Metastability in condensing zero-range processes

Inés Armendáriz

Universidad de Buenos Aires

in collaboration with A. de Masi, S. Grosskinsky, M. Loulakis and E. Presutti

July 4, 2016

Condensation phenomena in stochastic systems, BATH
The zero-range process

**Lattice:** $\Lambda$ of size $L$

**State space:** $X_L = \{0, 1, \ldots \}^\Lambda$

**Jump probabilities:** $p(x, y) \in [0, 1]$

**Jump rates:** $g_x : \{0, 1, \ldots \} \to [0, \infty)$, $g_x(k) = 0 \iff k = 0$

**Generator:** $f \in C_0(X_L)$

$$\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} g_x(\eta_x)p(x, y) \left( f(\eta^{x,y}) - f(\eta) \right)$$

[Spitzer (1970), Andjel (1982)]
The zero-range process

\[ g_x(k) = k \quad \Rightarrow \quad \text{independent identical particles} \]

\[ g_x(k) = g_x \quad \Rightarrow \quad \text{network of M/M/1 server queues} \]
The zero-range process

\[ g_x(k) = k \quad \Rightarrow \quad \text{independent identical particles} \]
\[ g_x(k) = g_x \quad \Rightarrow \quad \text{network of M/M/1 server queues} \]

Condensation phenomena

- \textbf{spatial heterogeneity}
  \[ \Rightarrow \quad \text{condensation on the 'slowest' site} \]

The zero-range process

\[ g_x(k) = k \quad \Rightarrow \quad \text{independent identical particles} \]
\[ g_x(k) = g_x \quad \Rightarrow \quad \text{network of M/M/1 server queues} \]

Condensation phenomena

- **spatial heterogeneity**
  \[ \Rightarrow \quad \text{condensation on the 'slowest' site} \]

- **effective attraction** of particles due to \( g(k) \downarrow \)
  \[ \Rightarrow \quad \text{condensation on a random site} \]
  \[ g(n) \simeq 1 + \frac{b}{n}, \quad b > 2 \quad [\text{Evans (2000)}] \]
  \[ g(n) \simeq ne^{-2n} \]
Evans model

**Lattice:** $\Lambda$ of size $L$

**State space:** $X_L = \{0, 1, \ldots\}^\Lambda$

$\eta = (\eta_x)_{x \in \Lambda}$

**Jump rates:** $p(x, y) g(\eta_x)$

- choose $g(k) = \left(\frac{k}{k-1}\right)^b \simeq 1 + \frac{b}{k}$ with $b > 0$
- $g(0) = 0$, $g(1) = 1$

- choose $p(x, y) = \frac{1}{2} \delta_{y,x+1} + \frac{1}{2} \delta_{y,x-1}$

**Generator:** $\mathcal{L}f(\eta) = \sum_{x \in \Lambda_L} g(\eta_x)\left(\frac{1}{2}f(\eta^x,x+1) + \frac{1}{2}f(\eta^x,x-1) - f(\eta)\right)$

[Spitzer '70; Andjel '82; Evans '00]
Canonical measures and condensation

fixed number of particles $N$: $\mu_{L,N}[\cdot] = \frac{\nu_\phi[\cdot]}{\nu_\phi[\sum_x \eta_x = N]}$, $\nu_\phi[\eta_x = k] \propto \frac{\phi^k}{k!}$
Canonical measures and condensation

fixed number of particles $N$: 

$$\mu_{L,N}[\cdot] = \frac{\nu_{\phi}[\cdot]}{\nu_{\phi}[\sum_x \eta_x = N]}, \quad \nu_{\phi}[\eta_x = k] \propto \frac{\phi^k}{k!}$$

Equivalence of ensembles

In the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$

$$\mu_{L,N} \to \nu_{\phi} \quad \text{where} \quad \left\{ \begin{array}{l} \phi \leftrightarrow \rho \ , \ \rho \leq \rho_c \\ \phi = \phi_c \ , \ \rho \geq \rho_c \end{array} \right. \ .$$

[Jeon, March, Pittel '00; Grosskinsky, Schütz, Spohn '03; Ferrari, Landim, Sisko '07; A., Loulakis '09]
Metastability: dynamics of the condensate

**Potential theoretic approach:** [Bovier, Gayrard, Eckhoff, Klein ’01, ’02,…]  
[Bovier, den Hollander, Metastability - a potential theoretic approach (2016)]

**Martingale approach:** [Beltrán, Landim ’10, ’11, ’15]
Metastability: dynamics of the condensate

**Potential theoretic approach:** [Bovier, Gayrard, Eckhoff, Klein ’01, ’02, …]

[Bovier, den Hollander, Metastability - a potential theoretic approach (2016)]

**Martingale approach:** [Beltrán, Landim ’10, ’11, ’15]

**Trace process**  
- metastable wells

\[ \mathcal{E}^x := \left\{ \eta_x \geq N - \rho_c L - \alpha_L, \eta_y \leq \beta_L, y \neq x \right\} ; \]

\[ \mathcal{E} = \bigcup_x \mathcal{E}^x \]

\[ \Delta = \mathcal{X}_{L,N} \setminus \mathcal{E} \]
Trace process \( \eta^E \)

- \( \eta^E \) is a Markov process on \( E = \bigcup_{x \in \Lambda} E^x \) with generator \( \mathcal{L}^E \) and rates

\[
r^E(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) P_{\zeta}[T_E = T_\xi]
\]
Trace process $\eta^E$

- $\eta^E$ is a Markov process on $\mathcal{E} = \bigcup_{x \in \Lambda} \mathcal{E}^x$ with generator $L^E$ and rates

$$r^E(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) \mathbb{P}_\zeta[T^E = T^\xi]$$
Trace process $\eta^E$

- $\eta^E$ is a Markov process on $\mathcal{E} = \bigcup_{x \in \Lambda} \mathcal{E}^x$ with generator $\mathcal{L}^E$ and rates

$$r^{E}(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) \mathbb{P}_\zeta[T_E = T_\xi]$$
Trace process $\eta^E$

- $\eta^E$ is a Markov process on $E = \bigcup_{x \in \Lambda} E^x$ with generator $L^E$ and rates

$$r^E(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) P_\zeta[T_E = T_\xi]$$
Trace process $\eta^\mathcal{E}$

- $\eta^\mathcal{E}$ is a Markov process on $\mathcal{E} = \bigcup_{x \in \Lambda} \mathcal{E}^x$ with generator $\mathcal{L}^\mathcal{E}$ and rates

$$r^\mathcal{E}(\eta, \xi) = r(\eta, \xi) + \sum_{\zeta \in \Delta} r(\eta, \zeta) \mathbb{P}_{\zeta}[T_{\mathcal{E}} = T_{\xi}]$$

- Invariant measure

$$\mu[\cdot] = \mu_{L,N}[\cdot \mid \mathcal{E}]$$
Main result

**Theorem.** A., Grosskinsky, Loulakis [arXiv:1507.03797]

The ZRP with \( b > 20 \), as \( L, N \to \infty \), \( N/L \to \rho > \rho_c \), exhibits metastability w.r.t. the rescaled condensate location

\[
Y_t^L := \psi_L(\eta^E(\theta_L t)) := \frac{1}{L} \sum_{x \in \Lambda} x 1_{E_x}(\eta^E(\theta_L t)) \in T \quad \text{on the scale } \theta_L = L^{1+b} .
\]

The ZRP with $b > 20$, as $L, N \to \infty$, $N/L \to \rho > \rho_c$, exhibits metastability w.r.t. the rescaled condensate location

$$Y^L_t := \psi_L(\eta^E(\theta_L t)) := \frac{1}{L} \sum_{x \in \Lambda} x \mathbb{1}_{\mathcal{E}_x}(\eta^E(\theta_L t)) \in \mathbb{T} \quad \text{on the scale } \theta_L = L^{1+b}.$$

For all initial conditions $\eta^L(0) \in \mathcal{E}^0$ we have weakly on pathspace

$$(Y^L_t : t \geq 0) \Rightarrow (Y_t : t \geq 0) \quad \text{with } \quad Y_0 = 0,$$

where $(Y_t : t \geq 0)$ is a Lévy-type jump process on $\mathbb{T}$ with generator

$$\mathcal{L}^\mathbb{T} f(u) = K_{b,\rho} \int_{\mathbb{T} \setminus \{0\}} \frac{1}{d(v,u)} (f(v) - f(u)) \, dv,$$

where $d(v,u) = |v-u| (1 - |v-u|)$ is the distance in $\mathbb{T}$.
Proof

- \((Y_t^L : t \geq 0)\) is **tight** on \(D([0, T], \mathbb{T})\)
- identify limit points \((Y_t : t \geq 0)\) as solutions of the **martingale problem**

\[
 f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^T f(Y_s) \, ds \quad \text{is a martingale.} \tag{1}
\]
Proof

- \( (Y_t^L : t \geq 0) \) is **tight** on \( D([0, T], \mathbb{T}) \)
- identify limit points \( (Y_t : t \geq 0) \) as solutions of the **martingale problem**

\[
    f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}_T f(Y_s) \, ds \quad \text{is a martingale}.
\]

Introduce the **auxiliary process** \( \mathcal{L}^\Lambda \) on \( \Lambda \) with averaged rates

\[
r^\Lambda(x, y) = \frac{1}{\mu[E_x]} \sum_{\eta \in E_x, \xi \in E_y} \mu[\eta] r^E(\eta, \xi)
\]

and write

\[
\begin{align*}
    \int_0^t \left( \mathcal{L}_T f(Y_s^L) - \theta_L \mathcal{L}^E(f \circ \psi_L)(\eta^E(\theta_L s)) \right) \, ds \\
    = \int_0^t \left( \mathcal{L}_T f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L) \right) \, ds + \theta_L \int_0^t \left( \mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^E(f \circ \psi_L)(\eta^E(\theta_L s)) \right) \, ds
\end{align*}
\]
Proof

- \((Y_t^L : t \geq 0)\) is **tight** on \(D([0, T], \mathbb{T})\)
- identify limit points \((Y_t : t \geq 0)\) as solutions of the **martingale problem**

\[
f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^T f(Y_s) \, ds\quad \text{is a martingale}. \tag{1}
\]

Introduce the **auxiliary process** \(\mathcal{L}^\Lambda\) on \(\Lambda\) with averaged rates

\[
r^\Lambda(x, y) = \frac{1}{\mu[\mathcal{E}x]} \sum_{\eta \in \mathcal{E}x, \xi \in \mathcal{E}y} \mu[\eta] r^\mathcal{E}(\eta, \xi)
\]

and write

\[
\int_0^t \left( \mathcal{L}^T f(Y_s^L) - \theta_L \mathcal{L}^\mathcal{E}(f \circ \psi_L)(\eta^\mathcal{E}(\theta_L s)) \right) \, ds
\]

\[
= \int_0^t \left( \mathcal{L}^T f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L) \right) \, ds + \theta_L \int_0^t \left( \mathcal{L}^\Lambda f(Y_s^L) - \mathcal{L}^\mathcal{E}(f \circ \psi_L)(\eta^\mathcal{E}(\theta_L s)) \right) \, ds
\]

1. Prove equilibration within wells on a scale \(t_{\text{mix}} \ll \theta_L = L^{1+b}\)
2. Prove convergence of averaged dynamics on the scale \(\theta_L\)
3. central Lemma: **uniform bounds** on exit rates
1 – Equilibration within a well

Restricted process to a well $\mathcal{E}^x$ by ignoring jumps outside, $\mu^x = \mu[ \cdot | \mathcal{E}^x ]$

- bound on relaxation time $t_{\text{rel}}$, mixing time $t_{\text{mix}}(\epsilon)$

\[
t_{\text{rel}} \leq C L^4 \quad \text{and} \quad t_{\text{mix}}(\epsilon) \leq t_{\text{rel}} \log \left( \frac{1}{\epsilon \mu_{\text{min}}} \right) \leq C L^5 \log \left( \frac{1}{\epsilon} \right)
\]
1 – Equilibration within a well

Restricted process to a well $\mathcal{E}^x$ by ignoring jumps outside, $\mu^x = \mu[\cdot | \mathcal{E}^x]$

- bound on relaxation time $t_{\text{rel}}$, mixing time $t_{\text{mix}}(\epsilon)$

$$t_{\text{rel}} \leq CL^4 \quad \text{and} \quad t_{\text{mix}}(\epsilon) \leq t_{\text{rel}} \log \left( \frac{1}{\epsilon \mu_{\min}} \right) \leq CL^5 \log \left( \frac{1}{\epsilon} \right)$$

- ergodic $L^2$ bound for functions with $\mu^x(h) = 0$, $x \in \Lambda$

$$\mathbb{E}_\mu \left| \int_0^t h(\eta^x_u) \, du \right|^2 \leq 24t \, t_{\text{rel}} \sum_{x \in \Lambda} \mu[\mathcal{E}^x] \, \mu^x(h^2), \quad (2)$$

[J. Beltrán and C. Landim '15, *Martingale approach to metastability*]
1 – Equilibration within a well

Restricted process to a well $\mathcal{E}^x$ by ignoring jumps outside, $\mu^x = \mu \cdot |\mathcal{E}^x|

- bound on relaxation time $t_{\text{rel}}$, mixing time $t_{\text{mix}}(\epsilon)$

$$t_{\text{rel}} \leq CL^4 \quad \text{and} \quad t_{\text{mix}}(\epsilon) \leq t_{\text{rel}} \log \left( \frac{1}{\epsilon \mu_{\text{min}}} \right) \leq CL^5 \log \left( 1/\epsilon \right)$$

- ergodic $L^2$ bound for functions with $\mu^x(h) = 0$, $x \in \Lambda$

$$\mathbb{E}_\mu \left| \int_0^t h(\eta^\mathcal{E}_u) \, du \right|^2 \leq 24t \, t_{\text{rel}} \sum_{x \in \Lambda} \mu [\mathcal{E}^x] \, \mu^x(h^2), \quad (2)$$

[J. Beltrán and C. Landim '15, Martingale approach to metastability]

- Apply (2) + 3. + bounds on $\sum_{y \neq x} r^\Lambda(x, y)$ from 2. to $h = r^\mathcal{E} - r^\Lambda$ to get

$$\sup_{\eta \in \mathcal{E}} \mathbb{E}_\eta \left| \theta_L \int_0^t \left( \mathcal{L}^\Lambda f(Y^L_s) - \mathcal{L}^\mathcal{E} (f \circ \psi_L)(\eta^\mathcal{E}(\theta_L s)) \right) ds \right| \to 0$$
2 – Mean rates as capacities

\[ \mu[\mathcal{E}^{A_1}]r^\Lambda(A_1, A_2) = \mu[\mathcal{E}^{A_1}] \frac{1}{|A_1|} \sum_{x \in A_1, y \in A_2} r^\Lambda(x, y) \quad A_1, A_2 \subset \Lambda \]

\[ = \frac{1}{2} \left( \text{cap}(\mathcal{E}^{A_1}, \mathcal{E} \setminus \mathcal{E}^{A_1}) + \text{cap}(\mathcal{E}^{A_2}, \mathcal{E} \setminus \mathcal{E}^{A_2}) - \text{cap}(\mathcal{E}^{A_1 \cup A_2}, \mathcal{E} \setminus \mathcal{E}^{A_1 \cup A_2}) \right) \]

Prove bounds

\[ \theta_L \text{cap}(\mathcal{E}^{A_1}, \mathcal{E} \setminus \mathcal{E}^{A_1}) \leq K(b, \rho) (1 + \bar{\epsilon}_L) \sum_{x \in A, y \notin A} \text{cap}_\Lambda(x, y) \]

\[ \theta_L \text{cap}(\mathcal{E}^{A_1}, \mathcal{E} \setminus \mathcal{E}^{A_1}) \geq K(b, \rho) (1 - \epsilon_L) \sum_{x \in A, y \notin A} \text{cap}_\Lambda(x, y) \]

where \( \text{cap}_\Lambda(x, y) = \frac{1}{|x-y|(L-|x-y|)} \) capacities of symmetric rw on \( \Lambda \).
2 – Regularization

- Total exit rate from a well $\propto \log L$
- Upper and lower bounds for rates $r^\Lambda(x, y)$ do not match

see also [A. Bovier, R. Neukirch '14]
2 – Regularization

- Total exit rate from a well $\propto \log L$

- Upper and lower bounds for rates $r^\Lambda(x, y)$ do not match

  see also [A. Bovier, R. Neukirch ’14]

- Coarse graining in $\Lambda$ & Lipschitz test functions to regularize

  $$\theta_L \mathcal{L}^\Lambda f(x) = \sum_{m=1}^{\bar{L}} r^\Lambda(V_0, V_m) \left( f\left( \frac{x + \ell m}{L} \right) - f\left( \frac{x}{L} \right) \right) + o(1)$$

  with $|V_i| = \ell \propto \alpha_L \log^3 L \to \infty$, $\bar{L} = L/\ell$.

  $\left( \to \text{ leads to choice of } \alpha_L = L^{1/2+5/(2b)} \right)$

- Matching bounds from capacity representation for $r^\Lambda(V_0, V_m)$

  $$\sup_{\eta \in \mathcal{E}} \mathbb{E}_\eta \left| \int_0^t \left( \mathcal{L}^T f(Y_s^L) - \theta_L \mathcal{L}^\Lambda f(Y_s^L) \right) ds \right| \to 0$$
3 – Coupling to a branching system of BD processes

\[ m = \lceil 2^b \rceil \quad \text{largest possible arrival rate for ZRP} \]

\[ x \in \Lambda, \text{ couple } (\eta_x(t) : t \geq 0) \text{ with a growing system of BD chains } \zeta_x^k, \]

indexed by the \( m \)-regular tree \( R_m \)

- Each chain \( \zeta_x \) has birth rate 1 and death rate \( g(\zeta_x) \).
  Arrival events for \( \eta_x(t) \) are used only for one of the coupled chains
- At any time \( t \), only \( m \) of the chains are coupled to \( \eta_x(t) \), and the rest are evolving independently.
3 – Coupling to a branching system of BD processes

\[ m = \lceil 2^b \rceil \] largest possible arrival rate for ZRP

\( x \in \Lambda \), couple \( (\eta_x(t) : t \geq 0) \) with a growing system of BD chains \( \zeta^k_x \), indexed by the \( m \)-regular tree \( R_m \)

- Each chain \( \zeta_x \) has birth rate \( 1 \) and death rate \( g(\zeta_x) \).
- Arrival events for \( \eta_x(t) \) are used only for one of the coupled chains
- At any time \( t \), only \( m \) of the chains are coupled to \( \eta_x(t) \), and the rest are evolving independently.
- Number of chains grows linearly with time
- \( \max_k \zeta^k_x(t) \geq \eta_x(t) \) for all times \( t \geq 0 \).

Uniform exit rate bound:

\[
\sup_{\eta\in\varepsilon^x} \sum_{\xi\notin\varepsilon^x} r^\varepsilon(\eta, \xi) \leq C \frac{1}{L^5(\log L)^2}
\]
3 – Coupling to a branching system of BD processes

Example for $m = 2$

- arrows $\rightarrow$: identical copies
- coupled chains: red encircled
- independent chains: in blue

- coupled at generation $n = 1$ (top)
- particle arrives at $x$ (middle)
  - chains in 1st gen. turn independent
  - 2 descendants on top coupled
- second particle arrives, etc.
Lattice: complete graph $\Lambda$ of size $L$

State space: $X_L = \{0, 1, ..\}^\Lambda$

$\eta = (\eta_x)_{x \in \Lambda}$

Jump rates: $p(x, y) g(\eta_x)$

choose $g(k) = ke^{-2(k-1)\frac{L-1}{L}}$ and $p(x, y) = \frac{1}{L-1}$

Grand canonical measures: $\frac{1}{g!(k)} = e^{k^2-k}/k!$, do not exist

Canonical measure: $\frac{1}{Z} \prod_x \frac{e^{\eta_x^2}}{\eta(x)!} 1{\sum \eta_x = N}$

Number of particles: $N = (1 + \gamma) \log L$, $N/L \to 0$. 

Metastability

- **Metastable wells**

\[ E^x = \{ \eta_x \geq N - \alpha_L, \eta_y \leq \beta_L, y \neq x \} \]
Metastability

- **Metastable wells**
  \[ E^x = \{ \eta_x \geq N - \alpha_L, \eta_y \leq \beta_L, y \neq x \} \]

- **Trace process** \( (\eta_{t}^{E}, t \geq 0) \)
Metastability

- **Metastable wells**

\[ \mathcal{E}^x = \{ \eta_x \geq N - \alpha_L, \eta_y \leq \beta_L, y \neq x \} \]

- **Trace process** \((\eta^\mathcal{E}_t, t \geq 0)\)

- **Time scale** \(\theta_L : \left| \frac{\log \theta_L}{N^2} - \frac{(1+2\gamma)^2}{4(1+\gamma)^2} \right| \to 0\)
Metastability

- **Metastable wells**

\[ \mathcal{E}^x = \{ \eta_x \geq N - \alpha_L, \eta_y \leq \beta_L, y \neq x \} \]

- **Trace process** \( (\eta_t^\mathcal{E}, t \geq 0) \)

- **Time scale** \( \theta_L : \left| \frac{\log \theta_L}{N^2} - \frac{(1+2\gamma)^2}{4(1+\gamma)^2} \right| \to 0 \)

**Theorem.**

A., de Masi, Presutti

The ZRP with as \( L, N \to \infty , \quad N = (1 + \gamma) \log L, \quad \gamma \in \left(0, \frac{1}{\sqrt{2}}\right)\), exhibits metastability w.r.t. the rescaled condensate location

\[ Y_t^L := \psi_L(\eta^\mathcal{E}(\theta_L t)) := \frac{1}{L} \sum_{x \in \Lambda} x \mathbb{1}_{\mathcal{E}^x}(\eta^\mathcal{E}(\theta_L t)) \in \mathbb{T} \quad \text{on the scale } \theta_L . \]

For all \( \eta^L(0) \in \mathcal{E}^1 \), \( (Y_t^L : t \geq 0) \Rightarrow (Y_t : t \geq 0, Y_0 = 0) \), rate 1, uniform on \( \mathbb{T} \).
Metastability

- Condensates $\eta^x$: $\eta^x_x = N$, $\eta^x_y = 0$, $y \neq x$
Metastability

- Condensates $\eta^x: \eta^x_x = N, \eta^x_y = 0, y \neq x$
- Fluid $\mathcal{F} = \{\eta: \eta_y \leq 1, y = 1, \ldots, L\}$
Metastability

- Condensates $\eta^x : \eta^x_x = N, \eta^x_y = 0, y \neq x$
- Fluid $\mathcal{F} = \{\eta : \eta_y \leq 1, y = 1, \ldots, L\}$

Theorem continued

A., de Masi, Presutti

The time scale is

$$\theta_L = \frac{1}{r^\Lambda(\eta^1, \mathcal{F})}, \quad r^\Lambda(\eta^1, \mathcal{F}) = \frac{1}{\mu_{L,N}(\eta^1)} \text{cap}(0, N),$$

where $\text{cap}(0, N)$ are the capacities of a BD chain on $0, \ldots N$ with

$$b(k, k + 1) = \frac{N - k}{L}, \quad k \leq N - 1, \quad d(k, k - 1) = ke^{-2(k-1)} \frac{L - N + k}{L}, \quad k \geq 1$$

and invariant measure

$$\nu_{L,N} = \frac{1}{Z} \left(\frac{L - 1}{N - k}\right) e^{k^2 - k} \frac{k!}{k!}$$

With probability 1 as $L \to \infty$, the trajectory between two condensate $\eta^x$ and $\eta^y$, $x \neq y$, passes through $\mathcal{F}$.
The end