Do all sensible Galerkin methods converge for the standard 2nd kind boundary integral equations on Lipschitz domains?

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Joint work with:
Euan Spence (Bath)

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Overview of the talk

1. **Lipschitz domains** - the setting for modern PDE theory
   - What are Lipschitz domains?
   - What is the Lipschitz character of a Lipschitz domain?
2. **Potential theory** and **2nd kind boundary integral equations** (BIEs)
   - A Dirichlet problem and 2nd kind BIE formulation
   - The Galerkin approximation to the BIE
   - **A long-standing open problem: do all "sensible" Galerkin methods converge?**
3. The **Hilbert space** theory of **Galerkin methods**
   - Definitions of bounded, compact, coercive
   - Galerkin methods and their convergence
4. **Do all sensible Galerkin methods converge for the standard 2nd kind BIEs?**
   - Previous results
   - **Solving the open problem**: Constructing Ω for which $A = I - D$ is not coercive + compact so not all sensible Galerkin methods converge

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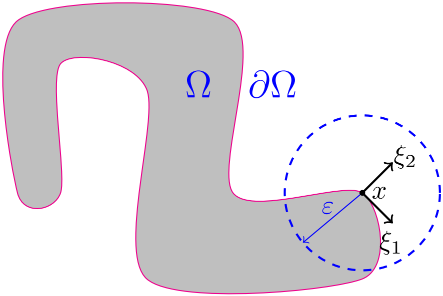
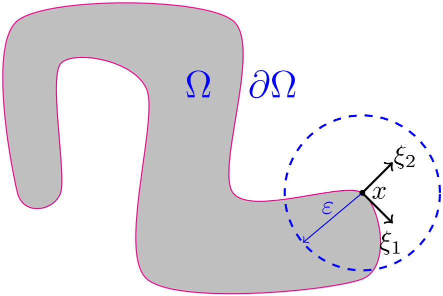


A bounded domain $\Omega \subset \mathbb{R}^2$ is **Lipschitz** if, in a neighbourhood of each point $x \in \partial\Omega$, $\partial\Omega$ is the graph of a **Lipschitz continuous function** f, with respect to some rotated coordinate system $0\xi_1\xi_2$, with Ω on precisely one side of $\partial\Omega$.



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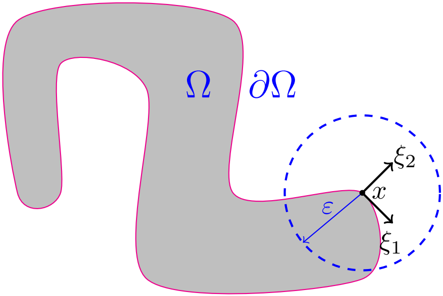
In equations,

$$\partial\Omega \cap B_\epsilon(x) = \{(\xi_1, f(\xi_1)) : \xi_1 \in \mathbb{R}\} \cap B_\epsilon(x),$$

for some f that satisfies, for some $L > 0$ (the **Lipschitz constant**)

$$|f(s) - f(t)| \leq L|s - t|, \quad \text{for } s, t \in \mathbb{R}. \quad (*)$$



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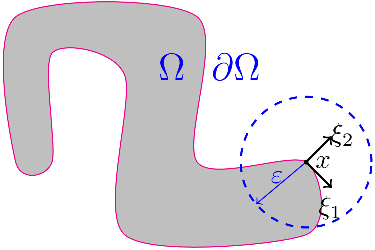
A Lipschitz continuous function is differentiable almost everywhere. Indeed, $(*)$ holds iff $|f'(s)| \leq L$, for almost all $s \in \mathbb{R}$.



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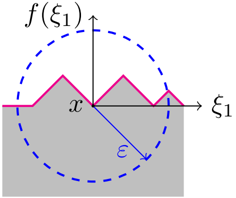
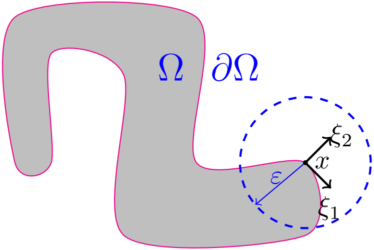
This **allows corners**, e.g. this f has $L = 1$ ...





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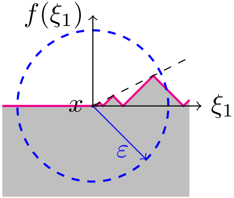
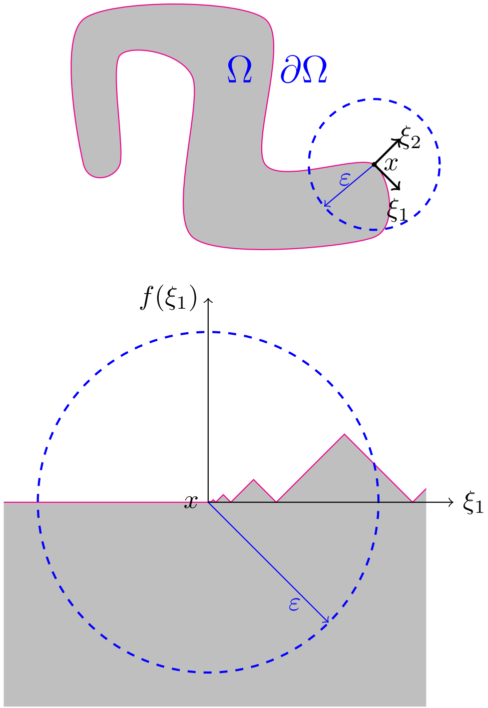
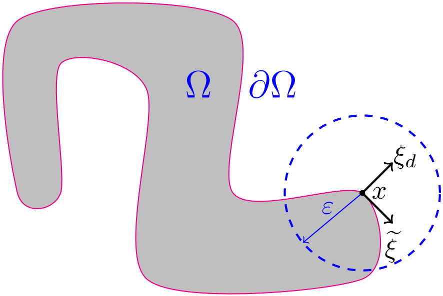
Indeed it allows **infinitely many corners**, e.g. this f also has $L = 1$ ...





Ω
∂Ω
ξ2
x
ε
ξ1
f(ξ1)
x
ξ1
ε



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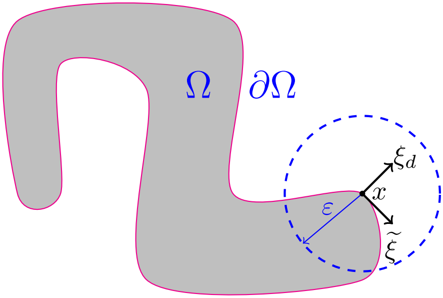

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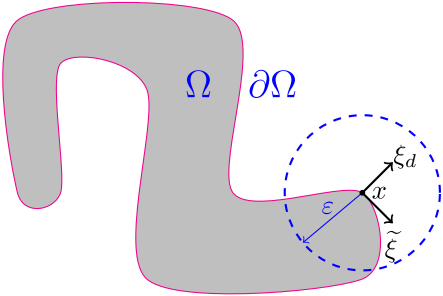

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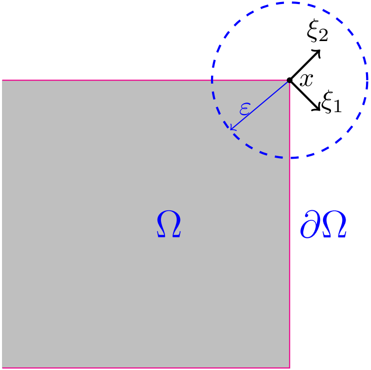
The **Lipschitz character** of Ω, char(Ω), is the infimum of the L such that for every $x \in \partial\Omega$ there exists $\varepsilon > 0$ and coordinates $0\xi_1\xi_2$ such that (X) and $(*)$ hold.



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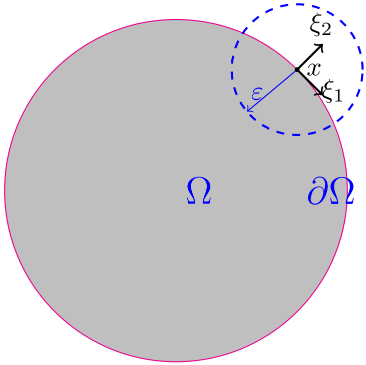
A **square** has **Lipschitz character** char$(\Omega) = 1$ (and a **cube** has char$(\Omega) = \sqrt{2}$).



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A **smooth (C^1) boundary** has **Lipschitz character** char(Ω) $= 0$.

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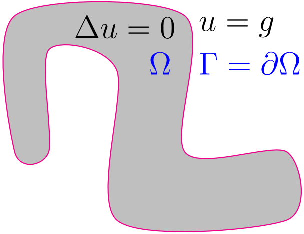
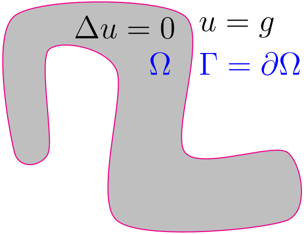


Assume that $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) is **bounded** and **Lipschitz**, and $g \in L^2(\Gamma)$.



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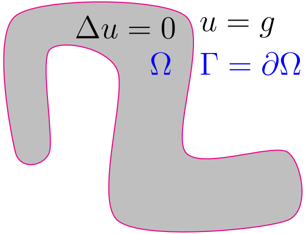
BVP: Find $u \in C^2(\Omega)$ such that $\Delta u = 0$ in Ω and $u = g$ on Γ.



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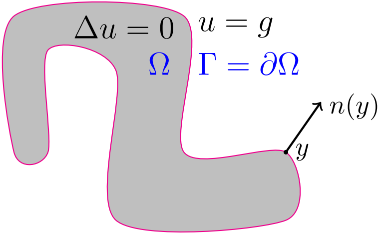
Define the **fundamental solution**

$$G(x,y) := \begin{cases} -\dfrac{1}{\pi}\log|x-y|, & d=2, \\ (2\pi|x-y|)^{-1}, & d=3, \end{cases}$$



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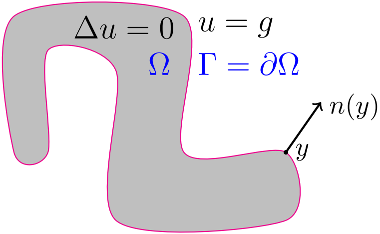
Look for a solution as the **double-layer potential** with density $\phi \in L^2(\Gamma)$ (which satisfies $\Delta u = 0$ in Ω):

$$u(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\,\mathrm{d}s(y), \quad x \in \Omega.$$



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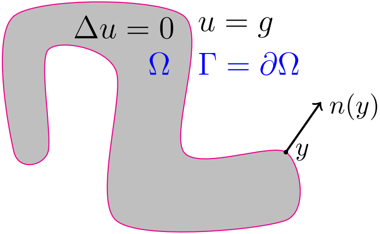
This satisfies the BVP iff ϕ satisfies the **boundary integral equation (BIE)**

$$\phi(x) - \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y) = -2g(x), \quad x \in \Gamma.$$



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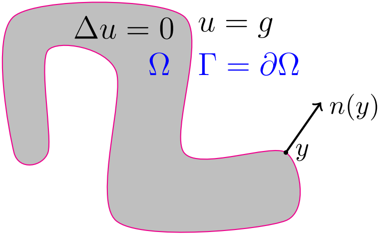
in operator form

$$\phi - D\phi = -2g \quad \text{or } A\phi = -2g,$$

where $A = I - D$, I is the identity operator, and D is the **double-layer potential operator** given by

$$D\phi(x) = \int_\Gamma \frac{\partial G(x,y)}{\partial n(y)} \phi(y)\, \mathrm{d}s(y), \quad x \in \Gamma,\ \phi \in L^2(\Gamma).$$



The double-layer potential satisfies the BVP iff ϕ satisfies the **BIE**

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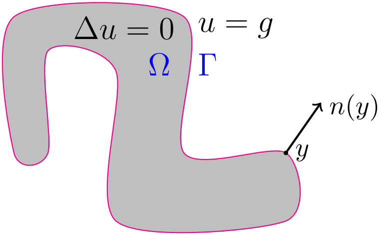
where $A = I - D$. The **Galerkin method** for solving the BIE numerically is: choose a basis $v_1, \ldots, v_N$ for a linear subspace V_N of $L^2(\Gamma)$ and approximate

$$\phi \approx \phi_N := \textstyle\sum_{n=1}^N \alpha_n v_n,$$

choosing the coefficients $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ so that

$$\int_\Gamma A\phi_N \overline{v_m}\, \mathrm{d}s = -2 \int_\Gamma g\overline{v_m}\, \mathrm{d}s, \quad m = 1, \ldots, N.$$



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Long-standing open problem. Do all sensible Galerkin methods converge?

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H is a **complex Hilbert space** with inner product (u, v) and norm $\|u\| = \sqrt{(u,u)}$, e.g.

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Suppose that A is a **bounded linear operator** on H, i.e.

$$A(\lambda u) = \lambda Au, \quad A(u + v) = Au + Av, \quad \forall \lambda \in C,\ u, v \in H,$$

and, for some $C \geq 0$,

$$\|Au\| \leq C\|u\|, \quad \forall u \in H.$$

The **norm** of A is

$$\|A\| := \sup_{u \in H \setminus \{0\}} \frac{\|Au\|}{\|u\|}.$$

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A is **compact** if, for some sequence of finite rank operators $A_1, A_2, ...$, it holds that $\|A - A_n\| \to 0$ as $n \to \infty$.

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E.g. if $A = I - B$, where I is the identity operator and B is bounded,

$$\begin{aligned} (Au,u) = (u - Bu, u) &= (u,u) - (Bu,u) \\ &= \|u\|^2 - (Bu,u) \\ &\geq \|u\|^2 - |(Bu,u)| \end{aligned}$$

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So $A = I - B$ is coercive if $\|B\| < 1$, with $\gamma = 1 - \|B\|$.

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If $\{v_1, ..., v_N\}$ is a basis for V_N, in which case $u_N = \sum_{n=1}^{N} \alpha_n v_n$, for some $\alpha_n \in \mathbb{C}$, then (G) is equivalent to

$$\sum_{n=1}^{N} (Av_n, v_m)\, \alpha_n = (g, v_m), \quad m = 1, ..., N.$$

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We will say that V **converges to** H if, for every $u \in H$,

$$\inf_{v \in V_N} \|u - v\| \to 0 \quad \text{as} \quad N \to \infty.$$

It is easy to see that a **necessary condition** for the convergence of the Galerkin method is that V converges to H.

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Part a). If A is invertible then there exists a sequence $V = (V_1, V_2, ...)$ for which the Galerkin method converges.

This is interesting theoretically, but not helpful for computation.

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This is almost as strong a result as Part b), with weaker requirements on A.

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What is known about the double-layer potential operator D and $A = I - D$ when Ω is Lipschitz? Remember the BIE in operator form is $A\phi = -2g$.

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The answer is **NO** in each case (C-W & Euan 2018), so no guarantee of convergence of Galerkin methods.

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Here $W_{\text{ess}}(A)$ denotes the **essential numerical range** of A, defined by

$$W_{\text{ess}}(A) := \bigcap_{K \text{ compact}} \overline{W(A+K)},$$

where, for a bounded linear operator B, $W(B)$ denotes the **numerical range** or **field of values** of B, given by

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If $A = I - D$ and D is the double-layer potential operator, is $0 \in W_{\text{ess}}(A)$? Equivalently, is $1 \in W_{\text{ess}}(D)$?

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For a bounded linear operator B on H

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Theorem. $W(B)$ is **convex** and $\mathrm{spec}(B) \subset \overline{W(B)}$, so the **convex hull**

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Special case. If $H = \mathbb{C}^N$ with norm $\|u\| = \sqrt{\sum_{m=1}^{N} |u_n|^2}$ and B is an $N \times N$ matrix, then $W(B) = \overline{W(B)}$ and $\mathrm{spec}(B) = \{$eigenvalues of $B\} \subset W(B)$.

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Theorem. $W(B)$ is **convex** and $\mathrm{spec}(B) \subset \overline{W(B)}$, so the **convex hull**

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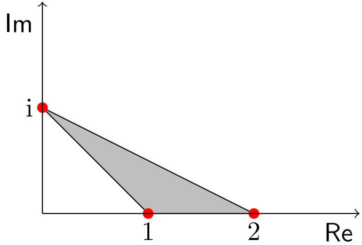

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$$W(D) = \{(D\phi, \phi) : \phi \in L^2(\Gamma),\ \|\phi\| = 1\}, \quad W_{\mathrm{ess}}(D) = \bigcap_{K \text{ compact}} \overline{W(D+K)}.$$

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Thus $1 \notin W_{\mathrm{ess}}(D)$ and $A = I - D =$ coercive + compact if $\mathrm{char}(\Omega)$ is small enough.

Proof. Uses localisation arguments plus standard bounds for the norm of D on Lipschitz graphs.

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A couple of simple lemmas.

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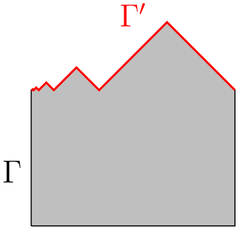

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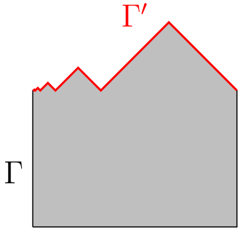

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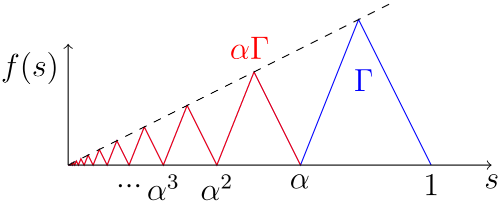

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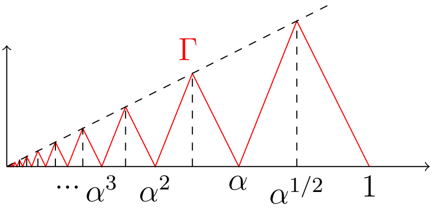

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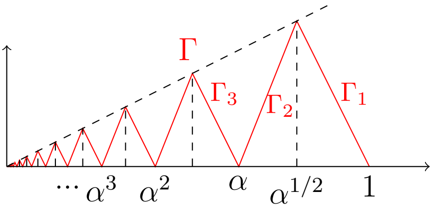

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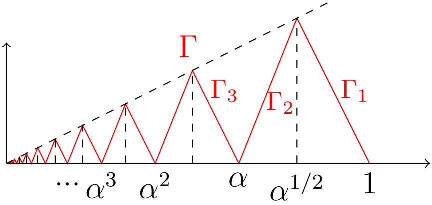

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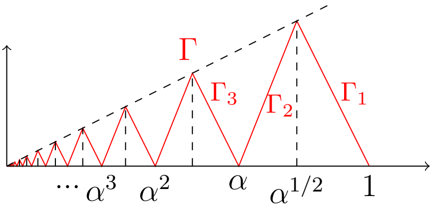

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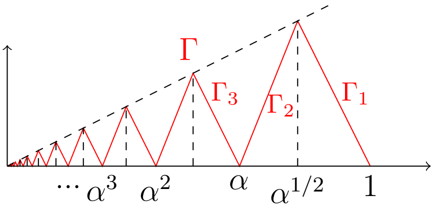

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as $\alpha \to 1^-$. **So every neighbourhood of $W(D)$ contains $W(A_N)$ if α is close enough to 1.**

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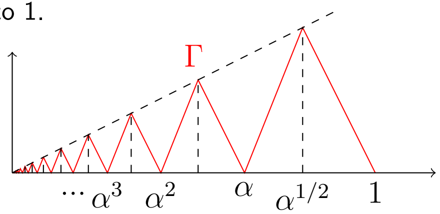

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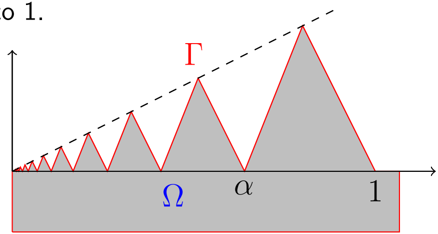

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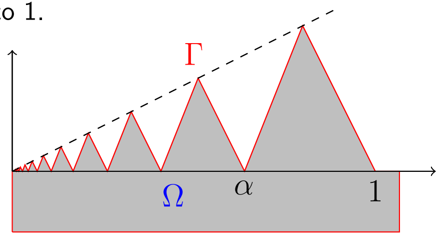

Corollary. For this domain Ω, $A = I - D$ is not coercive + compact if α is close enough to 1.

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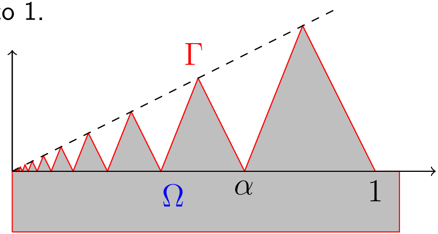

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Summary of the talk

1. **Lipschitz domains** - the setting for modern PDE theory
   - What are Lipschitz domains?
   - What is the Lipschitz character of a Lipschitz domain?
2. **Potential theory** and **2nd kind boundary integral equations** (BIEs)
   - A Dirichlet problem and 2nd kind BIE formulation
   - The Galerkin approximation to the BIE
   - **A long-standing open problem: do all "sensible" Galerkin methods converge?**
3. The **Hilbert space** theory of **Galerkin methods**
   - Definitions of bounded, compact, coercive
   - Galerkin methods and their convergence
4. **Do all sensible Galerkin methods converge for the standard 2nd kind BIEs?**
   - Previous results
   - **Solving the open problem**: Constructing Ω for which $A = I - D$ is not coercive + compact so not all sensible Galerkin methods converge