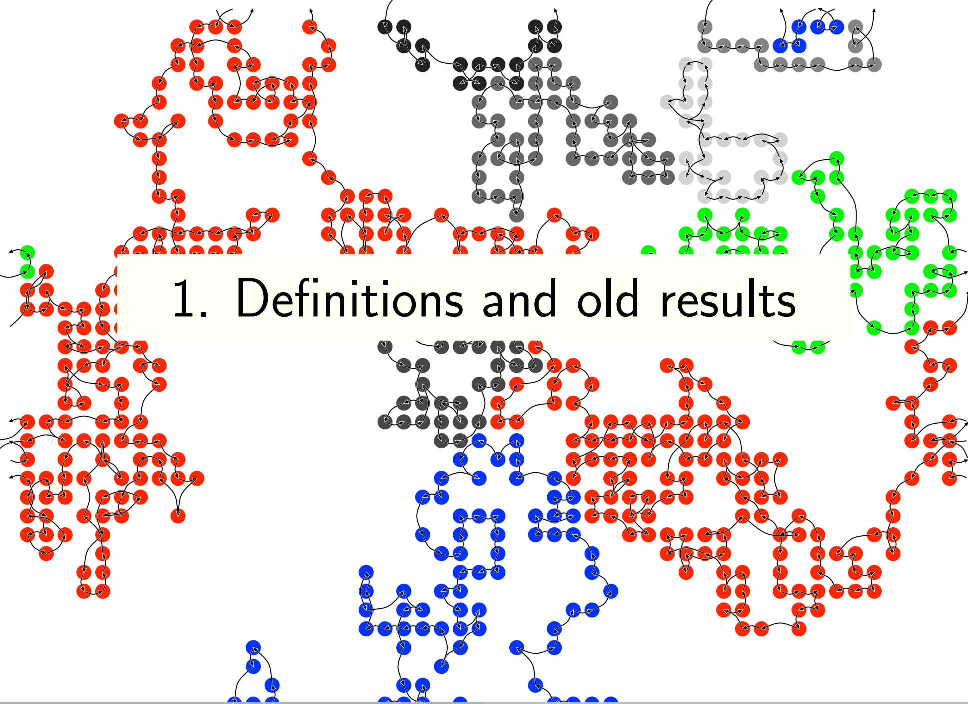
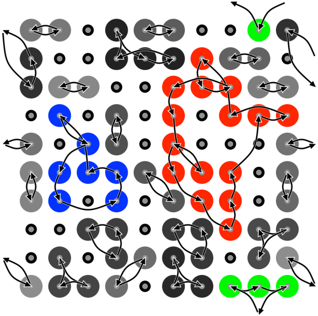


1. Definitions and old results

Spatial random permutations: finite volume



- $X \subset \mathbb{R}^d$ locally finite set,
  $\Lambda \subset \mathbb{R}^d$ bounded domain,
  $X_\Lambda = X \cap \Lambda$.
- Periodic boundary conditions.
- $\mathcal{S}_\Lambda$ = set of permutations
  $\pi : X_\Lambda \to X_\Lambda$.
- Typical example for a measure on $\mathcal{S}_\Lambda$:

$$\mathbb{P}_\Lambda(\{\pi\}) = \frac{1}{Z(\Lambda)} \exp\Big(- \alpha \sum_{x \in \Lambda} |\pi(x) - x|^2 \Big).$$

- Penalization parameter α determines expected jump length.

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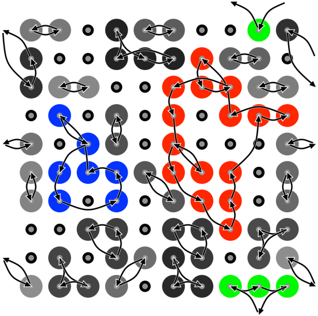

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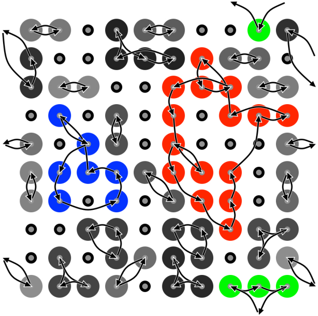

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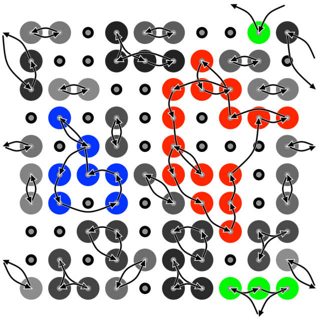

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- First question: Existence of the infinite volume limit.
- Exciting questions: Existence and geometry of long cycles.

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- None of the Gibbs measures techniques work!

Infinite cycles: a phase transition

$$\mathbb{P}_\Lambda(\pi) = \frac{1}{Z(\Lambda)} \exp\Big(-\alpha \sum_{x \in X_\Lambda} \big|\pi(x) - x\big|^2 \Big)$$

- Fix a point x (e.g. the origin). Write $C_x(\pi)$ for the cycle of π containing x.
- Question: How long is C_x typically?

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$$\exists \alpha_c > 0 : \quad p_\alpha := \lim_{K \to \infty} \liminf_{|\Lambda| \to \infty} \mathbb{P}_\Lambda(|C_x| > K) > 0 \quad \text{iff } \alpha < \alpha_c$$

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- We do not even know monotonicity of p_α.
- Only result so far: in $d = 1$ with convex potential, there is no (nontrivial) phase transition. [Biskup, Richthammer 2014].

SRP: what is known

$$\mathbb{P}_\Lambda(\pi) = \frac{1}{Z(\Lambda)} \exp\Big(- \alpha \sum_{x \in X_\Lambda} \xi(\pi(x) - x)\Big)$$

- [B. 14]: Existence of the infinite volume limit if X is a regular lattice with periodic bc, and if for some $\delta > 0$:

$$\sum_{x \in X} \mathrm{e}^{-(\alpha-\delta)\xi(x)} < \infty$$

- [B., Ueltschi 09]: Absence of infinite cycles for large α.
- [Biskup, Richthammer 14]: Rather complete theory for $d = 1$ and convex ξ.
- [B., U. 09-11]: Phase transition for the **annealed model**:

$$\mathbb{P}_{L,N}(\pi) = \frac{1}{Z(L)N!} \int_{[-L,L]^{dN}} \exp\Big(-\alpha \sum_{i=1}^{N} \xi(x_{\pi(i)} - x_i)\Big) \prod_{i=1}^{N} \mathrm{d}x_i$$

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Assume positivity of the Fourier transform of $\mathrm{e}^{-\xi}$.

Define $\varepsilon(k)$ through $\mathrm{e}^{-\varepsilon(k)} = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} kx} \, \mathrm{e}^{-\xi(x)} \, \mathrm{d}x$,

$\ell^{(j)}(\pi) =$ the length of the j-th longest cycle in π.

Critical density: $\rho_c := \int_{\mathbb{R}^d} \frac{1}{\mathrm{e}^{\varepsilon(k)} - 1} \, \mathrm{d}k \leqslant \infty.$

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Theorem: [B.-Ueltschi 2011]

a) The expected fraction of points in infinite cycles is

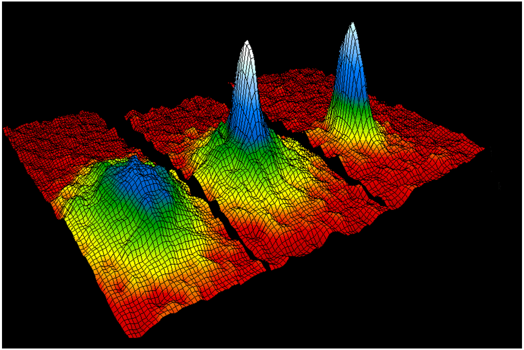
$$\lim_{K \to \infty} \lim_{V,N \to \infty, N/V = \rho} \mathbb{E}\Big(\frac{1}{N} \sum_{j : \ell^{(j)} > K} \ell^{(j)} \Big) = \nu = \max\Big(0, 1 - \frac{\rho_\mathrm{c}}{\rho}\Big).$$

b) For $\nu > 0$, long cycles are Poisson-Dirichlet distributed:

$$\lim_{V,N \to \infty} \Big(\frac{\ell^{(1)}}{\nu N}, \frac{\ell^{(2)}}{\nu N}, \dots \Big) = \mathrm{PD}(1) \qquad \text{in distribution.}$$

2. From BEC to SRP (and back ?)

Bose-Einstein condensation



Very cold quantum gases (e.g. ^{23}Na) behave radically different from classical gases:
A finite fraction of particles will be in the quantum state with momentum 0. (Bose-Einstein Kondensation)
Classical gases: Boltzmann-distribution.

Many body quantum mechanics at positive temperature

- Hamilton-Operator for N particles with pair potential U on $\Lambda^N \subset \mathbb{R}^{dN}$, periodic b.c.:

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i<j \leqslant N} U(x_i - x_j).$$

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- At positive temperature $1/\beta$ the **density matrix** $e^{-\beta \boldsymbol{H}}$ describes the system: the expected value of an observable A (self-adjoint operator) is given by

$$\langle A \rangle_\beta = \frac{\mathrm{Tr}_{\,\mathrm{Symm}}(A\,\mathrm{e}^{-\beta \boldsymbol{H}}\,)}{\mathrm{Tr}_{\,\mathrm{Symm}}(\,\mathrm{e}^{-\beta \boldsymbol{H}}\,)} = \frac{\mathrm{Tr}\,(S A\,\mathrm{e}^{-\beta \boldsymbol{H}}\,)}{\mathrm{Tr}\,(S\,\mathrm{e}^{-\beta \boldsymbol{H}})}$$

(S is symmetrisation operator, A commutes with S.)

From BEC to SRP: trace formula

We want an expression for $\mathrm{Tr}\ \mathrm{e}^{-\beta H}$ for all $\beta > 0$.

Trace formula:

$$\mathrm{Tr}\,(\,\mathrm{e}^{-\beta H}\,) = \int K_\beta(x,x)\,\mathrm{d}x,$$

where K_β is the integral kernel of $\mathrm{e}^{-\beta H}$:

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Symmetrisation: $\boldsymbol{H}$ on $L^2_{\mathrm{symm}}(\Lambda^N)$,
integral kernel $K_\beta(\boldsymbol{x},\boldsymbol{y})$ of $\mathrm{e}^{-\beta \boldsymbol{H}}$, $\boldsymbol{x} = (x_1, \dots, x_N)$:

$$\mathrm{Tr}_{\,\mathrm{Symm}}(\,\mathrm{e}^{-\beta \boldsymbol{H}}\,) = \mathrm{Tr}\,(S\,\mathrm{e}^{-\beta \boldsymbol{H}}\,) = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \int_{\Lambda^N} K_\beta(\boldsymbol{x}_\pi, \boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

with $\boldsymbol{x}_\pi = (x_{\pi(1)}, \dots, x_{\pi(N)})$.

From BEC to SRP: Feynman-Kac formula

For a Schrödinger operator $H = -\Delta + V$ with e.g. $V \in L^\infty$:

$$\mathrm{e}^{-\beta H}(x,y) = \frac{1}{(8\pi\beta)^{3/2}} \mathrm{e}^{-|x-y|^2/8\beta} \int \mathrm{e}^{-\int_0^{4\beta} V(\omega_s)\,\mathrm{d}s}\, \widehat{\mathcal{W}}_{x,y}^{4\beta}(\mathrm{d}\omega),$$

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$$\mathrm{Tr}\,(S\,\mathrm{e}^{-\beta \boldsymbol{H}}) = \frac{1}{N!(8\pi\beta)^{dN/2}} \sum_{\pi\in\mathcal{S}_N} \int_{\Lambda^N} \mathrm{e}^{-\frac{1}{8\beta}\sum_{i=1}^N |x_i - x_{\pi(i)}|^2} \,\mathrm{e}^{H_{\mathrm{I}}(\boldsymbol{x},\pi)} \prod_{i=1}^N \mathrm{d}x_i,$$

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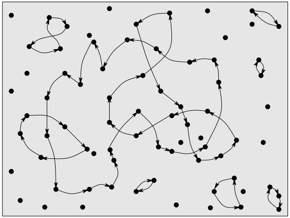
with $\Lambda = [-L, L]^d$ and

$$\mathrm{e}^{H_{\mathrm{I}}(\boldsymbol{x},\pi)} = \Big[\prod_{i=1}^{N} \int \mathrm{d}\widehat{W}^{4\beta}_{x_i, x_{\pi(i)}}(\omega_i)\Big] \, \mathrm{e}^{-\sum_{1 \leqslant i<j \leqslant N} \int_0^{4\beta} U(\omega_i(s)-\omega_j(s))\mathrm{d}s} .$$

Connection to SRP



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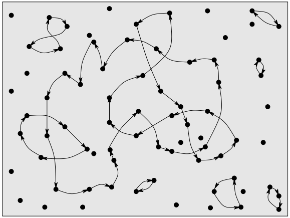

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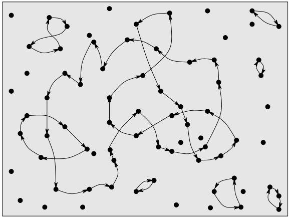

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$C_1(\pi) :=$ Length of the cycle containing 1. Feynmans claim:

$$\text{BEC} \;\Leftrightarrow\; \exists \varepsilon > 0 : \liminf_{N\to\infty} \mathbb{P}_N(C_1 > \varepsilon N) > 0.$$

For $U = 0$ we can show this, what about $U \neq 0$?

Expected fraction of particles in (free) ground state

Number operator wrt. $\phi \in L^2(\Lambda)$:

$$[\boldsymbol{N}_\phi \psi](x_1, \ldots, x_N) = \sum_{j=1}^{N} \phi(x_j) \Big\langle \phi, \psi(x_1, \ldots, x_{j-1}, \cdot, x_{j+1}, \ldots, x_N \Big\rangle_{L^2(\Lambda)}$$

measures 'total overlap' of particles in ψ with ϕ.

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$$\text{BEC} \quad \Leftrightarrow g_{\rho,\beta} \neq 0.$$

Permutations with open cycles

Daniel Ueltschi [PRL 97, 170601 (2006)] observed:

$$\mathrm{Tr}\,(\boldsymbol{N}_\phi\, \mathrm{e}^{-\beta H}\, S) = \frac{1}{\rho V^2} \int_{\Lambda^2} \mathrm{d}x \mathrm{d}y\, Y_{x\to y}(\beta, N, V),$$

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We have

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ODLRO, open cycles and infinite cycles

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If things are nice, we expect:

$g_{\rho,\beta} > 0 \Leftrightarrow \dfrac{Y_{x\to y}(\beta,N,V)}{Y(\beta,N,V)}$ does not decay as $|x-y| \to \infty$

(this is ODLRO in a different language)

$\Leftrightarrow$ The large N asymptotics of the two partition functions are comparable uniformly in $|x-y|$

$\Leftrightarrow$ Cycles connecting x and y are not rare even when not enforced, uniformly in $|x-y|$.

$\Leftrightarrow$ Annealed SRP has infinite cycles

There is no rigorous proof of these connections.

Back to lattice SRP: lattice Bosons

Force the Bosons to live on a lattice $\mathbb{Z}^d \cap \Lambda$:

$$\boldsymbol{H} = -\sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i<j \leqslant N} U(x_i - x_j),$$

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Feynman-Kac-representation:

$$\mathbb{P}_N(\{\pi\}) := \frac{1}{Z_N N!} \sum_{\boldsymbol{x} \in \Lambda^N \cap \mathbb{Z}^{Nd}} \prod_{i=1}^{N} p_\beta(x_i, x_{\pi(i)}) \, \mathrm{e}^{H_{\mathrm{I}}(\boldsymbol{x},\pi)} \,.$$

$p_\beta(x, y)$ ist the transition kernel of continuous time RW.

Balint Toth (93): Representation of the hard core Bose-Gas via an ensemble of self-avoiding random walks.

A radical simplification

$$\mathbb{P}_N(\{\pi\}) := \frac{1}{Z_N N!} \sum_{\boldsymbol{x} \in \Lambda^N \cap \mathbb{Z}^{Nd}} \prod_{i=1}^{N} p_\beta(x_i, x_{\pi(i)})\, \mathrm{e}^{H_\mathrm{I}(\boldsymbol{x},\pi)} .$$

with

$$\mathrm{e}^{H_\mathrm{I}(\boldsymbol{x},\pi)} = \Big[\prod_{i=1}^{N} \int \mathrm{d}\widehat{Q}^{2\beta}_{x_i, x_{\pi(i)}}(\omega_i)\Big]\, \mathrm{e}^{-\sum_{1 \leqslant i<j \leqslant N} \int_0^{2\beta} U(\omega_i(s)-\omega_j(s))\mathrm{d}s} .$$

Radical Simplification: Replace the term $\mathrm{e}^{H_I(\boldsymbol{x},\pi)}$ by the condition that the particles do not meet at the beginning and the end of the run time β only (see also Feynman 1953!).

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$$\tilde{\mathbb{P}}_N(\{\pi\}) := \frac{1}{Z_N N!} \sum_{\boldsymbol{x} \in A_N} \prod_{i=1}^{N} p_\beta(x_i, x_{\pi(i)}),$$

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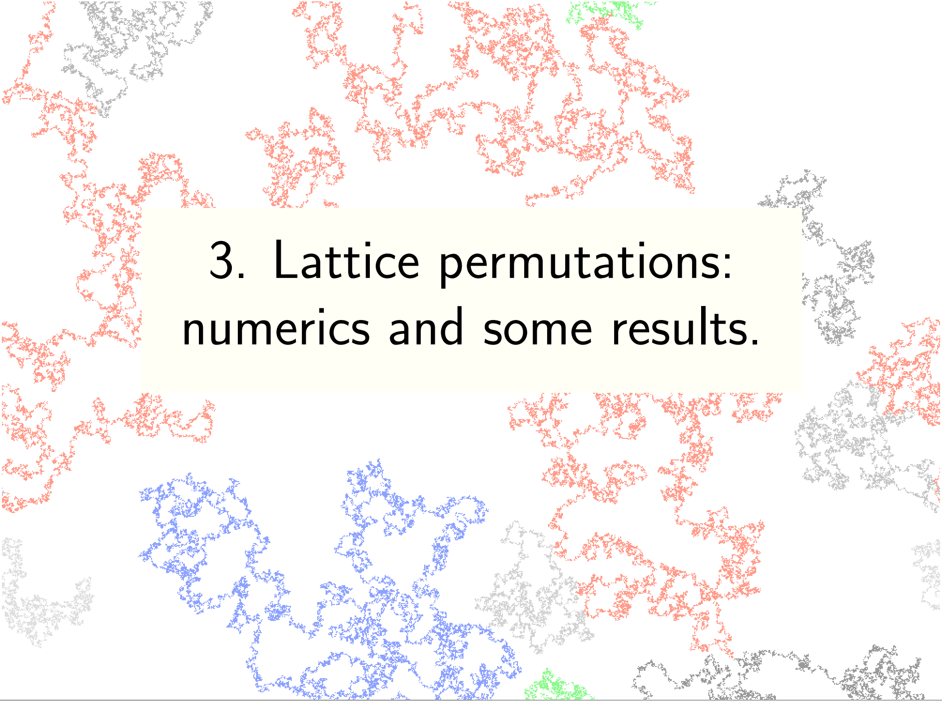
Lattice SRP is this model at 'full filling', i.e. exactly as many particles as there are places.

3. Lattice permutations: numerics and some results.



SRP and Self-avoiding random walks

- Nearest neighbor SRP with forced long cycle, $\Lambda = [-L, L]^d$

$$\mathbb{P}_\Lambda(\{\pi\}) = \frac{1}{Z(\Lambda)} \exp\Big(-\alpha \sum_{x\in\Lambda} |\pi(x) - x|^2\Big) 1_{\{|\pi(x)-x| \leqslant 1\}},$$

with the condition that $\pi((L/2, 0, \dots, 0)) = (0, \dots, 0)$.

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γ self-avoiding path of length $|\gamma|$ from 0 to $\boldsymbol{L}$,

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- [Duminil-Copin, Kozma, Yadin '12]: γ is 'weakly space filling' as $L \to \infty$ if $\mathrm{e}^{\alpha} < \mu =$ connective constant of SARW.
- [B., Taggi '16]: $\exists \, \alpha_0$ with $\mathrm{e}^{\alpha_0} < \mu$, such that $\forall \alpha > \alpha_0$, there are no infinite cycles in the **standard** nearest neighbor SRP.
- [Kovchegov '02]: For $\mathrm{e}^{\alpha} > \mu$, the SARW from 0 to L converges to a Brownian Bridge in **diffusive scaling**.
- [B., Taggi '16]: Nearest neighbor SRP with a forced cycle does the same for large enough α.

Geometry of SRP in two dimensions

Λ a finite box in $\mathbb{Z}^2$

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- Numerical results [Gandolfo, Ruiz, Ueltschi 07] show: The origin (or any point) is not in an infinite cycle with probability one.

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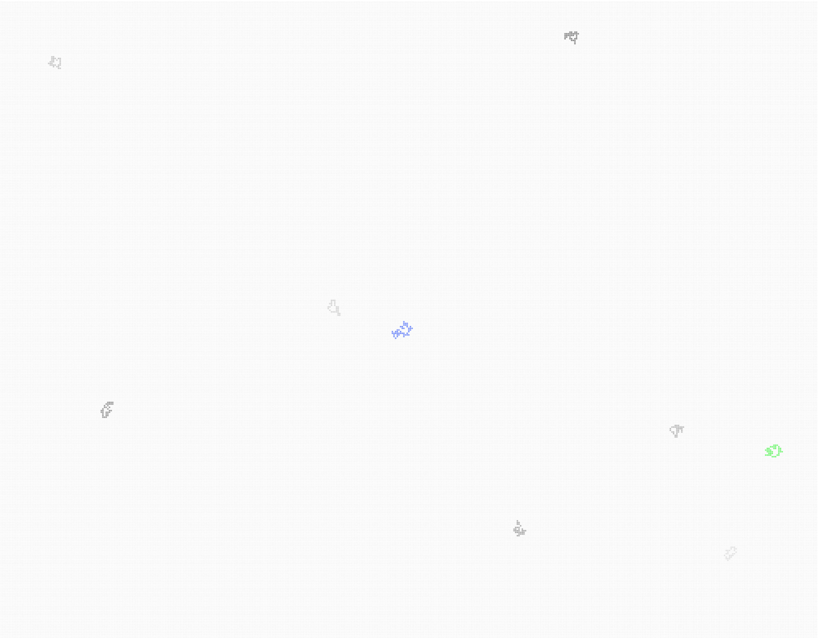
- Numerical results [Gandolfo, Ruiz, Ueltschi 07] show: The origin (or any point) is not in an infinite cycle with probability one.
- But if we focus on the *longest* cycle or force cycles through the system, interesting things happen!
- We show a snapshot of the equilibrated Metropolis dynamics in a box of side length 1000.
- The **10 longest cycles** are shown, color coded in (red, blue, green, black, dark gray, not so dark gray, etc).

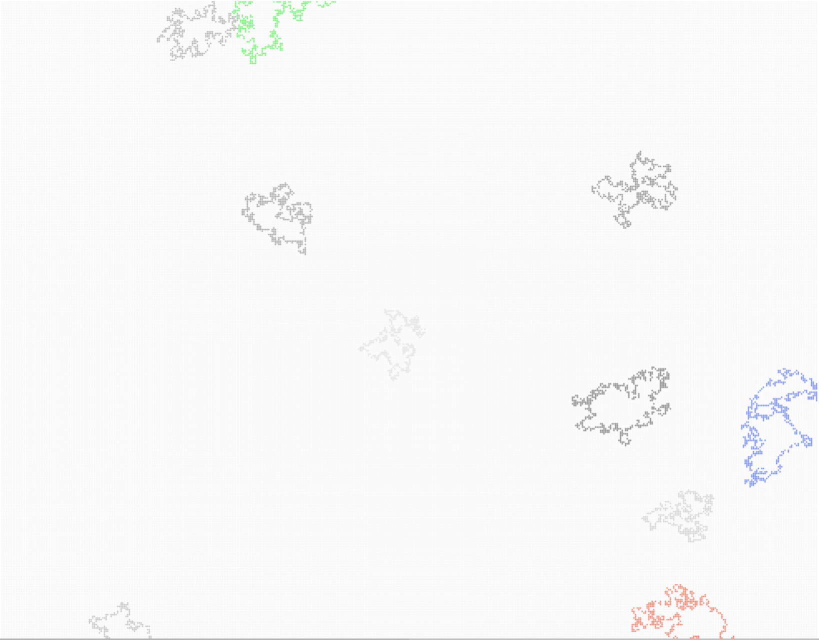
SRP for parameter $\alpha = 1.1$



SRP for parameter $\alpha = 1.0$

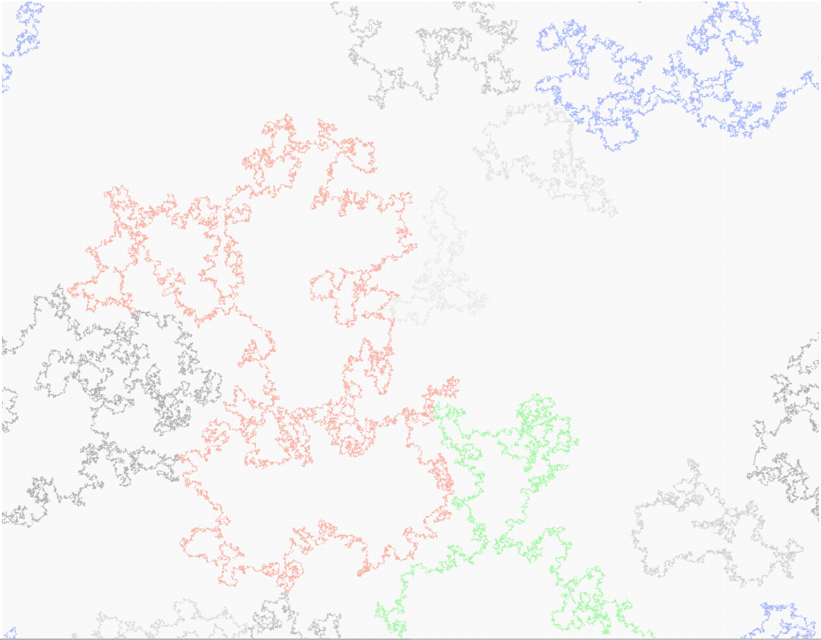
SRP for parameter $\alpha = 0.9$

SRP for parameter $\alpha = 0.8$

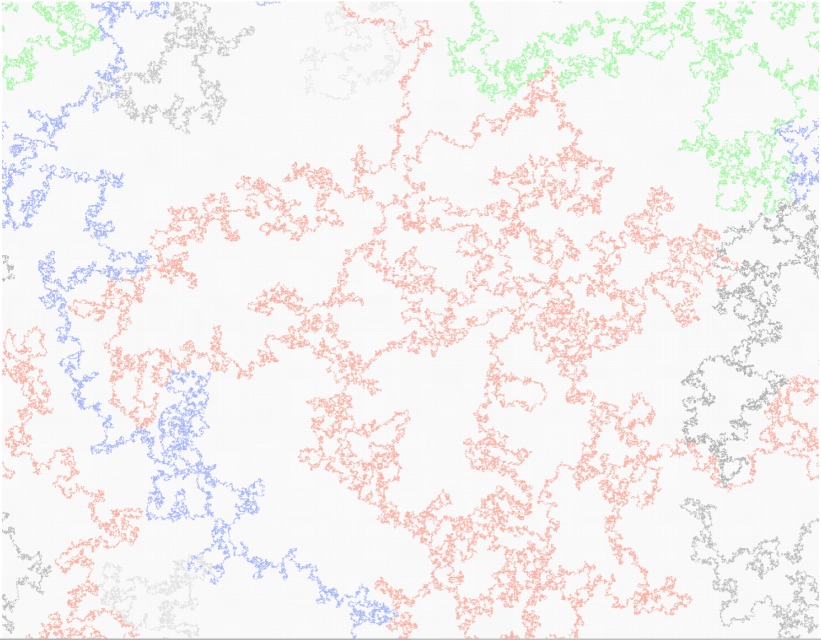


SRP for parameter $\alpha = 0.75$

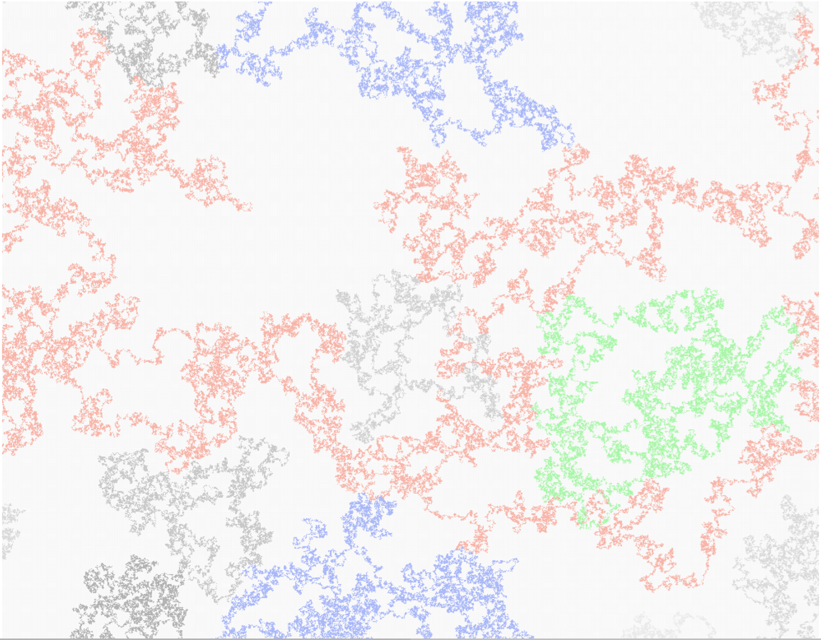
SRP for parameter $\alpha = 0.7$



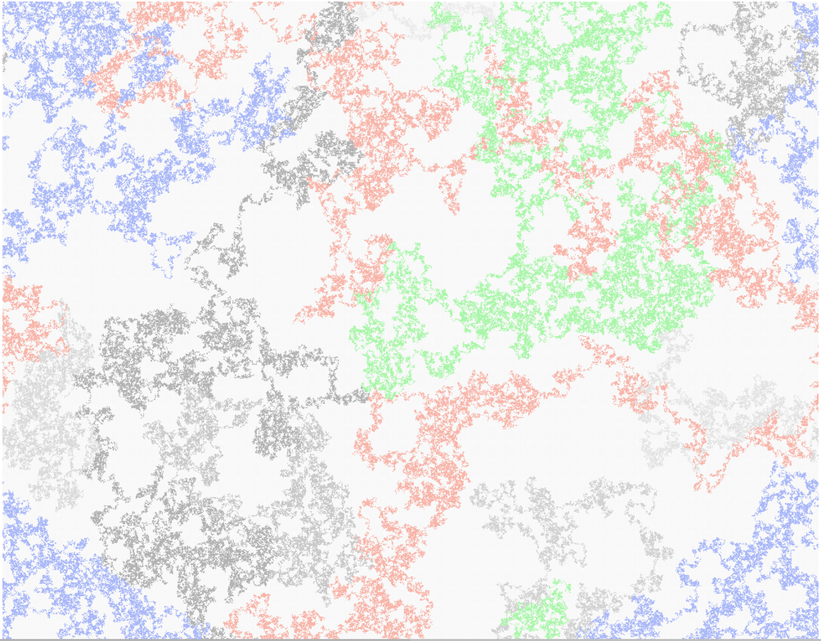
SRP for parameter $\alpha = 0.6$



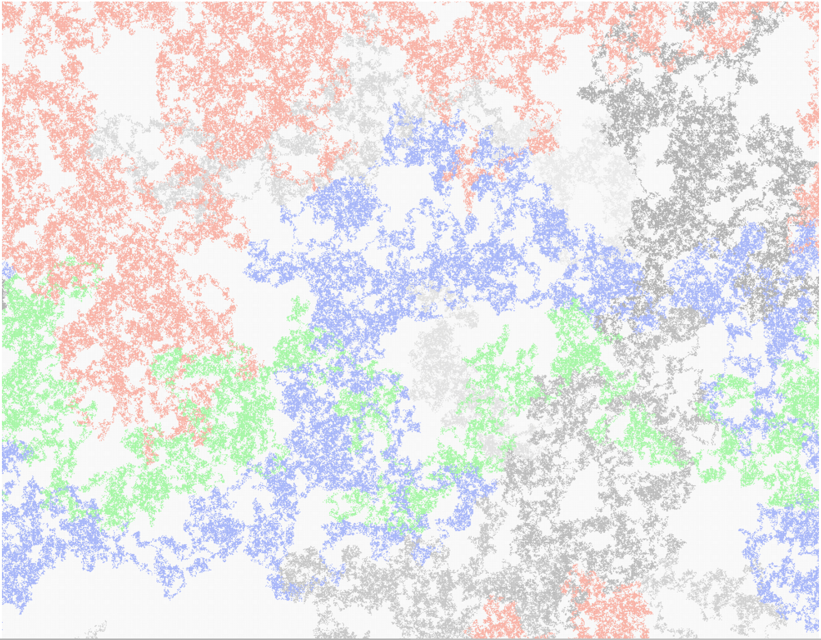
SRP for parameter $\alpha = 0.5$



SRP for parameter $\alpha = 0.4$

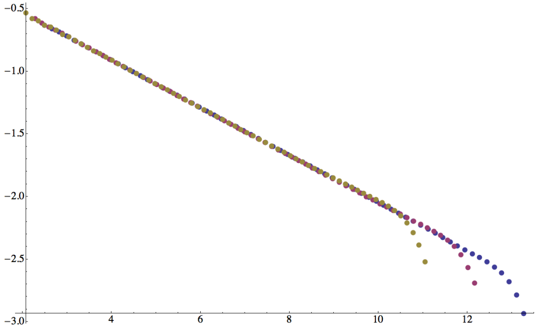


SRP for parameter $\alpha = 0.3$



Kosterlitz-Thouless transition

- The rate of decay of $\mathbb{P}(|C_x| > K)$ changes from exponential to algebraic: $\mathbb{P}(|C_x| > K) \sim K^{-p(\alpha)}$.
- $K \mapsto \mathbb{P}(C_x > K)$ is algebraic iff the two-point function $\mathbb{P}(y \in C_x)$ decays algebraically in $|x - y|$.
- Kosterlitz-Thouless phase transition, known from 2d models with a continuous symmetry.



Log-log-plot of $K \mapsto \mathbb{P}(|C_x| > K)$ for $\alpha = 0.5$
and box side length $1000, 2000, 4000$.

Kosterlitz-Thouless transition: Numerics

In [B. 14] the decay behaviour of $\phi(K) = \mathbb{P}(|C_x| > K)$ is investigated systematically, in order to estimate the critical parameter α_c.

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Amazing **universality predictions** by general (physics) KT-theory:

- For $\alpha < \alpha_c$, $\phi(K) \sim K^{-p(\alpha)}$, **and** $p(\alpha)$ is approximately linear and $\lim_{\alpha \to \alpha_c} p(\alpha) = 0.25$.
- For $\alpha > \alpha_c$, $\phi(K) \sim \mathrm{e}^{-r(\alpha)K}$, **and** there exist constants D, γ such that $r(\alpha) = D \exp(-\frac{\gamma}{|\alpha - \alpha_c|^{1/2}})$.

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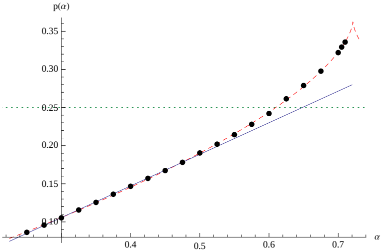
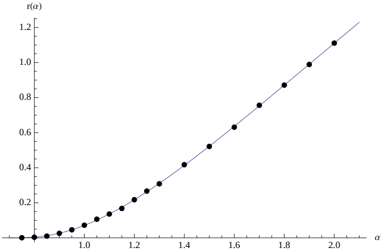
Here are the numerical results, predicting $\alpha_\mathrm{c} \approx 0.64$:



Measured power law $p(\alpha) \approx 0.019 + 0.415\alpha$.



Measured inverse correlation length (exp decay rate) $r(\alpha) \approx 20.99 \exp(-3.434/|\alpha - \alpha_\mathrm{c}|^{1/2})$.

Fractal dimension

- Compute the box-counting dimension:

$$d_{\text{box}} = \lim_{\varepsilon \to 0} \frac{\ln(\#\text{ of }\varepsilon\text{-boxes needed to cover longest cycle})}{\ln(1/\varepsilon)}$$

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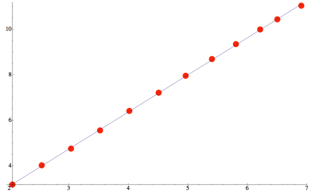

Loglog plot of the number of boxes needed to cover the longest cycle vs the box side length

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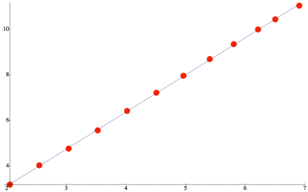
- Linear fitting gives $d_{\text{box}}(\alpha) \approx 2 - \frac{7}{10}\alpha$ for small α.



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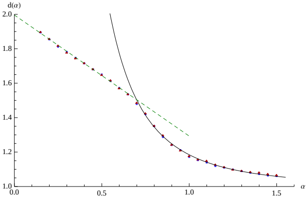
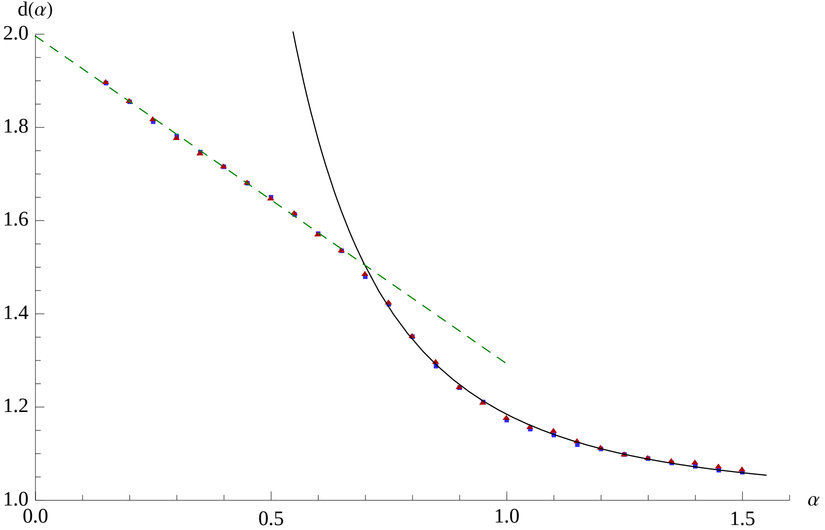
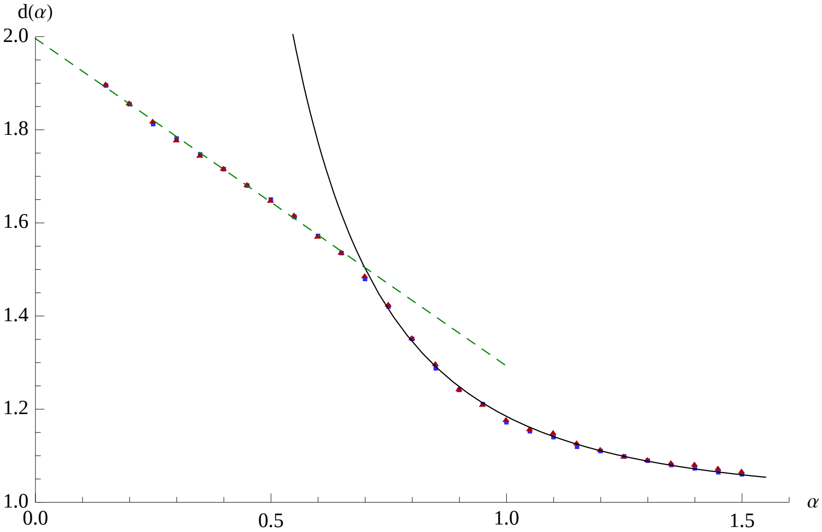
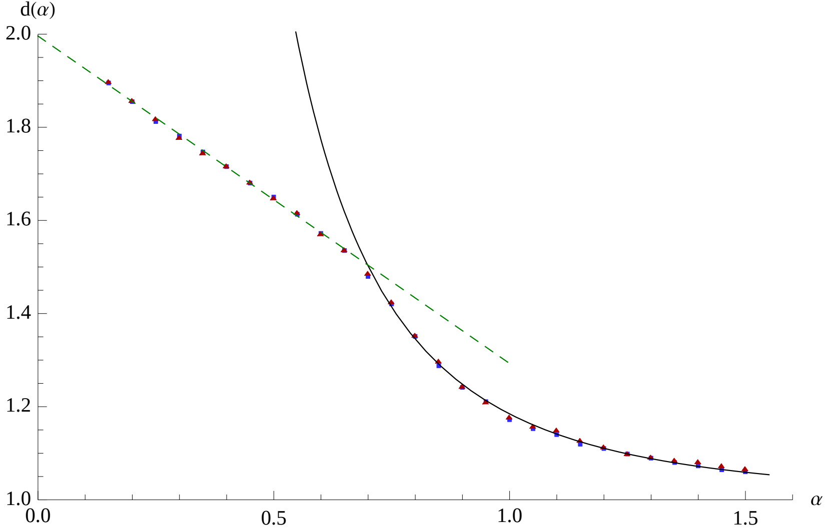


Box counting dimension as function of the temperature


d(α)
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.5
1.0
1.5
α



- Good agreement for square and triangular lattice; domain Markov property; symmetries;



- Good agreement for square and triangular lattice; domain Markov property; symmetries;
- Conjecture: two-dimensional SRP cycles are distributed like SLE curves, at least for $\alpha < \alpha_c$.

Thank you for your attention!