CONDITIONED RANDOM WALKS AND SPATIALLY-EXTENDED CONDENSATION

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Condensation phenomena in stochastic systems

July 5, 2016, Bath, UK

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Outline

1. Nonequilibrium path ensembles and their (in)equivalence
2. Condensation phenomena in random walk bridges
3. Random walk conditioned on fixed area and local time
4. Large deviation theory of spatially-extended (interaction-driven) condensation

Nonequilibrium path ensembles

Stochastic thermodynamics

- a new paradigm: extending the notion of statistical ensembles to dynamical trajectories
  [Maes (1999), Crooks (2000), Seifert (2005), Lecome et al (2007), Harris and Schütz (2007), Jack and Sollich (2010), Chetrite and Touchette (2013)]
- closely connected to the mathematical theory of large deviations
- for a given stochastic process, one is interested in
  - calculating statistics of time-integrated observable $A_T[x]$

$$P(A_T = a) = \int \mathcal{D}[x] P[x] \delta(A_T[x] - a),$$

  - understanding how the event $A_T[x] = A$ occured

Examples

$$A_T[x] = \frac{1}{T}\int_0^T f(x_t)dt + \begin{cases} \frac{1}{T}\sum_{\substack{0\le t\le T \\ \Delta X(t)\neq 0}} g(X(t^-),X(t^+)) & \text{jump} \\ \frac{1}{T}\int_0^T g(X(t))\circ dx_t & \text{diffusion} \end{cases}$$

- action functional: $f=0$, $g(x,y)=\ln\frac{w(x,y)}{w(y,x)}$
  [Lebowitz and Spohn (1999)]
- probability history: $f=0$, $g(x,y)=\ln\frac{w(x,y)}{\sum_y w(x,y)}$
  [Lecomte et al (2007)]
- particle current in driven diffusive systems
  [Bodineau and Derrida (2004)]
- dynamical activity: $f=0$, $g(x,y)=1$
  [Merolle et al (2005); Garrahan et al (2005); Jack et al (2006)]

Path ensembles: microcanonical vs. canonical

Microcanonical path ensemble:

$$P[X|A_T = a] = \frac{P[X, A_T = a]}{P(A_T = a)}$$

- in general difficult to work with

Canonical path ensemble:

$$P_s[x] = \frac{P[x]e^{sA_T[x]}}{\langle e^{sA_T} \rangle}$$

- s plays the role of inverse temperature
- other names: s-ensemble, driven or tilted or biased process

Path ensembles and their (in)equivalence

- mathematical proof by [Chetrite and Touchette (2014), (2015)]

$$\lim_{T\to\infty} \frac{1}{T} \ln \frac{P[x|A_T = a]}{P_s[x]} = 0,$$

Conditions:

- A_T satisfies large deviation principle

$$P(A_T = a) \asymp e^{-TI(a)}, \quad T \to \infty$$

- $I(a)$ is a convex function of a
- if $I(a)$ is differentiable, then $s = I'(a)$

inequivalence $\overset{?}{\longleftrightarrow}$ condensation

Random walk bridges

Random walk bridges: definition

- a discrete-time and continuous-space random walk:

$$X_t = X_{t-1} + \eta_t, \quad X_0 = 0,$$

- jump probability density is $\phi(\eta_t)$ and

$$\mathbb{E}_\phi[\eta_t] = \mu, \qquad \mathrm{Var}_\phi[\eta_t] = \sigma^2$$

- random walk bridge: conditioning on fixed

$$A_T = \frac{X_T - X_0}{T} = a$$

Random walk bridges: path probability

- path probability density:

$$
\begin{aligned}
P[X|X_T = aT] &= \frac{1}{P(X_T = aT)} \prod_{t=1}^{T} w(X_t|X_{t-1})\delta\left(X_T - aT\right) \\
&= \frac{1}{P(X_T = aT)} \prod_{t=1}^{T} \phi(X_t - X_{t-1})\delta\left(X_T - aT\right) \\
&= \frac{1}{P\left(\sum_{t=1}^{T} \eta_t = aT\right)} \prod_{t=1}^{T} \phi(\eta_t)\delta\left(\sum_{t=1}^{T} \eta_t - aT\right)
\end{aligned}
$$

$\rightarrow$ same as factorised steady states in mass-transfer models

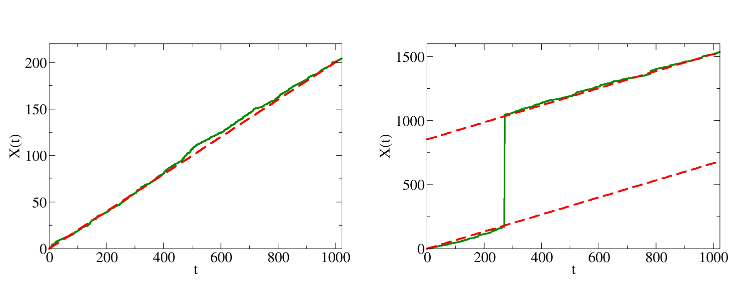
Random walk bridges: standard condensation

- condensation occurs when $\phi(\eta_t)$ is heavy tailed, i.e. when

$$\int \mathrm{d}\eta \phi(\eta) e^{k\eta} = \infty \quad \text{for all } k > 0,$$

- sums of iid heavy-tailed random variables [Linnik (1961), Nagaev (1969)]:

$$P(X_T/T = a) = T\phi(a-\mu), \quad T \to \infty,$$

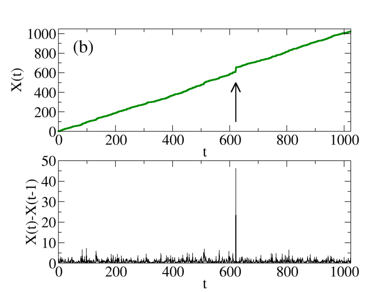


Random walk bridges: constraint-driven condensation

- $\phi(\eta_t)$ can be light-tailed if there are two constraints:

$$X_T = \sum_{t=1}^{T} \eta_t = aT$$

$$\underbrace{\sum_{t=1}^{T}(X_t - X_{t-1})^2}_{\text{realised variance}} = \sum_{t=1}^{T} \eta_t^2 = bT$$



- mechanism: [JSN, Evans and Majumdar (2014)]

$$\phi(\eta_t) \underset{exp.tilting}{\longrightarrow} \phi(\eta_t)\mathrm{e}^{-r\eta_t} \underset{\xi_t=\eta_t^2}{\longrightarrow} \phi(\xi_t^{1/2}) \underbrace{\mathrm{e}^{-r\xi_t^{1/2}}}_{\text{Weibull tail}} \longrightarrow \text{condensation}$$

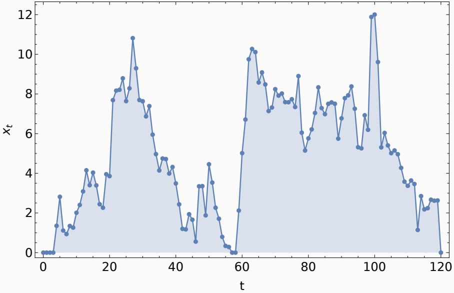
Random walks conditioned on fixed area and local time

Reflected random walk

- definition:

$$x_t = \max\{0, x_{t-1} + \eta_t\}$$

η_t are iid random variables with probability density $\phi(\eta_t)$



- (generalised) transition probability density:

$$w(x_t|x_{t-1}) = \delta(x_t) \int_{-\infty}^{-x_{t-1}} \phi(\eta) \mathrm{d}\eta + \phi(x_t - x_{t-1})$$

Conditioning on fixed area and local time

- area $A_T[x]$ under the path x:

$$A_T[x] = \sum_{t=1}^{T} x_t = A \equiv (\sigma/\mu)T$$

- local time $l_T[x]$ (number of returns to the origin):

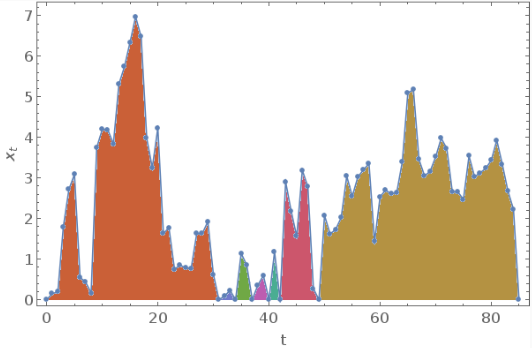
$$l_T[x] = \sum_{t=1}^{T} \delta(x_t) = N \equiv (1/\mu)T$$

- path probability density for paths conditioned on fixed value of A_T and l_T:

$$P[x|A_T = A, l_T = N] = \frac{1}{Z_N(A,T)} \prod_{t=1}^{T} w(x_t|x_{t-1})\delta(l_T - N)\delta(A_T - A)$$

Fixing local time explicitly

- definitions:
  - t_i is time of i-th return to the origin, $i = 1, \ldots, N$
  - excursion is path between to successive returns to the origin



  - i-th excursion has duration τ_i and area a_i,

$$\tau_i = t_{i+1} - t_i, \qquad a_i = \sum_{t=t_i}^{t_{i+1}} x_t$$

Random walk excursions under two constraints

- we can now rewrite the partition function in terms of random walk excursions:

$$\begin{aligned} Z_N(A,T) &= \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_T \prod_{i=1}^{T} w(x_t|x_{t-1})\delta(I_T - N)\delta(A_T - A) \\ &= \sum_{\{\tau_i\}} \int_0^\infty \mathrm{d}a_1, \ldots \mathrm{d}a_N \prod_{i=1}^{N} f(a_i,\tau_i)\delta\left(\sum_{j=1}^{N} a_j - A\right)\delta\left(\sum_{k=1}^{N}\tau_k - T\right) \end{aligned}$$

  $f(a_i, \tau_i)$ is joint probability density for a_i and τ_i

- key simplification: pairs of random variables (a_i, τ_i) are mutually independent!
- difficulty: $f(a, \tau)$ explicitly known only for the simple lattice random walk [Takács (1993)] and Brownian motion [Majumdar and Comtet (2005)]

Explicit calculation for the Laplace jump distribution

- consider the random walk starting from $x_0 = x \geq 0$ and let $f(x, a, \tau)$ denotes the corresponding joint probability density
- integral equation for $f(x, a, \tau)$:

$$f(x,a,\tau) = \begin{cases} \delta(x-a)\int_{-\infty}^{0} dx_1 \phi(x_1 - x), & \tau = 1 \\ \int_0^\infty \mathrm{d}x_1\, \phi(x_1 - x) f(x_1, a-x, \tau-1), & \tau > 1 \end{cases}$$

- the Laplace transform/moment-generating function is given by:

$$g(x,p,z) = \int_0^\infty \mathrm{d}a\, \mathrm{e}^{-pa} \sum_{\tau=1}^{\infty} f(x,a,\tau) z^\tau .$$

Explicit calculation for the Laplace jump distribution (cont'd)

- the integral equation for $g(x,p,z)$:

$$g(x,p,z) = z\mathrm{e}^{-px}\int_0^\infty \mathrm{d}x_1\,\phi(x_1-x)G(x_1,p,z) + z\mathrm{e}^{px}\int_{-\infty}^0 \mathrm{d}x_1\phi(x_1-x).$$

- key simplification for $\phi(x)=\exp(-|x|)/2$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathrm{e}^{-|y-x|} = \mathrm{e}^{-|y-x|} - 2\delta(x-y)$$

- differential equation for $g(x,p,z)$:

$$\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} - 2p\frac{\mathrm{d}g}{\mathrm{d}x} + (p^2-1+z\mathrm{e}^{px})g = 0 \to g(p,z) = \frac{z^{1/2}J_{2/p}(2z^{1/2}/p)}{J_{2/p-1}(2z^{1/2}/p)}.$$

Properties of $f(a, \tau)$

- Laplace transform:

$$\int_0^\infty \mathrm{d}a \, \mathrm{e}^{-pa} f(a, \tau) = \frac{4^\tau}{p^{2\tau - 1}} \sigma_\tau(2/p - 1)$$

  $\sigma_\tau(\nu)$ is the Rayleigh function [Kishore (1963)]

- marginal $f(\tau)$ (the Sparre-Andersen theorem)

$$f(\tau) = \frac{1}{2^{2\tau-1}\tau} \binom{2\tau - 2}{\tau - 1}$$

- scaling limit [Takacś (1993), Denisov et al (2015)]:

$$f(a, \tau) \approx \frac{1}{2\sqrt{2\pi}\tau^3} f_{\mathrm{Airy}} \left(\frac{a}{2^{1/2}\tau^{3/2}} \right), \qquad \tau \to \infty$$

  $f_{\mathrm{Airy}}(x)$ is the Airy distribution [Majumdar and Comtet (2005)]

Analysis of the partition function: saddle point equations

- Laplace transform of the partition function:

$$\mathcal{Z}_N(p,z) = \sum_{T=0}^{\infty} z^T \int_0^\infty \mathrm{d}A\, \mathrm{e}^{-pA} Z_N(T,A) = [g(p,z)]^N$$

- the partition function is then given by:

$$Z_N(A,T) = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}p}{2\pi i}\, \mathrm{e}^{pA} \oint_\gamma \frac{\mathrm{d}z}{2\pi i} \frac{[g(p,z)]^N}{z^{T+1}},$$

$\rightarrow$ for N large we can try the saddle point method

- amounts to solve the following saddle point equations

$$\mu = z\frac{\partial}{\partial z}\ln g(p,z), \qquad \sigma = -\frac{\partial}{\partial p}\ln g(p,z)$$

Analysis of the partition function: condensation transition

- define auxiliary probability density $\omega(a, \tau)$

$$\omega(a, \tau; p, z) = \frac{f(a, \tau) z^{\tau} \mathrm{e}^{-pa}}{g(p, z)}$$

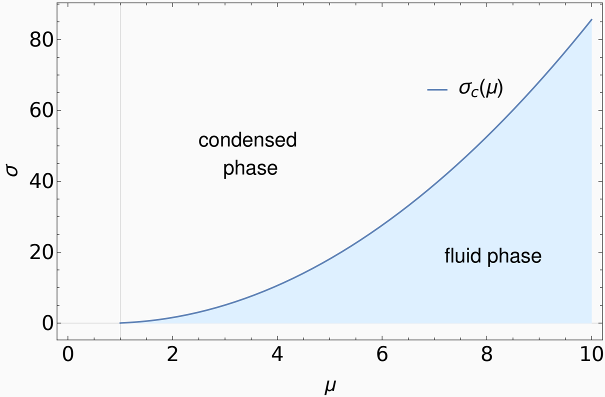
- the saddle point equations then become

$$\mu = \mathbb{E}_{\omega}[\tau](p, z), \qquad \sigma = \mathbb{E}_{\omega}[a](p, z)$$

- the first equation can be solved for any $\mu > 1 \rightarrow$ gives $z_0(p; \mu)$
- the second equation has no solution for $\sigma > \sigma_c$, where σ_c is given by

$$\sigma_c = \mathbb{E}_{\omega}[a](0, z_0(0, \mu)) < \infty$$

Analysis of the partition function: phase diagram



$$\sigma_c(\mu) = \frac{1}{12\mu\left[\ln\left(\frac{\mu}{\mu-1}\right) - \frac{1}{2\mu-1}\right]}$$

Nature of the condensate

- the microcanonical partition function is given by:

$$Z_N(A,T) = \int_0^\infty \mathrm{d}a_1 \ldots \mathrm{d}a_N \sum_{\{\tau_i\}} \prod_{i=1}^N f(a_i,\tau_i)\delta\left(\sum_{k=1}^N \tau_k - T\right)\delta\left(\sum_{j=1}^N a_j - A\right)$$

- the canonical partition function is given by:

$$\mathcal{Z}_N(p,z) = \int_0^\infty \mathrm{d}a_1 \ldots \mathrm{d}a_N \sum_{\{\tau_i\}} \prod_{i=1}^N f(a_i,\tau_i)\mathrm{e}^{-pa_i} z^{\tau_i} = [g(p,z)]^N$$

- condensation means that $Z_N(A,T)$ is not equivalent to $\mathcal{Z}_N(p,z)$ for $\sigma > \sigma_c$

Nature of the condensate (cont'd)

- for $\sigma > \sigma_c$, $Z_N(A, T)$ is equivalent to the mixed canonical-microcanonical partition function $\mathcal{Y}_N(A, z)$

$$\begin{aligned}\mathcal{Y}_N(A, z_0) &= \int_0^\infty \mathrm{d}a_1 \ldots \mathrm{d}a_N \left[\prod_{i=1}^N \sum_{\{\tau_i\}} f(a_i, \tau_i) z_0^{\tau_i}\right] \delta\left(\sum_{j=1}^N a_j - A\right) \\ &= [g(0, z_0)]^N P\left(\sum_{i=1}^N a_i = A\right)\end{aligned}$$

$$P\left(\sum_{i=1}^N a_i = A\right) = \int_0^\infty \mathrm{d}a_1, \ldots \mathrm{d}a_N \prod_{i=1}^N \omega(a_i) \delta\left(\sum_{j=1}^N a_j - A\right)$$

$$\omega(a) = \sum_{\tau=1}^\infty \omega(a, \tau; 0, z_0) = \frac{\sum_{\tau=1}^\infty f(a, \tau) z_0^\tau}{g(0, z_0)}, \quad \int_0^\infty \mathrm{d}a\, a\omega(a) = \sigma_c$$

Nature of the condensate: tail of $\omega(a)$

- tail of $\omega(a)$: an open problem
- heuristic argument that $\omega(a)$ has a Weibull-like tail:
  - one can show that $f(a,\tau)$ behaves for large a as

    $$f(a,\tau) = c_\tau \mathrm{e}^{-\frac{2a}{\tau-1}} - \mathrm{O}\left(\mathrm{e}^{-\frac{2a}{\tau-2}}\right), \quad \tau \geq 2, \quad a \to \infty$$

    where the coefficient c_τ is given by

    $$c_\tau = \frac{(\tau-1)^{2\tau-3}}{4^{\tau-1}[(\tau-1)!]^2}, \qquad \tau \geq 2.$$

  - the largest contribution to $\omega(a)$ is then

    $$\sum_{\tau=2}^{\infty} c_\tau \mathrm{e}^{-2a/(\tau-1)} z_0^\tau \sim \mathrm{e}^{-\kappa\sqrt{a}}, \quad a \to \infty,$$

Large deviation theory of spatially-extended condensation

Pair-factorised steady states

- generalisation to the zero-range process: hopping rate $u(m_{i-1}, m_i, m_{i+1})$ depends on the surrounding environment [Evans, Zia and Majumdar (2006)]
- if $u(m_{i-1}, m_i, m_{i+1}) = \alpha(m_{i-1}, m_i)\beta(m_i, m_{i+1})$ and

$$\alpha(l, m) = \frac{g(l, m-1)}{g(l, m)}, \qquad \beta(m, n) = \frac{g(m-1, n)}{g(m, n)}$$

  then the steady state has a pair-factorised probability

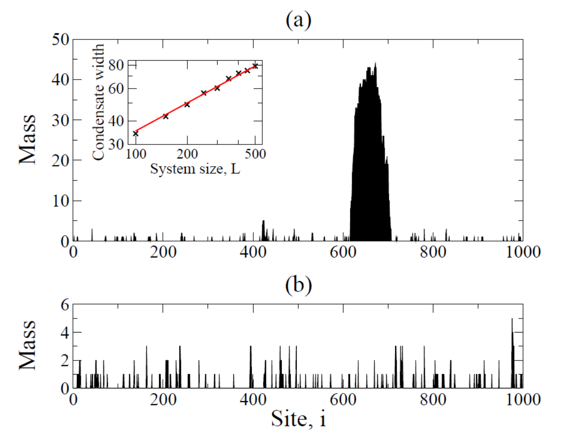
$$P[\{m_i\}] = \frac{1}{\mathcal{Z}_L(M)} \prod_{i=1}^{L} g(m_i, m_{i+1}) \delta\left(\sum_{j=1}^{L} m_j - M\right)$$

- choice: $g(m_i, m_{i+1}) = \mathrm{e}^{-J|m_{i+1}-m_i| + \frac{1}{2}U_0(\delta_{m_i,0} + \delta_{m_{i+1},0})}$

Spatially-extended (interaction-driven) condensation

$\rho > \rho_c$

$\rho < \rho_c$



$$\rho_c = \frac{1}{e^{2J}(1 - e^{-U})^2 - 1}$$

[Majumdar, Evans and Zia (2005)]

Mapping to the reflected (lattice) random walk paths

- the partition function:

$$\mathcal{Z}_L(M) = \sum_{\{m_i \geq 0\}} \left[\prod_{i=1}^{L} \mathrm{e}^{-J|m_{i+1}-m_i|} \right] \mathrm{e}^{U \sum_{j=1}^{L} \delta(m_j)} \delta\left(\sum_{j=1}^{L} m_i - M \right).$$

- transition probability for the (discrete) Laplace distribution:

$$w(m_{i+1}|m_i) = \underbrace{\left(\frac{\mathrm{e}^J - 1}{\mathrm{e}^J + 1} \right) \mathrm{e}^{-J|m_{i+1}-m_i|}}_{\phi(m_{i+1}-m_i)} \mathrm{e}^{R\delta(m_{i+1})}, \quad R = J - \ln(\mathrm{e}^J - 1)$$

Mapping to the reflected (lattice) random walk paths (cont'd)

- altogether gives the following partition function:

$$\mathcal{Z}_L(M) = \sum_{\{m_i \geq 0\}} \prod_{i=1}^{L} w(m_{i+1}|m_i) \mathrm{e}^{(U-R)\sum_{j=1}^{L} \delta(m_j)} \delta\left(\sum_{j=1}^{L} m_i - M\right).$$

- we assume that $P(I_L = \lambda N)$ satisfies large deviation principle

$$P(I_L = \lambda N) \sim \mathrm{e}^{-LI(\lambda)}, \qquad L \to \infty,$$

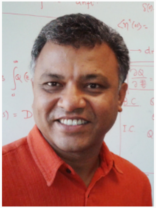
- then one can choose λ, $I'(\lambda) = U - R$, such that

$$\mathcal{Z}_L(M) \sim Z_L(M, N)$$

$$Z_L(M, N) = \sum_{\{m_i \geq 0\}} \prod_{i=1}^{L} w(m_{i+1}|m_i) \delta\left(\sum_{j=1}^{L} \delta(m_j) - N\right) \delta\left(\sum_{j=1}^{L} m_i - M\right),$$

Collaborators:

Martin Evans



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Funding:

EPSRC
Engineering and Physical Sciences Research Council

Grant Number: EP/J007404/1

Thank you for your attention.