The Role of Density Dependent Diffusion in Reaction-Diffusion Systems

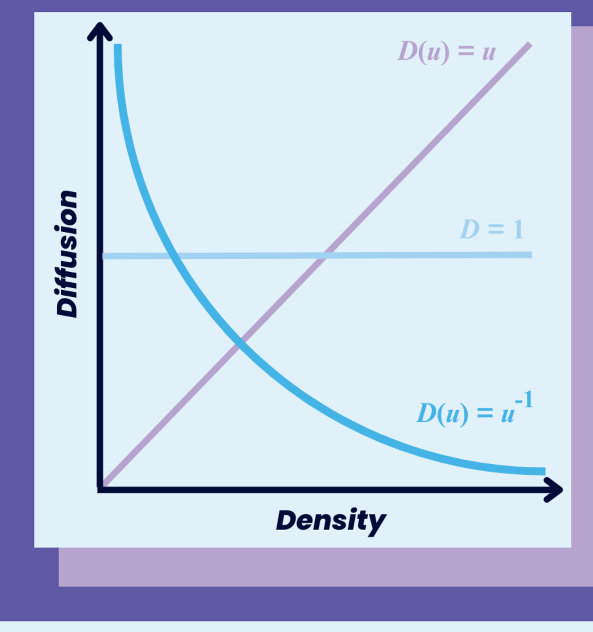
Background

For a population with density $u(x,t)$, the general form of a reaction-diffusion equation with density dependent diffusion and logistic growth is given by:

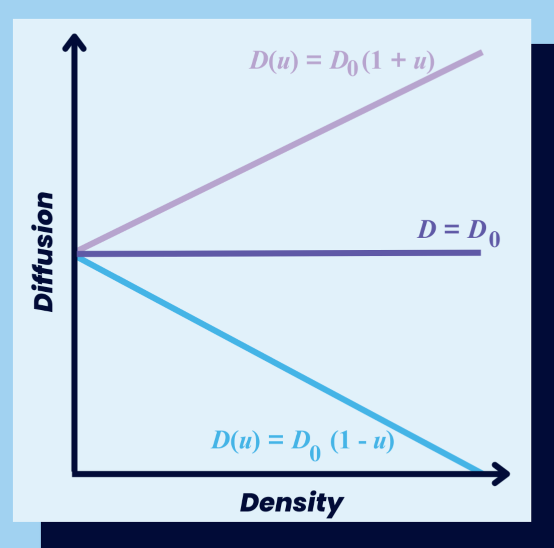
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + ru(1-u)$$

This reduces to the FKPP when both $D(u)$ and r are constant with value 1 [1,2], which has been extensively studied due to it's biological interpretation as a weakly invasive species.

Positive and negative density dependent diffusion in specific forms have also been studied [3,4,5], but these often lead to solutions which have unrealistic biological interpretations.



Motivation



Despite the limitations of some cases, density dependent diffusion still provides us with a greater understanding of real-world population invasion and dispersal.

Positive density dependent diffusion is often thought of as a populations response to overcrowding, whereas negative density dependent diffusion can be a useful interpretation in the context of mate searching and pursuit behaviour.

We investigate **non-degenerate** cases of both positive and negative density dependent diffusion. In particular, the effects they have on the **speed** and **shape** of the front, and on the underlying dynamics.

This is with the aim of developing a deeper mathematical understanding of biologically realistic reaction-diffusion systems and hence provide an enhanced explanatory power of the real-world systems they describe.

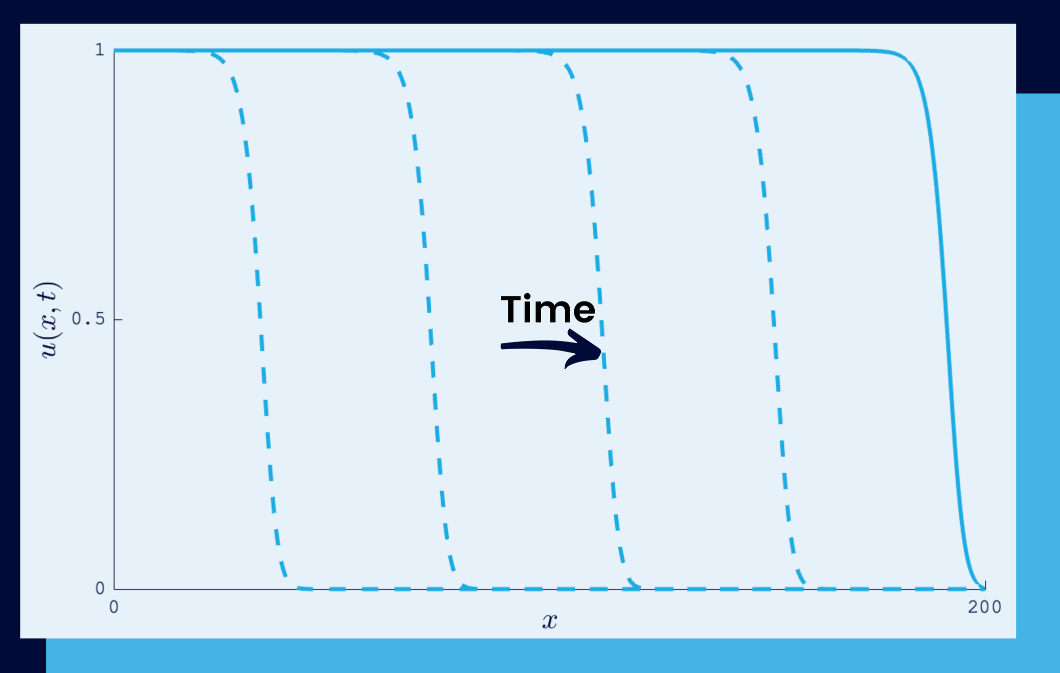
Speed of a Travelling Wave

A travelling wave solution is one which travels without change of shape, and therefore propagates at a constant speed [6].

It is well-known that for constant diffusion, this constant travelling wavespeed is given by:

$$c^* = 2\sqrt{rD}$$

Which reduces to a minimum stable wavespeed of $c^* = 2$ in the case of the FKPP [6].



Assuming that the travelling wave solution with density dependent diffusion is also of the pulled type [7], we can obtain an equation for their speed similarly, from the **linearised** version of the PDE:

$$D(0)\frac{d^2U}{dz^2} + c\frac{dU}{dz} + rU = 0$$

Following the same stability arguments used in [6] for the case of constant diffusion, there exists a family of travelling wave solutions with wavespeeds:

$$c \geq 2\sqrt{rD(0)}$$

It remains to determine if the stable wavespeed is also the minimum as it is in the case of constant diffusion.

This result indicates that the non-linearity in the diffusion rates are as irrelevant as the non-linearity in the growth rates when determining speed and long-time behaviour.

It further highlights why the degenerate cases of density dependent diffusion given above are biologically unrealistic.

Link to Microscopic Models

Starting from a discrete description of the model, we can formulate a chemical master equation to describe the evolution of the joint probability of the number of particles in each cell. Diffusive transition rates are made up of both local and non-local contributions. They can be:

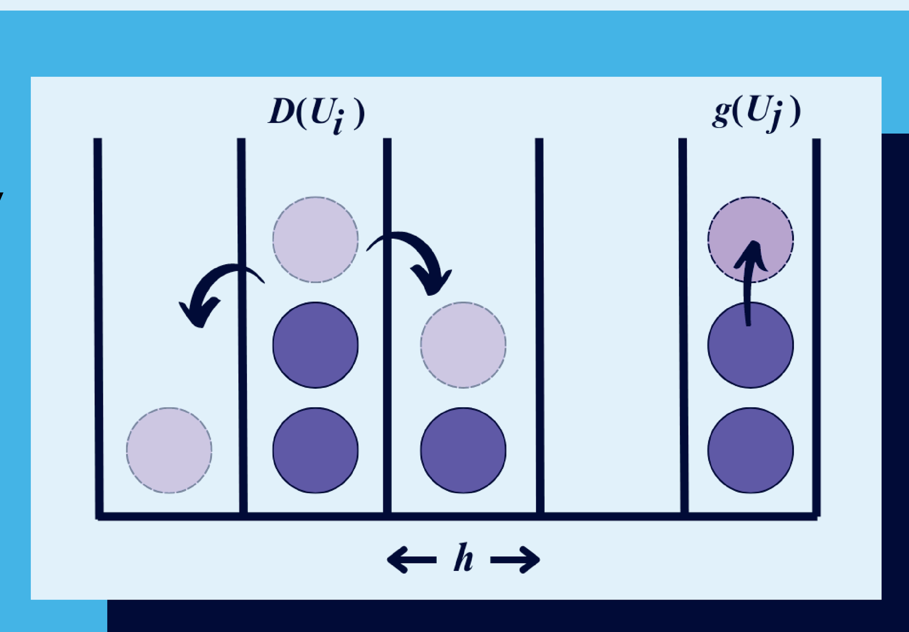
additive or multiplicative

$$T_i^{\pm} = f(n_i) + g(n_{i\pm1}) \quad \text{or} \quad T_i^{\pm} = f(n_i)g(n_{i\pm1})$$

Following the steps outlined in [8,9], in either case we can derive the continuous PDE description of the system via a mean-field approximation.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(f(u) + g(u) + uf'(u) - ug'(u))\frac{\partial u}{\partial x}\right] + ru(1-u)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(f(u)g(u) + uf'(u)g(u) - uf(u)g'(u))\frac{\partial u}{\partial x}\right] + ru(1-u)$$



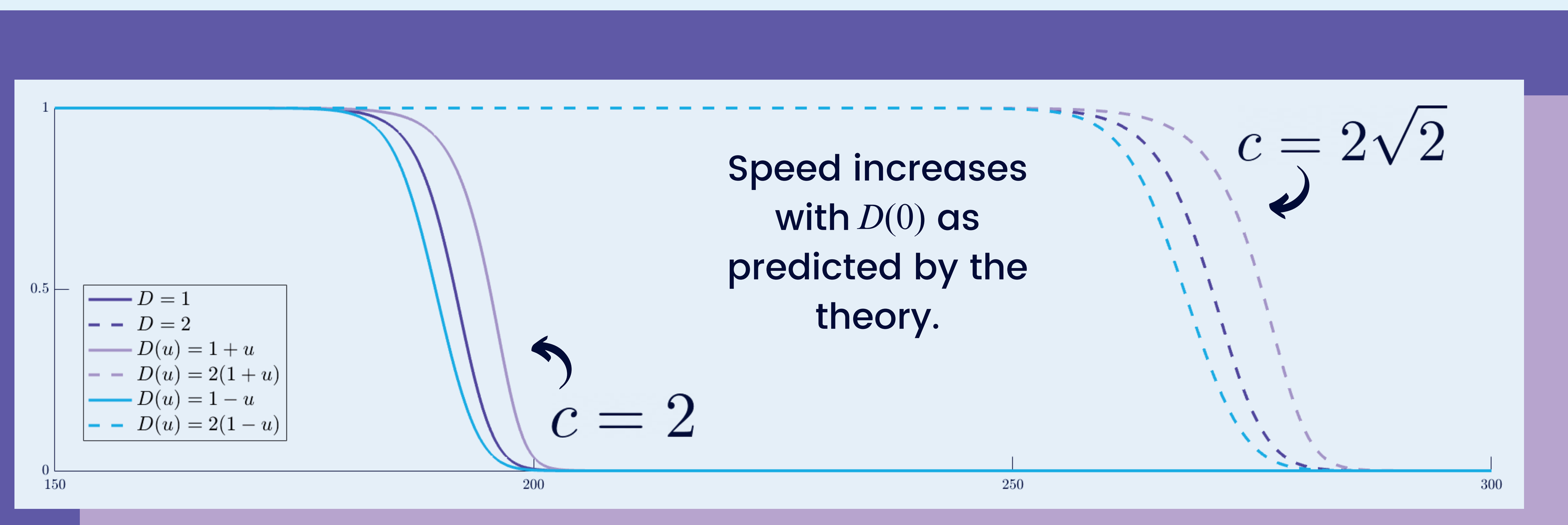
This result indicates that **we cannot simply view density dependent movement rates to be the same as density dependent diffusion.**

Further steps are required to reconcile the movement rates in the discrete system with a single density dependent diffusion function $D(u)$. We have two simplifying cases:

1. Assume equal local and non-local contributions, then in the additive and multiplicative cases respectively, we require $D(u) = \frac{1}{2}f(u)$ or $D(u) = f^2(u)$
2. Consider only local contributions which, in either case results in the relation given below. This can be solved for well-behaved, integrable functions.

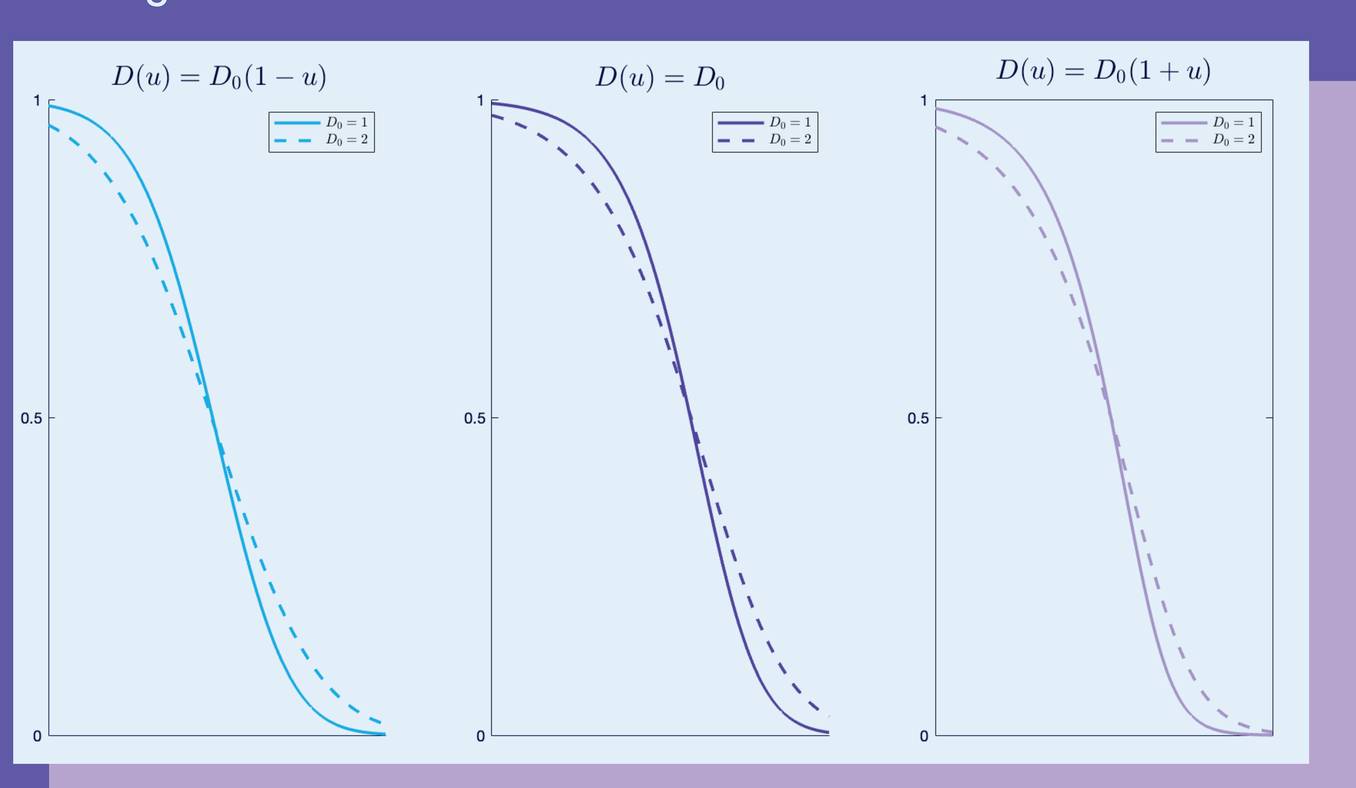
$$D(u) = f(u) + uf'(u)$$

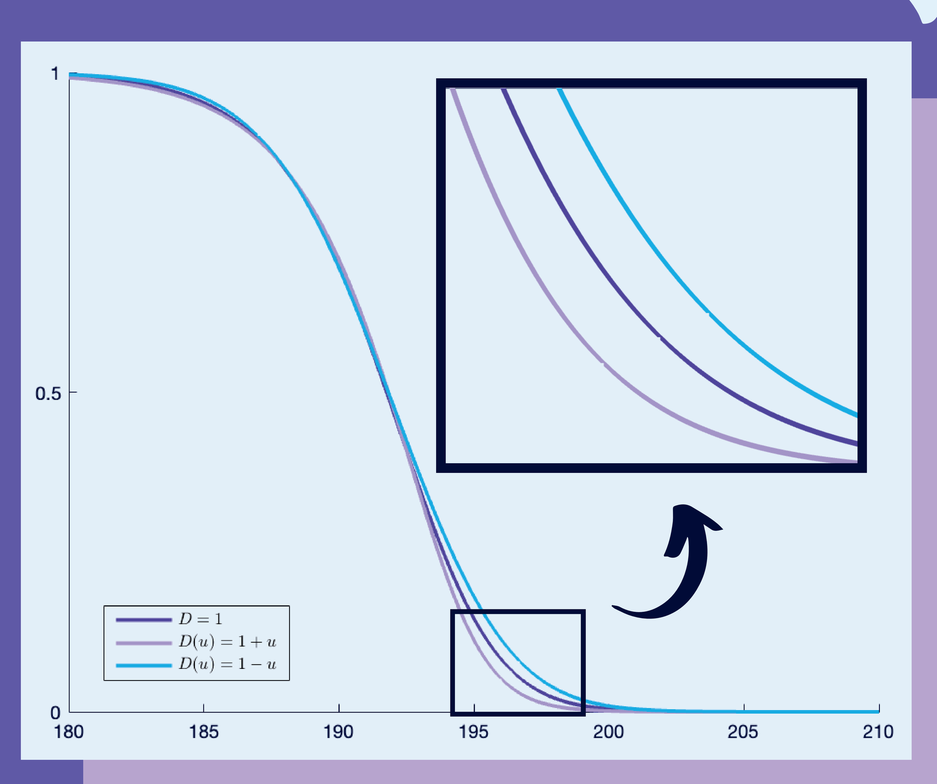
Results



$D(0)$ alone can affect the speed and hence, the shape of the travelling wave.

Negative density dependent diffusion leads to higher diffusion in low density regions.





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