An invitation to analytic combinatorics and lattice path counting

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Abstract

The term "Analytic Combinatorics", coined by P. Flajolet and B. Sedgewick [6], combines powerful analytic methods from complex analysis with the field of enumerative combinatorics. The link between these fields is provided by generating functions, which are a priori only defined as formal power series. However, in many applications they can also be interpreted as analytic power series with a non-zero radius of convergence. Analyzing the analytic properties enables us to gain insight into the underlying coefficients.

Let $(a_n)_{n\geq 0}$ be a sequence of non-negative integers. The associated generating function $A(z) = \sum_{n\geq 0} a_n z^n$ is a formal power series in $\mathbb{N}[[z]]$. In many applications it can be interpreted as a convergent power series in $\mathbb{C}(z)$. In this case methods from analytic combinatorics yield formulae for the asymptotic growth of a_n for $n \to \infty$ including bounds on the rate of convergence.

After a brief introduction into these methods, we will apply them to the classical study of lattice paths. The enumeration of lattice paths is a classical topic in combinatorics which is still a very active field of research. They have many applications in chemistry, physics, mathematics and computer science. For example lattice paths are used as the solution of integer programming problems, in cryptanalysis, in crystallography and as models in queueing theory.

A lattice path is a sequence of vectors $v = (v_1, v_2, \dots, v_n)$, such that $v_j \in \mathcal{S}$, where $\mathcal{S}$ is a given, finite set of vectors called the step set. For our purposes we will deal with $\mathcal{S} \subset \mathbb{Z}^2$ and the lattice paths will start at the origin $(0, 0)$. One of the most known examples are Dyck-paths constructed from the step set $\mathcal{S} = \{(1, 1), (1, -1)\}$.

Contents

1 Introduction

The enumeration of lattice paths is a classical topic in combinatorics which is still a very active field of research. Our fascination for this topic is founded in the fact that despite the easily understood construction of lattice paths, most of their properties remain unproven or even unknown. Figure 1 gives an intuition of this statement: In the small scale, lattice paths look like mathematical doodles, but when taking a few steps further away, they exhibit a completely different behavior. A fractal structure becomes visible, giving glimpse of the difficulties encountered in lattice path combinatorics.

The aim of this mini course is to give an introduction to lattice path combinatorics. A unified framework is derived in order to present all methods and ideas as easily accessible as possible. All necessary derivations are made explicit and connections to other parts in the literature are added.

For a more detailed introduction, we refer to the master's thesis of the second author [15]. It gives an introduction into three well studied families of lattice paths (directed paths, walks confined to the quarter plane, self avoiding walks) and recent developments in the field.

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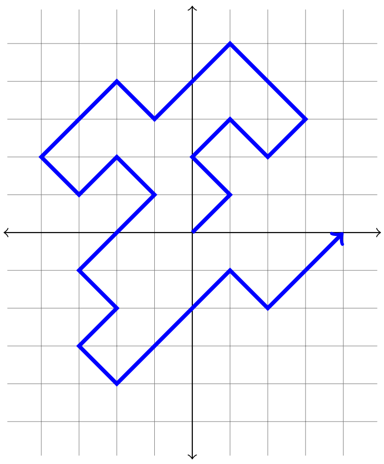
1.1 A historical introduction: the ballot problem

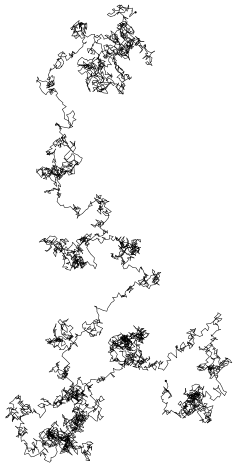
The so-called ballot problem is formulated as follows:

> We suppose that two candidates have been submitted to a vote in which the number of voters is μ. Candidate A obtains n votes and is elected; candidate B obtains $m = \mu - n$ votes. We ask for the probability that during the counting of the votes, the number of votes for A is at all times greater than the number of votes for B.

In 1887, Joseph Louis François Bertrand published an answer to this question in the Comptes Rendus de l'Academie des Sciences: The probability is simply $(2n - \mu)/\mu = (n - m)/(n + m)$. This result is now known as the first Ballot theorem.



(a) A Random Walk on a Euclidean Lattice



(b) A Random Walk with 5000 steps

Figure 1: Examples of two Random Walks in the Euclidean Plane

His “proof” was a rather non-rigorous argument based on a recurrence relation that is fulfilled by the numbers counting sequences of votes that have the desired property. The first formal proof was given by Désiré André (see the Exercises).

In this context, it is very helpful to represent sequences of votes with the help of paths in the Euclidean plane and the ballot problem can be seen as the birth hour of lattice path theory. We start at the origin $(0, 0)$ and move one step for every vote: if the vote is for candidate A we move one unit to the right and one unit up, if the vote is for candidate B we move one unit to the right and one unit down. If there are n votes for A and $m = \mu - n$ votes for B this means that we end up in the point $(\mu, 2n - \mu)$ (which lies in the first quadrant since A wins the election). Then the property that the number of votes for A is at all times greater than the number of votes for B is simply translated into the fact that the lattice path may never touch the x-axis (except at the beginning).

In Figure 2, the black lattice path corresponds to the sequence of votes AABAABAB-BAAB and the red one to the sequence ABBAABAAABBA. In both cases A wins the election but only the black path has the property that A is always ahead of B.

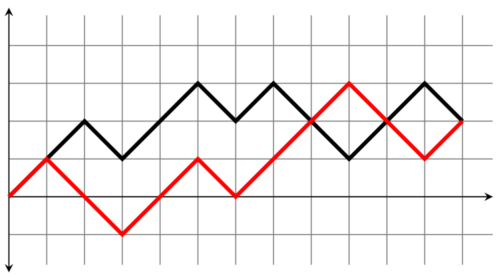


Figure 2: Lattice paths used to represent sequences of votes

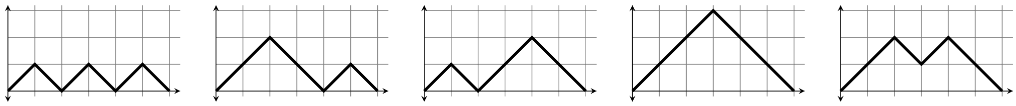


Figure 3: The five Dyck paths consisting of six steps

1.2 Variants and special cases

In the original problem one has to find the probability that candidate A is always strictly ahead of B in the vote count. If one is interested in sequences of votes where B is never ahead of A, this means that the corresponding lattice paths may never go below the x-axis but are allowed to touch it. In this case, the probability is $(n - m + 1)/(n + 1)$.

If we consider the special case that ties are allowed and that A and B both obtain the exact same number of votes we obtain an important class of lattice paths called *Dyck paths*[1]. The five Dyck paths of length six are represented in Figure 3. These paths will occur at various occurrences throughout this mini-course.

One can also consider variants of the ballot problem where the two options have different weights. For instance, consider the following scenario known as *Duchon's club model* [5]: A club opens in the evening and closes in the morning. People arrive by pairs and leave in threesomes. What is the possible number of scenarios from dusk to dawn as seen from the club's entry? This problem translates into lattice paths starting at the origin and never going below the x-axis with $(1, 2)$ and $(1, -3)$ as possible steps.

Another related problem is the so-called *gambler's ruin problem:* Two players, Alice and Bob, play a coin tossing game. Alice starts the game with a pennies and Bob with b pennies; the game ends as soon as one of the players has gone broke. The rules are as follows: The players take turns tossing a coin and each player has a 50% chance of winning with each flip of the coin. At each round, the winner gets one penny from the loser. Such a game can be described as a random lattice path starting at $(0, a)$, never going above the horizontal line $y = a + b$ and never going below the x-axis. At each stage, the probability of a step up and of a step down is the same. The question is when the path reaches the line $y = a + b$ (Alice wins) or the x-axis (Bob wins) for the first time. The answer is simple: The probability that Alice loses is $b/(a + b)$ and the probability that Bob loses is $a/(a + b)$. One can of course also consider variants of this game where player one wins each toss with probability p, and player two wins with probability $q = 1 - p$, where $p \neq q$. In this case a step up occurs with probability p and a step down with probability q.

1.3 Other objects counted by the same numbers as Dyck paths

Dyck paths are probably the most famous example of lattice paths and will occur at several occasions throughout this course. As we will see later on, Dyck paths are counted by the Catalan numbers. In his newly published book *Catalan numbers* [12], Richard Stanley presents 214 different kinds of objects counted by them. Here is a short list of some famous objects counted by the Catalan numbers:

- Expressions containing n pairs of parentheses which are correctly matched

[1] named after Walther von Dyck, 6.12.1856–5.11.1934

- Different ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines
- Rooted binary trees with n internal nodes ($n+1$ leaves)
- Permutations of the set $\{1, 2, \ldots, n\}$ that avoid the pattern 321. A permutation π avoids the pattern 321 if we cannot find a subsequence xyz of π such that $x > y > z$.

2 Preliminaries

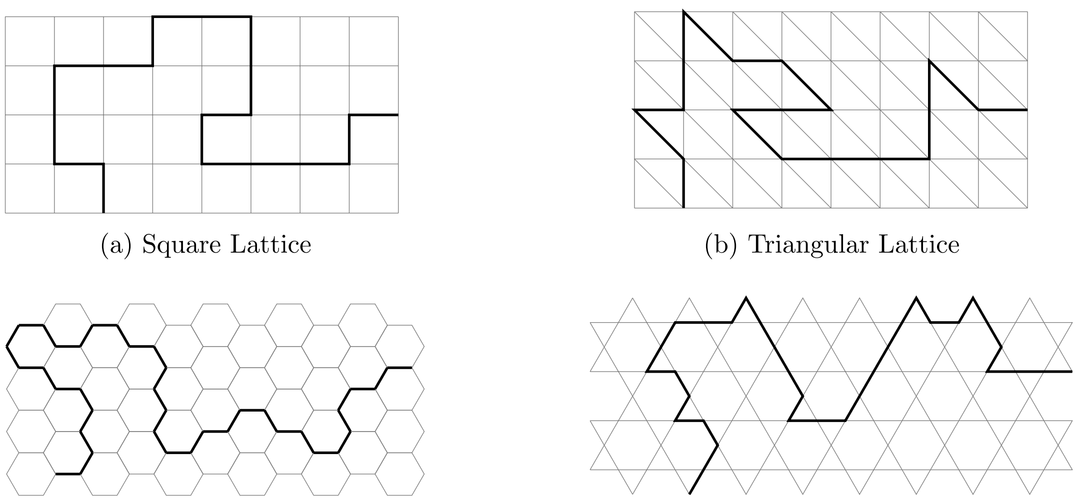
2.1 What is a Lattice Path?

The central topic of investigation of this mini-course are lattice paths. As the name suggests, they depend on a lattice, which can be described informally as a regular arrangement of points in the Euclidean space $\mathbb{R}^n$. Lattice maths can be used to encode various combinatorial objects such as trees, maps, permutations, lattice polygons, Young tableaux, queues etc. Moreover, lattice paths have many applications, for instance in physics and computer science, where they are used for the solution of integer programming problems, cryptanalysis as well as crystallography and sphere packing.

We start with a general and for our purpose suitable definition of the term *lattice*. Note that there are various ways of defining this term.

Definition 2.1. *A lattice $\Lambda = (V, E)$ is a mathematical model of a discrete space. It consists of a set $V \subset \mathbb{R}^n$ of vertices, and a set $E \subset \{\{v_1, v_2\} : v_1, v_2 \in V\}$ of edges. If two vectors are connected via an edge, we call them nearest neighbors.*

Some examples are shown in Figure 4. The expression "lattice" actually stems from physics. In mathematics and computer science lattices are also called *graphs* or *networks*.



(a) Square Lattice

(b) Triangular Lattice

(c) Hexagonal Lattice

(d) Kagomé Lattice

Figure 4: Examples of Lattices

On a lattice we want to look at walks that connect the vertices of the lattice. The basic component of a walk is a step, which essentially is nothing else than an edge.

Definition 2.2. *Let $\Lambda = (V, E)$. An n-step lattice path or lattice walk or walk from $s \in V$ to $x \in V$ is a sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of elements in V, such that*

1. $\omega_0 = s$, $\omega_n = x$,
2. $(\omega_i, \omega_{i+1}) \in E$.

The length $|\omega|$ of a lattice path is the number n of steps (edges) in the sequence ω.

During this course we are going to work on the *Euclidean lattice*, which consists of the vertices $\mathbb{Z}^d$. On this lattice an alternative definition of the edges via the so-called step set can be used. The step set $\mathcal{S}$ is a subset of $\mathbb{Z}^d$ and defines how one can move from one vertex to another. We are mainly going to work with a special kind of step set, namely small steps.

Definition 2.3. *If the step set $\mathcal{S}$ is a subset of $\{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, then we say $\mathcal{S}$ is a set of small steps.*

In order to simplify notation, it is sometimes more convenient to use a more intuitive terminology by representing a step set by the corresponding points on a compass or by a small picture. In Figure 5 the full set of small steps is depicted. In this special case moving from $(1, 0)$ counterclockwise corresponds to **E**, **NE**, **N**, **NW**, **W**, **SW**, **S** and **SE**.

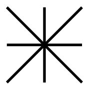


Figure 5: The full set of small steps

We can now give an alternative definition of lattice paths on the Euclidean lattice:

Definition 2.4. *An n-step lattice path or lattice walk or walk from $s \in \mathbb{Z}^d$ to $x \in \mathbb{Z}^d$ relative to $\mathcal{S}$ is a sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of elements in $\mathbb{Z}^d$, such that*

1. $\omega_0 = s$, $\omega_n = x$,
2. $\omega_{i+1} - \omega_i \in \mathcal{S}$.

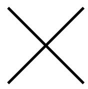
The length $|\omega|$ of a lattice path is the number n of steps in the sequence ω.

Comparing Definitions 2.2 and 2.4 we see that in the second case $V = \mathbb{Z}^d$ and the set of possible edges E is implicitly defined over the set of allowed steps. The edge $(x, y) \in E$ exists if and only if $(y - x) \in \mathcal{S}$. Note that the step set is defined globally for all vertices, i.e., the lattice has the same structure at every vertex . Thus, the lattice paths on the lattices (a) and (b) in Figure 4 can be defined with the help of a step set: The square lattice corresponds to the small step set $\mathcal{S} = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ and the triangular lattice to the small step set $\mathcal{S} = \{(1, 0), (0, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)\}$. However lattice paths on the lattices (c) and (d) cannot be defined with the help of a step set as can be seen easily. The advantage of the second definition is its compact form, which is why we are going to choose this one from now on.

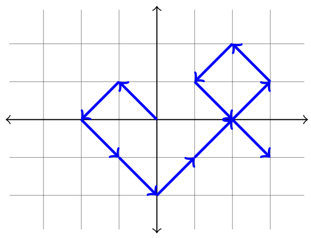
Remark 2.5: In the remainder of this course, we are going to work in the Euclidean plane only. Moreover, we will restrict Definition 2.4 and impose that lattice paths always start at the origin $s = (0, 0)$. But this fact will not represent a restriction to our discussion, as we are going to consider *homogeneous* lattices. These are lattices for which the number of n-step walks starting from s is independent of the starting point s for all values of n.

For more details on the basic properties of lattices we refer to [7]. ∎

In the Euclidean plane, we can also describe a lattice path by a polygonal line. An example is shown in Figure 6, where an unrestricted walk on the lattice $\mathbb{Z}^2$ and the set of small steps from which it was constructed, is shown. In this context unrestricted means that there are no boundaries on the domain (lattice) that we allow self-intersections and that the walk ends at an arbitrary point.



(a) $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}, \mathbf{NW}, \mathbf{SW}\}$



(b) Unrestricted Walk with Loops and 11 steps

Figure 6: Unrestricted Path with Loops in $\mathbb{Z}^2$

Obviously, another equivalent way of representing a walk with a fixed start point is by providing the sequence of performed steps. For example, the walk in Figure 6b is given by the sequence

$$(\mathbf{NW}, \mathbf{SW}, \mathbf{SE}, \mathbf{SE}, \mathbf{NE}, \mathbf{NE}, \mathbf{NE}, \mathbf{NW}, \mathbf{SW}, \mathbf{SE}, \mathbf{SE})$$

or

$$(\nwarrow, \swarrow, \searrow, \searrow, \nearrow, \nearrow, \nearrow, \nwarrow, \swarrow, \searrow, \searrow).$$

The concept of steps is also useful for introducing weights on paths, which are needed for many applications.

Definition 2.6. *For a given step set $\mathcal{S} = \{s_1, \ldots, s_k\}$ we define the respective system of weights as $\Pi = \{w_1, \ldots, w_k\}$ where $w_j > 0$ is the weight associated to step s_j for $j = 1, \ldots, k$. The weight of a path is defined as the product of the weights of its individual steps.*

Some useful choices are:

- $w_j = 1$: Combinatorial paths in the standard sense;
- $w_j \in \mathbb{N}$: Paths with colored steps, i.e. $w_j = 2$ means that the associated step has two possible colors;
- $\sum_j w_j = 1$: Probabilistic model of paths, i.e. step s_j is chosen with probability w_j.

Example 2.7. The gambler's ruin problem where Alice starts with a pennies and Bob with b pennies and where Alice has the probability p of winning a round and Bob has the probability $q = 1 - p$ can be modelled with the help of weighted lattice paths. If the lattice path represents the number of pennies that Alice has it starts at $(0, a)$ and the possible steps are: $s_1 = (1, 1)$ with $w_1 = p$ and $s_2 = (1, -1)$ with $w_s = q$. ∎

2.2 Formal Power Series

Formal power series are the central object of investigation. For a ring R we denote by $R[z]$ the ring of polynomials in z with coefficients in R.

Definition 2.8. *Let R be a ring with unity. The ring of formal power series $R[[z]]$ consists of all formal sums of the form*

$$\sum_{n\geq 0} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots,$$

with coefficients $a_n \in R$.

The sum of two formal power series $\sum_{n\geq 0} a_n z^n$, $\sum_{n\geq 0} b_n z^n$ is defined by

$$\sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n = \sum_{n\geq 0} (a_n + b_n) z^n$$

and their product by

$$\sum_{n\geq 0} a_n z^n \cdot \sum_{n\geq 0} b_n z^n = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

Definition 2.9. *Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series. We define the linear operator $[z^n]A(z)$ as*

$$[z^n]A(z) = a_n,$$

called the coefficient extraction operator.

The coefficient extraction operator satisfies the following identity for all suitable k, i.e. all expressions have to be well-defined:

$$[z^{n-k}]A(z) = [z^n] z^k A(z). \tag{1}$$

Let us recall some important power series expansions:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, \qquad e^x = \sum_{n\geq 0} \frac{1}{n!} x^n,$$

$$\log(1+x) = \sum_{n\geq 0} \frac{(-1)^{n+1}}{n!} x^n, \qquad (1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n,$$

where $\binom{\alpha}{n} = \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$.

2.3 Asymptotic Notation

These definitions are drawn from [6, Chapter A.2], where more examples can be found.

Let S be a set and $s_0 \in S$. We assume a notion of neighborhood to exist in S, e.g. $S = \mathbb{C}$ and $s_0 = 0$. Two functions $f, g : S \setminus \{s_0\} \to \mathbb{R}(\mathbb{C})$ are given.

- *$\mathcal{O}$-notation:* Denote

$$f(s) \underset{s\to s_0}{=} \mathcal{O}(g(s)),$$

if the ratio $f(s)/g(s)$ stays bounded as $s \to s_0$ in S. In other words, there exists a neighbourhood V of s_0 and a constant $C > 0$, such that

$$|f(s)| \leq C|g(s)| \qquad s \in V,\ s \neq s_0.$$

This is also known as "Big-Oh-notation".

- *$\sim$-notation:* Denote

$$f(s) \underset{s\to s_0}{\sim} g(s),$$

if the ratio $f(s)/g(s)$ tends to 1 as $s \to s_0$ in S. One also says *f and g are asymptotically equivalent* (as s tends to s_0). We will mostly use this notation for $s_0 = \infty$.

- *o-notation:* Denote

$$f(s) \underset{s\to s_0}{=} o(g(s)),$$

if the ratio $f(s)/g(s)$ tends to 0 as $s \to s_0$ in S. In other words, for any $\varepsilon > 0$, there exists a neighbourhood V of s_0, such that

$$|f(s)| \leq \varepsilon|g(s)| \qquad s \in V,\ s \neq s_0.$$

This is also known, as "little-Oh-notation".

3 Analytic Combinatorics

"Combinatorics, the branch of mathematics concerned with the theory of enumeration, or combinations and permutations, in order to solve problems about the possibility of constructing arrangements of objects which satisfy specified conditions."[2]

The focus of this mini-course with respect to the preceding definition lies on the enumeration of objects which are mostly described by recursions and boundary conditions, namely lattice paths. A standard tool in this context are generating functions which were introduced as formal power series whose coefficients give the sizes of a sought family of objects with respect to a parameter encoded in the exponent. A very colorful description from Wilf[3] [16] says

"A generating function is a clothesline on which we hang up a sequence of numbers for display."[4]

[2] CollinsDictionary.com, `http://www.collinsdictionary.com/dictionary/english/combinatorics`, accessed 12/08/2013.

[3] Herbert Wilf, 13.6.1931-7.1.2012

[4] Wilf, generatingfunctionology, p. 1

It describes quite vividly the idea of generating functions. This tool has led to many new insights in the field of combinatorics, by introducing new possible solution strategies. Their importance can be seen in the vast amount of available literature, like Stanley's[5] books *Enumerative combinatorics, I and II* [13, 14].

Furthermore, generating functions serve as a link for interdisciplinary applications of techniques from different branches of mathematics. One very important field, which found entrance to combinatorics, is complex analysis. This revolutionized the field and led to the new branch of *Analytic Combinatorics*. The fathers of this development are Flajolet[6] and Sedgewick[7] in their highly recommendable book *Analytic Combinatorics* [6]. They interpret the formerly only algebraically investigated formal power series as complex analytic functions on their radii of convergence. This allows the extraction of the asymptotic behavior and much more.

The structure of the subsequent chapter was inspired by [8, Chapter 4] and gives an introduction to symbolic methods, using [6, 13, 14, 16].

3.1 Combinatorial Classes and Ordinary Generating Functions

Following [6, pp. 16] we give a short introduction to the symbolic method. In particular, we emphasize on the topics important for lattice path combinatorics.

Definition 3.1. *A combinatorial class, or simply a class, is a finite or denumerable set on which a size function is defined, satisfying the following conditions:*

1. *the size of an element is a non-negative integer;*
2. *the number of elements of any given size is finite.*

If $\mathcal{A}$ is a class, the size of an element $\alpha \in \mathcal{A}$ is denoted by $|\alpha|$, or $|\alpha|_{\mathcal{A}}$ in the few cases where the underlying class is not clear from the context. Using this size function, we decompose $\mathcal{A}$ into disjoint subclasses $\mathcal{A}_n$, which contain all elements of $\mathcal{A}$ of size n and we denote the cardinality of these subsets by $a_n = \operatorname{card}(\mathcal{A}_n)$.

In accordance with this definition we define the class $\mathcal{W} = \mathcal{W}_{\mathcal{S},\Lambda}$ to be the set of all walks on a lattice Λ with respect to the step set $\mathcal{S} = \mathcal{S}_\Lambda$. Here, $|\omega|$ is the length of a walk $\omega \in \mathcal{W}$.

Definition 3.2. *The counting sequence of a combinatorial class $\mathcal{A}$ is defined as the sequence of integers $(a_n)_{n\geq 0}$.*

Definition 3.3. *Two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ are said to be (combinatorially) isomorphic, which is written $\mathcal{A} \cong \mathcal{B}$, if and only if their counting sequences are identical. This condition is equivalent to the existence of a bijection from $\mathcal{A}$ to $\mathcal{B}$ that preserves size. One also says $\mathcal{A}$ and $\mathcal{B}$ are bijectively equivalent.*

Note that such a bijection, despite it needs to exist, is not always easy to find. Also, such bijections do not necessarily have to behave in a nice or natural manner. For example, it is straightforward to give a bijection between Dyck paths and bracketings but it is less obvious to provide such a correspondence between Dyck paths and 321-avoiding permutations.

The enumerative information of a class is stored in the formal power series $A(z)$.

[5] Richard Peter Stanley, 23.6.1944-
[6] Philippe Flajolet, 1.12.1948-22.3.2011
[7] Robert Sedgewick, 20.12.1946-

Definition 3.4. *The ordinary generating function (OGF) of a sequence $(a_n)_{n\geq 0}$ is the formal power series*

$$A(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The OGF of a combinatorial class $\mathcal{A}$ is the generating function for the counting sequence $a_n = card(\mathcal{A}_n),\ n \geq 0$. Equivalently, the combinatorial form

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|},$$

is employed. We say the variable z marks the size in the generating function.

Note that there are two special classes:

Class	Nr. of elements	Weights	OGF
Empty class $\mathcal{E}$	1	0	$E(z) = 1$
Atomic class $\mathcal{Z}$	1	1	$Z(z) = z$

Here is a brief summary of the introduced naming convention:

Class	Subclasses of elements of size n	Cardinality of subclasses	OGF
$\mathcal{A}$	$\mathcal{A}_n$	a_n	$A(z)$

Generating functions are elements of the ring of formal power series $\mathbb{C}[[z]]$, thus they can be manipulated algebraically. Two basic operations are the sum and the Cauchy product.

Firstly, let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint classes. Their union $\mathcal{C} = \mathcal{A} \,\dot\cup\, \mathcal{B}$ represents a new class with size defined consistently as

$$|\gamma|_{\mathcal{C}} = \begin{cases} |\gamma|_{\mathcal{A}}, & \text{if } \gamma \in \mathcal{A}, \\ |\gamma|_{\mathcal{B}}, & \text{if } \gamma \in \mathcal{B}. \end{cases}$$

This translates naturally into $c_n = a_n + b_n$ which leads to the following generating function for $\mathcal{C}$:

$$C(z) = A(z) + B(z) = \sum_{n\geq 0} (a_n + b_n) z^n.$$

Secondly, their Cartesian product $\mathcal{C} = \mathcal{A} \times \mathcal{B} = \{\gamma = (\alpha, \beta) \mid \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$ represents a new class with size defined consistently as

$$|\gamma|_{\mathcal{C}} = |\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}.$$

In this case we have to consider all possibilities in the manner of a Cauchy product, hence $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, and we conclude as anticipated

$$C(z) = A(z) \cdot B(z) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

These two constructions are enough to derive many fundamental constructions that build upon the set-theoretic union and product. For instance, we can use sum and product in order to define the sequence class. If $\mathcal{B}$ is a class then the *sequence* class $\mathrm{SEQ}(\mathcal{B})$ is defined as the infinite sum

$$\mathrm{SEQ}(\mathcal{B}) = \mathcal{E} + \mathcal{B} + (\mathcal{B} \times \mathcal{B}) + (\mathcal{B} \times \mathcal{B} \times \mathcal{B}) + \dots, \tag{2}$$

with $\mathcal{E}$ the empty class containing one element of size 0. Note that this construction makes only sense if $\mathcal{B}$ contains no element of size 0. Otherwise the union would contain an infinite number of elements of size 0. Using the sum and product as introduced before, we obtain the following relation for the generating function $\mathcal{A} = \mathrm{SEQ}(\mathcal{B})$

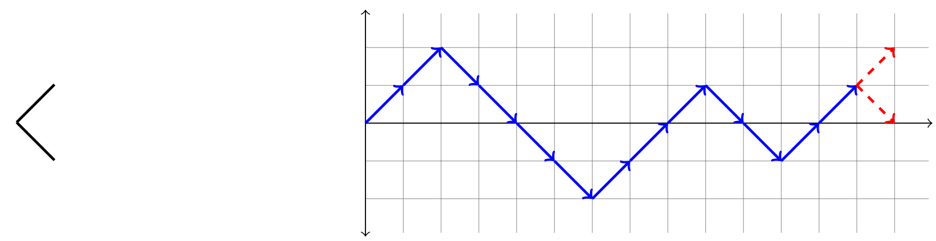
$$A(z) = \frac{1}{1 - B(z)}.$$

More constructions can be derived with the same ideas, see e.g. [6, Theorem I.1].

The true power resulting from the *symbolic method* is best understood by examples. Let's consider two cases in which we apply the above definitions and operations.

Example 3.5 (Unrestricted Paths). Consider the class $\mathcal{W}$ of unrestricted lattice paths employing the step set $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}\}$ as illustrated in Figure 7a. There are many ways to describe the construction of lattice paths. The most natural way is a step-by-step construction, from which one can deduce a recursive definition for the number of sought lattice paths. Let w_n denote the number of paths of length n. Then $w_{n+1} = w_n \cdot 2$ since there are two ways of extending a path of length n to a path of length $n+1$: we can either take a step up or a step down. Since $w_0 = 1$, it follows that $w_n = 2^n$.

Alternatively, one can describe the construction of the combinatorial class and translate this into the language of generating functions, which we want to demonstrate here. In this case, the direct approach is much simpler but the combinatorial construction-approach serves as a simple first example and should help to get accustomed with the symbolic method.



(a) $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}\}$ (b) Two possible extensions of an unrestricted path with a **NE**- or **SE**-step

Figure 7: Unrestricted **NE**-/**SE**-Path

A member of the class $\mathcal{W}$ is either the empty path or a path of non-zero length n. In the latter case we can construct a path of length $n+1$ by extending the path by one step out of the step set $\mathcal{S}$ and the resulting path is also a member of $\mathcal{W}$. This informal description is visualized in Figure 7b and translates into

$$\mathcal{W} = \underbrace{\mathcal{E}}_{\text{empty path}} \ \dot\cup \ \underbrace{\mathcal{W} \times \mathcal{Z}_{\mathbf{NE}}}_{\text{append } \mathbf{NE}\text{-step}} \ \dot\cup \ \underbrace{\mathcal{W} \times \mathcal{Z}_{\mathbf{SE}}}_{\text{append } \mathbf{SE}\text{-step}}.$$

As we do not distinguish between **NE**- and **SE**-steps it holds that $\mathcal{Z}_{\mathbf{NE}} \cong \mathcal{Z}_{\mathbf{SE}} \cong \mathcal{Z}$. Hence, we are able to apply the symbolic method by translating this equation into an equation on the corresponding generating functions:

$$W(z) = 1 + zW(z) + zW(z) = 1 + 2zW(z). \tag{3}$$

This equation can be solved algebraically and we get the solution

$$W(z) = \frac{1}{1-2z}. \tag{4}$$

In this case we extract the coefficients easily and get that the number of n-step unrestricted lattice paths with the step set $\mathcal{S}$ starting from the origin is

$$w_n = [z^n]W(z) = [z^n]\frac{1}{1-2z} = [z^n]\sum_{k\geq 0} 2^k z^k = 2^n.$$

Note that in this case it was quite easy to solve the functional equation (3). But in most general cases we are not able to deduce such a simple form for the solution and all we get is a relation on the functional equation. One main objective of Analytic Combinatorics is to develop different techniques on how to deal with these cases and how to extract enough information from this equation, in order to decide on certain properties of the solution. ■

Remark 3.6: From Algebra we know that solutions of algebraic equations are unique up to multiplicity of roots. Recalling the definition of combinatorial isomorphic classes this gives us an easy way of checking such isomorphisms. If the generating functions of two classes satisfy the same functional equation, then the coefficient sequences satisfy the same recursion.

In order to prove isomorphism, all that is left is to check the "start values'. This can also be achieved by comparison of the first "few" (depending on the order of the recursion/equation) terms of the sequence. Note that it is important to perform this check. A straightforward example of two classes whose generating functions fulfil the same functional equation are the empty class $\mathcal{E}$ and the atomic class $\mathcal{Z}$. Both OGF satisfy the equation $A(z)^2 = A(z)$, but they are not the same, as $E(z) = 1$ and $Z(z) = z$, respectively. ■

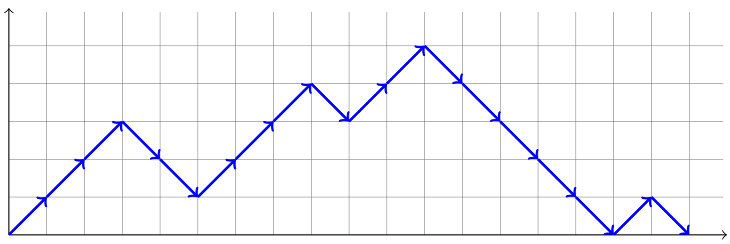


Figure 8: Dyck Path of length 18

Example 3.7 (Dyck Paths, [6, pp. 319]). Dyck paths were already defined in the Introduction. Let us recall their definition: They are paths on the same step set $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}\}$ as before but with the restriction that they start at the origin, never leave the first quadrant and end on the x-axis. An example is shown in Figure 8.

As before, we are able to construct a functional equation for the OGF $D(z)$ of Dyck Paths using the introduced operations: The technique we will apply is known as *First passage decomposition.* Basically it decomposes an arbitrary path $\omega \in \mathcal{D}$ into two (possibly empty) paths also belonging to $\mathcal{D}$.

A member of the class $\mathcal{D}$ is either the empty path or a path of non-zero length. If it is of non-zero length, after the initial point of contact at the origin, there will be another point of contact with the x-axis. Denote the first such second point as x_0. Now consider the path from the origin to x_0 without the initial **NE**- and the final **SE**-step. This (possible empty) sub-path is also a legitimate Dyck path that belongs to $\mathcal{D}$. (Recall that the empty path is also a member of $\mathcal{D}$.) After the “first passage”, which ends at x_0, there will be another path starting at x_0 and ending on the x-axis. This path could be empty as well, but it is, as before, again a Dyck Path. The described procedure is depicted in Figure 9.

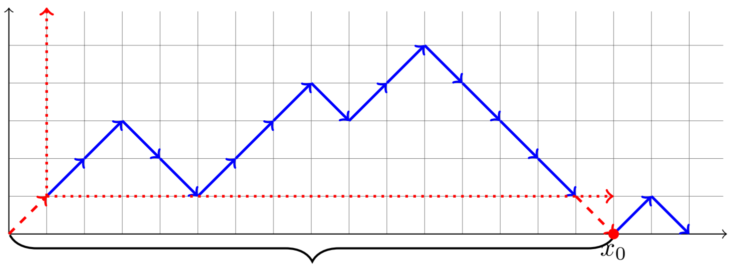


Figure 9: First Passage Decomposition of Dyck Path

This informal description translates into

$$\mathcal{D} = \underbrace{\mathcal{E}}_{\text{empty walk}} \;\dot{\cup}\; \underbrace{\mathcal{Z}_{\mathbf{NE}} \times \mathcal{D} \times \mathcal{Z}_{\mathbf{SE}}}_{\text{first passage}} \times \mathcal{D}.$$

Let d_n be the number of Dyck paths with $2n$ steps (one can e.g. map every up step to a down step and count them as one). The symbolic method gives with the same reasoning as before

$$D(z) = 1 + z\,(D(z))^2\,. \tag{5}$$

Here we obtained a quadratic functional equation, which has the two possible solutions

$$D_{\pm}(z) = \frac{1 \pm \sqrt{1-4z}}{2z}.$$

Taking a closer look at $D_+(z)$, we see that it possesses a singularity at 0, which corresponds to the constant term of the formal power series, and ought to be 1. Hence, we can dismiss this branch and arrive at the final solution

$$D(z) = \frac{1 - \sqrt{1-4z}}{2z}. \tag{6}$$

After using Newton’s expansion theorem for general exponents and some elementary manipulations of binomial coefficients (see the Exercises) we get

$$d_n = [z^n]D(z) = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

the n-th *Catalan number* (*OEIS* **A000108**[8]), as the number of n-step Dyck paths. ■

In the last two examples we have seen that the sought-after OGFs may be the solutions of algebraic equations, compare (3) and (5). But in the case of our first example, the OGF is even a rational function, see (4). Naturally the question for a general classification of all possible generating functions arises. Stanley introduces in [13, Chapter 6] a suitable hierarchy.

Recently, a lot of research was conducted on such classifications for "big" classes of lattice paths, see e.g. for walks in the quarter plane [3, 10]. Especially in computer algebra such a classification is of interest, as there exists efficient algorithms for problems in these specific classes. However, we will not pursue this direction here.

3.2 Multivariate Generating Functions

So far we have only considered univariate formal power series, but this concept can easily be generalized to multivariate formal power series. In the same manner OGFs generalize to *multivariate generating functions* (MGFs). As Flajolet and Sedgewick put it [6, Chapter III], the main advantage of several variables is the possibility to keep track of a collection of parameters defined for combinatorial objects. Multivariate generating functions are applicable to many combinatorial settings since the powerful symbolic method can be transferred to several variables in a straightforward way. Indeed, we can use the symbolic method not only to count combinatorial objects but also to *quantify* their properties.

In the case of lattice path combinatorics we will need the notion of a *bivariate generating function* (BGF), with the first parameter encoding the length of a lattice path, and the second parameter keeping track of the final height. This translates into

$$B(z,u) = \sum_{n,k\geq 0} b_{n,k} z^n u^k,$$

where $b_{n,k}$ is the number of lattice paths of length n and where the studied parameter is equal to k. We say that the variable z keeps track of the size (length) and the variable u of the additional parameter, the final height. Note that it can also be interpreted as a formal power series in z with coefficients in $\mathbb{Q}[u]$, where for all n, almost all coefficients $b_{n,k}$ are zero. This interpretation closes the circle and links MGFs with OGFs.

We just want to remark that yet another generalization is the usage of formal Laurent series instead of formal power series. All definitions and observations stay the same and can be adapted to this new case in a straightforward way.

Example 3.8. We will continue the analysis started in Example 3.5 of unrestricted paths $\mathcal{W}$ starting from the origin and using the step set $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}\}$. We derived the following construction of the combinatorial class $\mathcal{W}$

$$\mathcal{W} = \mathcal{E} \,\dot\cup\, \mathcal{W} \times \mathcal{Z}_{\mathbf{NE}} \,\dot\cup\, \mathcal{W} \times \mathcal{Z}_{\mathbf{SE}}.$$

The difference now, is that we have to distinguish between **NE**- and **SE**-steps. A **NE**-step increases the height by one and hence corresponds to the generating function u and a **SE**-step decreases the height by one and hence corresponds to $\frac{1}{u}$. Additionally, both steps increase the length by 1. Note that we will work in the ring of formal Laurent series $\mathbb{Z}[[u, \frac{1}{u}]]$. Let's define

[8] Catalan numbers; `http://oeis.org/A000108`, accessed 15/11/2015.

the BGF associated with $\mathcal{W}$ as A **NE**-step increases the height by one and hence corresponds to the generating function u and a **SE**-step decreases the height by one and hence corresponds to $\frac{1}{u}$. Additionally, both steps increase the length by 1. Note that we will work in the ring of formal Laurent series $\mathbb{Z}[[u, \frac{1}{u}]]$. Let's define the BGF associated with $\mathcal{W}$ as

$$W_2(z,u) = \sum_{\substack{n\geq 0\\ k\in\mathbb{Z}}} w_{n,k} z^n u^k.$$

This gives

$$W_2(z,u) = 1 + uzW_2(z,u) + \frac{z}{u}W_2(z,u).$$

Solving this equation for $W_2(z,u)$ results in

$$W_2(z,u) = \frac{1}{1 - z\left(u + \frac{1}{u}\right)}.$$

Next we will perform a coefficient extraction in order to get $w_{n,k}$, the number of walks of length n stopping at height k:

$$[z^n]W_2(z,u) = \left(u + \frac{1}{u}\right)^n.$$

This is a Laurent polynomial in u. Now we apply the shift identity of the coefficient extraction (1) to get

$$\begin{aligned} w_{n,k} &= [u^k]\left(u + \frac{1}{u}\right)^n = [u^{n+k}]\left(u^2+1\right)^n \\ &= \begin{cases} 0, & \text{for } n+k \equiv 1 \mod (2) \text{ or } k > n, \\ \binom{n}{\frac{n+k}{2}}, & \text{for } n+k \equiv 0 \mod (2). \end{cases} \end{aligned}$$

Note that the BGF can be easily transformed into the OGF we found in Example 3.5, by substituting $u = 1$. This action sums over all possible heights at fixed length n:

$$\begin{aligned} W_2(z,1) &= \frac{1}{1-2z} = W(z) \\ \sum_{k\in\mathbb{Z}} w_{n,k} &= \sum_{k=-n,-n+2,\dots n} \binom{n}{\frac{n+k}{2}} = \sum_{k=0}^{n}\binom{n}{k} = 2^n \end{aligned}$$

In general, we have to be careful here. We are only dealing with formal power series, which is the reason why insertion of special values for variables is in general not well-defined. So, we have to ensure that all operations are legitimate, e.g.: there are no singularities and all sums are finite, etc. ■

Often it is not so easy to extract the exact value of the coefficients. However, it is often possible to get their asymptotics, even without knowing them explicitly. The next section introduces some powerful tools for this purpose.

3.3 Coefficient Asymptotics

The *Gamma function* extends the factorial function to non-integral arguments. It was introduced by Euler as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt.$$

The integral converges provided $\Re(s) > 0$. Using integration by parts one immediately derives the basic functional equation of the Gamma function,

$$\Gamma(s+1) = s\Gamma(s).$$

Since $\Gamma(1) = 1$ one directly gets $\Gamma(n+1) = n!$. The special value $\Gamma(1/2) = \sqrt{\pi}$ proves to be very important. Also its asymptotic properties will be needed:

Proposition 3.9 (Stirling's formula)**.** *The factorial function admits the asymptotic expansion:*

$$x! \equiv \Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right), \qquad (x \to +\infty).$$

Example 3.10. A direct consequence of Stirling's formula is the asymptotic expansion

$$C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n^3}}\left(1 - \frac{9}{8n} + \frac{145}{128n^2} - \frac{1155}{1024n^3} + \cdots\right), \tag{7}$$

of the Catalan numbers C_n. ■

At the heart of the asymptotic enumeration lie the following fundamental theorems. The main idea is to treat generating functions not only as formal power series, but as converging power series on a certain radius of convergence. By doing so, one may utilize the wealth of results from complex analysis to derive formulas on the asymptotics of the coefficients. For the proofs and more details, the interested reader is refereed to [6] and the literature mentioned therein.

Theorem 3.11 ([6, Theorem VI.1], Standard function scale)**.** *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1-z)^{-\alpha}$$

admits for large n a complete asymptotic expansion in descending powers of n,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right),$$

where e_k is a polynomial in α of degree $2k$. In particular:

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)\right).$$

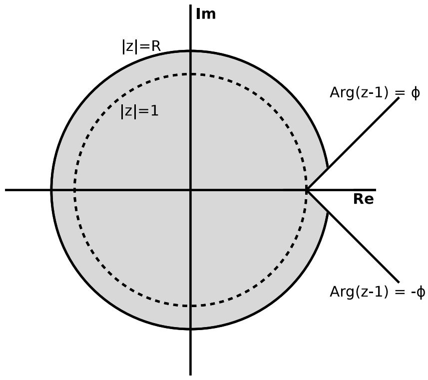


Figure 10: Sketch of a Δ-domain

Proof (Sketch). The proof idea consists of using Cauchy's coefficient formula and a properly chosen contour, a so called Hankel contour. Then, asymptotic estimates lead to the result. □

Theorem 3.12 ([6, Theorem VI.2], Standard function scale, logarithms)**.** *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1-z)^{-\alpha} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^{\beta}$$

admits for large n a complete asymptotic expansion in descending powers of n,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta} \left(1 + \sum_{k=1}^{\infty} \frac{C_k}{(\log n)^k} \right),$$

where $C_k = \binom{\beta}{k} \Gamma(\alpha) \left. \frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \right|_{s=\alpha}$.

The asymptotic results of the previous Theorem for some standard functions are summarized in Figure 11. These results will mostly suffice in the subsequent examples.

In order to also transfer the error terms of the coefficient asmyptotics we need the next (technical) definition.

Definition 3.13 (Δ-analytic)**.** *Given two numbers ϕ, R with $R > 1$ and $0 < \phi < \frac{\pi}{2}$, the open domain $\Delta(\phi, R)$ is defined as*

$$\Delta(\phi, R) = \{ z \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi \}.$$

A domain is a Δ-domain at 1 *if it is a $\Delta(\phi, R)$ for some R and ϕ. For a complex number $\zeta \neq 0$, a* Δ-domain at ζ *is the image by the mapping $z \mapsto \zeta z$ of a Δ-domain at 1. A function is* Δ-analytic *if it is analytic in some Δ-domain.*

For an illustration of a Δ-domain, see Figure 10.

Theorem 3.14 ([6, Theorem VI.3], Transfer, Big-Oh and little-oh)**.** *Let α, β be arbitrary real numbers, $\alpha, \beta \in \mathbb{R}$ and let $f(z)$ be a function that is Δ-analytic.*

1. *Assume that $f(z)$ satisfies in the intersection of a neighborhood of* 1 *with its* Δ*-domain the condition*

$$f(z) = \mathcal{O}\left((1-z)^{-\alpha}(\log\frac{1}{1-z})^{\beta}\right).$$

Then one has: $[z^n]f(z) = \mathcal{O}(n^{\alpha-1}(\log n)^{\beta})$.

2. *Assume that $f(z)$ satisfies in the intersection of a neighborhood of* 1 *with its* Δ*-domain the condition*

$$f(z) = o\left((1-z)^{-\alpha}(\log\frac{1}{1-z})^{\beta}\right).$$

Then one has: $[z^n]f(z) = o(n^{\alpha-1}(\log n)^{\beta})$.

The last three theorems lie at the heart of coefficient asymptotics and define the so-called *singularity analysis.* The next proposition summarizes this process and presents an algorithm to deal with such functions. We want to emphasize the fact that the structure of the generating function is used to derive results on its coefficients.

Proposition 3.15 ([6, Chapter VI.4], Process of singularity analysis)**.** *Let $f(z)$, the functions whose coefficients are to be analyzed, be analytic at* 0*.*

1. Preparation: *Locate dominant singularities and check analytic continuation.*
   (a) Locate singularities: *Determine the dominant singularities of $f(z)$ and check that $f(z)$ has a single singularity ρ on its circle of convergence.*
   (b) Check continuation: *Establish that $f(z)$ is analytic in some Δ-domain around ρ.*

2. Singular expansion: *Analyze the function $f(z)$ as $z \to \rho$ in the Δ-domain and determine an expansion of the form*

$$f(z) = \sigma(z/\rho) + \mathcal{O}(\tau(z/\rho)), \qquad \textit{with} \qquad \tau(z) = o(\sigma(z)), \qquad \textit{for } z \to \rho.$$

The functions σ and τ should belong to the standard scale of functions given by the set $\mathcal{S} = \{(1-z)^{-\alpha}\lambda(z)^{\beta}\}$, with $\lambda(z) := z^{-1}\log(1-z)^{-1}$.

3. Transfer: *Translate the main term of $\sigma(z)$ using the catalogs provided by Theorems 3.11 and 3.12. Transfer the error term using Theorem 3.14 and conclude that*

$$[z^n]f(z) = \rho^{-n}\sigma_n + \mathcal{O}(\rho^{-n}\tau_n^{\star}), \qquad \textit{for } n \to \infty,$$

where $\sigma_n = [z^n]\sigma(z)$ and $\tau_n^{\star} = [z^n]\tau(z)$ provided the corresponding exponent $\alpha \neq \mathbb{Z}_{\leq 0}$ (otherwise the factor $1/\Gamma(\alpha) = 0$ should be dropped).

Example 3.16. Using Theorem 3.11 there is another possibility to derive the asymptotic expansion of Catalan numbers. From (6) we know the generating function of Catalan numbers. Its dominant singularity is at $\rho = \frac{1}{4}$. Thus, we get the asymptotic expansion

$$D(z) = 2 - 2\sqrt{1-4z} + 2(1-4z) + \mathcal{O}\left((1-4z)^{3/2}\right).$$

Thus, applying Theorems 3.11 and 3.14 we recover the first term of (7):

$$[z^n]D(z) = \frac{4^n}{\sqrt{\pi n^3}}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Note that the full asymptotic expansion can also be derived from this expression. ■

4 Łukasiewicz Paths

As an introduction to lattice path theory, we are going to consider *directed paths.* These are paths with a fixed direction of increase which we choose to be the positive horizontal axis. This is described by the allowed steps: if $(i, j) \in \mathcal{S}$ then $i > 0$. One first important observation is that the geometric realization of the path always lives in the right half-plane $\mathbb{Z}_+ \times \mathbb{Z}$. This essentially means that directed paths are one-dimensional objects.

The following chapter mainly focuses on the expositions of Banderier[9] and Flajolet given in [2].

Definition 4.1. *Along these restrictions, we introduce the following classes (see Table 1):*

- *A bridge is a path whose end-point ω_n lies on the x-axis;*
- *A meander is a path that lies in the quarter plane $\mathbb{Z}_+^2$;*
- *An excursion is a path that is at the same time a meander and a bridge, i.e. it connects the origin with a point lying on the x-axis and involves no point with negative y-coordinate.*

Additionally, we call a family of paths or steps to be simple *if each allowed step in $\mathcal{S}$ is of the form $(1, b)$ with $b \in \mathbb{Z}$. In this case, we denote $\mathcal{S} = \{b_1, \ldots, b_k\}$.*

A Łukasiewicz path *is a simple path, its associated step set $\mathcal{S}$ is a subset of* $\{-1, 0, 1, \ldots\}$, *and* $-1 \in \mathcal{S}$.

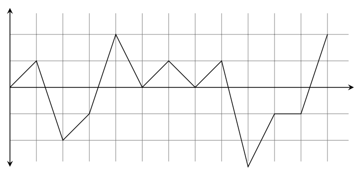
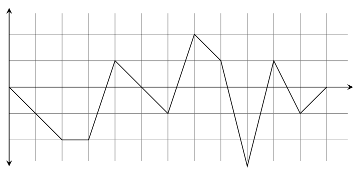
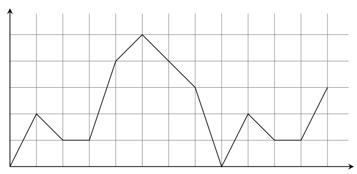
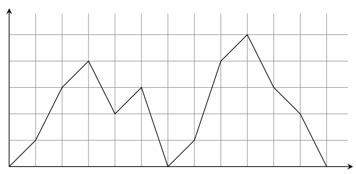
	ending anywhere	ending at 0
unconstrained (on $\mathbb{Z}$)	 walk/path ($\mathcal{W}$) $W(z) = \frac{1}{1-zP(1)}$	 bridge ($\mathcal{B}$) $B(z) = z\frac{u_1'(z)}{u_1(z)}$
constrained (on $\mathbb{Z}_+$)	 meander ($\mathcal{M}$) $M(z) = \frac{1-u_1(z)}{1-zP(1)}$	 excursion ($\mathcal{E}$) $E(z) = \frac{u_1(z)}{p_{-1}z}$

Table 1: The four types of paths: walks, bridges, meanders and excursions, and the corresponding generating functions for Łukasiewicz paths [2, Fig. 1].

In the remainder of this section, we will always consider Łukasiewicz paths.

[9]Cyril Banderier, 19.5.1975-

4.1 Walks and Bridges

The first cases we are going to consider are the unconstrained walks and bridges. First we introduce the algebraic structures associated with the previous definitions. The *characteristic polynomial* of $\mathcal{S}$ is defined as the polynomial in u, u^{-1} (a Laurent polynomial)

$$P(u) := \sum_{j=1}^{m} p_j u^{s_j},$$

where p_j is the weight associated to the step s_j. Let $c := -\min_j s_j$ and $d := \max_j s_j$ be the two extreme jump sizes, and assume throughout $c, d > 0$. Note that for Łukasiewicz paths we have $c = 1$. In order to count the number of walks, one sets all weights to 1. Thereby every walk has weight 1.

Let $W_{n,k}$ be the number of paths ending after n steps at altitude k. We define the associated generating function as

$$W(z, u) := \sum_{n \geq 0, k \in \mathbb{Z}} W_{n,k} z^n u^k.$$

Note that we are mainly interested in solving the counting problem, i.e. determining the numbers $W_{n,k}$ for certain families of paths (compare e.g. Figure 1). The generating function encodes all information we are interested in. The following variant of [2, Theorem 1] makes these explicit.

Theorem 4.2. *The bivariate generating function of paths (z marking size and u marking final altitude) relative to a simple step set $\mathcal{S}$ with characteristic polynomial $P(u)$ is a rational function. It is given by*

$$W(z, u) = \frac{1}{1 - zP(u)}.$$

The generating function of bridges is an algebraic function given by

$$B(z) = z \frac{u_1'(z)}{u_1(z)}, \tag{8}$$

where $u_1(z)$ is the unique solution of the kernel equation $1 - zP(u) = 0$ with $\lim_{z \to 0} u_1(z) = 0$.

Example 4.3 (Dyck Bridges). The step set $\mathcal{S} = \{\mathbf{NE}, \mathbf{SE}\} = \{+1, -1\}$ corresponds to the walks of *Dyck bridges*. The characteristic polynomial is $P(u) = u^{-1} + u$, and hence the kernel equation reads

$$1 - z\left(\frac{1}{u} + u\right) = 0.$$

We see immediately from the step set that $c = 1$ and $d = 1$. Therefore, the kernel equation is of degree 2:

$$u - z(1 + u^2) = 0.$$

There exists one small branch and one large branch. In this case, they can be easily computed, by solving the equation of degree 2:

$$u_1(z) = \frac{1-\sqrt{1-4z^2}}{2z} \underset{z\to 0}{\sim} z, \qquad v_1(z) = \frac{1+\sqrt{1-4z^2}}{2z} \underset{z\to 0}{\sim} \frac{1}{z} \tag{9}$$

We used the fact that $\sqrt{1-4z^2} = \sum_{n\geq 0} \binom{1/2}{n}(-4)^n z^{2n}$ in a small neighborhood of 0. Now we apply Theorem 4.2 which gives the GF for bridges:

$$B(z) = z\frac{u_1'(z)}{u_1(z)} = \frac{1}{\sqrt{1-4z^2}} = 1 + 2z^2 + 6z^4 + 70z^8 + 252z^{10} + \dots$$

The coefficients are known as *OEIS* **A000984**[10]

$$[z^n]B(z) = \binom{2n}{n} = [t^n](1+t^2)^n \tag{10}$$

and called *central binomial numbers.* They are closely related to the Catalan numbers. This result can be explained very easily: In order to uniquely characterize a Dyck bridge consisting of n **NE**-steps and n **SE**-steps, we simply need to choose the positions of the **NE**-steps (or equivalently of the **SE**-steps). For this, there are $\binom{2n}{n}$ possibilities. ∎

4.2 Meanders and excursions

Let $F_{n,k}$ be the number of paths ending after n steps at altitude k. We define the associated generating function as

$$F(z,u) := \sum_{n,k} F_{n,k} z^n u^k = \sum_{k\geq 0} F_k(z) u^k = \sum_{n\geq 0} f_n(u) z^n.$$

Firstly, the generating functions $F_k(z)$ represent walks ending at altitude k, i.e. $F_k(z) = \sum_{n\geq 0} F_{n,k} z^n$. Thus, the generating function of excursions is equal to $F_0(z)$. Secondly, the polynomials $f_n(u)$ represent walks of length n. The powers of u encodes their possible final altitudes.

Theorem 4.4. *Let $\mathcal{S}$ be the step set of a Łukasiewicz path, and $P(u)$ be the associated step polynomial. The bivariate generating function of meanders (where z marks length, and z marks final altitude) and excursions, respectively, are*

$$F(z,u) = \frac{1-u_1(z)/u}{1-zP(u)} \qquad \text{and} \qquad E(z) = \frac{u_1(z)}{p_{-1}z}, \tag{11}$$

where $u_1(z)$ is the unique small solution of the implicit equation

$$1 - zP(u) = 0,$$

which fulfills $\lim_{z\to 0} u_1(z) = 0$.

[10] Central binomial coefficients; `http://oeis.org/A000984`, accessed 15/11/2015.

Proof. A meander or excursion of length n is either empty, or it is constructed from a walk of length $n-1$ by appending a possible step from $\mathcal{S}$. However, a walk is not allowed to go below the x-axis, thus at altitude $u=0$ it is not allowed to use the step -1. This translates into

$$f_0(u) = 1, \qquad\qquad f_{n+1}(u) = \{u^{\geq 0}\}\left(P(u)f_n(u)\right),$$

where $\{u^{\geq 0}\}$ is the linear operator extracting all terms in the power series representation containing non-negative powers of u. Multiplying by z^{n+1} and summing over all $n \geq 0$ we derive the following functional equation where $F_0(z) = E(z)$

$$F(z,u) = 1 + zP(u)F(z,u) - \frac{p_{-1}z}{u}F_0(z),$$

and we get

$$\underbrace{(1 - zP(u))}_{=:K(z,u)} F(z,u) = 1 - \frac{p_{-1}z}{u}F_0(z), \tag{12}$$

where $K(z,u)$ is called the *kernel.* This functional equation is under-determined as their are two unknown functions, namely $F(z,u)$ and $F_0(z)$. However, the special structure on the left hand-side will resolve this problem and leads us to the *kernel method.*

From the theory of Newton–Puiseux expansions, the fundamental result in the theory of algebraic curves [1, 11], we know that the *kernel equation*

$$1 - zP(u) = 0, \tag{13}$$

has $d+1$ ($c=1$) distinct solutions in u, with 1 of them being called "small branch", as it maps 0 to 0 and is in modulus smaller than the other d "large branches" which grow in modulus to infinity while approaching 0. We call the small branch $u_1(z)$ and the large ones $v_1(z), \ldots, v_d(z)$. For this functions to be well-defined we restrict our attention to the complex plane slit along the negative real axis. Inserting the small branch into (12) we get

$$F_0(z) = \frac{u_1(z)}{p_{-1}z}. \tag{14}$$

Using this result we can solve (12) for $F(z,u)$ to get the final result. □

Remark 4.5 (Brief history of the kernel method): The main idea of the proof of Therorem 4.4 was to solve the functional equation (12) by the *kernel method*, which consists of binding z and u in such a way that the left hand side vanishes. Compare with [9, Exercise 2.2.1.1-4] as the first source, or with [4] for a combinatorial and analytic treatment, or with [2, p. 56] for the strongly related Wiener-Hopf approach from probability theory. For more details we refer to the summary of historical notes at the end of [2, Section 2.3]. ■

Example 4.6 (Dyck paths and the Ballot Problem). Continuing Example 4.3, Dyck paths are excursions with the step set $\mathcal{S} = \{-1, 1\}$. The associated step polynomial is

$$P(u) = \frac{1}{u} + u.$$

We may directly apply Theorem 4.4 as we have already computed the small and large branch in (9) and recover the generating function of Dyck paths from (6):

$$E(z) = \frac{1-\sqrt{1-4z^2}}{2z^2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} z^{2n} = \sum_{n\geq 0} C_n z^{2n},$$

where the coefficients C_n are the Catalan numbers.

Recall from the introduction that the *ballot problem* asks for the probability in a two candidate election between A and B that eventually ends in a tie, while A is dominating B throughout the poll. The fact that it ends in a tie, translates into a walk that ends on the x-axis, and the condition of A dominating B is modeled by the restriction that the walk must not leave the first quadrant. Hence, we are dealing with a Dyck Path.

The total number of possible walks from $(0,0)$ to $(2n,0)$ is $\binom{2n}{n}$, which are the number of bridges with respect to this step set, compare (10). Thus,

$$\mathbb{P}(\text{tie, } A \text{ dominates } B \text{ throughout}) = \begin{cases} \frac{1}{n+1}, & \text{2n votes,} \\ 0, & \text{2n+1 votes,} \end{cases}$$

is the asked probability. ■

Example 4.7. Consider the step set $\mathcal{S} = \{-1, 0, 1, 2\}$. There will be one small branch of order 1 and two large branches of order $-1/2$. The entire version of the characteristic equation is

$$u - z\left(1 + u + u^2 + u^3\right) = 0.$$

The one small branch is given by

$$u_1(z) = z + z^2 + 2z^3 + 5z^4 + 13z^5 + 36z^6 + 104z^7 + 309z^8 + \dots.$$

The first few terms of the GF for excursions are easily computed by (11)

$$E(z) = \frac{u_1(z)}{z} = 1 + z + 2z^2 + 5z^3 + 13z^4 + 36z^5 + 104z^6 + 309z^7 + \dots,$$

and similarly for meanders

$$M(z) = \frac{1-u_1(z)}{1-4z} = 1 + 3z + 11z^2 + 42z^3 + 163z^4 + 639z^5 + \dots.$$

Obviously the second representations for $E(z)$ and $M(z)$ in terms of the large branches lead to the same result, but are much more complicated to calculate in this case. ■

Let us end this chapter with a summary of some well-known lattice path enumeration problems. We state the specific step set, kernel, and GF of excursions.

- *Dyck Paths:* $\mathcal{S} = \{(1,-1),(1,1)\}$ with the kernel $K(z,u) = u - zu^2 - z$,

$$E(z) = \frac{1-\sqrt{1-4z^2}}{2z^2}.$$

- *Motzkin Paths:* $\mathcal{S} = \{(1,-1),(1,1),(1,0)\}$ with the kernel $K(z,u) = u - zu^2 - z - zu$,

$$E(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

- *Schröder Paths:* $\mathcal{S} = \{(1,-1),(1,1),(2,0)\}$ with the kernel $K(z,u) = u - zu^2 - z - z^2u$,

$$E(z) = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2}.$$

- *Delannoy Paths:* $\mathcal{S} = \{(1,0),(0,1),(1,1)\}$ with the kernel $K(z,u) = 1 - z - zu - u$,

$$F(z,u) = \frac{z + zu + u}{1 - z - zu - u}, \qquad E(z) = \frac{1}{1-z}.$$

5 Basic parameters for Dyck paths

5.1 Arches and contacts

Define an *arch* as an excursion of size > 0 whose only contact with the x-axis is at its end points and let $\mathcal{A}$ be the set of arches. The set $\mathcal{D}$ of excursions satisfies the combinatorial equation

$$\mathcal{D} = \mathrm{SEQ}(\mathcal{A}),$$

where SEQ denotes the combinatorial construction that forms sequences, compare (2). By the symbolic method this translates directly into the generating function equation

$$E(z) = \frac{1}{1 - A(z)}, \qquad \text{or equivalently} \qquad A(z) = 1 - \frac{1}{E(z)}. \tag{15}$$

Define a vertex of an excursion not equal to one of the end points to be a *contact* if its altitude is 0. Then, $A(z)^{k+1}$ is the generating function of excursions having k contacts.

The next theorem gives the result for the number of contacts. We will encounter a *negative binomial distribution*, where we say that a random variable X is distributed according to $\mathrm{NegBinom}(r,p)$ if

$$\mathbb{P}(X = k) = \binom{k + r - 1}{k} p^k (1-p)^r.$$

It represents the number of unsuccessful trials k until the r^{th} success in independent Bernoulli experiments with probability p.

Theorem 5.1. *The probability that a random Dyck path of size n has k contacts is for any fixed k of the form*

$$\frac{1}{4}(k+1)\left(\frac{1}{2}\right)^k + \mathcal{O}\left(\frac{1}{n}\right).$$

The number of contacts is thus asymptotically distributed like a negative binomial distribution with parameters $\mathrm{NegBinom}(2, 1/2)$.

Proof. The probability that a Dyck path of length n chosen uniformly at random has k contacts is

$$\frac{1}{C_n}[z^n]A(z)^{k+1}.$$

As we are interested in the probability for large n, we want to derive the asymptotics of these numbers for fixed k. The asymptotics of the Catalan numbers C_n has been well studied before in e.g. (7). Thus, what remains is to apply singularity analysis to $A(z)^{k+1}$. From (15) and the result for Dyck paths from Example 4.6 we get that

$$A(z)^{k+1} = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4z}\right)^{k+1} = \frac{1}{2^{k+1}} - (k+1)\frac{1}{2^{k+1}}\sqrt{1-4z} + \mathcal{O}(1-4z).$$

Its dominant singularity is at $1/4$, with the above singular expansion at that point. Thus, Theorems 3.11 and 3.14 combined with Figure 11 directly yield the result. □

5.2 Expected final altitude

Let us consider *Dyck meanders*. Thereby we understand paths constructed from the step set $\mathcal{S} = \{-1, 1\}$ constrained to be above the x-axis. In other words we drop the condition of Dyck paths to end on the x-axis, and consider meanders instead of excursions.

Theorem 5.2. *The generating function $G(z,u)$ ($U(z,u)$) of Dyck meanders of even (odd) length, with z marking twice the steps (twice minus 1 the steps), and u marking the final altitude is*

$$G(z,u) = \frac{D(z)}{1 - z(uD(z))^2}, \qquad U(z,u) = uD(z)G(z,u),$$

where $D(z) = \frac{1-\sqrt{1-4z}}{2z}$ is the generating function of Dyck paths.

Proof. Let us start with even length. First note that paths of even length must end on even altitude.

We uniquely decompose the path by the last times it leaves a given altitude. This is a so-called *last passage decomposition* (compare first passage decomposition in Example 3.7). Note that in $D(z)$ the power of z counts the number of pairs of up and down steps. In order to reach an even altitude a walk must have an even number more up than down steps. Thus, we group 2 consecutive of these last up steps and count them by z. Then the number of steps remains twice the power of z. Going from altitude k to altitude $k+2$ where the first jump leaves altitude k for the last time can be modeled by $z(uD(z))^2$.

The last passage decomposition shows that a meander is thus given by a Dyck path followed by a sequence of the previous objects. This yields the formula for $G(z,u)$.

Moreover, a path ending on an odd altitude, can be uniquely decomposed into a Dyck path followed by an up jump followed by a Dyck meander ending on an even altitude. Thus, we get the formula for $U(z,u)$. □

Let us now consider a probability distribution on the set of meanders of length n. The combinatorial probability model (or uniform distribution among elements of size n) is given by drawing uniformly at random an element of the given class.

Let X_n be the random variable of paths of length n ending on altitude k. Then we have

$$\mathbb{P}(X_{2n} = k) = \frac{[z^n u^k]G(z,u)}{[z^n]G(z,1)}, \qquad \mathbb{P}(X_{2n+1} = k) = \frac{[z^n u^k]U(z,u)}{[z^n]U(z,1)}.$$

It is easy to see that the expected value and variance are equal to

$$\mathbb{E}(X_{2n}) = \frac{[z^n]G_u(z,1)}{[z^n]G(z,1)}, \quad \mathbb{V}(X_{2n}) = \frac{[z^n]G_{uu}(z,1)}{[z^n]G(z,1)} + \frac{[z^n]G_u(z,1)}{[z^n]G(z,1)} - \left(\frac{[z^n]G_{uu}(z,1)}{[z^n]G(z,1)}\right)^2,$$

where $G_u(z,1) := \frac{\partial}{\partial u} G(z,u)\big|_{u=1}$, and $G_{uu}(z,1) := \frac{\partial^2}{\partial u^2} G(z,u)\Big|_{u=1}$. Obviously, the same holds for X_{2n+1} with $G(z,u)$ replaced by $U(z,u)$.

Then singularity analysis directly gives

Theorem 5.3. *The number of Dyck meanders of length n is equal to*

$$M_n = \begin{cases} \binom{2n}{n}, & \text{for } n = 2k, \\ \binom{2n+1}{n}, & \text{for } n = 2k+1. \end{cases}$$

The expected value and the variance for the final altitude of meanders of length n are asymptotically equal to

$$\mathbb{E}(X_n) = \sqrt{\pi n} + \mathcal{O}(1), \qquad \mathbb{V}(X_n) = (4-\pi)n + \mathcal{O}(1).$$

Proof. Let us start with the asymptotic number of meanders. A straightforward computation gives

$$G(z,1) = \frac{1}{\sqrt{1-4z}}, \qquad U(z,1) = \frac{1}{2z}\left(\frac{1}{\sqrt{1-4z}} - 1\right).$$

Extracting the z^n-th coefficient gives the result.

We perform the computations for $G(z,u)$, the ones for $U(z,u)$ are analogous. We get

$$\begin{aligned} G_u(z,1) &= \frac{1}{1-4z} - \frac{1}{\sqrt{1-4z}}, \\ G_{uu}(z,1) &= \frac{2}{(1-4z)^{3/2}} - \frac{3}{1-4z} + \frac{1}{\sqrt{1-4z}}. \end{aligned}$$

Applying singularity analysis to each of these terms directly gives the result. □

It is noteworthy that the leading terms for $n \to \infty$ of the expected value and the variance do not depend on the parity of n. One can show that the limit distribution of the rescaled random variable $\frac{X_n}{\sqrt{n}}$ is a Rayleigh distribution with parameter $\sigma = \sqrt{2}$, see [2, Theorem 6].

6 Exercises

1. (a) The generating function of Dyck paths is given by

$$D(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{16}$$

Extract the coefficients of this generating function, i.e., provide the formula for $d_n = [z^n]D(z)$.

(b) What are the coefficients of $\frac{1}{\sqrt{1-4z}}$?

Apply Newton's generalized binomial theorem which states that

$$(x+y)^r = \sum_{k\geq 0} \binom{r}{k} x^{r-k} y^k,$$

where r is some complex number and the binomial coefficients are defined as:

$$\binom{r}{k} = \frac{r \cdot (r-1) \cdots (r-k+1)}{k!}$$

where $k \in \mathbb{N}$.

2. The formula for the number of Dyck paths consisting of $2n$ steps can also be derived in a more direct way using a counting argument that is now referred to as *André's reflection principle*. The idea is the following: We count lattice paths consisting of n steps to the **NE** and n steps to the **SE** and then subtract the number of such paths that are not Dyck paths.
   (a) How many lattice paths consisting of n steps to the **NE** and n steps to the SE are there in total?
   (b) Let p be a lattice path with n **NE**-steps and n **SE**-steps that is not a Dyck path. Then pick the first step that lies beneath the x-axis and change all NE-steps occurring afterwards into **SE**-steps and vice-versa. What can be said about these *reflected paths*? How many such paths are there?
   (c) Re-derive the formula for d_n.

3. Use the symbolic method described in Section 3 in order to specify the generating function of *Motzkin paths*. These are paths in the Euclidean lattice with the step set $\mathcal{S} = \{(1,1),(1,0),(1,-1)\}$ subject to the restriction that they may never go below the x-axis, and end on the x-axis, i.e. excursions.

   Additionally, derive the bivariate generating function for Motzkin meanders, z marking length and u marking final altitude.

   Note the following difference to Dyck paths: Since we allow **E**-steps, Motzkin do not have to be of even length.

4. Using the generating function derived in the previous question, do the following:
   (a) Provide an exact formula for M_n, the number of Motzkin paths with n steps. These numbers are called *Motzkin numbers* (OEIS **A001006**).

   When extracting coefficients, the so-called *Lagrange inversion formula* can be helpful. Let $F(z)$ and $\varphi(u)$ be formal power series which satisfy $F(z) = z\varphi(F(z))$ and $\varphi_0 = [u^0]\varphi(u) \neq 0$. Then one has

$$[z^n]g(F(z)) = \begin{cases} \frac{1}{n}[F^{n-1}]g'(x)(\varphi(F))^n, & n > 0 \\ [F^0]g(F), & n = 0, \end{cases}$$

   for every formal power series $g(x)$.
   (b) Provide an asymptotic estimate of M_n (do not use your answer of question (a)).

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388 *VI. SINGULARITY ANALYSIS OF GENERATING FUNCTIONS*

Function	*coefficients*
$(1-z)^{3/2}$	$\frac{1}{\sqrt{\pi n^5}}(\frac{3}{4}+\frac{45}{32n}+\frac{1155}{512n^2}+O(\frac{1}{n^3}))$
$(1-z)$	(0)
$(1-z)^{1/2}$	$-\frac{1}{\sqrt{\pi n^3}}(\frac{1}{2}+\frac{3}{16n}+\frac{25}{256n^2}+O(\frac{1}{n^3}))$
$(1-z)^{1/2}\,\mathrm{L}(z)$	$-\frac{1}{\sqrt{\pi n^3}}(\frac{1}{2}\log n+\frac{\gamma+2\log 2-2}{2}+O(\frac{\log n}{n}))$
$(1-z)^{1/3}$	$-\frac{1}{3\Gamma(\frac{2}{3})n^{4/3}}(1+\frac{2}{9n}+\frac{7}{81n^2}+O(\frac{1}{n^3}))$
$z/\,\mathrm{L}(z)$	$\frac{1}{n\log^2 n}(-1+\frac{2\gamma}{\log n}+\frac{\pi^2-6\gamma^2}{2\log^2 n}+O(\frac{1}{\log^3 n}))$
1	(0)
$\log(1-z)^{-1}$	$\frac{1}{n}$
$\log^2(1-z)^{-1}$	$\frac{1}{n}(2\log n+2\gamma-\frac{1}{n}-\frac{1}{6n^2}+O(\frac{1}{n^4}))$
$(1-z)^{-1/3}$	$\frac{1}{\Gamma(\frac{1}{3})n^{2/3}}(1+O(\frac{1}{n}))$
$(1-z)^{-1/2}$	$\frac{1}{\sqrt{\pi n}}(1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}+O(\frac{1}{n^4}))$
$(1-z)^{-1/2}\,\mathrm{L}(z)$	$\frac{1}{\sqrt{\pi n}}(\log n+\gamma+2\log 2-\frac{\log n+\gamma+2\log 2}{8n}+O(\frac{\log n}{n^2}))$
$(1-z)^{-1}$	1
$(1-z)^{-1}\,\mathrm{L}(z)$	$\log n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}+O(\frac{1}{n^6}))$
$(1-z)^{-1}\,\mathrm{L}(z)^2$	$\log^2 n+2\gamma\log n+\gamma^2-\frac{\pi^2}{6}+O(\frac{\log n}{n})$
$(1-z)^{-3/2}$	$\sqrt{\frac{n}{\pi}}(2+\frac{3}{4n}-\frac{7}{64n^2}+O(\frac{1}{n^3}))$
$(1-z)^{-3/2}\,\mathrm{L}(z)$	$\sqrt{\frac{n}{\pi}}(2\log n+2\gamma+4\log 2-4+\frac{3\log n}{4n}+O(\frac{1}{n}))$
$(1-z)^{-2}$	$n+1$
$(1-z)^{-2}\,\mathrm{L}(z)$	$n\log n+(\gamma-1)n+\log n+\frac{1}{2}+\gamma+O(\frac{1}{n})$
$(1-z)^{-2}\,\mathrm{L}(z)^2$	$n(\log^2 n+2(\gamma-1)\log n+\gamma^2-2\gamma+2-\frac{\pi^2}{6}+O(\frac{\log n}{n}))$
$(1-z)^{-3}$	$\frac{1}{2}n^2+\frac{3}{2}n+1$

Figure 11: A table from [6, Figure VI.5, p. 388] of some commonly encountered functions and the asymptotic forms of their coefficients. The following abbreviation is used: $L(z) := \log\frac{1}{1-z}$.