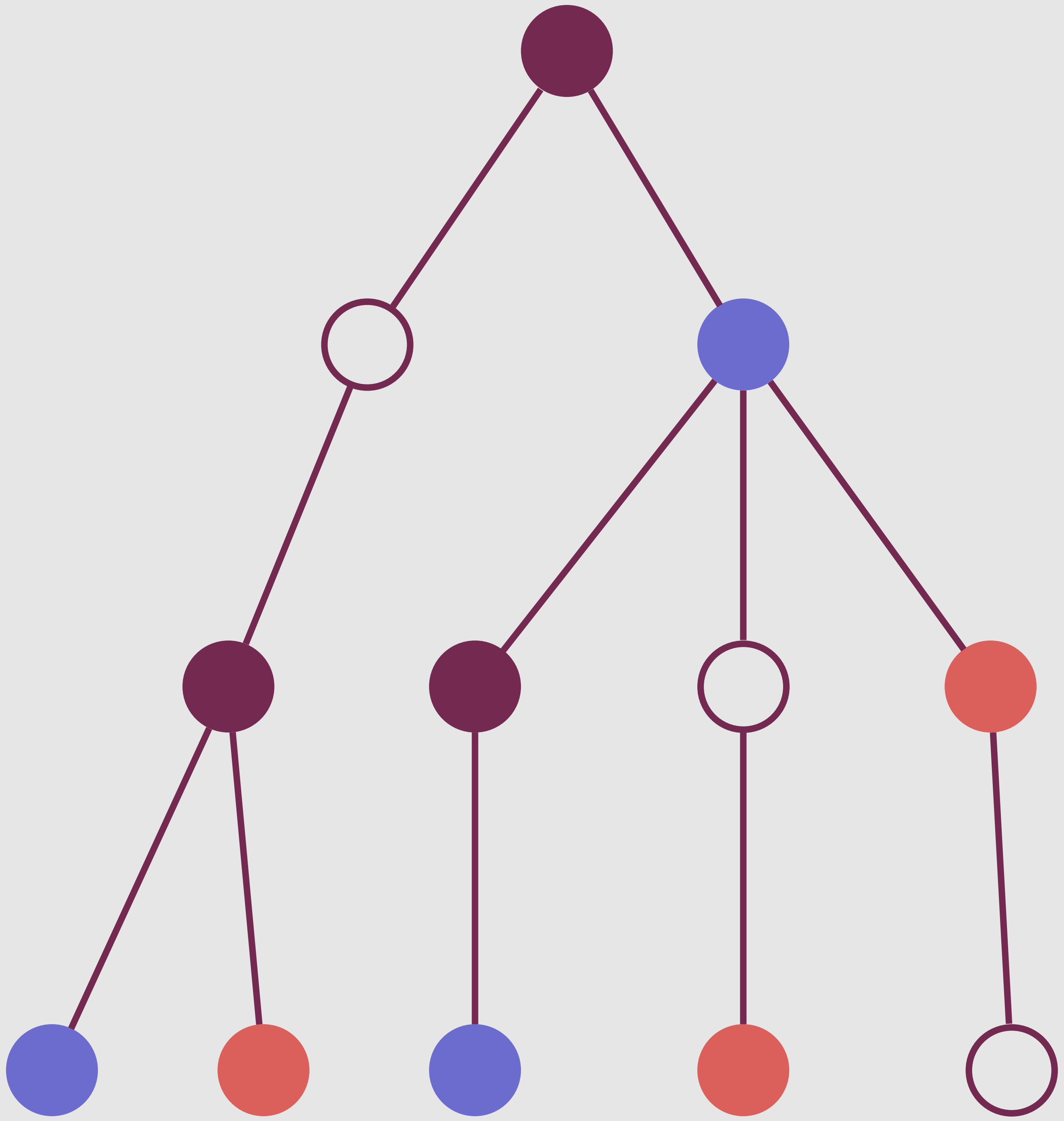
The Symbiotic Contact Process on Trees

Joint research with
Marcel Ortgiese and
Sarah Penington

The Process

The Process

- An augmented contact process (continuous time, discrete space) that models two diseases



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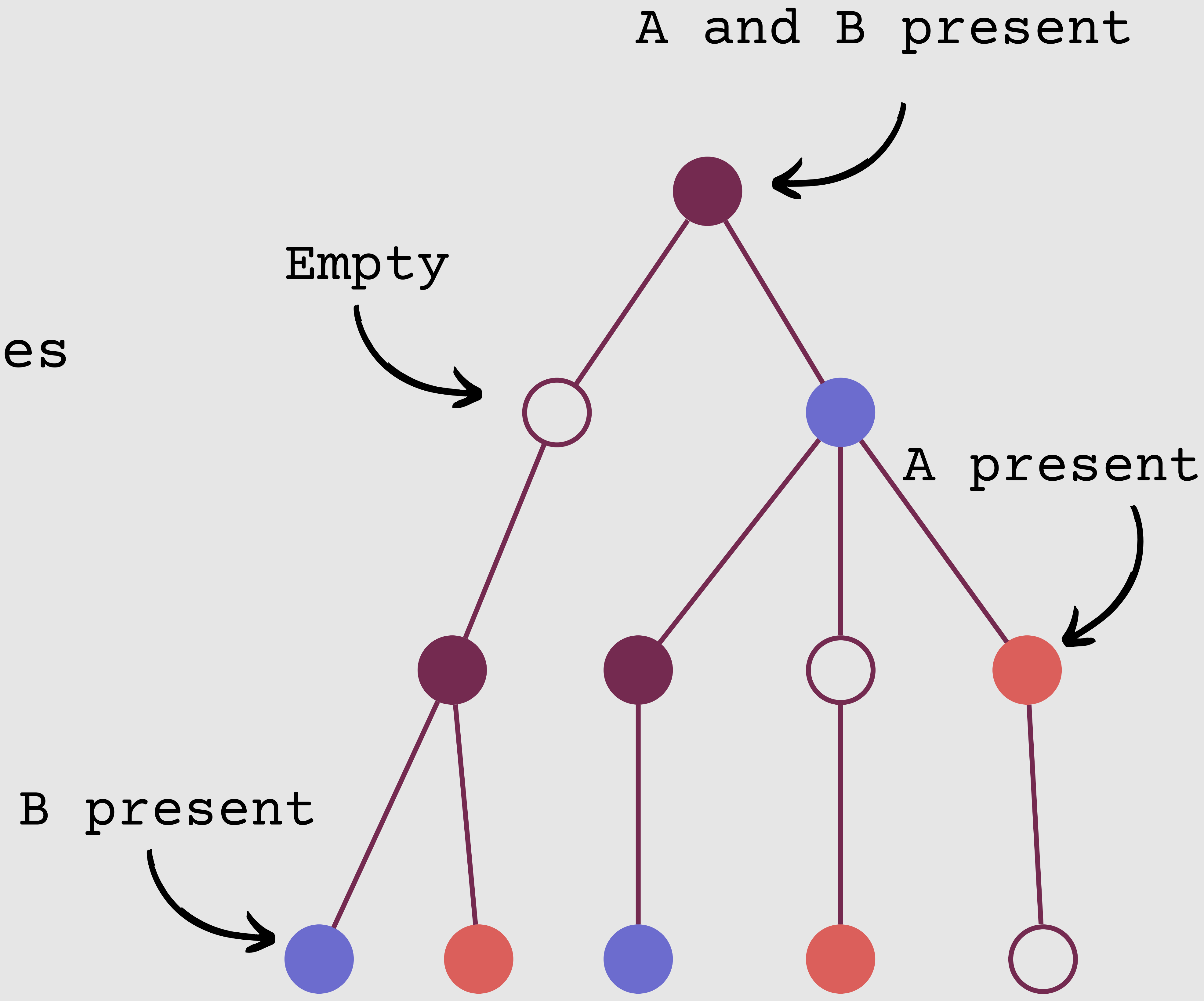

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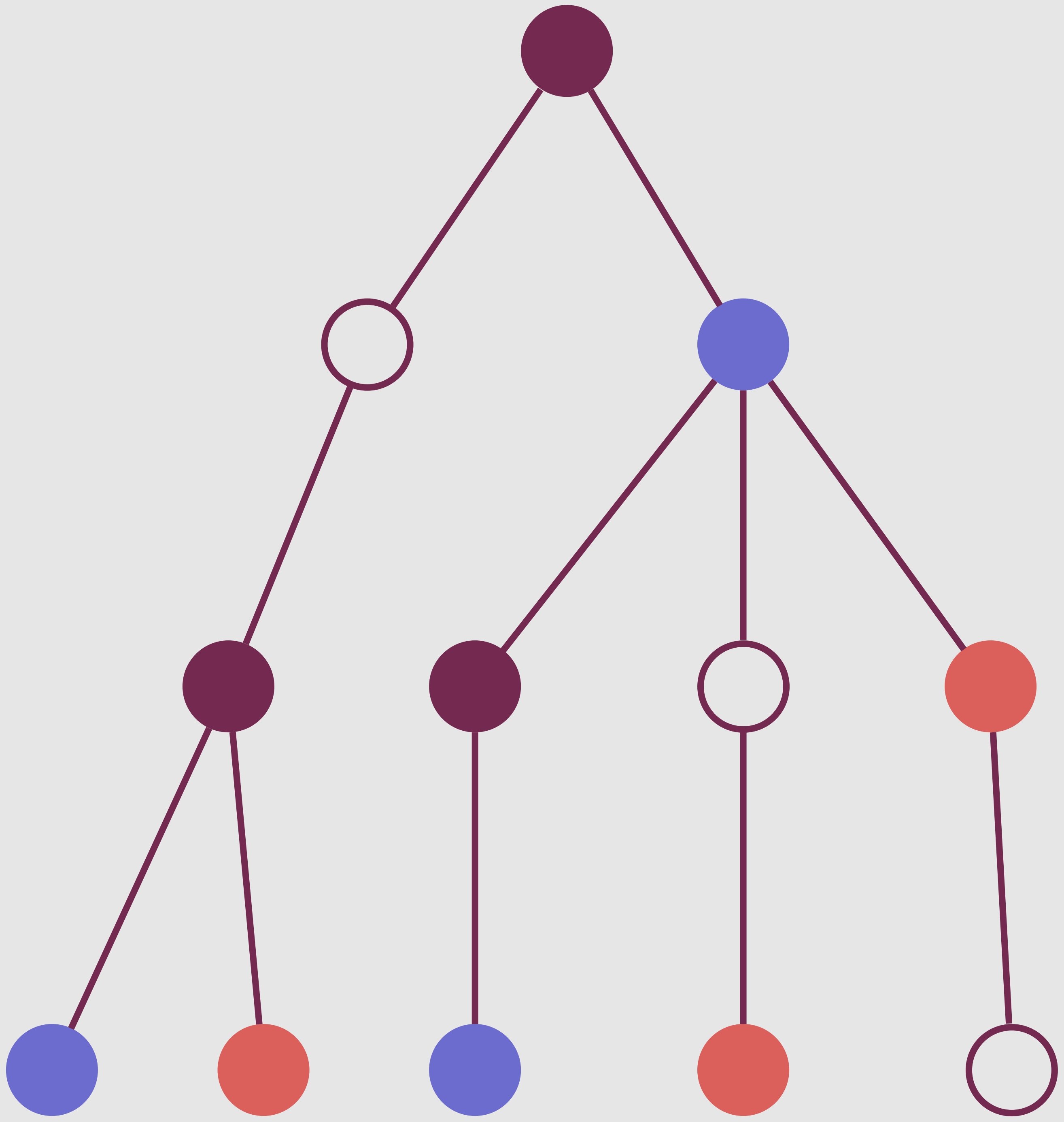

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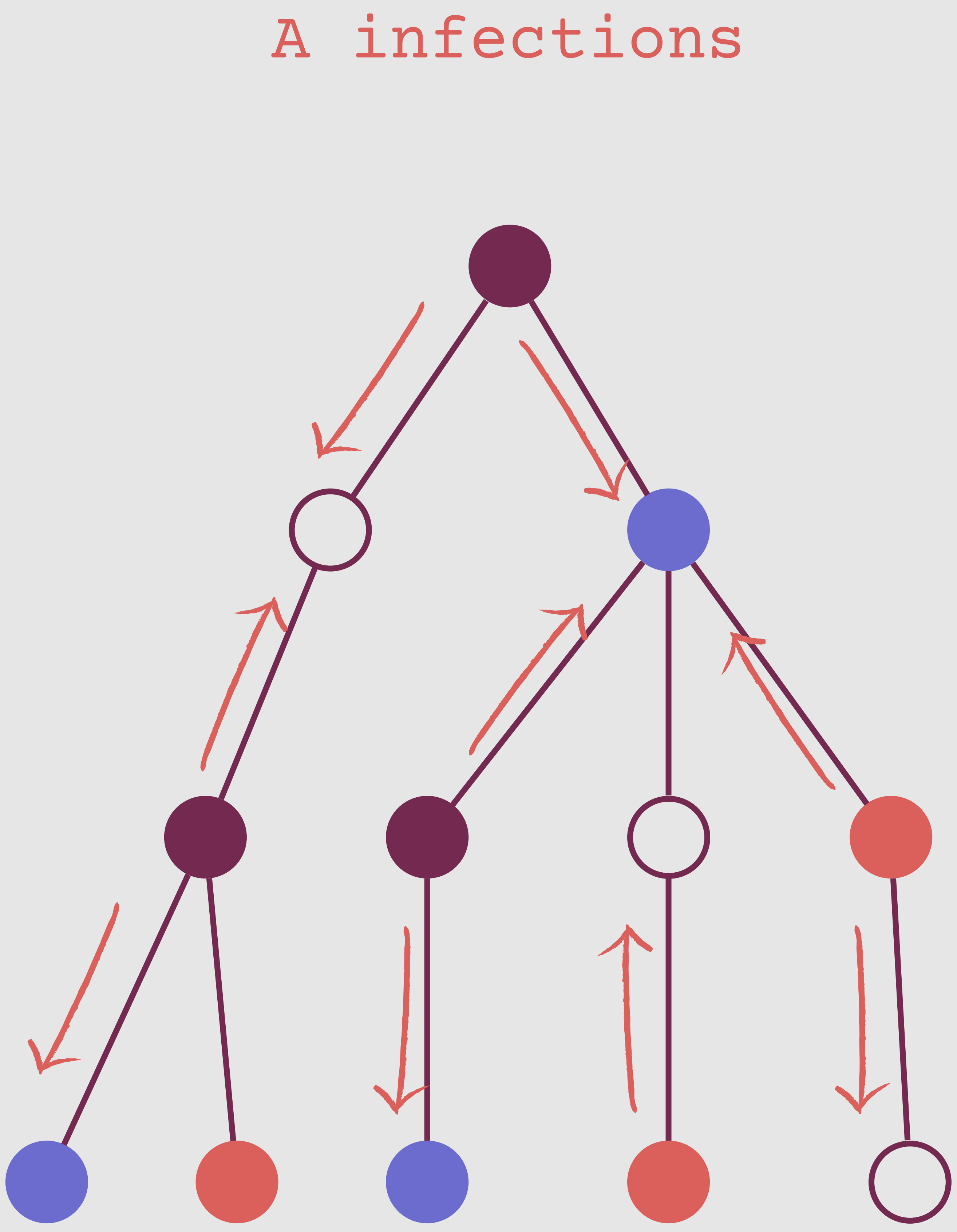

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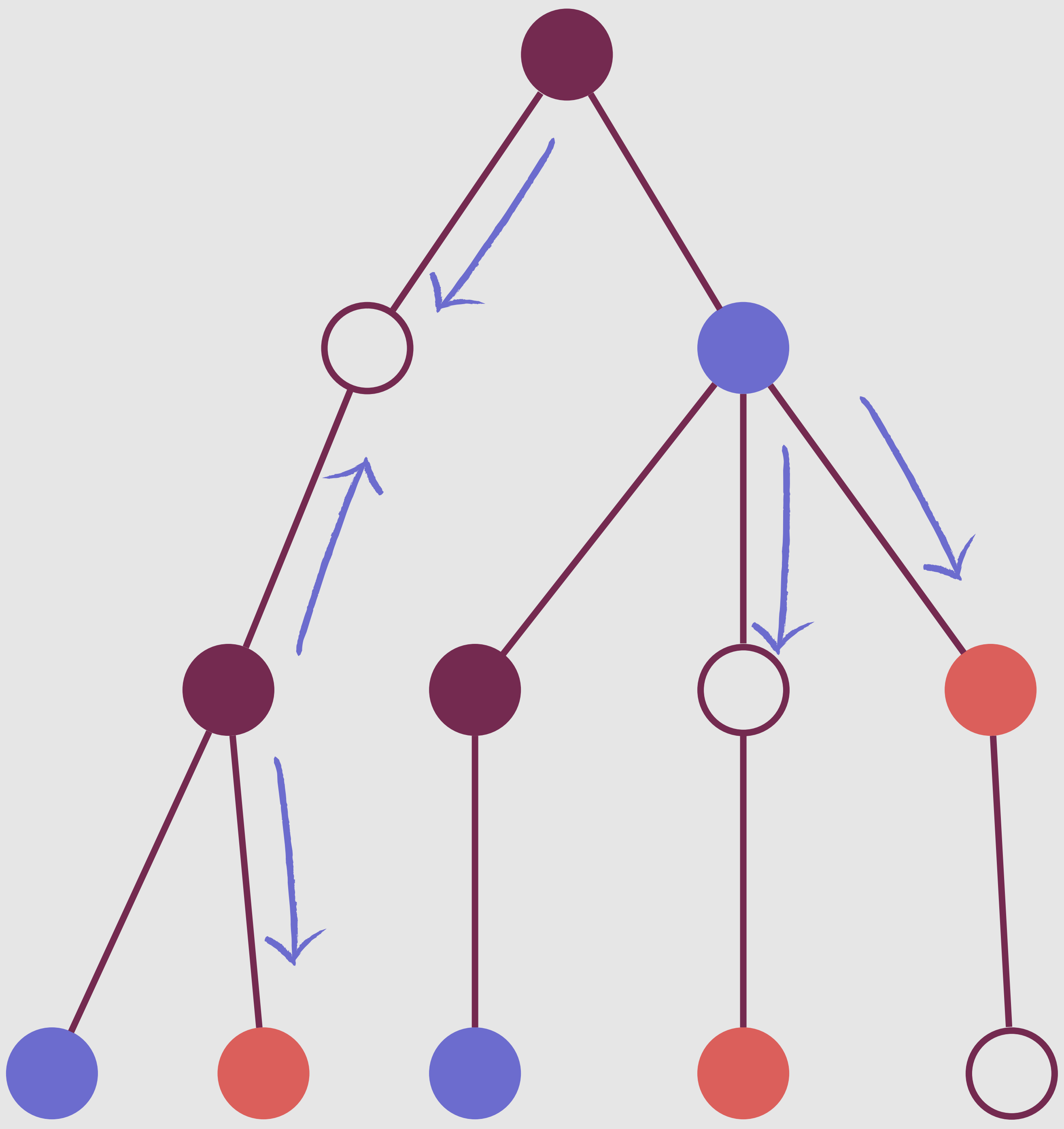
B infections



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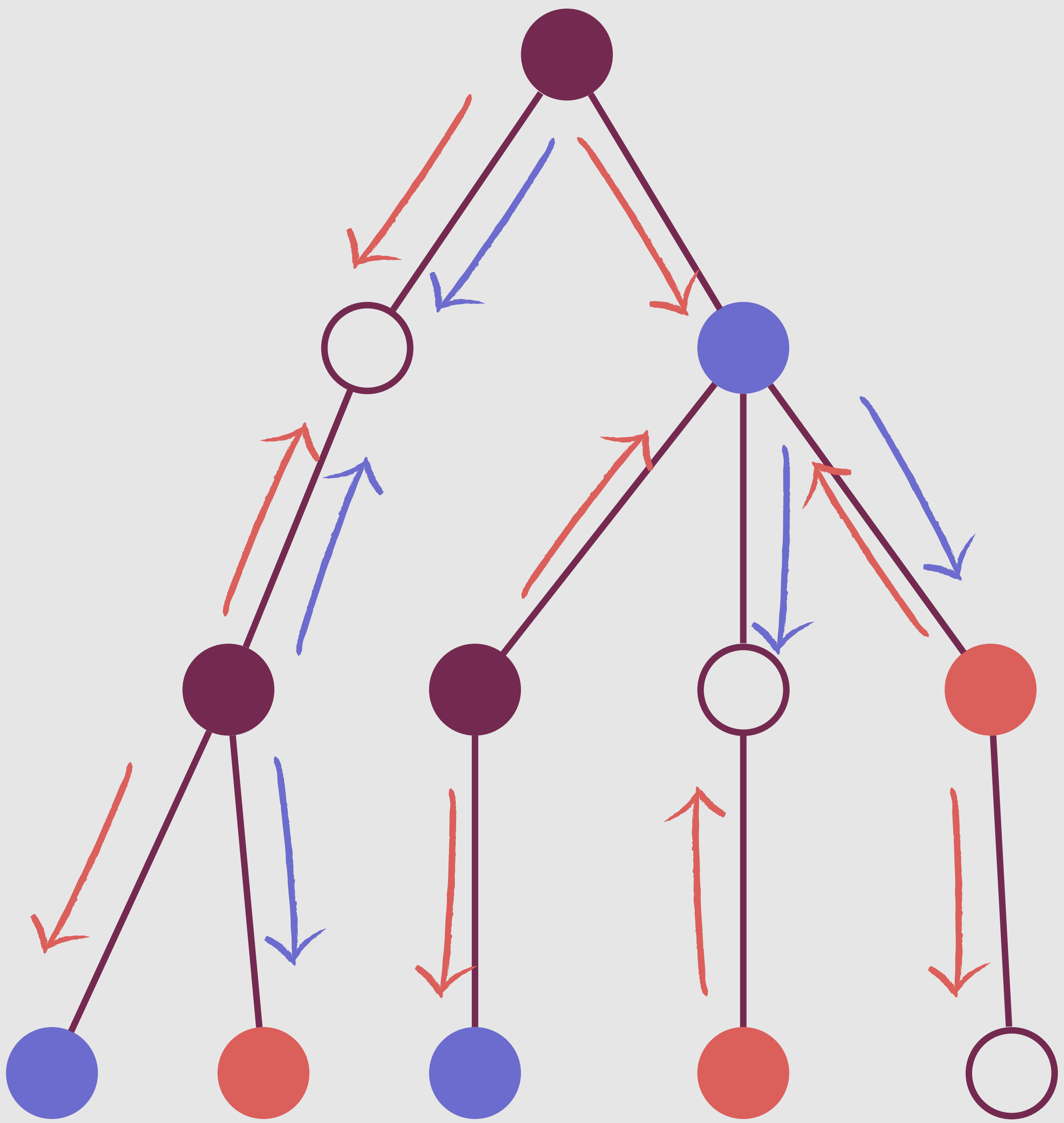
All infections



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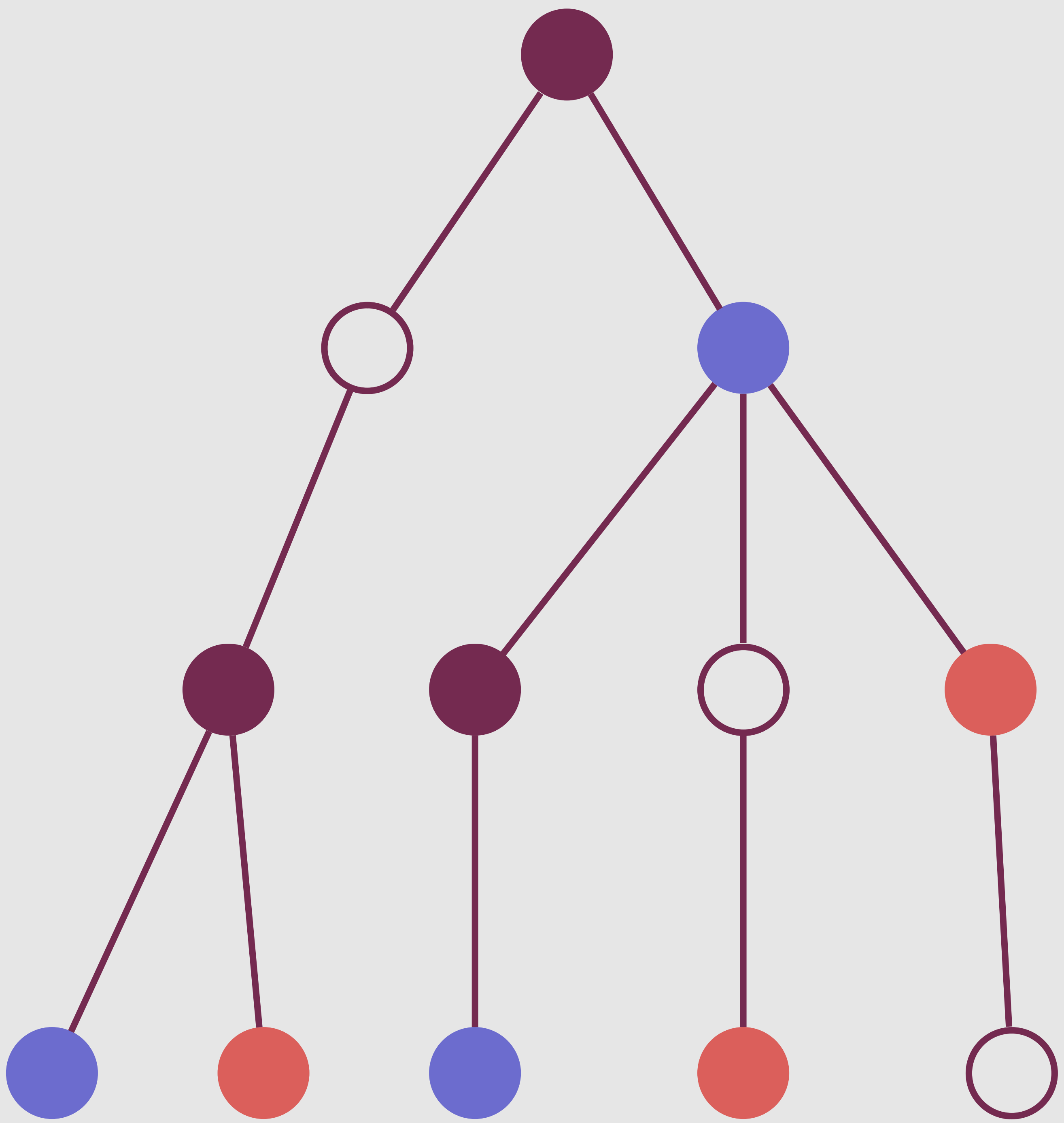
- Recoveries occur at rate 1 when a site has only 1 infection or rate μ when a site has both diseases



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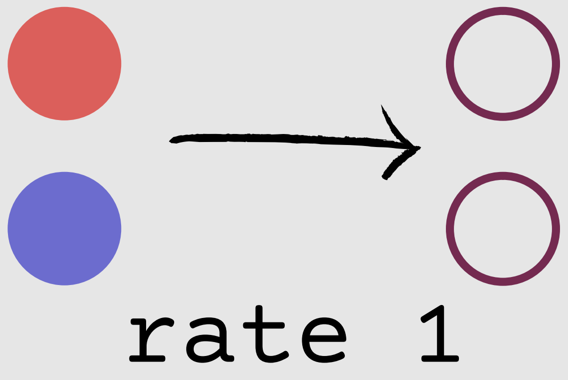

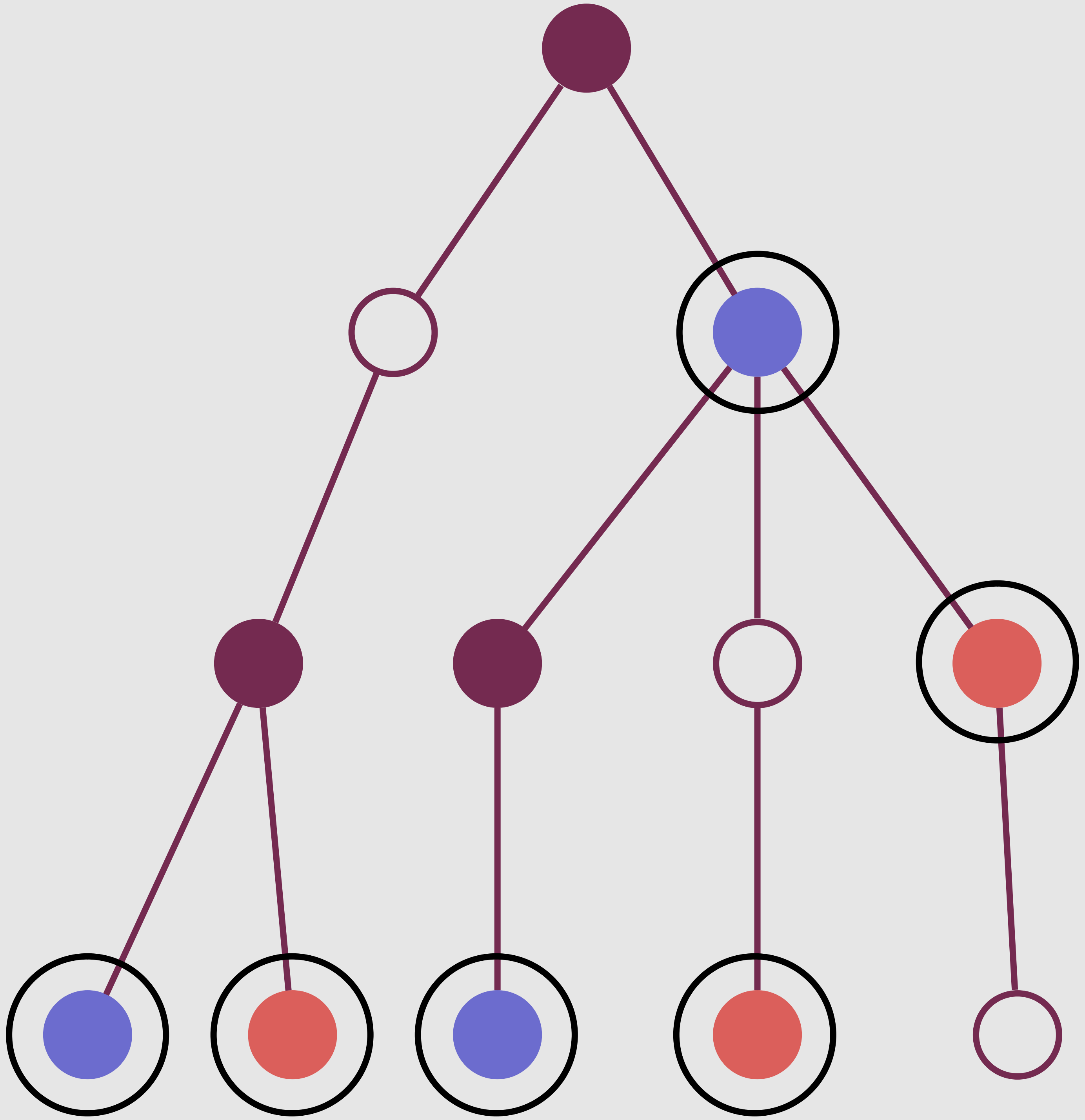

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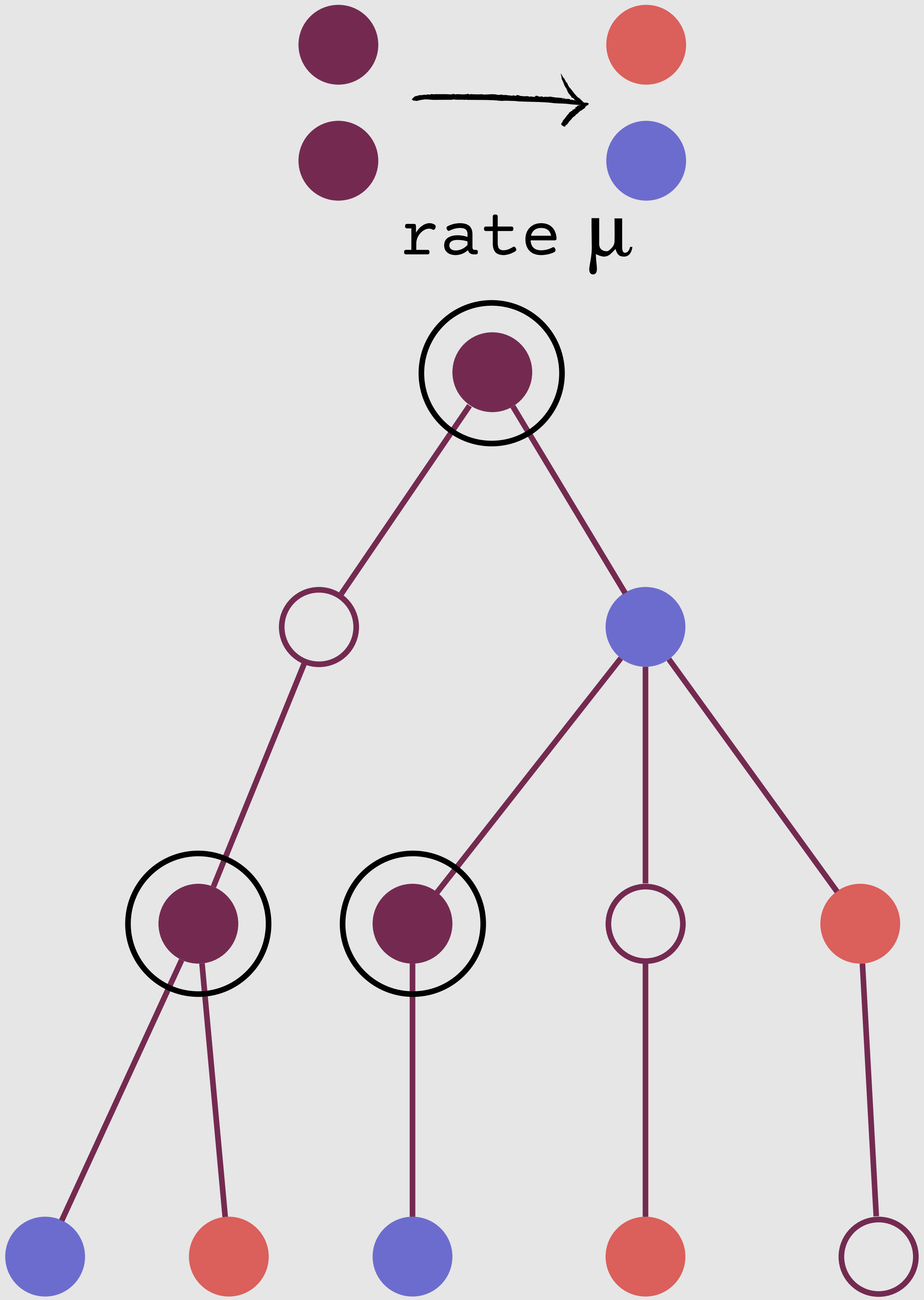

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We will be looking at the process on a Galton-Watson tree with offspring distribution ζ

When does our process survive?

Critical Values

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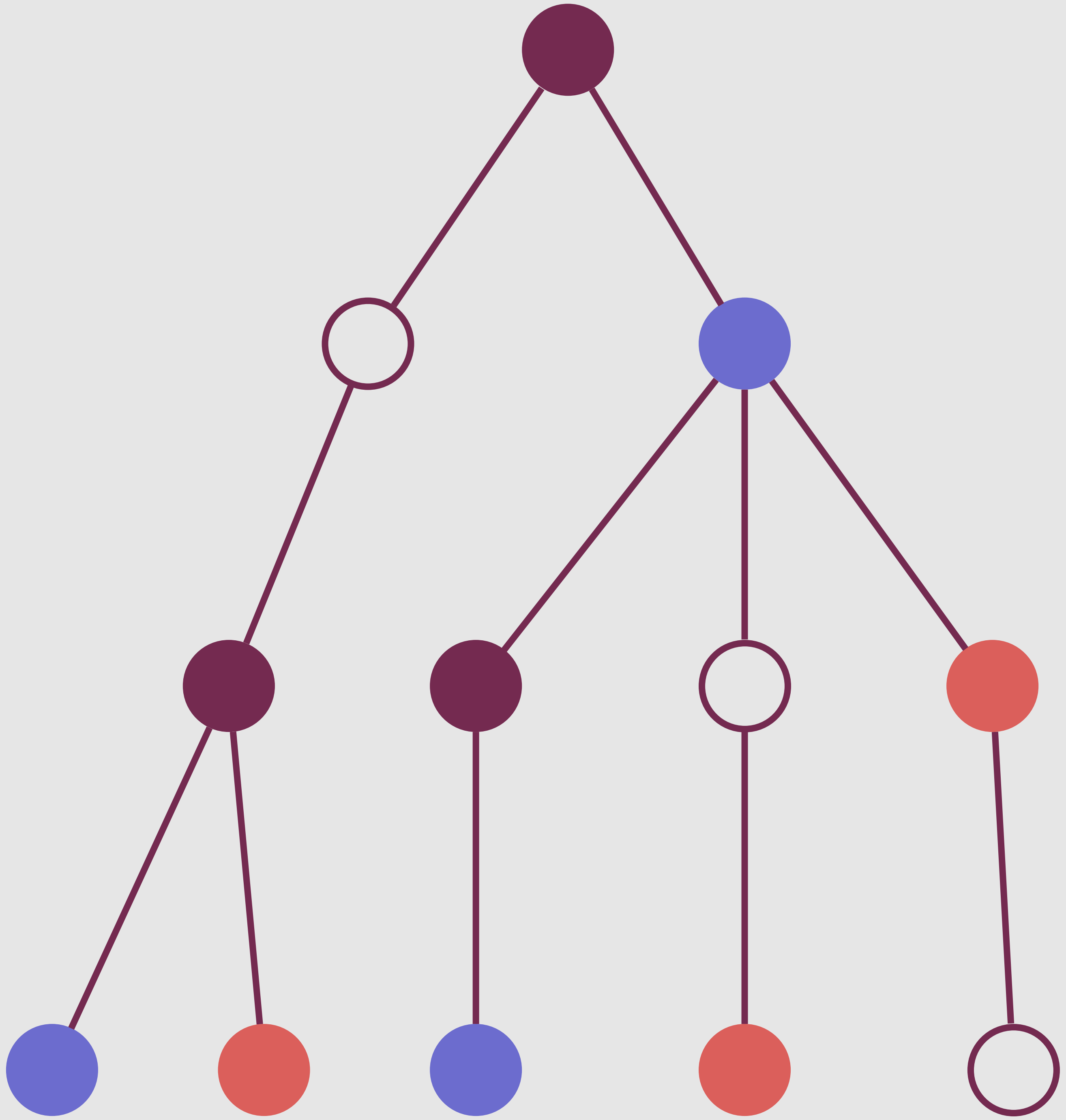
The symbiotic contact process has **survived weakly** if at all times there is an A infection and a B infection



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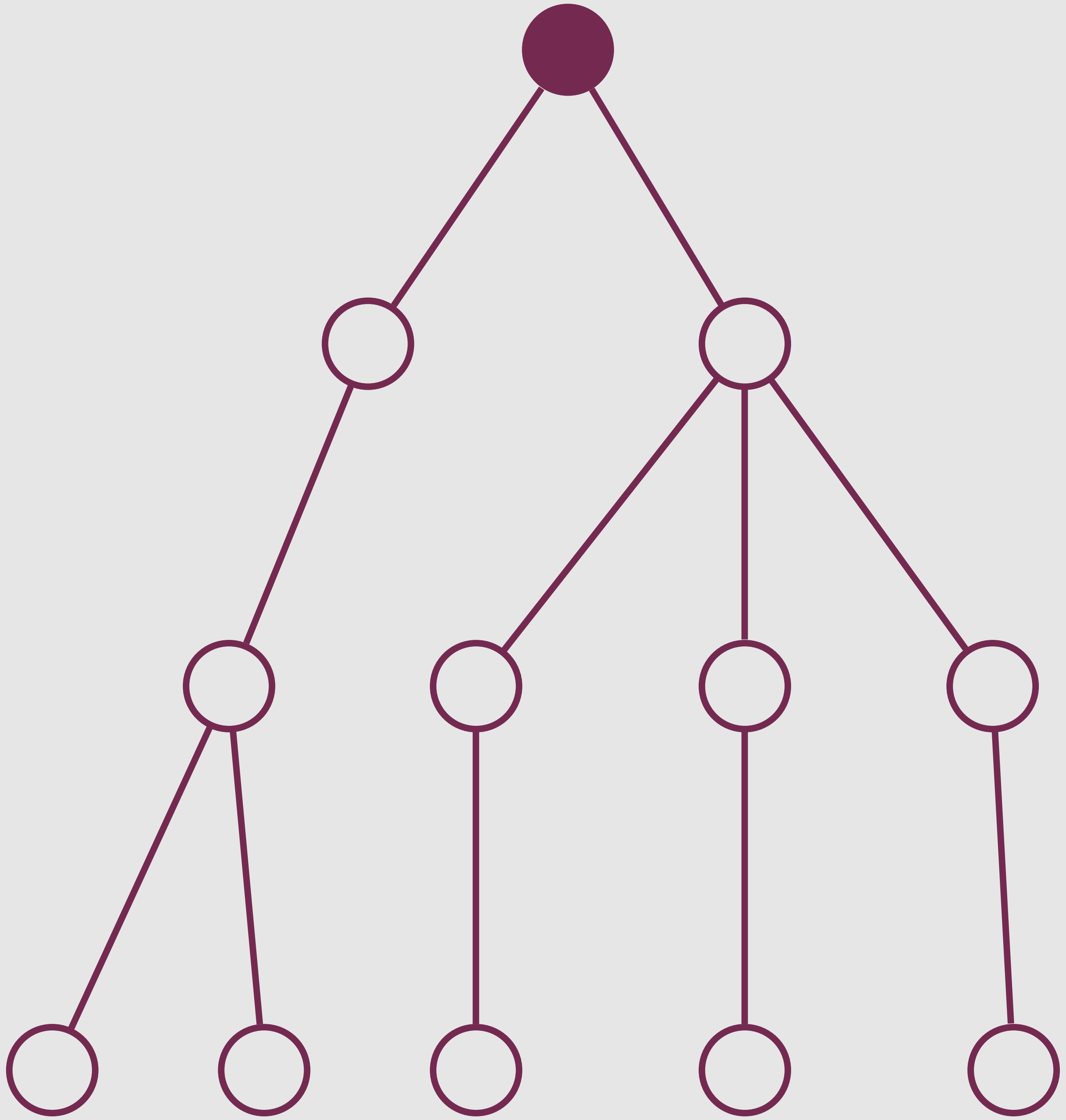

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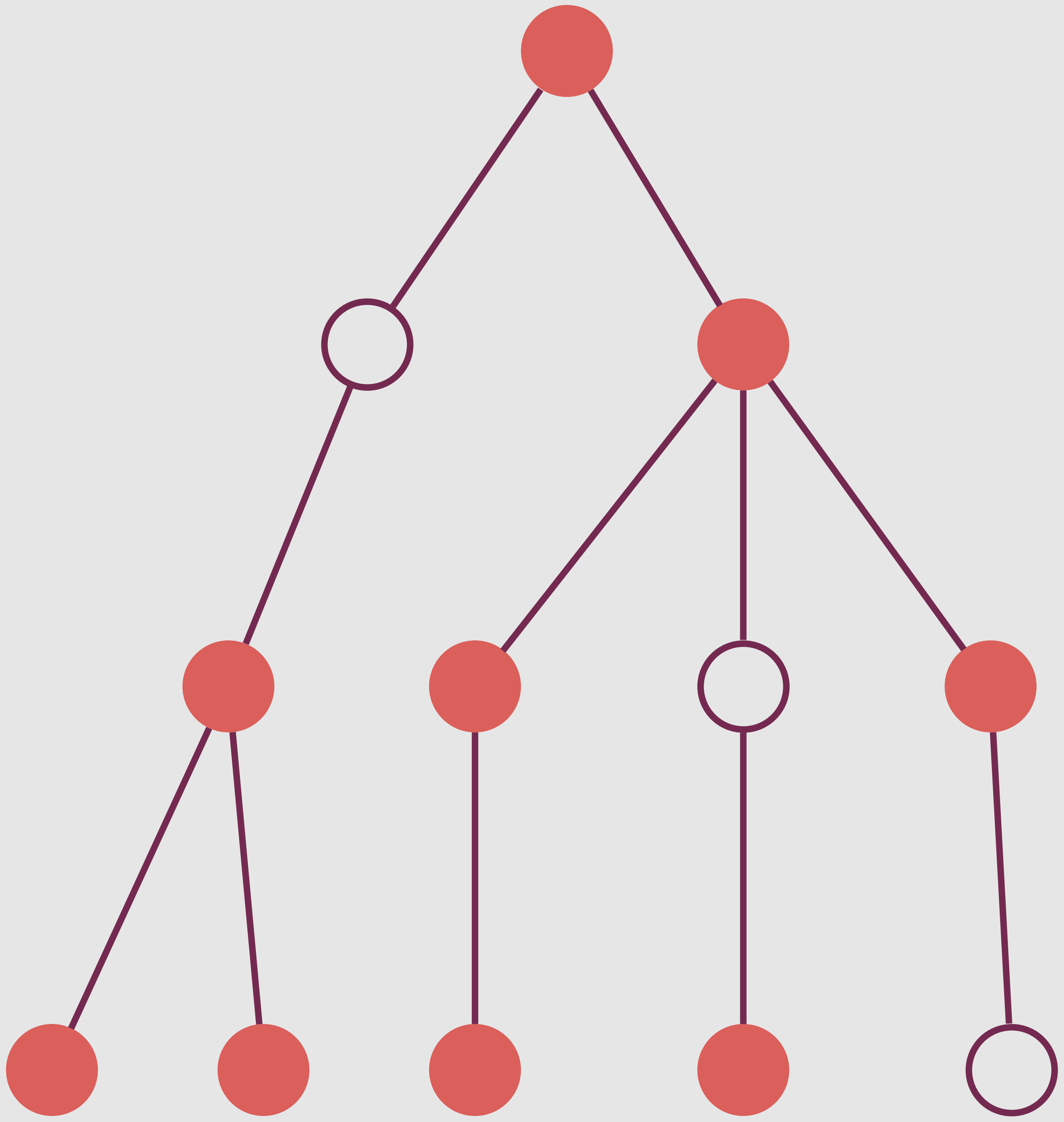

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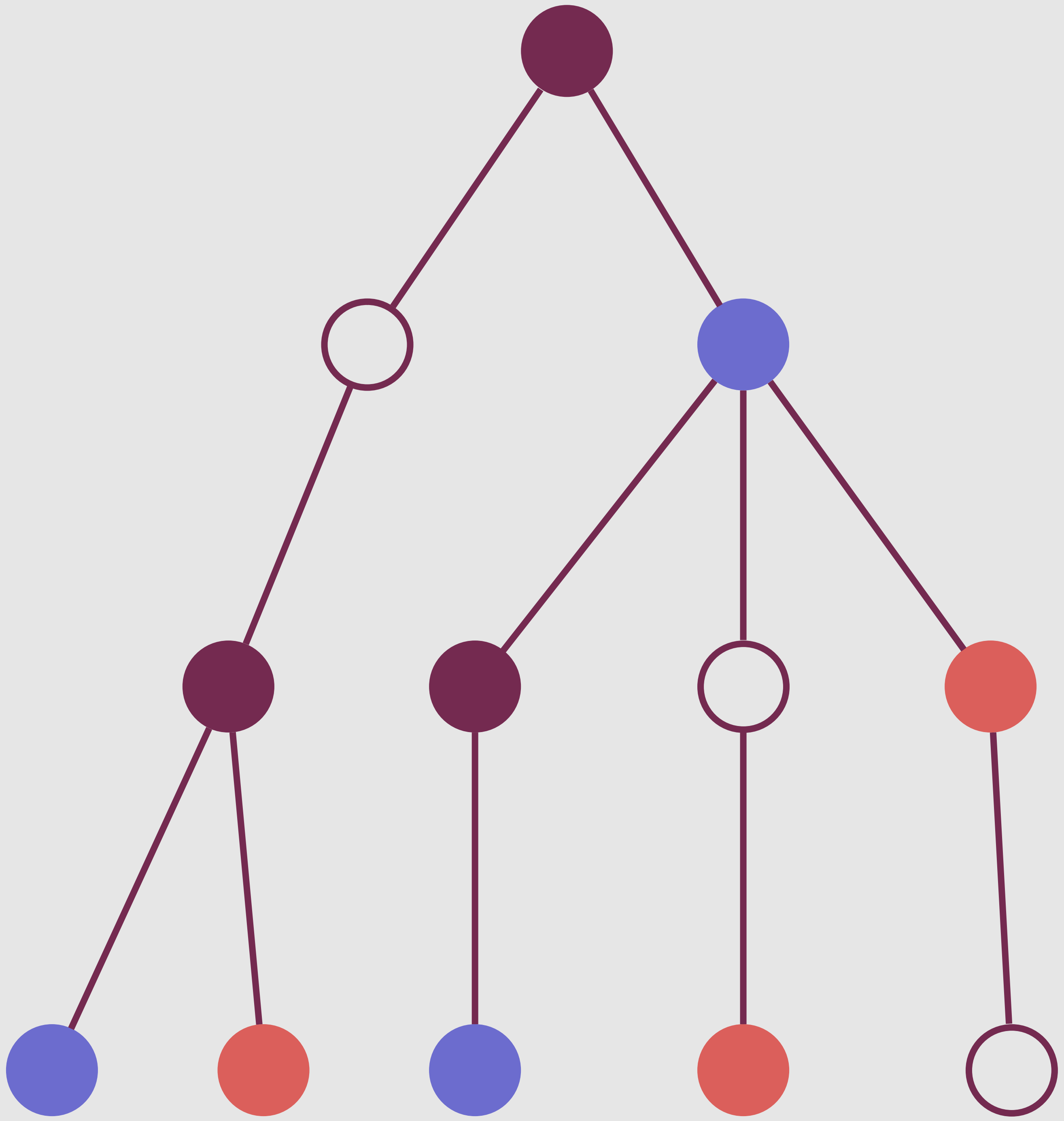

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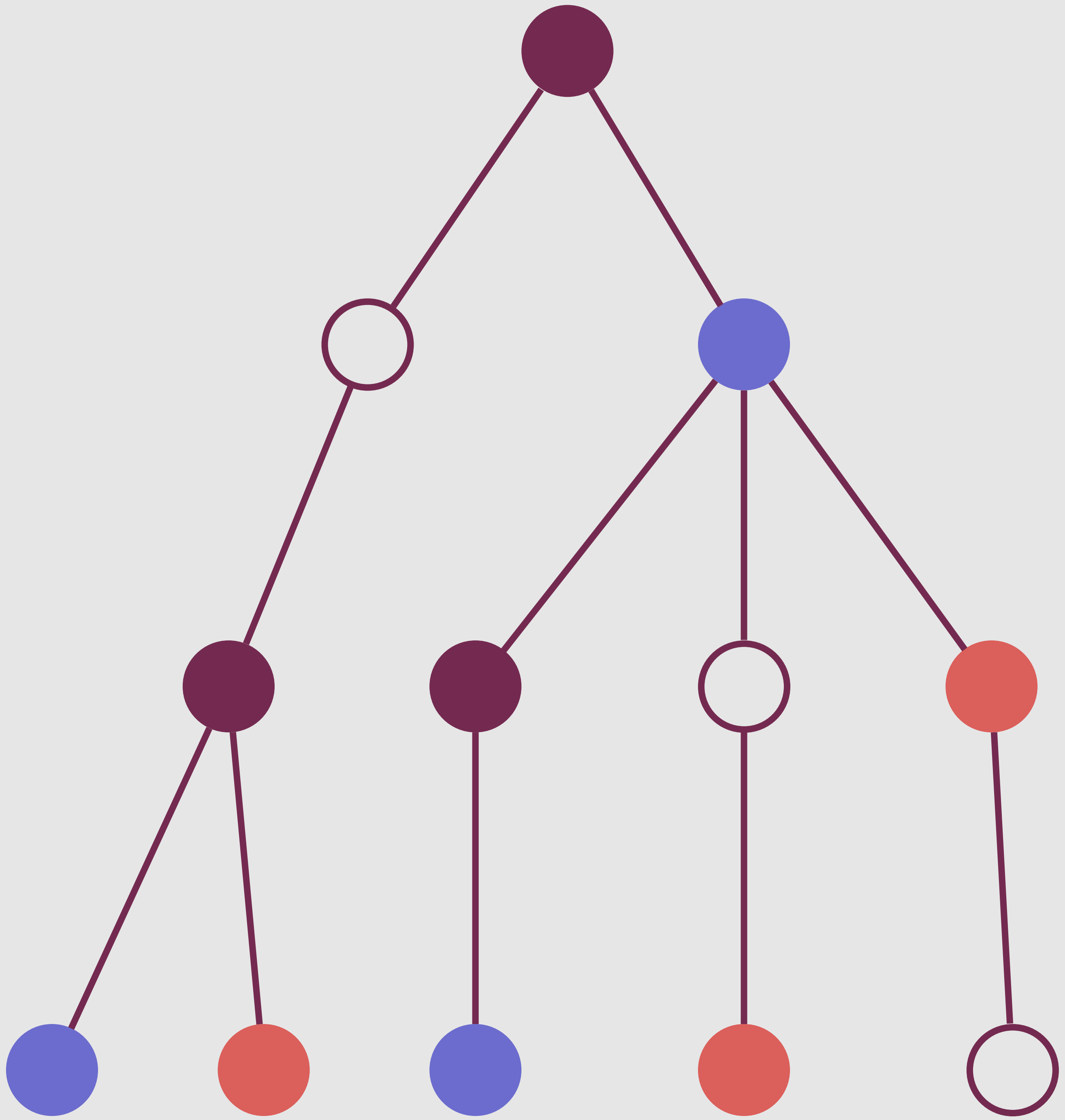

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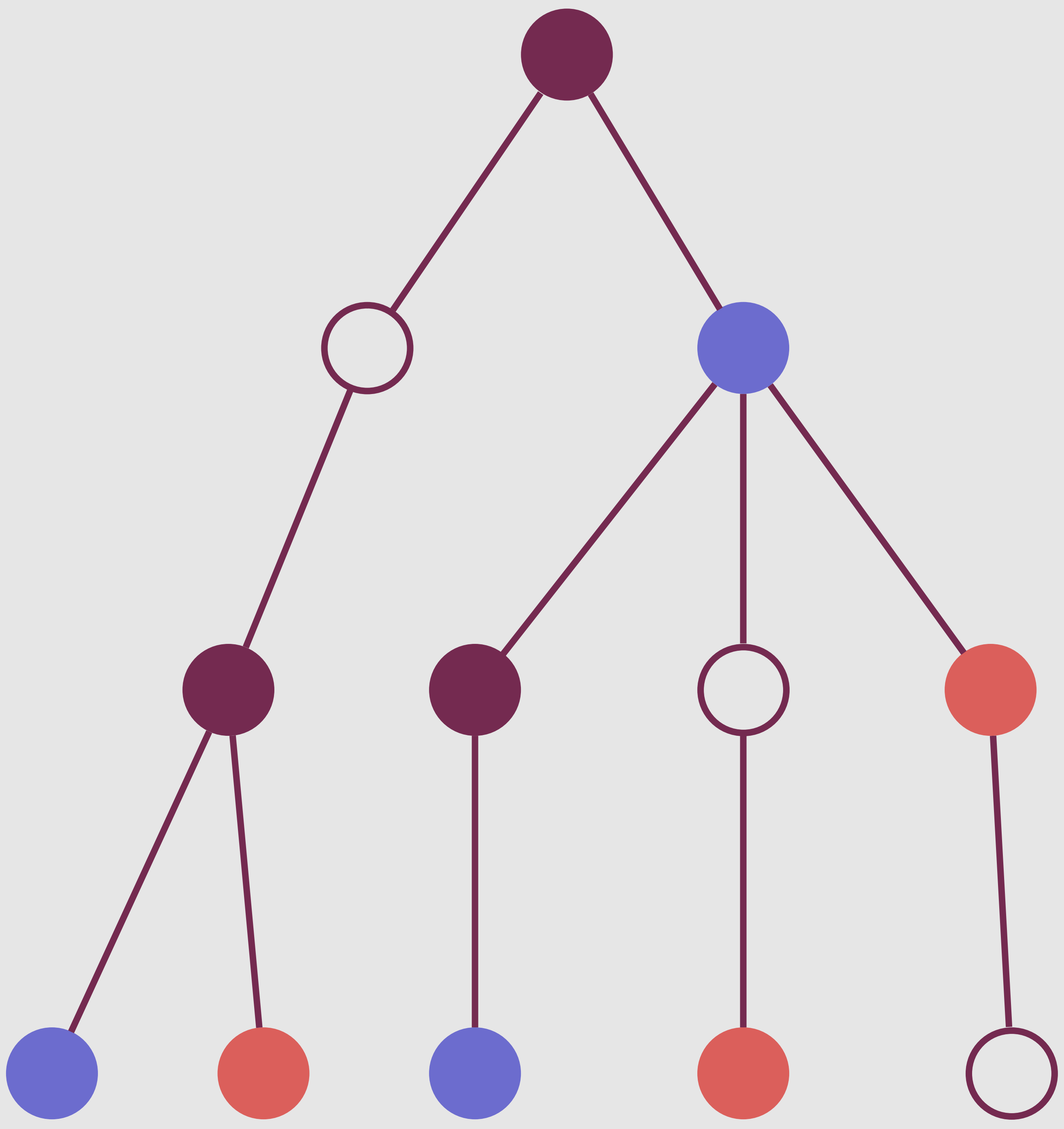

Comparisons

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By the Monotonicity of the process we have that

$$\lambda_1(\mu) \leq \lambda_1(1)$$

Comparisons

When $\mu = 1$ we have that the process can be split into two independent contact processes for the A particles and the B particles and thus

$$\lambda_1(\mu) \leq \lambda_1(1) = \lambda_1^{\mathrm{CP}}$$

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This means that the only interesting case is when the offspring distribution has an exponential tail

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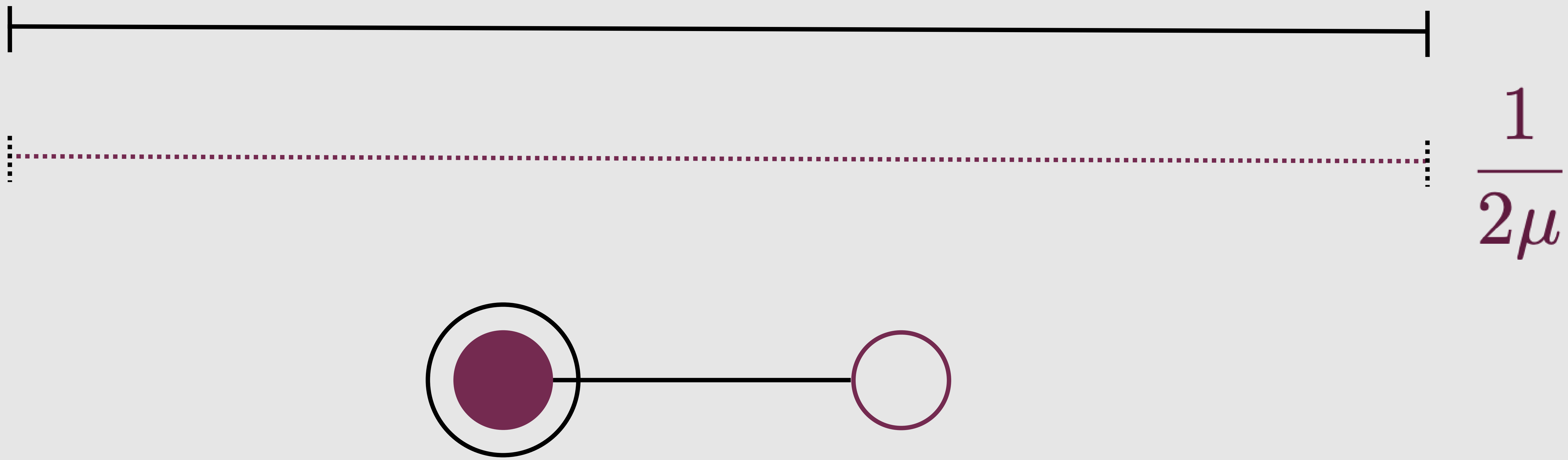

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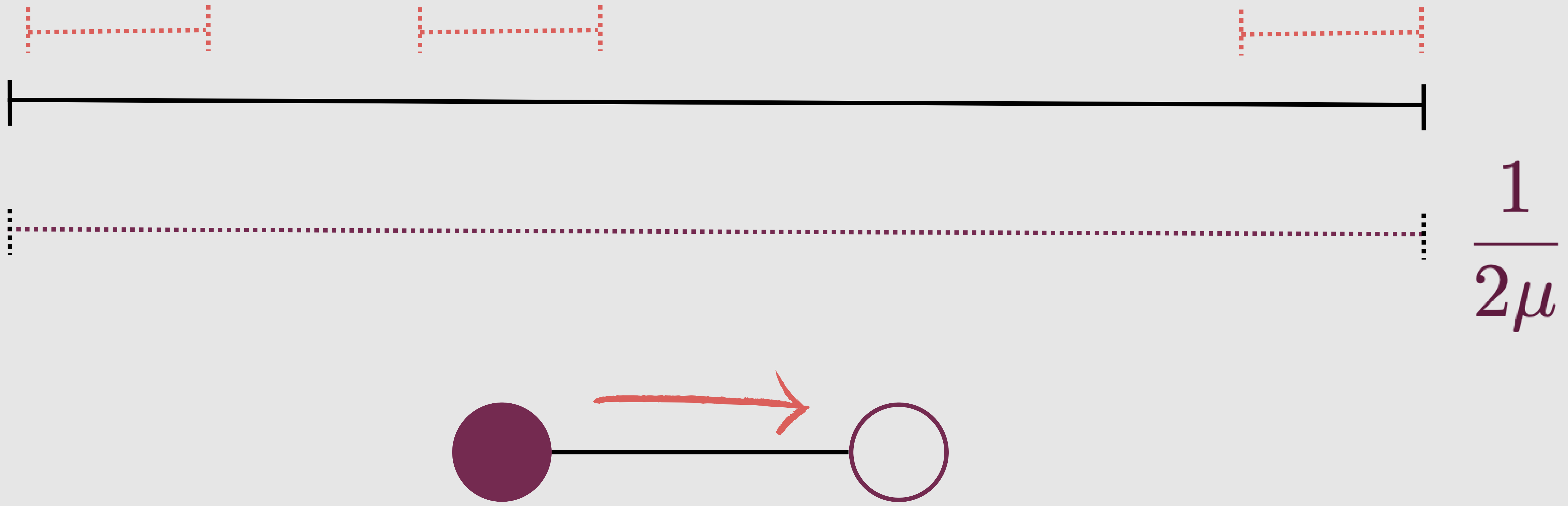

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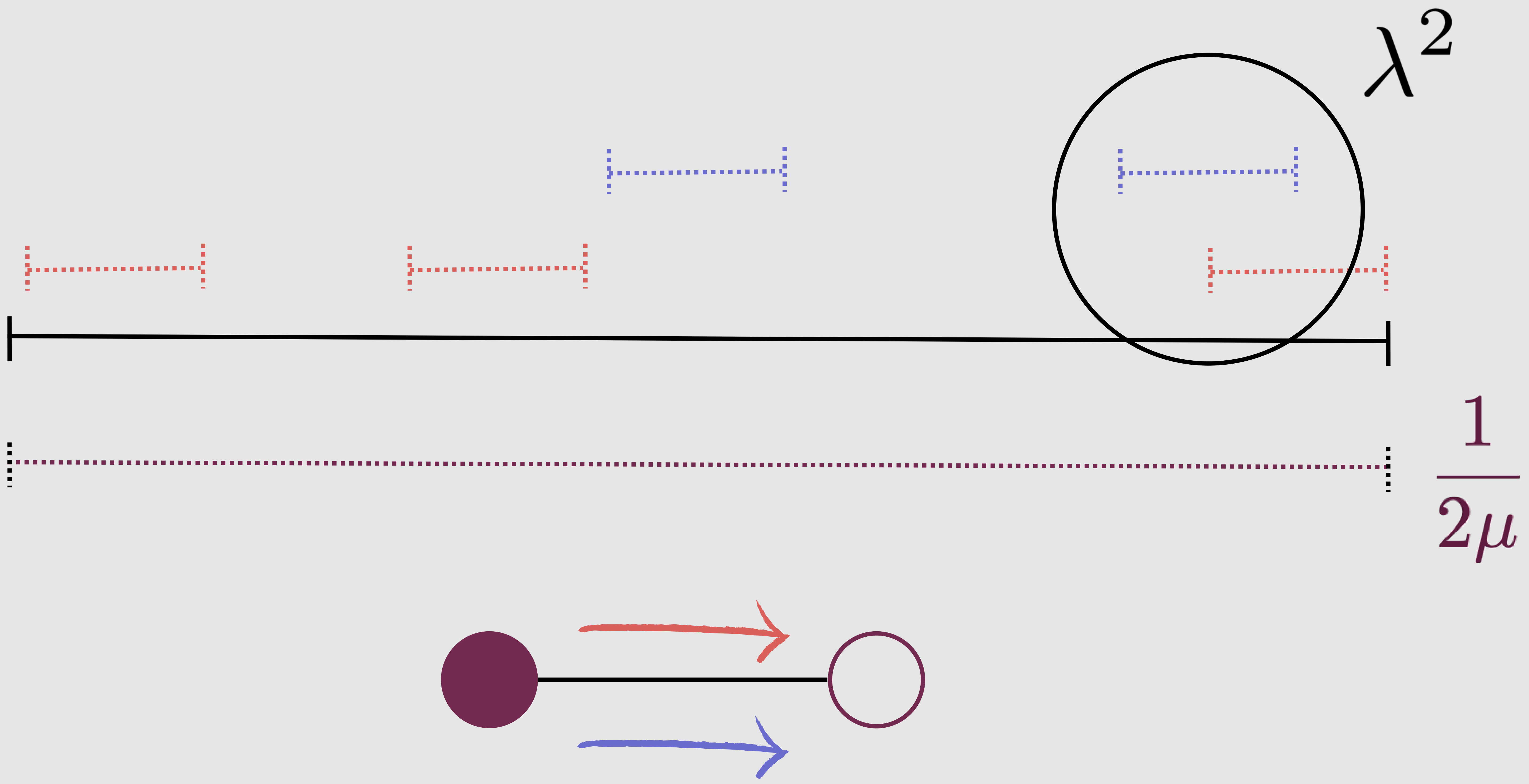
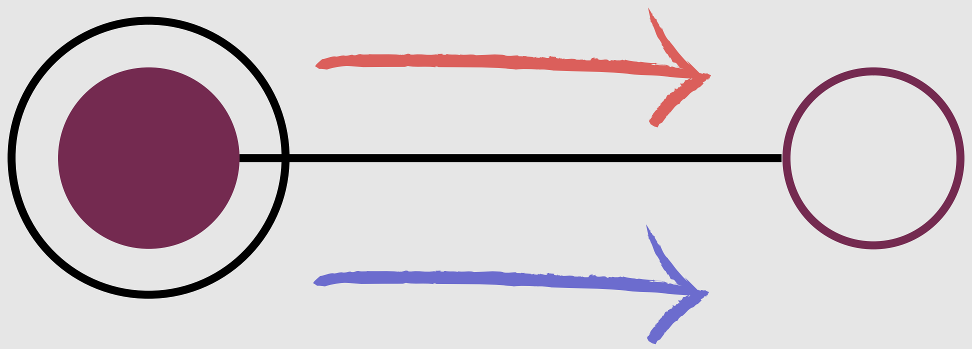


Passing an infection

$$p_{\text{inf}} = \frac{2\lambda^2}{2\lambda^2 + 2\mu(1+3\lambda) + 4\mu^2}$$



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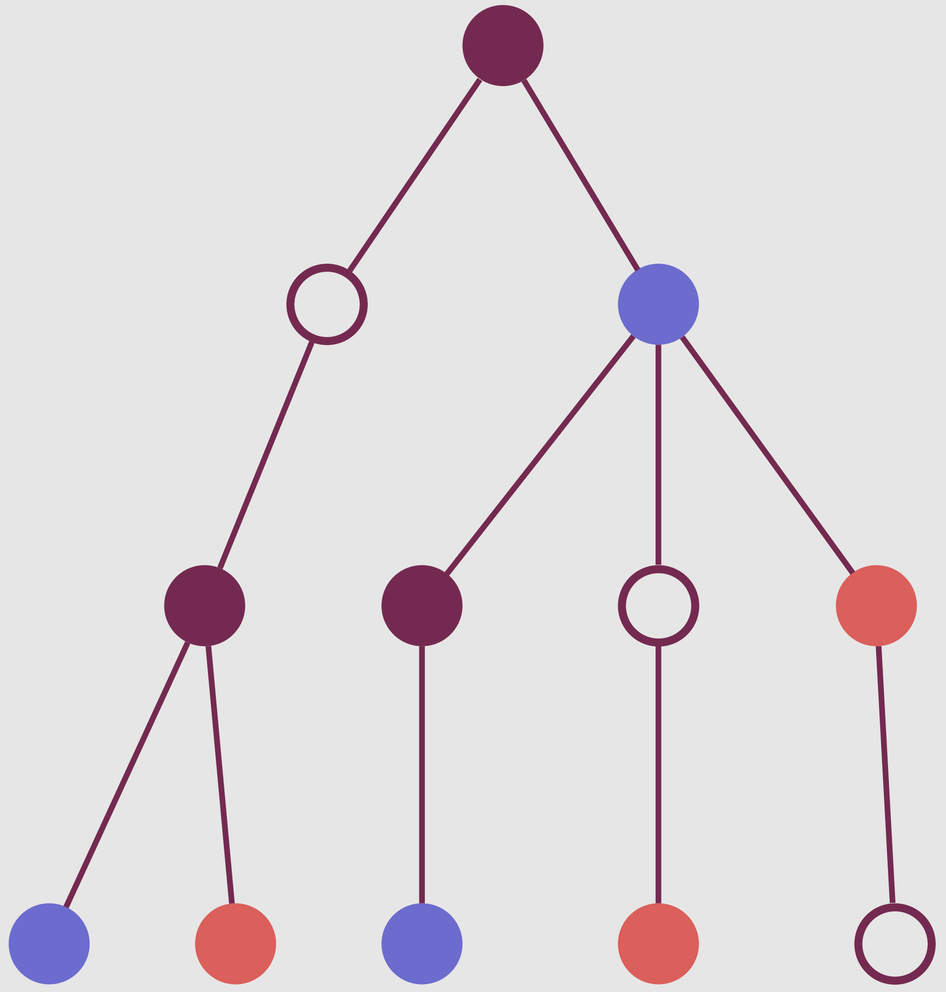
We can obtain an upper bound on λ_1 by finding any value at which the process must survive



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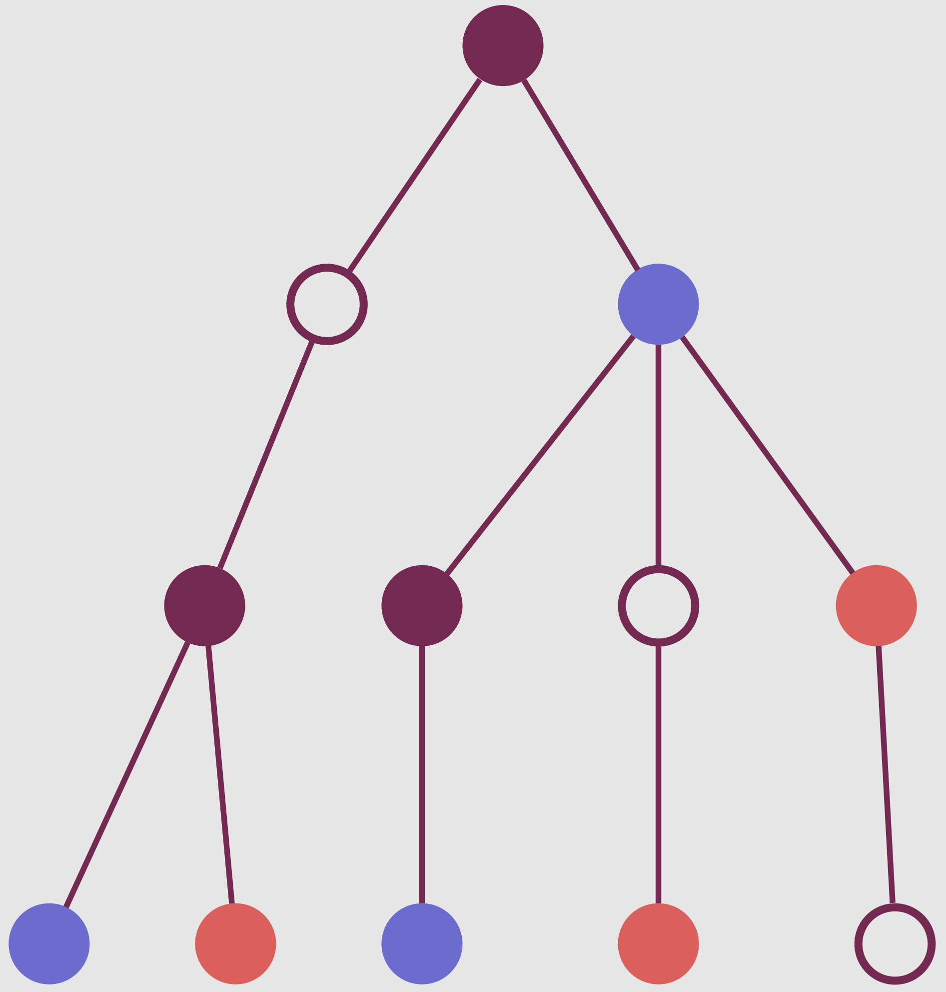

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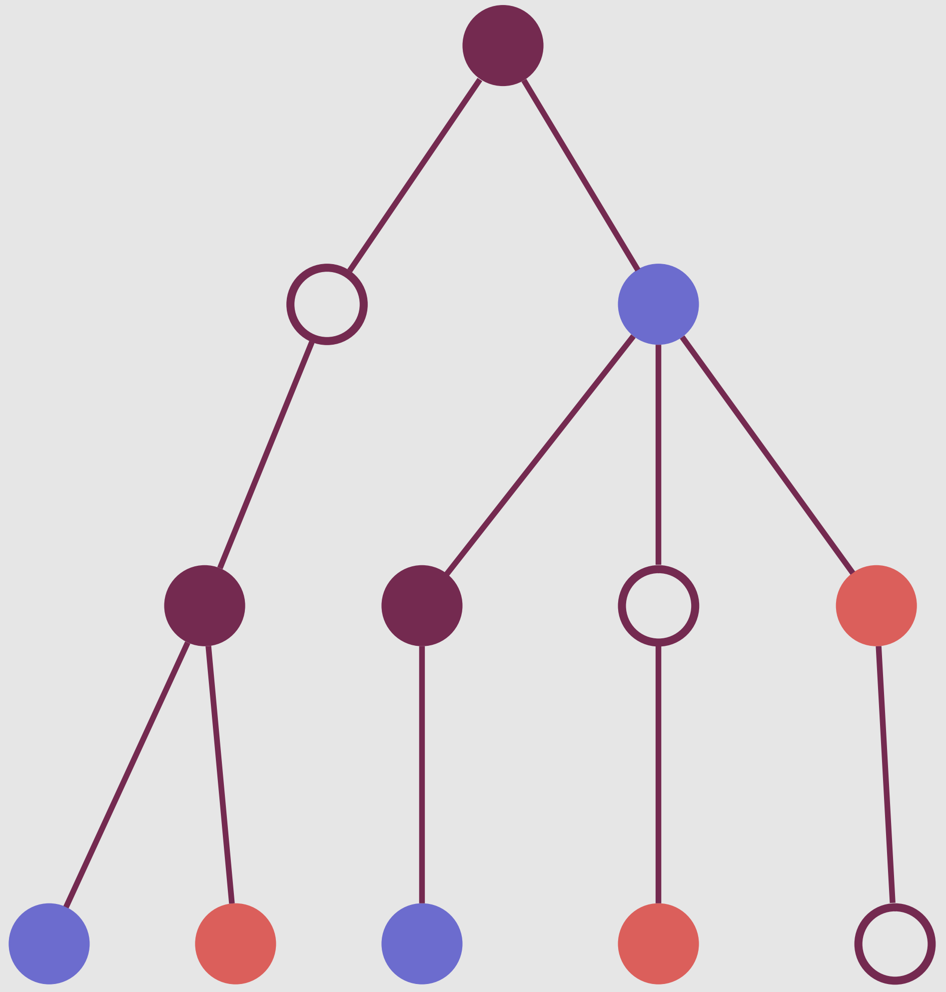

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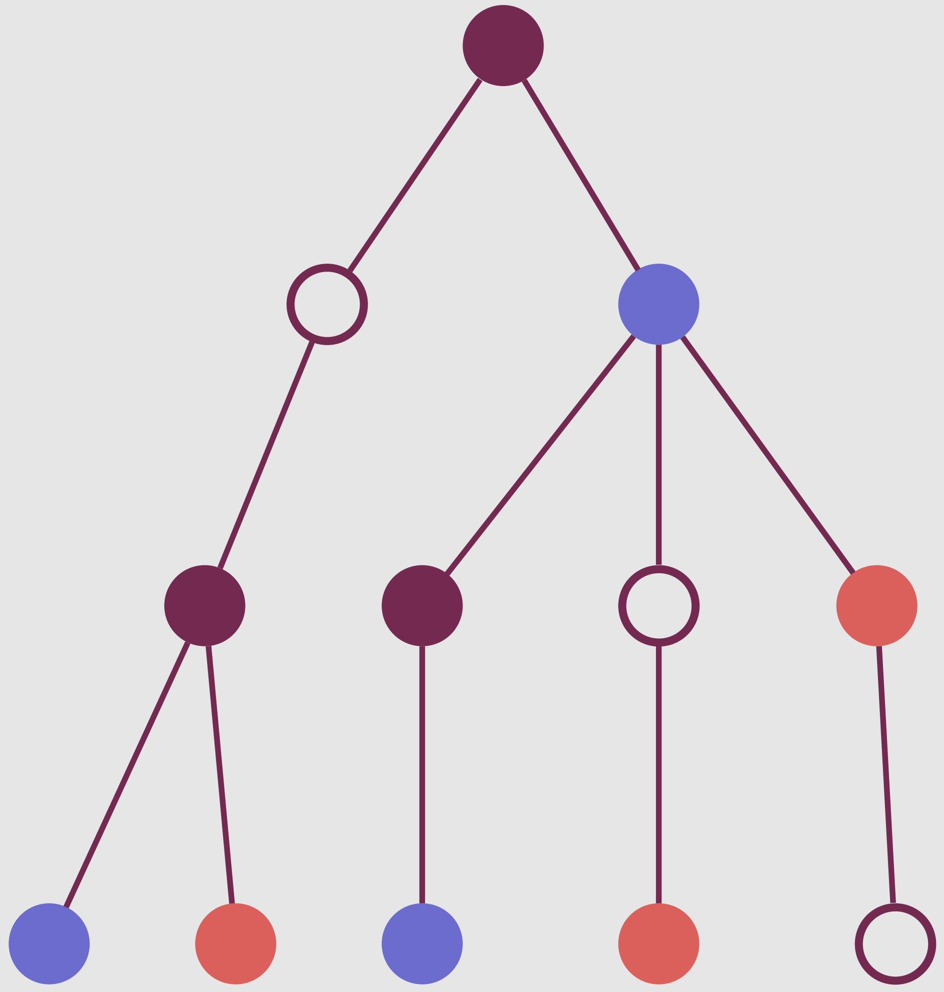
Thus we obtain survival if $p_{\text{inf}}\mathbb{E}[\zeta] > 1$



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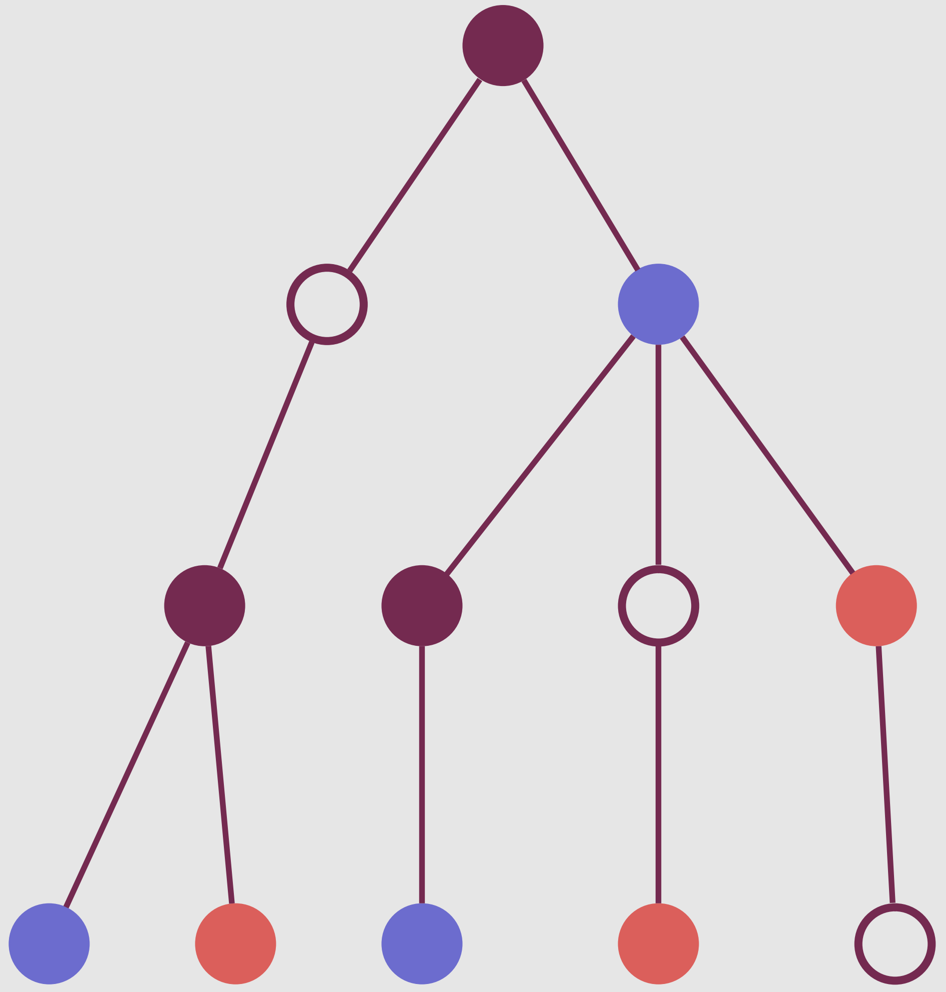
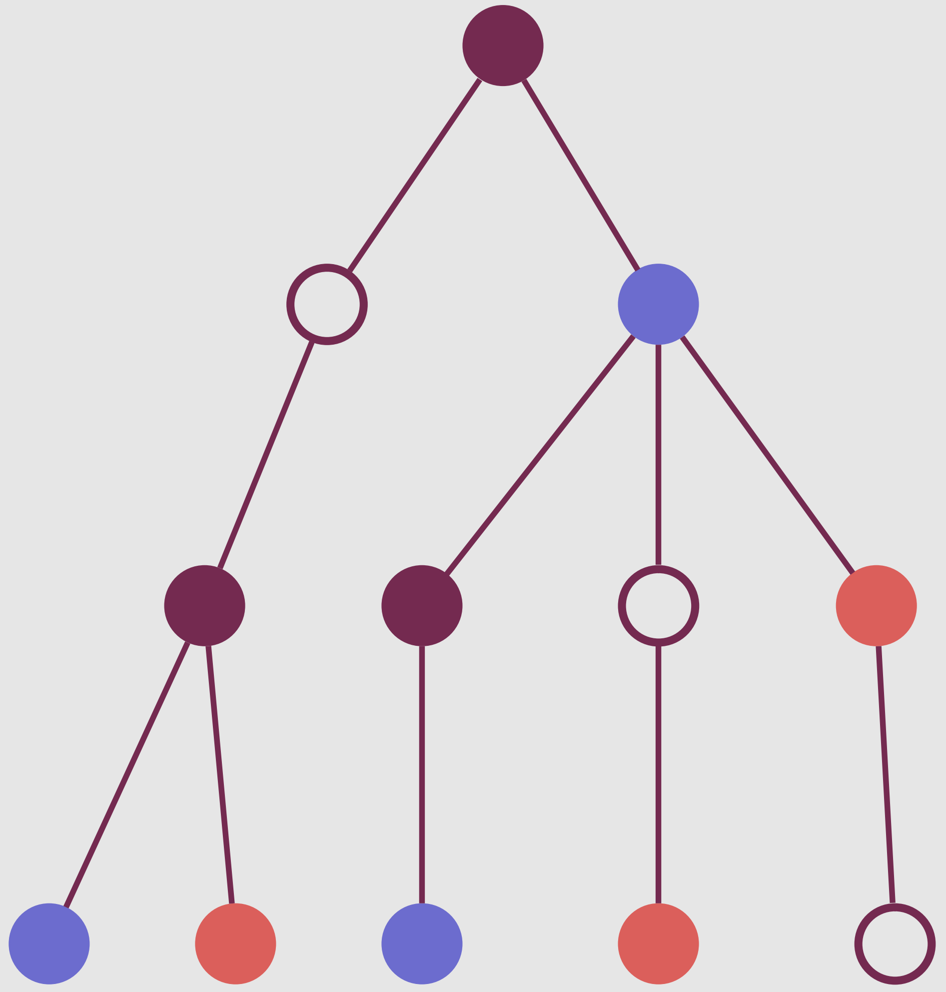


Proposition - Proof $\lambda_1(\mu) \leq \sqrt{C\mu}$

Setting $\lambda_1 = \sqrt{C\mu}$ we obtain:

$$p_{\text{inf}} = \frac{2C^2\mu^2}{2(C^2+2)\mu^2 + 2\mu(1+3\sqrt{C\mu})}$$

Which we can take arbitrarily close to 1 by choosing C large



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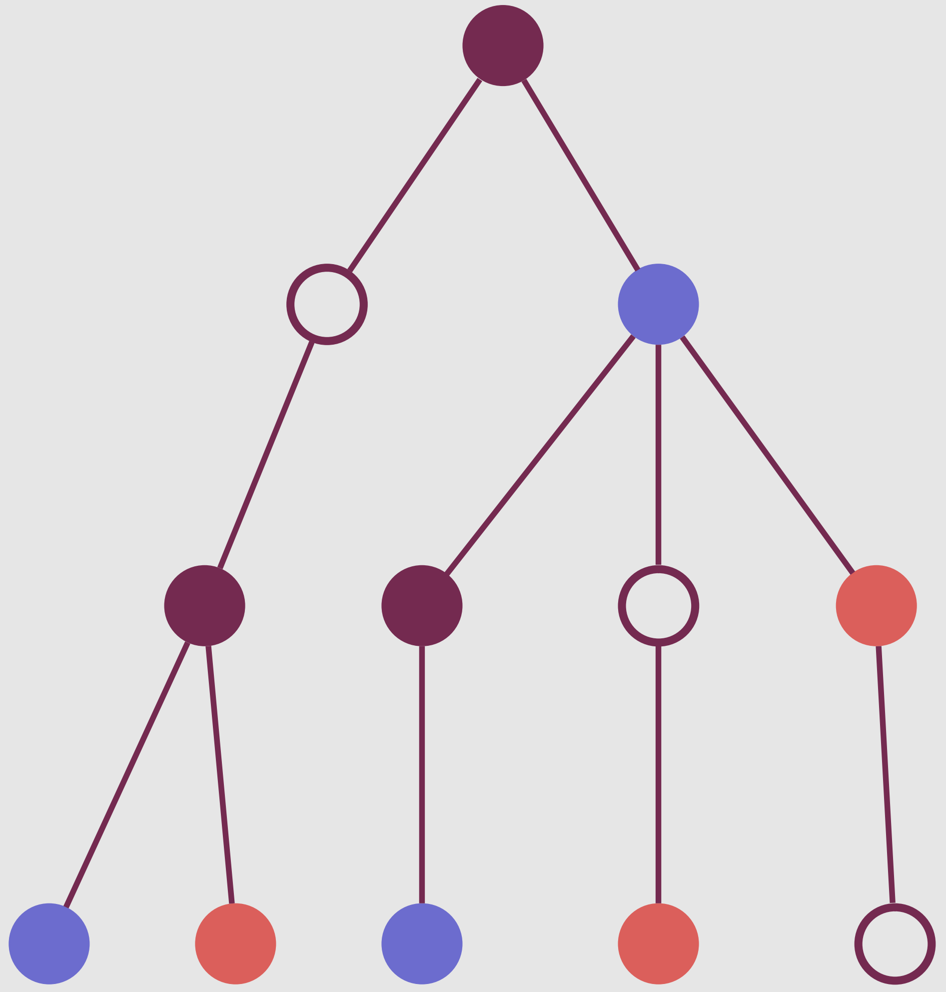
Noting that $\mathbb{E}[\zeta] > 1$ implies that we can find large C such that

$$p_{\text{inf}}\mathbb{E}[\zeta] > 1$$



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The main idea here is to use the survival time on a star with k leaves

$$S \sim e^{\frac{1}{10}\frac{\lambda^2}{\mu}k} \quad \text{for } \lambda < 1,\ \mu < \frac{1}{20}$$

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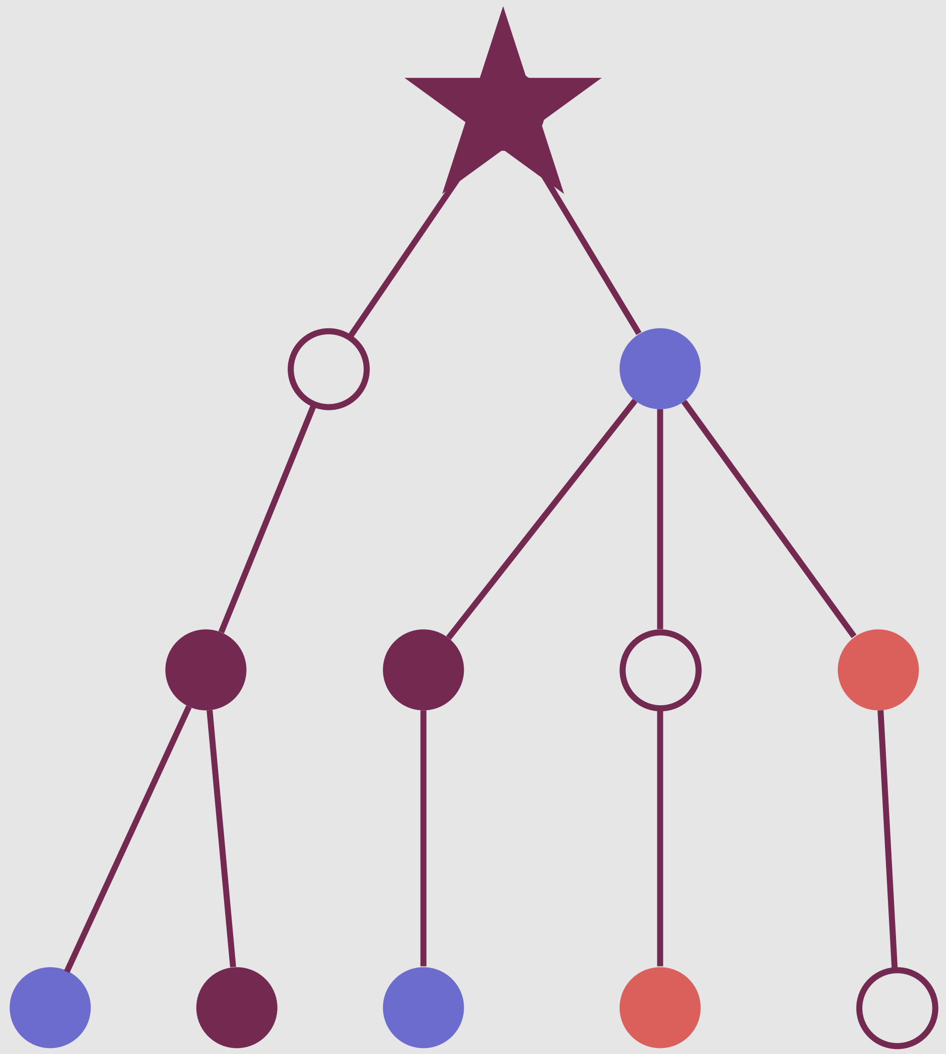

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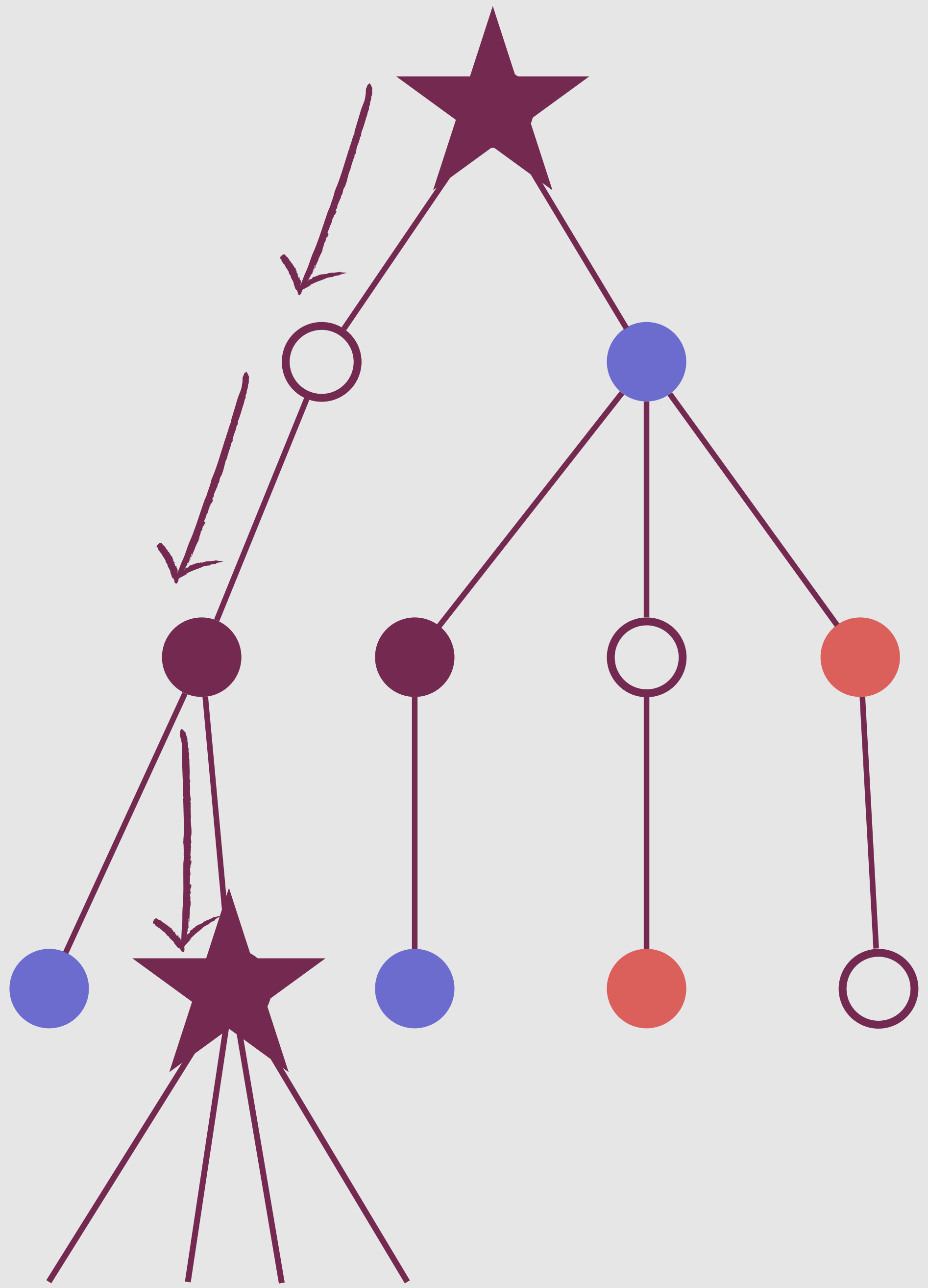

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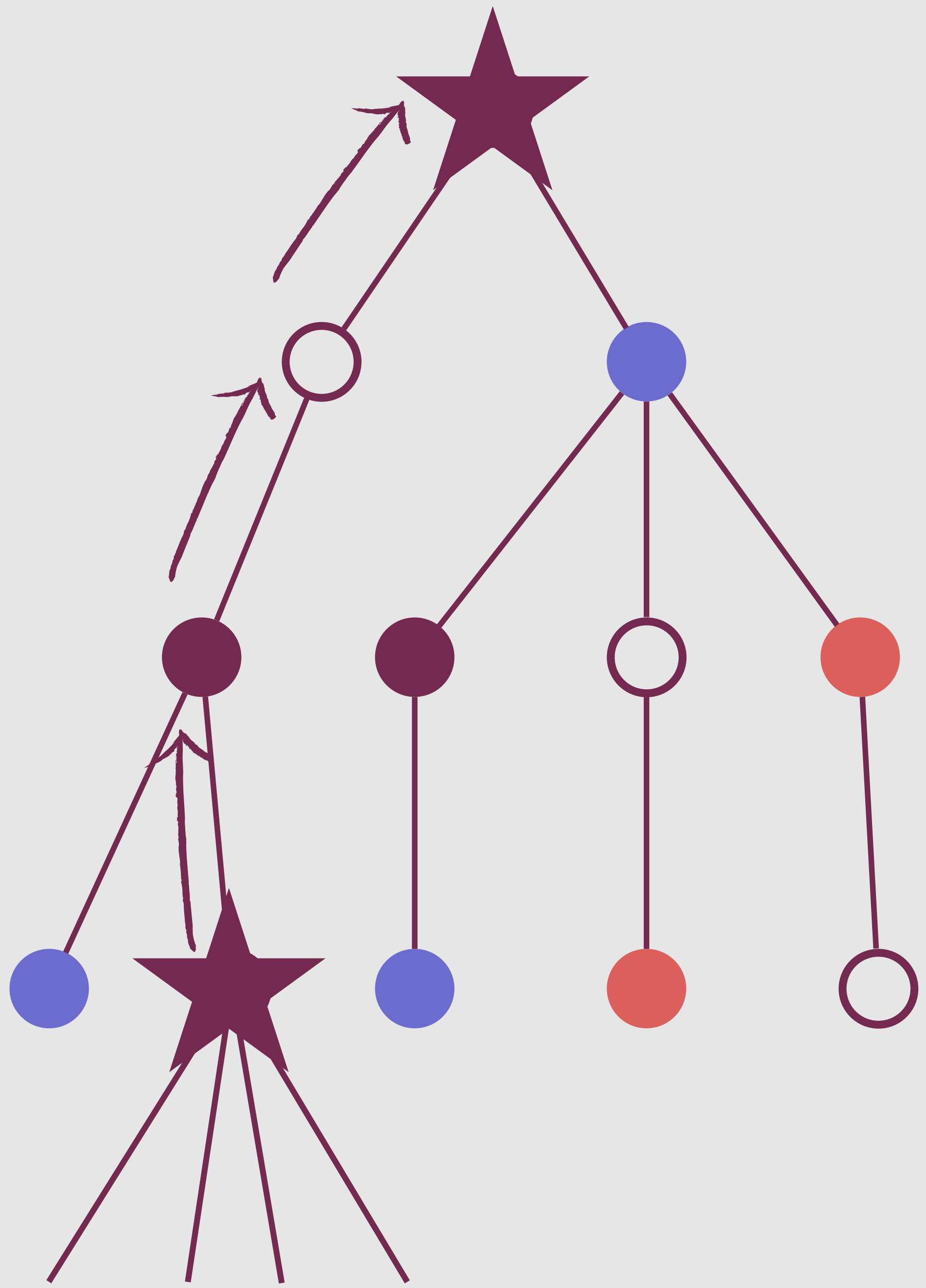

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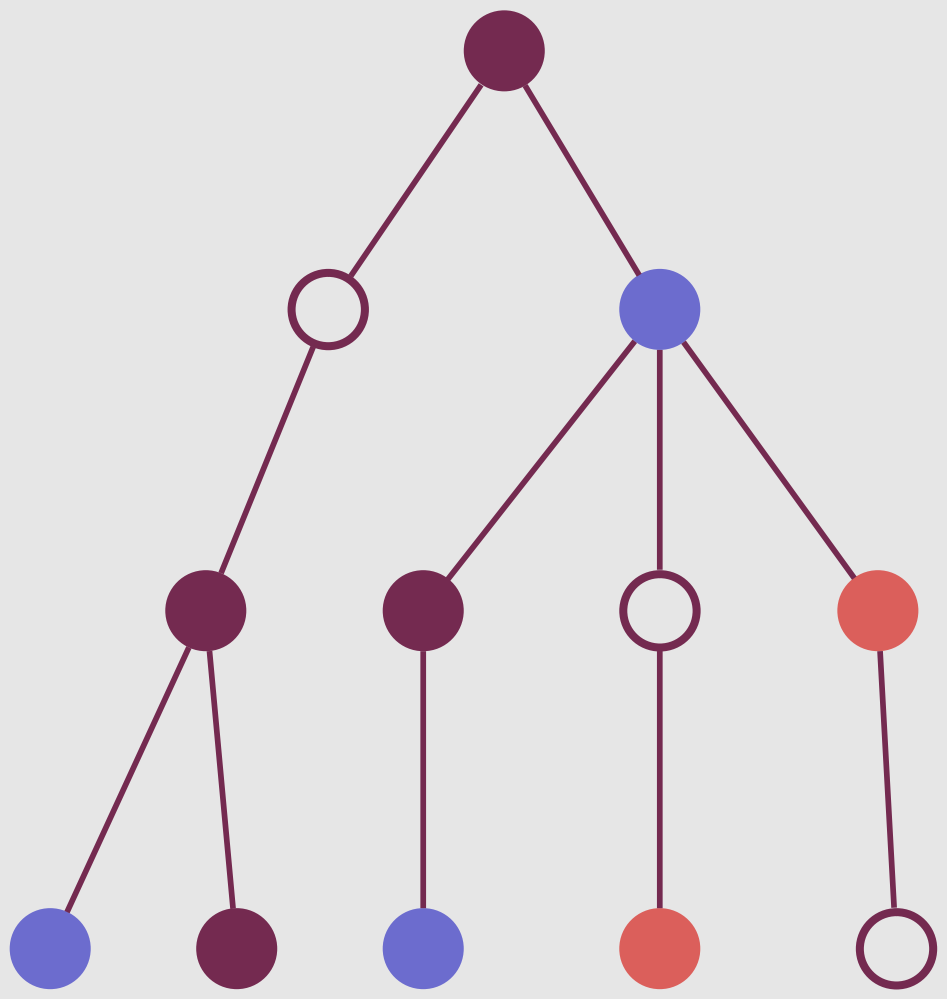

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It can be shown that strong survival occurs if

$$\log(p_{\text{inf}}) > -\frac{1}{r}\log(S)$$

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Using this method, we have been able to show that if $\mathbb{P}(\zeta = k) = e^{-\alpha k}$ then we have that there exists C such that for $\lambda < 1,\ \mu < \frac{1}{20}$

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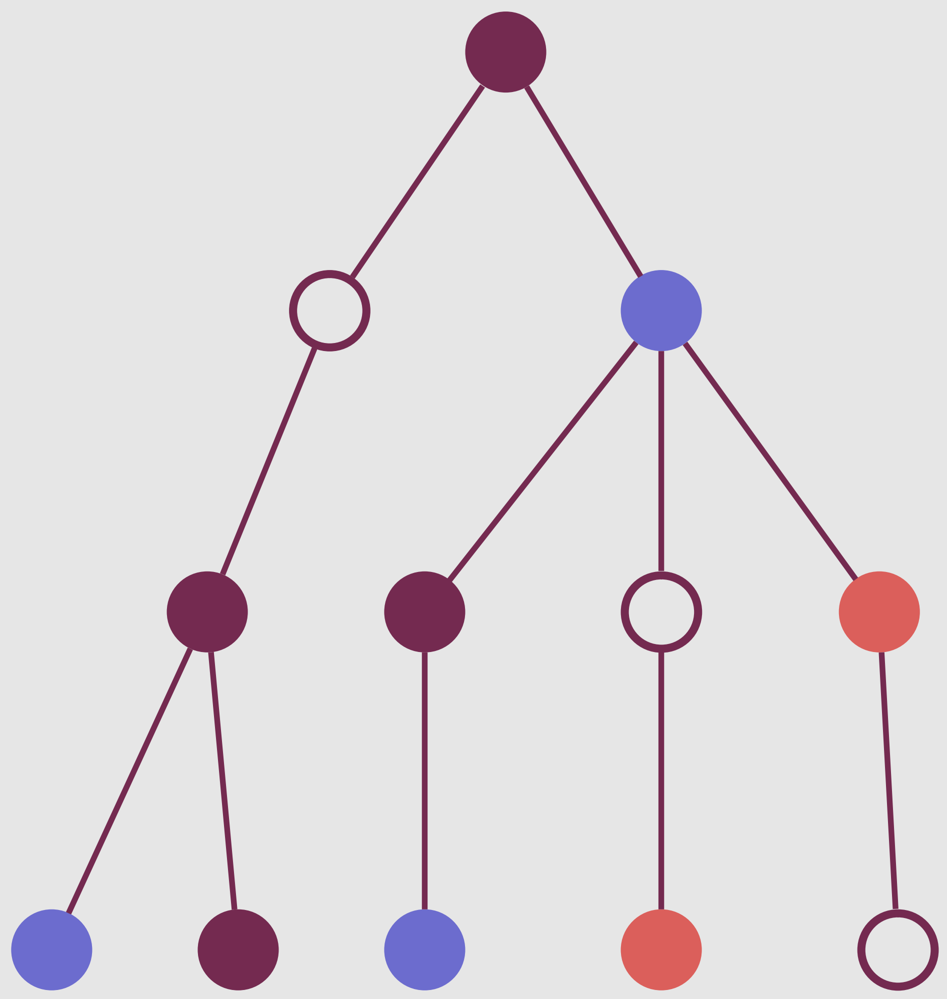
Comparison

We have a similar bound for $\lambda_1^{\mathbb{Z}}(\mu)$ which holds if $\mu < 1/1600$

$$\lambda_1^{\mathbb{Z}}(\mu) \leq 40\sqrt{\mu}$$

Durrett and Yao (2020)

Summary



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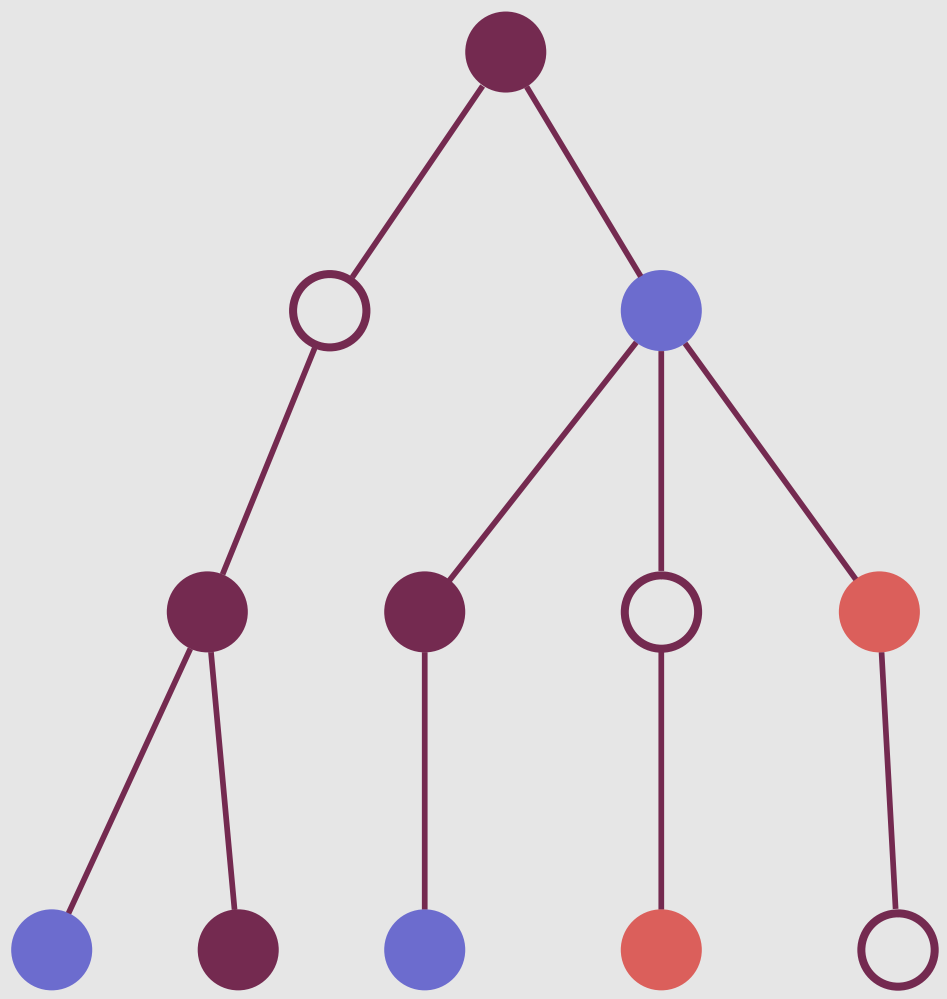

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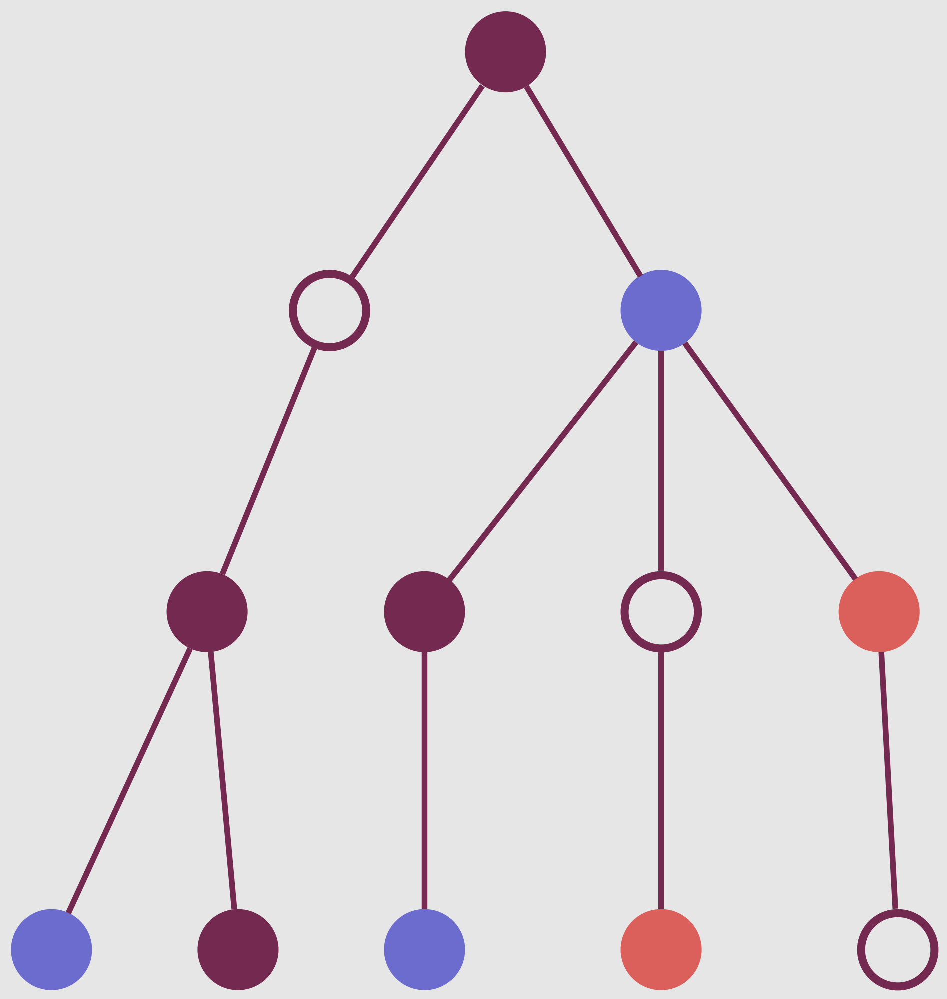

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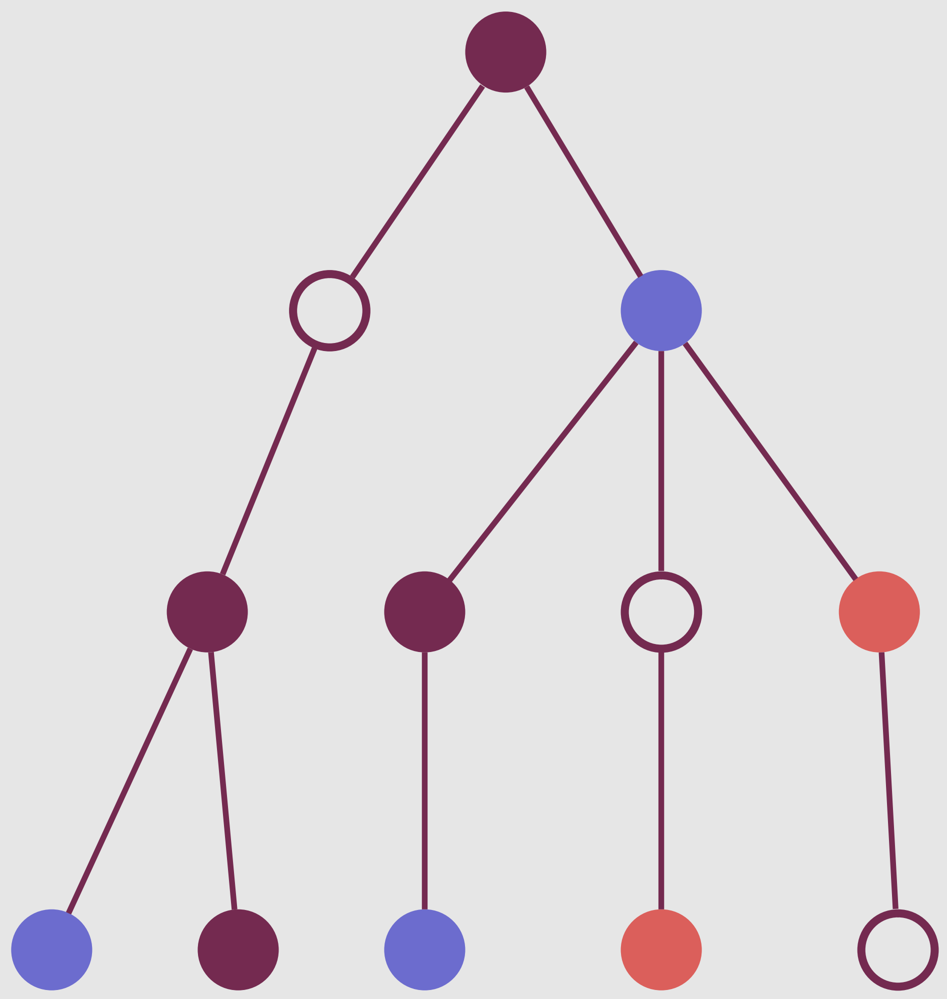

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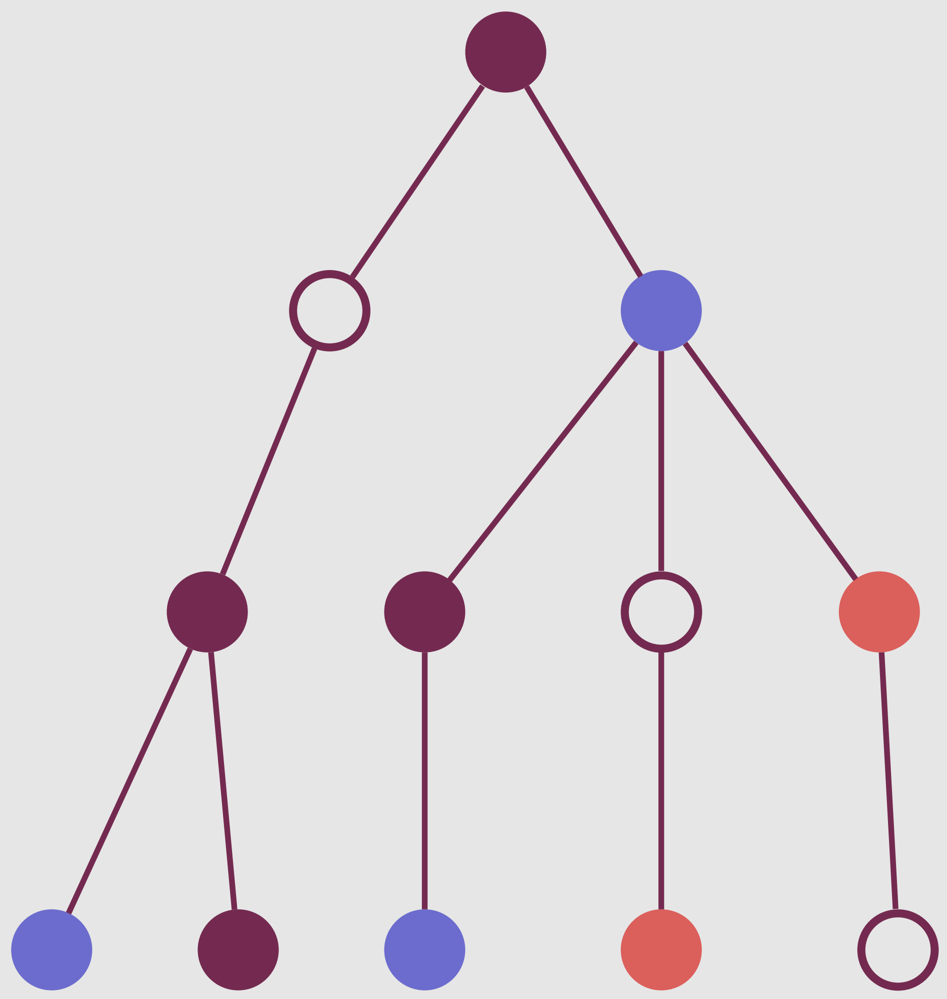

Any Questions?

References

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