Introduction to interacting particle systems

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Abstract

We will start by recalling some important and useful results from the theory of Markov Processes, e.g. semi-groups, infinitesimal generators, reversibility and sufficient conditions for the existence of stationary distributions. Once we have laid the foundations, we introduce two classical examples of interacting particle systems, the zero-range process and the simple exclusion process, and discuss some of their properties. Towards the end, we have a brief look at the scaling limit for the simple exclusion process.

1 Introduction

The particle systems we will look at were (amongst others) first introduced by Spitzer [5] in 1970. We will have a closer look at two of these systems. One of them is called the zero-range process and the other simple exclusion process.

The simplest (in fact trivial) models for interacting particle systems are systems with only one particle or many particles that don't interact with each other. For this, one can think of a family of independent copies of a random walk.

For another example of interacting particles, we can think of two Brownian motions that attract or repel each other. The figure below shows a plot with the simulation of two Brownian motions that attract each other if they are a certain distance apart from each other.

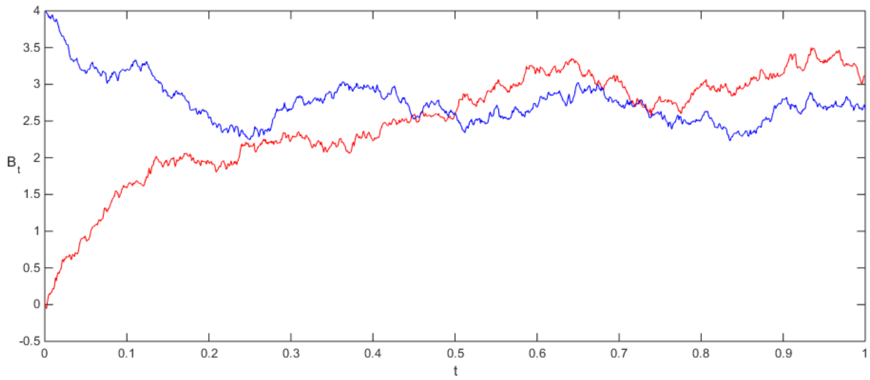

Figure 1: Two diffusion processes that attract each other.

One 'problem' that arises with coupled diffusion processes like in this example is that it is very hardy to keep track of them for a very large number of particles. Therefore a usual approach is to use a mean field approximation. This is a theory mainly used in physics and probability theory where one studies instead of many particles just one particle 'living' in a well-chosen environment (in the sense of an external field). This external field then replaces the interactions with the other particles.

An alternative approach is to use particles that jump on a grid as depicted in the figure below.

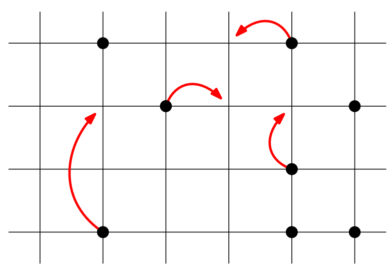

Figure 2: particles on a grid

Under the assumption that these particles are indistinguishable, we can keep track of the number of particles at each site only. The advantage of this approach is that we can (for models that are simple enough) keep track of the whole system and don't need to use main-field approximations.

It is of great interest in many areas such as physics and chemistry to have a good understanding of the macroscopic behaviour of certain materials in terms of microscopic properties. That means one wants to be able to explain and describe macroscopic observations in terms of properties that arise from (the limit of) particle systems. The mathematical term associated to this is called the *scaling* *limits* (of a processes). A well-known example which is not completely solved until now is Boltzmann's equation that describes (statistically) the movement of particles in a gas.

In the following we will not deal with Boltzmann's equation, but we will rather have a look at two systems defined on the lattice. The first model we will consider is the *zero-range process* and the second one is the *simple exclusion process*.

One last remark at this point: It often possible to consider different microscopic models that give rise to the same macroscopic behaviour in the sense that the limits coincide.

2 Review of Markov processes

2.1 Markov processes and their semigroups

We use this section to recall some properties related to Markov processes which yield the background for our following investigation of interacting particle systems. We recall the definition of a Markov process, the special case of a Feller process and point out the connection to the associated semigroup and the infinitesimal generator. This subclass has nice properties, for example Feller processes always admit a right-continuous modification and have the strong Markov property.

Notation and remarks.

- The notation in this chapter is mainly following Liggett [3]. Most of the statements can be found in Klenke [2] and Rudin [4].
- For a topological space X, we denote with $\mathcal{B}(X)$ the Borel σ-algebra generated by the open sets of X and equip X with this σ-algebra if not stated otherwise. $\mathcal{M}(X)$ denotes the space of Borel measures on X and $\mathcal{M}_1(X)$ is the subset of all probability measures.
- $C_0(X)$ is defined to be the space of continuous and real-valued functions that 'vanish at infinity', i.e

$$C_0(X) := \Big\{ f \in C(X) \;\Big|\; \text{for all } \epsilon > 0 \text{ exists a } K \subseteq X \text{ compact, s.t. } |f(x)| < \epsilon \text{ for all } x \in X \setminus K \Big\}.$$

  The space endowed with the supremum norm $\|\cdot\|_\infty$ is a Banach space. If X is compact, $(C_0(X), \|\cdot\|_\infty)$ coincides with the space of continuous functions $(C(X), \|\cdot\|_\infty)$.

Definition 2.1 (Polish space)**.** A **Polish space** X is a separable topological space (X, τ) s.t. there exists a complete metric that induces the topology on X. That means there exists a metric d, s.t. (X, d) is a complete separable metric space and the open balls $\{B_d(x, r) := \{y \in X : d(x, y) < r\}, x \in X, r > 0\}$ form a basis for the topology on τ. In the following, we will keep the dependence of X on its topology τ implicit.

Remark. The relevant examples here will be the natural numbers $\mathbb{N}_0$, the real numbers $\mathbb{R}$ and subsets of these.

Definition 2.2 (Markov process)**.** Let $I = [0,T)$ for some $T > 0$ and let X be a Polish space. A stochastic process $(\eta_t)_{t\in I}$, taking values in X, is called a **time-homogeneous Markov process** with distributions $(P^\eta)_{\eta\in X}$, if it satisfies the following three conditions:

(1) the map $X \times \mathcal{B}(X)^{\otimes I} \to [0,1]$, $(\eta, A) \mapsto P^\eta\left[(\eta_t)_{t\in I} \in A\right]$ is a stochastic kernel:

   – For all $A \in \mathcal{B}(X)^{\otimes I}$: The map $X \ni \eta \mapsto P^\eta\left[(\eta_t)_{t\in I} \in A\right]$ is measurable.

   – For all $\eta \in X$: $\mathcal{B}(X)^{\otimes I} \ni A \mapsto P^\eta\left[(\eta_t)_{t\in I} \in A\right]$ is a probability measure.

(2) For each $\eta \in X$ the process $(\eta_t)_{t\in I}$ is under the law P^η almost surely emitted from η :

$$P^\eta[\eta_0 = \eta] = 1$$

(3) $(\eta_t)_{t\in I}$ fulfills the time-homogeneous Markov property: For all $\eta \in X$ and all $A \in \mathcal{B}(X)$, we have that

$$P^\eta[\eta_{t+s} \in A|\mathcal{F}_s] = P^{\eta_s}[\eta_t \in A]\ P^\eta\text{−a.s.}$$

The associated semigroup for the Markov process $(\eta_t)_{t\in I}$ with start in $\eta \in X$ is for bounded functions $f \colon X \to \mathbb{R}$ defined as

$$P_t f(\eta) := E^\eta[f(\eta_t)] := \int_X f(\eta_t) dP^\eta, \tag{1}$$

where E^η denotes the expected value w.r.t. the measure P^η.

Lemma 2.1. *Under the assumptions from definition 2.2 equation* (1) *defines a Markov semigroup. That means that for all $t \in I$ the map $(\eta, A) \mapsto P_t\mathbf{1}_A(\eta)$ is a stochastic kernel (which is clear by definition 2.2), and further*

- *$P_0 = I$ the identity operator, and*
- *the Chapman-Kolmogorov-Equation is fulfilled, i.e. for all $s,t \geq 0$:*

$$P_{t+s} = P_t P_s \tag{2}$$

Proof. The first point follows directly from (2) in definition 2.2 and the second point is a straightforward application of the Markov property (3) in definition 2.2, since

$$P_{t+s}f(\eta) = E^\eta\left[E^{\eta_t}[f(\eta_s)]\right] = E^\eta[P_s f(\eta_t)] = P_t P_s f(\eta).$$

□

2.2 Feller processes

We obtained that every time-homogeneous Markov process induces a Markov semigroup. Conversely we have to restrict ourselves to the subclass of Feller semigroups. We will see in an instant that each

Feller semigroup induces a time-homogeneous Markov process (which then is referred to as Feller process).

Definition 2.3 (Feller semigroup)**.** A Markov semigroup $(P_t)_{t\geq 0}$ acting on a locally compact Polish space X is called a **Feller semigroup**, if

- $(P_t)_{t\geq 0}$ maps $C_0(X)$ into itself, i.e. for all $t \in I$:
$$f \in C_0(X) \;\Rightarrow\; P_t f \in C_0(X), \tag{3}$$
- for all $\eta \in X$ and $f \in C_0(X)$ the semigroup is right-continuous in t at $t = 0$:
$$P_t f(\eta) \to f(\eta), \text{ as } t \to 0. \tag{4}$$

Remarks.

- Given that (3) holds, the Chapman-Kolmogorov-equation (2) yields that the second condition in definition 2.3 is equivalent to right-continuity of the map $t \mapsto P_t f(\eta)$ for all $f \in C_0(X)$ and $\eta \in X$.
- By the linearity of the integral, we obtain that for each $t \geq 0$ that the map $f \to P_t f$ is linear. In other words, P_t is a linear operator. One further can see that $(P_t)_{t\in I}$ is a contraction semigroup, i.e. $\|P_t\| \leq 1$ for all $t \in I$. Hence $(P_t)_{t\in I}$ defines a family of bounded linear operators from the Banach space $(C_0(X), \|\cdot\|_\infty)$ to itself.

The following theorem can be found, with a slightly different notation, in Klenke [2] and yields us, as indicated before, that every Feller semigroup induces a Markov process. This Markov process further is càdlàg, i.e. is right-continuous and has left-limits. Additionally, the process satisfies the strong Markov property:

Theorem 2.1 (Klenke, [2], p.464)**.** *Let $(P_t)_{t\in I}$ be a Feller semigroup on the locally compact Polish space X. Then there exists a strong Markov process $(\eta_t)_{t\in I}$ with càdlàg paths and transition kernels $(P_t)_{t\in I}$. Such a process is called a Feller process.*

2.3 Infinitesimal generators of Feller processes

We can define the infinitesimal generator for Feller semigroups as follows:

Definition 2.4 (Infinitesimal generator)**.** Let $(P_t)_{t\in I}$ be a Feller semigroup on X. We define for $f \in C_0(X)$ and $\epsilon > 0$ the operators $\mathcal{A}_\epsilon$ as

$$\mathcal{A}_\epsilon f := \frac{1}{\epsilon}\left[P_\epsilon f - f\right].$$

The **infinitesimal generator** $\mathcal{A}$ is then defined as the linear operator

$$\mathcal{A} f := \lim_{\epsilon\to 0} \mathcal{A}_\epsilon f \tag{5}$$

on $\mathcal{D}(\mathcal{A}) := \{f \in C_0(X) \,|\, \text{the limit in (5) exists in } \|\cdot\|_\infty\}$, the domain of $\mathcal{A}$.

Remark. It is further possible to show that $\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \to C_0(X)$, i.e. that $\mathcal{A}f$ is continuous for all $f \in \mathcal{D}(\mathcal{A})$, see Theorem 13.36 on page 379 in Rudin [4].

Theorem 2.2 (Theorem 13.35, p.376, [4])**.** *Some properties of the semigroup and the infinitesimal generator:*

- *For all $f \in C_0(X)$ the map $t \mapsto P_t f$ is continuous from $[0,\infty)$ to $C_0(X)$.*
- *P_t restricted to $\mathcal{D}(\mathcal{A})$ is differentiable in t. More precisely, we have for all $f \in \mathcal{D}(\mathcal{A})$ that $\frac{d}{dt} P_t f = \mathcal{A} P_t f = P_t \mathcal{A} f$.*
- *The domain of the infinitesimal generator $\mathcal{D}(\mathcal{A}) \subseteq C_0(X)$ is dense and closed.*

We immediately obtain using proposition 2.1 below that the evolution of the semigroup can be written in terms of the generator as

$$P_t f = P_0 f + \int_0^t \mathcal{A} P_s f\, ds = f + \int_0^t \mathcal{A} P_s f\, ds.$$

The 'Martingale Problem' stated below goes back to Strook and Varadhan:

Proposition 2.1. *Let $(X_t)_{t\geq 0}$ be a Feller-process with semi-group $(P_t)_{t\geq 0}$ on X. For all $f \in \mathcal{D}(\mathcal{A})$:*

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{A} f(X_u)\, du \tag{6}$$

is a mean-zero martingale w.r.t. the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(X_t)_{t\geq 0}$.

2.4 Invariant measures and reverse dynamics

For $\mu \in \mathcal{M}_1(X)$, we define the measure with initial distribution μ as

$$P^\mu := \int_X P^\eta \, \mu(d\eta)$$

and denote the expected value w.r.t. P^μ with E^μ. For $\eta \in X$, we can henceforth interpret the measure P^η with this notation as the random distribution obtained from the Dirac measure δ_η with point mass in the state η.

We can extend this notation to Markov semigroups. For $(P_t)_{t\in I}$ a Markov semigroup, we define analogously the measure μP_t for $t \in I$ as

$$\int_X f \, d[\mu P_t] := \int_X (P_t f)\, d\mu = \int_X (P_t f)(\eta)\, \mu(d\eta). \tag{7}$$

The measure μP_t can be interpreted as the distribution at time t, if the process was started with distribution μ.

Remark. In fact μP_t is the push forward of μ under the map P_t, i.e. we have by definition that $\mu P_t = P_{t\#}\mu := \mu \circ P_t^{-1}$.

Definition 2.5 (Invariant measure)**.** We call $\mu \in \mathcal{M}_1(X)$ an **invariant measure** or **stationary distribution** for the Markov process $(\eta_t)_{t\in I}$, if $\mu P_t = \mu$ for all $t \in I$, or equivalently

$$\int_X (P_t f)\, d\mu = \int_X f\, d\mu. \tag{8}$$

We denote the set of stationary measure with $\mathcal{I}$. It is clear that $\mathcal{I}$ is convex.

Remark. Recall that for a discrete-time Markov chain with transition matrix P a measure is invariant if $\mu P = \mu$, and therefore by iteration for all $n \in \mathbb{N}_0$: $\mu P^n = \mu$. In a similar way we have here, by equation (7), that

$$\int_X f\, d[\mu P_t] = \int f\, d\mu$$

for all f, and therefore we have that $\mu P_t = \mu$.

In the theory of Markov processes, it is of great interest to characterise stationary measures. For Feller processes, we obtain the following equivalent criterion for invariant measures related to the infinitesimal generator:

Proposition 2.2. *Let $\mathcal{A}$ be the infinitesimal generator of a Feller semigroup. $\mu \in \mathcal{M}_1(X)$ is an invariant measure w.r.t. this process, if and only if $\mathcal{A}$ satisfies for all $f \in \mathcal{D}(\mathcal{A})$ the equation*

$$\int_X \mathcal{A}f\ d\mu = 0. \tag{9}$$

Proof. Let us assume first that (8) holds for all $f \in C_0(X)$. Note that for all $f \in \mathcal{D}(\mathcal{A}) \subseteq C_0(X)$:

$$\int_X \mathcal{A}f\, d\mu = \lim_{\epsilon\to\infty} \int_X \mathcal{A}_\epsilon f\, d\mu, \tag{10}$$

since on spaces with finite mass the L^1-norm is bounded by the L^∞-norm:

$$\left|\int_X \mathcal{A}f\, d\mu - \int_X \mathcal{A}_\epsilon f\, d\mu\right| \leq \int_X |\mathcal{A}f - \mathcal{A}_\epsilon f|\, d\mu \leq \|\mathcal{A}f - \mathcal{A}_\epsilon f\|_\infty \to 0, \text{ as } \epsilon \to 0.$$

Hence, we obtain from (8), that

$$\int_X \mathcal{A}_\epsilon f\, d\mu = \frac{1}{\epsilon}\left[\int_X (P_t f)\, d\mu - \int_X f\, d\mu\right] = 0,$$

since the expression in the square brackets in the line above is zero. This already implies together with equation (10) that (9) holds true.

Now, for the converse implication, we firstly obtain that by (9) for $\lambda \geq 0$ and $f \in \mathcal{D}(\mathcal{A})$:

$$\int_X f\, d\mu = \int_X (f - \lambda\mathcal{A}f)\, d\mu = \int_X (I - \lambda\mathcal{A})f\, d\mu,$$

and hence by inversion and linearity, that

$$\int_X (I - \lambda \mathcal{A})^{-1} f \, d\mu = \int_X f \, d\mu.$$

This formula extends to all $f \in C_0(X)$ since (by Proposition 2.8 (b) on page 15 in Liggett [3]) the mapping $I - \lambda\mathcal{A}$ is for all $\lambda \geq 0$ surjective, i.e. its range satisifes

$$\mathcal{R}(I - \lambda\mathcal{A}) = C_0(X).$$

Hence we obtain with the *Trotter-Kato formula* for the exponential representation of the semigroup, stating that on the whole domain (i.e. for all $f \in C_0(X)$):

$$P_t = \lim_{n\to\infty} \left(I - \tfrac{t}{n}\mathcal{A}\right)^{-n}$$

(see Theorem 2.9 on page 16 in Liggett [3] or Appendix 1.3 on page 323 in Kipnis-Landim [1]) that

$$\int_X P_t f \, d\mu = \lim_{n\to\infty} \int_X \left(I - \tfrac{t}{n}\mathcal{A}\right)^{-n} f \, d\mu = \int_X f \, d\mu.$$

$\square$

Given a stationary measure $\mu \in \mathcal{M}_1(X)$ for a Feller process with infinitesimal generator $\mathcal{A}$, the reverse dynamics are given by the adjoint operator $\mathcal{A}^*$ w.r.t. the inner product induced by the stationary measure μ, i.e. the unique operator that satisfies for all $f, g \in \mathcal{D}(\mathcal{A})$:

$$\langle f, \mathcal{A}g \rangle_{P^\mu} := E^\mu\big[f\mathcal{A}g\big] = E^\mu\big[(\mathcal{A}^* f)g\big] = \langle \mathcal{A}^* f, g \rangle_{P^\mu}.$$

Definition 2.6 (Detailed balance)**.** Let $\mathcal{A}$ be the generator of a Feller semigroup on a space X and $\mu \in \mathcal{M}_1(X)$. We say that μ satisfies **detailed balance**, if $\mathcal{A}$ is formally self-adjoint w.r.t. μ, i.e. if for all $f, g \in \mathcal{D}(\mathcal{A})$:

$$\langle f, \mathcal{A}g \rangle_{P^\mu} = \langle \mathcal{A}f, g \rangle_{P^\mu}.$$

In this case the process is also called **reversible**, **time reversible** or **time reversible invariant**. We denote the set of reversible measures with $\mathcal{R}$.

Similar as stationarity, reversibility can be stated in terms of the semigroup $(P_t)_{t\geq 0}$:

$$\langle f, P_t g \rangle_{P^\mu} = \langle g, P_t f \rangle_{P^\mu}$$

for all $t > 0$ and all $f, g \in C_0(X)$. For a proof, see page 91 in Liggett [3]. We can interpret this equation as follows: If we extend the process η_t for all times $t \in (-\infty, \infty)$, the stationary process η_t started in $\mu \in \mathcal{R}$ has the same law as the time reversed process η_{-t}.

Lemma 2.2. *If the state space S is countable and the generator is of the form*

$$\mathcal{A}f(\eta) = \textstyle\sum_{\eta' \in S} \big[f(\eta') - f(\eta) \big] \, c(\eta, \eta'),$$

we obtain that reversibility w.r.t. a measure $\mu \in \mathcal{M}_1(S)$ is equivalent to the 'common version' of the **detailed balance** *condition:*

$$c(\eta, \eta') \, \mu(\eta) = c(\eta', \eta) \, \mu(\eta').$$

Proof. Exercise. □

Lemma 2.3. *$\mathcal{R} \subseteq \mathcal{I}$, i.e. reversibility implies stationarity.*

Proof. First note that the constant function $\mathbf{1} : X \to \mathbf{R}$, $x \mapsto 1$ satisfies $\mathcal{A}\mathbf{1} = 0$ and thus in particular $\mathbf{1} \in \mathcal{D}(\mathcal{A})$. Hence, letting $f \equiv \mathbf{1}$, we obtain by the non-degeneracy of the inner product, that

$$\langle \mathbf{1}, \mathcal{A}g \rangle_{P^\mu} = E^\mu[(\mathcal{A}\mathbf{1})g] = \langle 0, g \rangle_{P^\mu} = 0.$$

□

3 Lattice systems

Let Λ be a graph / index set / subset of the grid $\mathbb{Z}^d$ (in $d > 1$ dimensions). Further let $x, y \in \Lambda$. We will denote with $\eta^{x,y}$ the configuration that is obtained from η by moving one particle from x to y (if possible, i.e. if there is a particle at x originally):

$$\eta^{x,y}(z) = \begin{cases} \eta(z) - 1, & \text{if } z = x \text{ and } \eta(x) > 0 \\ \eta(z) + 1, & \text{if } z = y \text{ and } \eta(x) > 0 \\ \eta(z), & \text{otherwise.} \end{cases}$$

The transition between states is governed by a transition function $p(\cdot, \cdot)$ which is here assumed to be translational invariant, i.e. for x, y we have $p(x, y) = p(0, y-x) =: p(y-x)$. Recall, that in this case 'double stochasticity' holds:

$$\sum_{x \in \Lambda} p(x-y) = \sum_{y \in \Lambda} p(x-y) = \sum_{z \in \Lambda} p(z) = 1.$$

The generator for this type of models then takes the form

$$\mathcal{A}f(\eta) = \sum_{x,y \in \Lambda} \big[f(\eta^{x,y}) - f(\eta) \big] \, \tilde{c}(\eta, \eta^{x,y}) p(y - x).$$

3.1 Zero-range process (ZRP)

For $d \geq 1$ let $\mathbb{T}_L^d$ be the discrete d-dimensional torus with $L \in \mathbb{N}$ sites. The state space for the ZRP is then given by $S = \mathbb{N}_0^{\mathbb{T}_L^d}$.

Fix $T > 0$ and let $I := [0, T]$. For $t \in I$, we interpret $\eta_t \in S_L$ as the vector containing the number of particles at each site, i.e. for $x \in \mathbb{T}_L^d$ the number of particles at site x at time t is given by $\eta_t(x) \in S$.

Since the torus has no boundary, there is no interaction with a boundary where particles are added or removed, and hence the number of particles is conserved, i.e. for all $t \in I$:

$$\sum_{x\in\mathbb{T}_L^d} \eta(x) = \sum_{x\in\mathbb{T}_L^d} \eta_t(x) \ P^{\eta}\text{−a.s.} \tag{11}$$

The rate at which particle jump from a site x is given by $g(\eta_t(x))$ for a monotonically increasing function $g : \mathbb{N}_0 \to [0,\infty)$ with $g(k) = 0$ if and only if $k = 0$.

The jump rate at which particles jump from x to another position is then given by the product $g(\eta(x))p(x,\cdot)$. Recall that this rate only depends on the number of particles at site x and is independent of the number of particles at any of the other positions. The name zero-range process is justified by the fact that the particles only depend on each other, if they have distance zero in the discrete topology.

Now we can define the ZRP on the state space S_L with transition rates and transition probabilities given by $g(\cdot)$ and $p(\cdot)$, respectively, as the unique continuous time Markov process on S_L with infinitesimal generator acting on cylinder functions $f\colon S_L \to \mathbf{R}$:

$$(\mathcal{A}_L f)(\eta) = \sum_{x,y\in\mathbb{T}_L^d} \big[f(\eta^{x,y}) - f(\eta)\big] g(\eta(x))p(y-x). \tag{12}$$

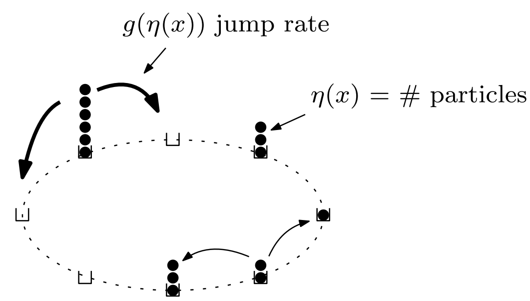

Figure 3: The zero-range process. The jump rates are dependent on $g(\eta(x))$.

We can interpret this as follows: The more particle are at position x (i.e. the bigger $\eta_t(x)$), the more likely is it that the next particle that performs a jump is located at site x before the jump takes place. Further $g(k) = 0$ if and only if $k = 0$ implies that there are no jumps from 'empty' sites, where no particles are located and particles cannot get trapped at any site.

Here the choice $g(k) = k$, $k \in \mathbb{N}_0$ recovers the case of particles moving independent of each other with jump rates equal to unity.

Clearly not every choice for $g(\cdot)$ is a good choice and we have to restrict the class of possible functions. In Kipnis-Landim [1] two classes of functions are stated:

(SLG) The jump-rate is of sub-linear growth, i.e. $\limsup_{k\to\infty} g(k)/k = 0$.

(FEM) The function $g(\cdot)$ is such that for all $\varphi \in [0,\infty)$: $\displaystyle\sum_{k=0}^{\infty} \frac{\varphi^k}{g(k)!} < \infty$.

We will see later that equivalently the partition function as defined in equation (13) is finite on $[0,\infty)$, i.e. the radius of convergence $\varphi^* = \infty$. This is in particular the case when $g(\cdot)$ has a super-linear tail behaviour.

Since we consider only monotonically increasing functions $g(\cdot)$, one of these conditions is always satisfied. **(FEM)** is satisfied if $g(\cdot)$ is unbounded and **(SLG)** if $g(\cdot)$ is bounded. See remark 5.6 on page 97 in Kipnis-Landim [1].

Sometimes one can also find the following assumption in the literature, that the increments are of bounded variation (which is only needed if $g(\cdot)$ is unbounded, i.e. in the case that **(FEM)** holds):

(BV) $g^* := \sup_{k\in\mathbb{N}_0} |g(k+1) - g(k)| < \infty$.

With $g(0)$, this implies that $g(k) \leq g^* k$ for all $k \in \mathbf{N}$.

In the following, we will assume that **(FEM)** and **(BV)** hold to avoid technicalities.

Invariant measures

It is well known that the invariant measure for the ZRP on the discrete torus $\mathbb{T}_L^d$ is a translational invariant product measure. In the following we will prove this result. For this, we define the one site *partition function*

$$z(\varphi) := z_g(\varphi) := \sum_{k=0}^{\infty} \frac{\varphi^k}{g(k)!}, \tag{13}$$

where the expression $g(k)!$ has to be understood as $g(k)! := \prod_{i=1}^{k} g(i)$ for $k \in \mathbb{N}_0$ with the usual convention for the empty product that $g(0)! = 1$. We define $\varphi^* := \sup\{\varphi > 0 \ : \ z(\varphi) < \infty\}$ to be the convergence radius of φ. It further is common to state the following growth assumption on $Z(\cdot)$, which in fact is a growth assumption on $g(\cdot)$ and is only needed in the case that $\varphi^* = \infty$, which is assumption **(FEM)**:

$$z(\varphi) \to \infty, \text{ as } \varphi \to \varphi^*.$$

For the ZRP on $\mathbb{T}_L^d$, there exists for every $\varphi \in [0, \varphi^*)$ an invariant measure which is translational invariant and in product form:

Definition 3.1. For $\varphi \in [0,\varphi^*)$, we define the product measure $\nu_{\varphi,g} := \nu_{\varphi,g}^L \in \mathcal{M}_1(S_L)$ via its marginals for $k \in \mathbb{N}_0$ and $x \in \mathbb{T}_L^d$ as

$$\nu_{\varphi,g}^L\big(\{\pi_x = k\}\big) := \frac{1}{z(\varphi)} \frac{\varphi^k}{g(k)!}. \tag{14}$$

By definition the measure is translational invariant.

Proposition 3.1. *For all $\varphi \in [0,\varphi^*)$ the measure $\nu_{\varphi,g}^L \in \mathcal{I}$. Further if $p(\cdot)$ is symmetric $\nu_{\varphi,g}^L \in \mathcal{R}$.*

Remark. For the choice $g \equiv \mathrm{Id}$ the identity, we recover that the stationary distribution for independent random walks on the torus is the product measure of Poisson distributions with parameter φ.

Next, we introduce a characterisation for the mean density and the mean jump rate for the one site distributions of the stationary measures. For this, we define the strictly monotonically increasing function

$$R(\varphi) := \frac{\varphi z'(\varphi)}{z(\varphi)} = \varphi \partial_\varphi \log(z(\varphi)). \tag{15}$$

We remark that R is equal to the mean number of particles at an arbitrary site w.r.t. the measure $\nu_{\varphi,g}$, i.e.

$$R(\varphi) = E^{\nu_{\varphi,g}}[\eta(0)]. \tag{16}$$

Another important property is that we can identify the measure $\nu_{\varphi,g}$ with the parameter φ, which is equal to the mean jump rate:

$$E^{\nu_{\varphi,g}}[g(\eta(0))] = \varphi.$$

3.2 Simple exclusion process (SEP)

The simple exclusion process, compared to the zero range process, allows at most one particle per site.

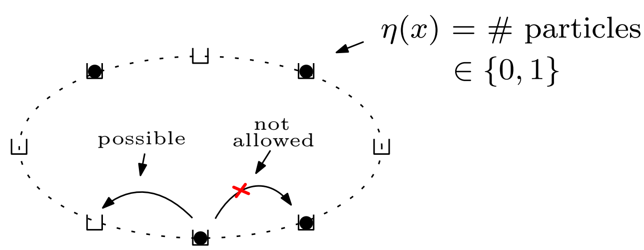

Figure 4: The simple exclusion process. Jumps are only allowed from occupied to non-occupied sites.

The state space is therefore given by $S = \{0,1\}^{\mathbb{T}_L^d}$ and the generator for this process takes the form

$$\begin{aligned}\mathcal{A}f(\eta) &= \sum_{x,y\in\mathbb{T}_L^d} \big[f(\eta^{x,y}) - f(\eta)\big]\mathbf{1}_{\{\eta(x)=1,\ \eta(y)=0\}}p(y-x) \\ &= \sum_{x,y\in\mathbb{T}_L^d} \big[f(\eta^{x,y}) - f(\eta)\big]\eta(x)(1-\eta(y))p(y-x).\end{aligned}$$

Invariant measures

A priori, we can identify two stationary measures for this process $\eta(x) = 1$ and $\eta(x) = 0$ for all $x \in \Lambda$, respectively. In fact we have for the product measures ν_α with $\nu_\alpha(\eta(x) = 1) = \alpha$ for all $x \in \Lambda$, that

Proposition 3.2. *$\nu_\alpha \in \mathcal{I}$ for all $\alpha \in [0,1]$.*

Proof. By a change of variables (replacing η by $\eta^{y,x}$), we obtain that

$$\begin{aligned}&\sum_{\eta\in S}\sum_{x,y\in\mathbb{T}_L^d} f(\eta^{x,y})\eta(x)(1-\eta(y))p(y-x)\nu_\alpha(\eta)\\ &=\sum_{\eta\in S}\sum_{x,y\in\mathbb{T}_L^d} f(\eta)\eta^{y,x}(x)(1-\eta^{y,x}(y))p(y-x)\nu_\alpha(\eta^{y,x})\\ &=\sum_{\eta\in S}\sum_{x,y\in\mathbb{T}_L^d} f(\eta)(1-\eta(x))\eta(y)p(y-x)\nu_\alpha(\eta),\end{aligned}$$

as $\nu_\alpha(\eta)=\nu_\alpha(\eta^{x,y})$ if $(1-\eta(x))\eta(y)=1$. This implies that

$$\begin{aligned}&\sum_{\eta\in S}\sum_{x,y\in\mathbb{T}_L^d}\big[f(\eta^{x,y})-f(\eta)\big]\eta(x)(1-\eta(y))p(y-x)\nu_\alpha(\eta)\\ &=\sum_{\eta\in S}f(\eta)\sum_{x,y\in\mathbb{T}_L^d}\Big[(1-\eta(y))\eta(x)-\eta(y)(1-\eta(x))\Big]p(y-x)\nu_\alpha(\eta)\\ &=\sum_{\eta\in S}f(\eta)\sum_{x,y\in\mathbb{T}_L^d}\big[\eta(x)-\eta(y)\big]p(y-x)\nu_\alpha(\eta)=0.\end{aligned}$$

□

Scaling limit

In the following we have a look at the scaling limit for the SEP. Scaling limits are of great interest in physics and in particular in mathematics. They explain how and why properties of microscopic systems can give rise to properties in macroscopic systems.

In the following, we think of the discrete torus $\mathbb{T}_L^d$ with L sites as an embedding in the ('continuous') torus $\mathbb{T}^d$ such that all sites are equidistant. As $L\to\infty$ the number of sites increases and the sites 'get closer together'.

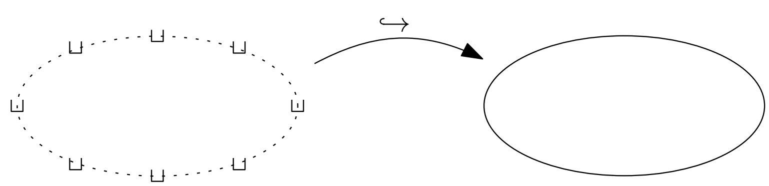

Figure 5: Embedding of the discrete torus $\mathbb{T}_L^d$ in the torus $\mathbb{T}_L^d$.

From now on we assume that the transition function $p(\cdot)$ is isotropic, i.e. that $p(\mathrm{e}_k)=1/(2d)$ for all directions (the canonical coordinates) e_k for $k=1,\cdots,d$. The empirical measure is defined as

$$\pi_t^L(du):=\pi^L(\eta_t,du):=\frac{1}{L^d}\sum_{x\in\mathbb{T}_L^d}\eta_t(x)\delta_{x/L}(du).$$

We can think of the empirical measure as a measure on T^d, and therefore can integrate a function

$h : \mathbb{T}^d \to \mathbb{R}$ w.r.t. this measure:

$$\langle \pi_t^L, h\rangle := \int_{\mathbb{T}^d} h(u)\, \pi_t^L(du) = \frac{1}{L^d} \sum_{x\in\mathbb{T}_L^d} h\big(\tfrac{x}{L}\big)\eta_t(x).$$

We obtain from the formula for the martingale problem (equation (6) on page 6), that we can represent this process by the formula

$$\langle \pi_t^L, h\rangle - \langle \pi_0^L, h\rangle = \int_0^t \mathcal{A}_L \langle \pi_s^L, h\rangle ds + M_t^L.$$

We here assume that $M_t^L \to 0$ as $L \to \infty$ (we write this as $M_t^L = o_L(1)$). By linearity, we further obtain, that

$$\mathcal{A}_L \langle \pi_t^L, h\rangle = \frac{1}{L^d} \sum_{x\in\mathbb{T}_L^d} h\big(\tfrac{x}{L}\big)\, \mathcal{A}_L \eta_t(x).$$

Therefore we have to calculate as next the value of $\mathcal{A}_L\eta(x)$, the projection on the x-coordinate:

$$\begin{aligned}
\mathcal{A}_L\eta(x) &= \frac{1}{2d}\sum_{k=1}^d \Big\{ \big[\eta(x+\mathrm{e}_k)(1-\eta(x)) - \eta(x-\mathrm{e}_k)(1-\eta(x))\big] \\
&\qquad\qquad - \big[\eta(x)(1-\eta(x+\mathrm{e}_k)) - \eta(x)(1-\eta(x-\mathrm{e}_k))\big]\Big\} \\
&= \frac{1}{2d}\sum_{k=1}^d \Big[\eta(x+\mathrm{e}_k) + \eta(x-\mathrm{e}_k) - 2\eta(x)\Big] = \Delta^L\eta(x),
\end{aligned}$$

where Δ^L is the discrete Laplacian (graph Laplacian). We therefore obtain that

$$\begin{aligned}
\langle \pi_t^L, h\rangle - \langle \pi_0^L, h\rangle &= \int_0^t \mathcal{A}_L \langle \pi_s^L, h\rangle ds + o_L(1) \\
&= \int_0^t \frac{1}{L^d} \sum_{x\in\mathbb{T}_L^d} h\big(\tfrac{x}{L}\big)\, \mathcal{A}_L \eta_s(x)\, ds + o_L(1) \\
&= \int_0^t \frac{1}{L^d} \sum_{x\in\mathbb{T}_L^d} h\big(\tfrac{x}{L}\big) \Delta^L \eta_s(x)\, ds + o_L(1).
\end{aligned}$$

By taking the so called ‘hydrodynamic limit with diffusive rescaling’, one can in fact show that one obtains a quantity ρ that satisfies a weak form of the heat equation

$$\partial_t \rho = c\Delta\rho,$$

where Δ is the usual Laplace operator and $c \in \mathbb{R}$ a constant. For more details on this, see for example the book of Kipnis-Landim [1].

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