The Zero-Range Process Conditioned on an Atypical Current

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1) The zero-range process (ZRP)

2) Current conditioning

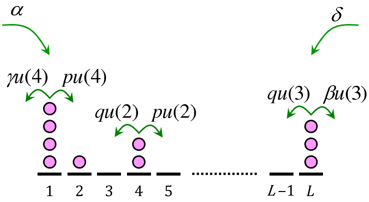
3) Effective dynamics

4) Exact density profiles and supercritical chain segments

5) Conclusions

1 The zero-range process (ZRP)

1.1 Definition [Spitzer (1970)]



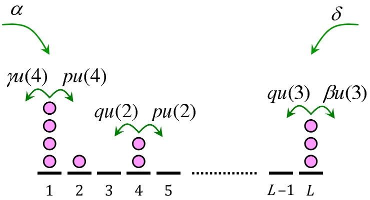
- Finite integer lattice $\Lambda := \{1, 2, \ldots, L\}$
- Local occupation variables $n_k \in \mathbb{N}$ for $k \in \Lambda$
- Configuration $\mathbf{n} = \{n_1, \ldots, n_L\} \in \mathbb{N}^L$
- Markovian jumps with rates $u(n)w^{\pm}$ (interaction and bias)
- Bulk driving field: $w^{\pm}_{bulk} = p, q$
- Open boundaries: $w^{\pm}_{edge} = \alpha, \beta, \gamma, \delta$ (particle exchange with reservoirs)
- Master equation for probability $P_{\mathbf{n}}(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{\mathbf{n}}(t) = \sum_{\mathbf{n}'} \left[w_{\mathbf{n},\mathbf{n}'} P_{\mathbf{n}'}(t) - w_{\mathbf{n}',\mathbf{n}} P_{\mathbf{n}}(t) \right]$$

Significance:

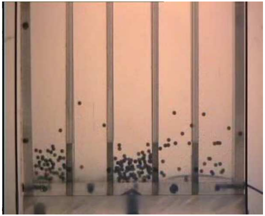
A) Toy model for far-from-equilibrium physics:

- Exactly soluble stationary distribution even in absence of detailed balance
- Rigorous derivation of nonlinear hydrodynamics
- Microscopic model for shock discontinuities
- Classical analog of BEC
- Coarsening dynamics and metastability



B) Applications of condensation transition:

- Granular media (Clustering)
- Networks (Hubs)
- Socio-economic systems (Aggregation of wealth)
- Traffic flow (Traffic jams)



1.2 Quantum Hamiltonian Formalism for master equation

Definitions:

- Probability vector $|P\rangle = \sum_{\mathbf{n}} P_{\mathbf{n}}|\,\mathbf{n}\,\rangle$ where $|\,\mathbf{n}\,\rangle = |n_1) \otimes \cdots \otimes |n_L)$ spans $(\mathbb{C}^\infty)^{\otimes L}$
- Dual basis $\langle\,\mathbf{n}\,|$ with orthogonality condition $\langle \mathbf{n}|\mathbf{n}'\rangle = \delta_{\mathbf{n},\mathbf{n}'}$
- Summation vector $\langle\,S\,| = \sum_{\mathbf{n}} \langle\,\mathbf{n}\,| \Rightarrow \langle S|P\rangle = 1$
- Particle creation and annihilation matrices

$$\hat{a}^+ = \sum_{n=1}^{\infty} |\,n\,)(\,n-1\,|, \quad \hat{a}^- = \sum_{n=1}^{\infty} u(n)|\,n-1\,)(\,n\,|$$

- Diagonal loss matrix and particle number operator

$$\hat{d} = \sum_{n=1}^{\infty} u(n)|\,n\,)(\,n\,|, \quad \hat{n} = \sum_{n=1}^{\infty} n|\,n\,)(\,n\,|$$

Generator for ZRP hopping dynamics:

$$\hat{H} = \hat{h}_0 + \sum_{k=1}^{L-1} \hat{h}_k + \hat{h}_L$$

Bulk hopping matrices: $\hat{h}_k = p(\hat{d}_k - \hat{a}_k^- \hat{a}_{k+1}^+) + q(\hat{d}_{k+1} - \hat{a}_k^+ \hat{a}_{k+1}^-)$

Boundary matrices: $\hat{h}_0 = \alpha(1 - \hat{a}_1^+) + \gamma(\hat{d}_1 - \hat{a}_1^-), \quad \hat{h}_L = \delta(1 - \hat{a}_L^+) + \beta(\hat{d}_L - \hat{a}_L^-)$

$\Rightarrow$ Master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|P(t)\rangle = -\hat{H}|P(t)\rangle$$

Solution: $|P(t)\rangle = \mathrm{e}^{-\hat{H}t}|P(0)\rangle$

Expectation of a function $F(\mathbf{n})$: $\langle\, S\,|\hat{F}|P(t)\rangle$

Diagonal matrix $\hat{F} = \sum_{\mathbf{n}} F(\mathbf{n})|\,\mathbf{n}\,\rangle\langle\,\mathbf{n}\,|$

1.3 Stationary distribution [Levine, Mukamel, GMS (2005)]

Stationary distribution: $H|\, P^* \,\rangle = 0$ (lowest right eigenvector with lowest eigenvalue 0)

- Product measure $|P^*\rangle = |P_1^*) \otimes |P_2^*) \otimes \ldots \otimes |P_L^*)$

- Marginals $P_k^* := \mathrm{Prob}\,[\, n_k = n\,] = \frac{z_k^n}{Z_k} \prod_{i=1}^n u(i)^{-1}$

- Local fugacities $z_k = \dfrac{[(\alpha+\delta)(p-q)-\alpha\beta+\gamma\delta]\left(\frac{p}{q}\right)^{k-1} - \gamma\delta + \alpha\beta\left(\frac{p}{q}\right)^{L-1}}{\gamma(p-q-\beta)+\beta(p-q+\gamma)\left(\frac{p}{q}\right)^{L-1}} =: \mathrm{e}^{\mu_k}$

- Local partition function $Z_k = \sum_{n=0}^{\infty} z_k^n \prod_{i=1}^n u(i)^{-1}$

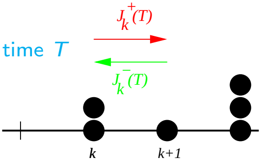
- Local mean density $\rho_k = \langle n_k \rangle = \frac{\mathrm{d}}{\mathrm{d}\mu_k} \ln Z_k$

- Steady-state current $j^* = (p-q)\dfrac{-\gamma\delta+\alpha\beta\left(\frac{p}{q}\right)^{L-1}}{\gamma(p-q-\beta)+\beta(p-q+\gamma)\left(\frac{p}{q}\right)^{L-1}}$

2 Current conditioning

Introduce time-integrated local directed current (fluctuating)

$J_k^+(T)$ = number of particle jumps from site k to $k+1$ up to time T



$\Rightarrow$ Define:

- Time-integrated local current $J_k(T) = J_k^+(T) - J_k^-(T)$
- Time-integrated total current $J(T) = \sum_{k=0}^{L} J_k(T)$
- Time-averaged currents $j_k(T) = J_k(T)/T$ and $j(T) = J(T)/T$
- Law of large numbers: limits $j_k = \lim_{T\to\infty} j_k(T)$ and j exist

Current is a fluctuating quantity, in which way can non-average current $j \neq j^*$ arise?

More precisely, what is the optimal (most likely) realization of the ZRP that generates an atypical current $j \neq j^*$? (Average density profile, correlations, ...)

▷ Macroscopic (MFT, Bertini et al. '02/'15, Derrida '07) $\Longleftrightarrow$ Microscopic (Exact solutions)

Answer from MFT: Consider macroscopic space and time scales (Law of large numbers and local equilibrium)

• Many microscopic realizations of the ZRP have the same $j = J(T)/T$ from the current distribution $\mathrm{Prob}\,[\,J(T) = J\,]$

• Large deviation form $P_J(T) \propto \mathrm{e}^{-Tf(j)}$ with $j = J/T$

• Large deviation function $f(j) = \min\limits_{\rho(x)} \int_0^1 \mathrm{d}x \, \frac{[j-(\nu\sigma(\rho)-D(\rho)\frac{\mathrm{d}\rho}{\mathrm{d}x})]^2}{2\sigma(\rho)}$

- Optimal profile $\rho(x)$ with boundary conditions $\rho(0) = \rho^-$, $\rho(1) = \rho^+$
- Static compressibility $\sigma(\rho)$
- Collective diffusion coefficient $D(\rho)$

• No information about correlations on microscopic scale

▶ Pursue microscopic approach

2.1 Canonical conditioning (current ensemble)

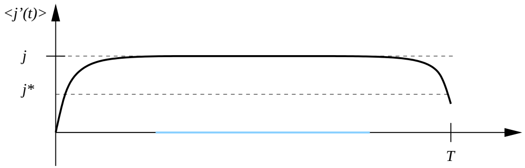
Fix $J(T) = J \Rightarrow$ Define two-time joint probability distribution

$$P_{\mathbf{n},J}(t,T) = \mathrm{Prob}\,[\,\mathbf{n}(t) = \mathbf{n}, J(T) = J\,] = \langle\, S, J\,|\mathrm{e}^{-\hat{\mathbf{H}}(T-t)}|\,\mathbf{n}\,\rangle\langle\,\mathbf{n}\,|\mathrm{e}^{-\hat{\mathbf{H}}t}|\,P(0), 0\,\rangle$$

- Normalization $P_J(T) = \sum_{\mathbf{n}} P_{\mathbf{n},J}(T,T) = \langle\, S, J\,|\mathrm{e}^{-\hat{\mathbf{H}}T}|\,P(0), 0\,\rangle$

  $=$ current distribution $\mathrm{Prob}\,[\,J(T) = J\,] \propto \mathrm{e}^{-Tf(j)}$

- Expected instantaneous current $\langle\, j'(t)\,\rangle = \mathrm{d}/\mathrm{d}t\langle\, J(t)\,\rangle$ at time t:



stationary regime

2.2 Grandcanonical conditioning (thrust ensemble)

- Generating function $Q_{\mathbf{n},s}(t,T) = \sum_J \mathrm{e}^{sJ} P_{\mathbf{n},J}(t,T)$
  $= \langle\, S\,|\mathrm{e}^{-\hat{H}(s)(T-t)}|\,\mathbf{n}\,\rangle\langle\,\mathbf{n}\,|\mathrm{e}^{-\hat{H}(s)t}|\,P(0)\,\rangle$

- Thrust s canonically conjugate to current J

- Weighted generator $\hat{H}(s)$: hopping rates $w_{\mathbf{n}',\mathbf{n}}\mathrm{e}^{\pm s}$, $\hat{H}(0) = \hat{H}$

- Current generating function $Q_s(T) = \sum_{\mathbf{n}} Q_{\mathbf{n},s}(T,T) = \sum_J \mathrm{e}^{sJ} P_J(T)$
  $= \langle\, S\,|\mathrm{e}^{-\hat{H}(s)T}|\,P(0)\,\rangle$

- Asymptotic moment generating function $g(s) := \lim_{T\to\infty} \frac{1}{T} \ln Q_s(T) = -\epsilon_0(s)$

with lowest eigenvalue $\epsilon_0(s)$ of $\hat{H}(s)$ (Legrende transform of $f(j)$)

$$\Rightarrow j(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\epsilon_0(s)$$

- Stationary conditional expectations $t = \alpha T$, $T \to \infty$:

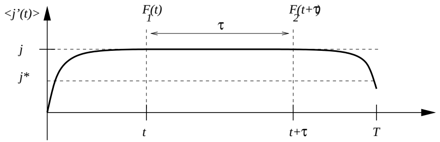
  $\langle\, F\,\rangle^* = \langle\,\Delta(s)\,|\hat{F}|\,\Gamma(s)\,\rangle$ with lowest right and left eigenvectors of $\hat{H}(s)$

2.3 Effective dynamics

Can one construct a process for which the large deviation of the current is typical?

$\Rightarrow$ Consider two-time conditional expectation

$$\langle\, F_2(t+\tau)F_1(t)\,\rangle_T = \frac{\langle\, S\,|\mathrm{e}^{-\hat{H}(s)(T-t-\tau)}\hat{F}_2\mathrm{e}^{-\hat{H}(s)\tau}\hat{F}_1\mathrm{e}^{-\hat{H}(s)t}|\,P(0)\,\rangle}{Q_s(T)}$$



- Study stationary regime far from 0 and T: $\mathrm{e}^{-\hat{H}(s)t} \to \mathrm{e}^{-\epsilon_0(s)t}|\,\Gamma(s)\,\rangle\langle\,\Delta(s)\,|$

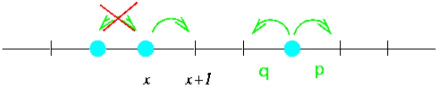
  $\Rightarrow \langle\, F_2(\tau)F_1(0)\,\rangle^* = \langle\,\Delta(s)\,|\hat{F}_2\mathrm{e}^{-[\hat{H}(s)-\epsilon_0(s)]\tau}\hat{F}_1|\,\Gamma(s)\,\rangle$

- Generalized Doob h-transform $\hat{G}(s) := \hat{\Delta}(s)\hat{H}(s)\hat{\Delta}^{-1}(s) - \epsilon_0(s)$

  $\Rightarrow \langle\, F_2(\tau)F_1(0)\,\rangle^* = \langle\, S\,|F_2\mathrm{e}^{-\hat{G}(s)\tau}\hat{F}_1|\,P^*(s)\,\rangle$ (stationary effective process $\hat{G}(s)$)

- Stationary distribution $P^*_{\mathbf{n}}(s) = \Delta_{\mathbf{n}}(s)\Gamma_{\mathbf{n}}(s)$

2.4 Mini review of conditioned ASEP



Periodic boundary conditions:

MFT for weak asymmetry $p - q = \mathcal{O}(1/L)$, any j:

$j < j^*$: Dynamical phase transition at j_c: Flat profile $\rightarrow$ travelling wave [Bodineau, Derrida (2005)]

$j > j^*$: Not accessible [Lazarescu (2013)]

Exact microscopic results for any asymmetry, but specific j:

$j < j^*$: Random walk of travelling wave, no correlations ($t = T$) [Belitsky, GMS (2013), GMS (2015)]

$j > j^*$: [Simon, Popkov, GMS (2010) ; Popkov, GMS (2011)]

- Flat with algebraically decaying correlations
- Change from KPZ to ballistic dynamical universality class
- Effective dynamics with long-range interactions

Open boundary conditions: Similar phenomena

$j < j^*$: [Bodineau, Derrida (2006), Simon (2009), Belitsky, GMS (2013), Lazarescu (2013)

$j > j^*$: Karevski, Dubail, GMS (work in progress)]

Study ZRP in the regime of an atypical current $j \neq j^*$:

MFT predicts optimal density profile to realize this rare event for weak asymmetry

$\Rightarrow$ Dynamical phase transition? Unclear! Questions beyond scope of MFT:

- Which process makes this large deviation typical?
- Which is the optimal profile for strong asymmetry?
- Microscopic structure of optimal profile?
- Role of condensation?

$\Rightarrow$ Compute lowest eigenvalue and eigenvectors (right and left) of weighted generator $\hat{H}(s)$

3 Effective dynamics

Weighted generator for local conditioning at bond $(k, k+1)$:

$$\hat{H}^{(k)}(s) = \sum_{l=0}^{k-1} \hat{h}_l + \hat{h}_k(s) + \sum_{l=k+1}^{L} \hat{h}_l.$$

with $\hat{h}_l = \hat{h}_l(0)$ and

$$\begin{aligned}
\hat{h}_0(s) &= -\left[\alpha(\mathrm{e}^s \hat{a}_1^+ - 1) + \gamma(\mathrm{e}^{-s} \hat{a}_1^- - \hat{d}_1)\right] \\
\hat{h}_k(s) &= -\left[p(\mathrm{e}^s \hat{a}_k^- \hat{a}_{k+1}^+ - \hat{d}_k) + q(\mathrm{e}^{-s} \hat{a}_k^+ \hat{a}_{k+1}^- - \hat{d}_{k+1})\right], \quad 1 \leq k \leq L-1 \\
\hat{h}_L(s) &= -\left[\delta(\mathrm{e}^{-s} \hat{a}_L^+ - 1) + \beta(\mathrm{e}^s \hat{a}_L^- - \hat{d}_L)\right]
\end{aligned} \tag{1}$$

Define the partial number operator $\hat{N}_k = \sum_{i=1}^{k} \hat{n}_i$

$\Rightarrow \hat{H}^{(0)}(s) = \mathrm{e}^{s\hat{N}_k} \hat{H}^{(k)}(s) \mathrm{e}^{-s\hat{N}_k}$

$\Rightarrow$ Lowest eigenvalue and effective dynamics independent of k!

$\Rightarrow$ Choose $k = 0$ (without loss of generality) and drop superscript (0)

3.1 Lowest left eigenvector

- Product ansatz [Harris, Rákos, GMS (2006)]: $\langle \Delta | = (y_1| \otimes (y_2| \otimes \ldots \otimes (y_L|$

- $(y_k|$ has components $y_k^n \Rightarrow \langle \Delta | = \langle S | \hat{\Delta}$ with $\hat{\Delta} = y_1^{\hat{n}} \otimes \ldots y_L^{\hat{n}}$

- Action of bulk hopping matrices:

$$-\langle \Delta | \hat{h}_k(s) = p(y_{k+1} - y_k)\hat{d}_k y_k^{-1} + q(y_k - y_{k+1})\hat{d}_{k+1} y_{k+1}^{-1}$$

Boundaries:

$$-\langle \Delta | \hat{h}_0(s) = \alpha(y_1 \mathrm{e}^s - 1) + \gamma(\mathrm{e}^{-s} - y_1)\hat{d}_1 y_1^{-1}$$

$$-\langle \Delta | \hat{h}_L(s) = \delta(y_L - 1) + \beta(1 - y_L)\hat{d}_L y_L^{-1}$$

$\Rightarrow$ Recursion for y_k

$$\begin{aligned} 0 &= p(y_{k+1} - y_k) + q(y_{k-1} - y_k) \\ 0 &= \gamma(\mathrm{e}^{-s} - y_1) + p(y_2 - y_1) \\ 0 &= q(y_{L-1} - y_L) + \beta(1 - y_L) \end{aligned}$$

$\Rightarrow y_k(s) = A(s) + B(s) a^{L+1-k}$ with $a = p/q$

$$A(s) = \frac{\gamma \mathrm{e}^{-s}(p-q-\beta) + \beta(p-q+\gamma)a^{L-1}}{\gamma(p-q-\beta) + \beta(p-q+\gamma)a^{L-1}}, \quad B(s) = \frac{\beta\gamma(\mathrm{e}^{-s}-1)a^{-1}}{\gamma(p-q-\beta) + \beta(p-q+\gamma)a^{L-1}}$$

3.2 Lowest right eigenvector

- Product ansatz $|\,\Gamma\,\rangle = |x_1) \otimes |x_2) \otimes \ldots \otimes |x_L)$
- $|x_k)$ has components $\prod_{j=1}^{n} \frac{x_k}{u(j)}$
- Action of weighted hupping matrices $\Rightarrow$ recursion for x_k $\Rightarrow$ $x_k(s) = C(s) + D(s)a^k$

$C(s) = \frac{\alpha\beta \mathrm{e}^s a^{L-1} - \gamma\delta}{\beta(p-q+\gamma)a^{L-1} + \gamma(p-q-\beta)}$, $D(s) = a^{-1} \frac{\alpha \mathrm{e}^s (p-q-\beta) + \delta(p-q+\gamma)}{\beta(p-q+\gamma)a^{L-1} + \gamma(p-q-\beta)}$

- Lowest eigenvalue

$$\epsilon_0(s) = (p-q)(\mathrm{e}^{-s} - 1)\frac{\alpha\beta a^{L-1}\mathrm{e}^s - \gamma\delta}{\gamma(p-q-\beta) + \beta(p-q+\gamma)a^{L-1}}$$

- Current $j(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\epsilon_0(s)$

$$j(s) = (p-q)\frac{\alpha\beta \mathrm{e}^s a^{L-1} - \gamma\delta \mathrm{e}^{-s}}{\beta(p-q+\gamma)a^{L-1} + \gamma(p-q-\beta)}$$

3.3 h-transform for effective dynamics with $j^*_{\mathrm{eff}} = j(s)$

$$\begin{aligned}
\hat{G} &= \hat{\Delta}\hat{H}\hat{\Delta}^{-1} - \epsilon_0 \\
&= -\sum_{k=1}^{L-1}\left[p\frac{y_{k+1}}{y_k}\left(\hat{a}_k^-\hat{a}_{k+1}^+ - \hat{d}_k\right) + q\frac{y_k}{y_{k+1}}\left(\hat{a}_k^+\hat{a}_{k+1}^- - \hat{d}_{k+1}\right)\right] \\
&\quad - \left[\alpha y_1 \mathrm{e}^s(\hat{a}_1^+ - 1) + \gamma y_1^{-1}\mathrm{e}^{-s}(\hat{a}_1^- - \hat{d}_1)\right] \\
&\quad - \left[\delta y_L(\hat{a}_L^+ - 1) + \beta y_L^{-1}(\hat{a}_L^- - \hat{d}_L)\right]
\end{aligned}$$

- Driven ZRP with spatially varying driving field $E_k(s) = \log a + 2\log y_{k+1}(s)/y_k(s)$
- Space-dependence even for non-interacting particles with $u(n) = wn$.
- Stationary distribution has no spatial correlations

$$|\, P^*(s)\,\rangle = |P_1^*(s)) \otimes |P_2^*(s)) \otimes \ldots \otimes |P_L^*(s))$$

Marginals

$$(P_k^*(s))_n = \frac{z_k^n}{Z_k}\prod_{i=1}^{n} u(i)^{-1}$$

with local fugacities $z_k = x_k y_k$, local partition function $Z_k = \sum_{n=0}^{\infty}\prod_{j=1}^{n}\frac{z_k}{u(j)}$

- Time-reversed process $\hat{G}^* := \hat{P}^* \hat{G}^T (\hat{P}^*)^{-1}$ with diagonal matrix $\hat{P}^*$ of stationary weights

$\Rightarrow \hat{G}^*(s) = \hat{\Gamma}(s)\hat{H}^T(s)\hat{\Gamma}^{-1}(s) - \epsilon_0(s)$

- Same stationary distribution, opposite stationary current

- Generator

$$
\begin{aligned}
\hat{G}^*(s) \quad = \quad & -\sum_{k=1}^{L-1} \left[q \frac{x_{k+1}}{x_k} \left(\hat{a}_k^- \hat{a}_{k+1}^+ - \hat{d}_k \right) + p \frac{x_k}{x_{k+1}} \left(\hat{a}_k^+ \hat{a}_{k+1}^- - \hat{d}_{k+1} \right) \right] \\
& - \left[\gamma x_1 \mathrm{e}^{-s} (\hat{a}_1^+ - 1) + \alpha x_1^{-1} \mathrm{e}^{s} (\hat{a}_1^- - \hat{d}_1) \right] \\
& - \left[\beta x_L (\hat{a}_L^+ - 1) + \delta x_L^{-1} (\hat{a}_L^- - \hat{d}_L) \right]
\end{aligned}
$$

- *h*-transform with right lowest eigenvector yields time-reversed effective dynamics

- Driven ZRP with spatially varying driving field $E_k(s) = -\log a + 2 \log x_{k+1}(s)/x_k(s)$

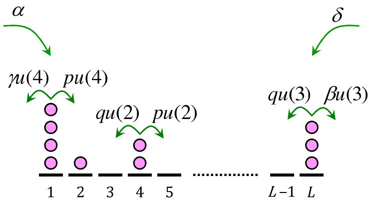
- Detailed balance $\hat{G}^*(s) = \hat{G}(s) \Longleftrightarrow j^*_{\mathrm{eff}}(s) = j(s) = 0$

4 Density profiles and supercritical chain segments

4.1 Barrier-free boundary conditions $\beta = p$, $\gamma = q$

Reservoir chemical potentials $\alpha = p\mathrm{e}^{\mu_-}$, $\delta = q\mathrm{e}^{\mu_+}$

$\bar{\mu} = (\mu^+ + \mu^-)/2$, $\bar{\delta} = (\mu^+ - \mu^-)/2$



Define $\tilde{\nu} = \frac{L+1}{2} \ln a$, $r_k = \frac{k}{L+1}$

- Lowest eigenvectors:

$$y_k = \frac{\mathrm{e}^{\tilde{\nu}(1-r_k)} \sinh(\tilde{\nu} r_k) + \mathrm{e}^{-s-\tilde{\nu} r_k} \sinh[\tilde{\nu}(1-r_k)]}{\sinh(\tilde{\nu})},$$

$$x_k = \frac{\mathrm{e}^{s+\mu^- + \tilde{\nu} r_k} \sinh[\tilde{\nu}(1-r_k)] + \mathrm{e}^{\mu^+ - \tilde{\nu}(1-r_k)} \sinh(\tilde{\nu} r_k)}{\sinh(\tilde{\nu})}$$

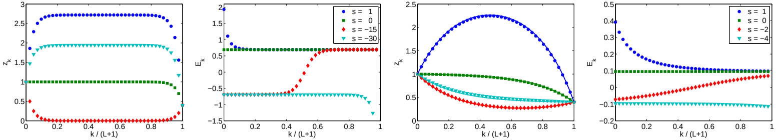
- Local driving field:

$$E_k = 2 \ln \left[\frac{\mathrm{e}^{s+\tilde{\nu}} \sinh(\tilde{\nu} r_{k+1}) + \sinh[\tilde{\nu}(1-r_{k+1})]}{\mathrm{e}^{s+\tilde{\nu}} \sinh(\tilde{\nu} r_k) + \sinh[\tilde{\nu}(1-r_k)]} \right] \Rightarrow \text{independent of reservoir chemical potentials}$$

4.2 Stationary fugacity profile

Define $\tilde{Q} = \cosh\left(s - \bar{\delta} + \tilde{\nu}\right))$

$$z_k = \mathrm{e}^{\bar{\mu}} \frac{\mathrm{e}^{-\bar{\delta}} \sinh^2[\tilde{\nu}(1-r_k)] + \mathrm{e}^{\bar{\delta}} \sinh^2(\tilde{\nu} r_k) + 2\tilde{Q} \sinh(\tilde{\nu} r_k) \sinh[\tilde{\nu}(1-r_k)]}{\sinh^2(\tilde{\nu})}$$



Fugacity profile Driving field Fugacity profile Driving field

asymmetric weakly asymmetric

- Stationary current $j^*_{\mathrm{eff}} = 2\sqrt{pq}\, \frac{\sinh(\frac{\tilde{\nu}}{L+1}) \sinh(\tilde{\nu} + s - \bar{\delta})}{\sinh(\tilde{\nu})}$

$\Rightarrow$ Same current could be generated by constant-field ZRP with reservoir chemical potentials

$\mu^{\mathrm{eff}}_{-} = \mu_{-} + s$, $\mu^{\mathrm{eff}}_{+} = \mu_{+} - s$ $\Rightarrow$ Not same fugacity profiles

Special cases:

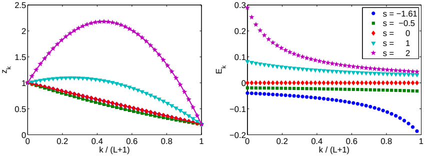
A) Symmetric ZRP ($p = q = 1$):

- Driving field $E_k = 2 \ln \left[\frac{e^{s/2} r_{k+1} + e^{-s/2}(1 - r_{k+1})}{e^{s/2} r_k + e^{-s/2}(1 - r_k)} \right]$

- Quadratic fugacity profile $z_k = e^{\mu_-}(1 - r_k)^2 + e^{\mu_+} r_k^2 + 2e^{\bar{\mu}} \cosh(s - \bar{\delta}) r_k (1 - r_k)$

- Stationary current $j^*_{\text{eff}} = \frac{e^{\mu_- + s} - e^{\mu_+ - s}}{L+1}$

$\Rightarrow$ Nontrivial conditioned process even for (unconditioned) equilibrium case $\mu_- = \mu_+$



B) General totally asymmetric ZRP ($p = 1$, $q = \gamma = \delta = 0$):

- Driving field $= \infty$
- Fugacity profile $z_k = \alpha \mathrm{e}^s$ ($k < L$), $z_L = \alpha \mathrm{e}^s / \beta$
- Stationary current $j^*_{\mathrm{eff}} = \alpha \mathrm{e}^s$

$\Rightarrow$ Conditioned process same as original process with $\alpha^{\mathrm{eff}} = \alpha \mathrm{e}^s$

- Lowest eigenvalue $\epsilon_0(s) = \alpha(1 - \mathrm{e}^s)$

$\Rightarrow$ Poissonian current distribution of original process $\mathrm{Prob}\,[\,J(T) = J\,] = \frac{(\alpha T)^J}{J!} \mathrm{e}^{-\alpha T}$

- Valid up to critical value j_c, temporary condensates beyond j_c [Harris, Rákos, GMS (2006)]

4.3 Condensation patterns

Tacit assumption: $Z_k(s) < \infty$ for all $k \in \Lambda$

However:

• Depending on interaction $u(n)$ the normalization $Z_k(s)$ may have finite radius z_c of convergence which does not depend on k

• $z_c < \infty$ for bounded $u(n)$, but $Z_k(s)$ generally unbounded
$\Rightarrow$ Construction valid only in finite interval $s_c^- \leq s \leq s_c^+$ ($j_c^- \leq j(s) \leq j_c^+$)

• Nonstationary condensation phenomena for $s \notin [s_c^-, s_c^+]$

• Supercritical regions rather than single supercritical boundary site as in typical behaviour of ZRP

• Equivalence of current and thrust ensemble?

• Condensation patterns?

4.4 Thermodynamic limit

Take introduce lattice constant $\lambda = 1/(L+1)$ and take thermodynamic limit $L \to \infty$, consider barrier-free boundaries

- Length of chain = 1, Lattice position $r_k \to r \in [0,1]$ (macroscopic position)

- Fixed asymmetry $a > 1$ (positive bias):

$\Rightarrow$ Fugacity profile $z(r) = z_- \mathrm{e}^s =: z^*$ for $r \neq 0, 1$, Current $j^* = (p-q)z^*$

$\Rightarrow$ Constant bulk profile with microscopic boundary layers of width $\xi = 1/\ln(a)$

$\Rightarrow$ Bulk: Conditioned process = Original process with effective injection rate $\alpha^{\mathrm{eff}} = \alpha \mathrm{e}^s$

- Symmetric hopping $a = 1$:

  – Driving field $E(r) = \frac{4\sinh(s/2)}{L[\mathrm{e}^{s/2} r + \mathrm{e}^{-s/2}(1-r)]}$

  – Current $j^*_{\mathrm{eff}} = \frac{1}{L}(z_- \mathrm{e}^s - z_+ \mathrm{e}^{-s})$

  – Fugacity profile $z(r) = \mathrm{e}^{\mu_-}(1-r)^2 + 2\cosh(s-\bar{\delta})\mathrm{e}^{\bar{\mu}} r(1-r) + \mathrm{e}^{\mu_+} r^2$

- Weakly asymmetric hopping $p = (1 + \nu/L)/2,\ q = (1 - \nu/L)/2$:

  - Driving field $E(r) = \frac{2\nu}{L} \times \frac{\mathrm{e}^{s+\nu}\cosh(\nu r) - \cosh[\nu(1-r)]}{\mathrm{e}^{s+\nu}\sinh(\nu r) + \sinh[\nu(1-r)]}$

  - Current $j^*_{\mathrm{eff}} = \frac{\nu \mathrm{e}^{\bar{\mu}}}{L} \times \frac{\sinh(s - \bar{\delta} + \nu)}{\sinh \nu}$

  - Fugacity profile

$$z(r) = \mathrm{e}^{\bar{\mu}}\,\frac{\mathrm{e}^{-\bar{\delta}}\sinh^2[\nu(1-r)] + \mathrm{e}^{\bar{\delta}}\sinh^2(\nu r) + 2Q\sinh(\nu r)\sinh[\nu(1-r)]}{\sinh^2(\nu)}$$

$\Rightarrow$ Agreement with MFT for weak asymmetry

$\Rightarrow$ Microscopic result for finite asymmetry: MFT with infinite asymmetry parameter $\nu = cL$

$\Rightarrow$ Microscopic result yields scale factor $c = 1/2\ln(p/q)$

5 Conclusions

- Generalized h-transform in thrust ensemble yields effective ZRP that makes current large deviation typical
- Space-dependent driving field
- Non-trivial optimal fugacity profile even for non-interacting particles and equilibrium ZRP
- No correlations
- Validity in current regime $[j_c^-, j_c^+]$
- Otherwise supercritical non-stationary regions where condensates can grow
- Agreement of fugacity profile with prediction of MFT (limited to weakly asymmetric case)
- MFT with infinite asymmetry parameter cL and judiciously chosen scale factor c yields conditioned ZRP with finite asymmetry

- Ensemble equivalence?
- Condensation patterns?