Collective Opinion Dynamics

Extremism, segregation and oscillatory states emerge in a novel agent-based model

Beth M. Stokes

bms58@bath.ac.uk
Department of Mathematical Sciences, University of Bath, UK

Tuesday 4th April 2023

University of Bristol

University of Bath

UWE Bristol University of the West of England

Co-authors and Acknowledgments

School of Mathematics,
University of Birmingham

The British Academy
SRG1920_101649

SAMBa CDT,
University of Bath

Samuel Jackson
(University of
Durham)

Galane J. Luo
(University of
Birmingham)

Philip Garnett
(University of
York)

Introduction

The aim: to build an agent-based model that provides insights into consensus formation in various contexts, synthesising several theoretical frameworks and introducing novel concepts.

The model should prescribe minimal rules for pairwise interactions and produce many emergent macroscopic behaviours including consensus, segregation and extremisation.

The model and analysis
1. Mathematical formulation

Let $D \in \{1, 2, 3, \dots\}$ and $N \in \{2, 3, 4, \dots\}$.
For all $i = 1, 2, 3, \dots, N$ and all $t = 0, 1, 2, \dots$, let $\mathbf{v}_i(t) \in \mathbb{R}^D$ be the D-dimensional 'opinion' of agent i at time t.

Every agent has a threshold, $\rho_i(t) \in [0, 1)$.

Every pair of agents shares an affinity, $a_{ij}(t) \in (0, 1]$, which measures the history of differences between i and j, and is parametrised by a memory capacity $\mu \in \{1, 2, 3, \cdots\}$. Inspired by bird-flock dynamics:[1]

$$a_{ij}(t; \mu) = \frac{1}{\left(1 + \sum_{\tau=\max\{t-\mu+1,0\}}^{t} \left\| \mathbf{v}_j(\tau) - \mathbf{v}_i(\tau) \right\|^2\right)^{1/2}},$$

At every t, agent i tries to align with agent j iff $a_{ij}(t) > \rho_i(t)$.

[1] Cucker, F., & Smale, S. (2007). IEEE Transactions on Automatic Control, 52(5), 852–862.

The model and analysis

1. Mathematical formulation

The evolution rule for $\mathbf{v}_i(t)$, inspired by bounded-confidence models and memory-based connectivity models:[2][3]

$$\mathbf{v}_i(t+1) = \mathbf{v}_i(t) + \frac{\sum_{j=1}^{N} H(a_{ij}(t) - \rho_i(t))\ a_{ij}(t)\left(\mathbf{v}_j(t) - \mathbf{v}_i(t)\right)}{\sum_{j=1}^{N} H(a_{ij}(t) - \rho_i(t))} \ ,$$

where H is the Heaviside step function with $H(0) = 0$.

Equivalently, defining $\boldsymbol{V} \in \mathbb{R}^{N \times D}$ with $v_{ij} = [\mathbf{v}_i]_j$,

$$\boldsymbol{V}(t+1) = \boldsymbol{M}(t)\boldsymbol{V}(t), \quad m_{ii} = 1 - \frac{\sum_{j \neq i} H(a_{ij} - \rho_i)a_{ij}}{\sum_{j=1}^{N} H(a_{ij} - \rho_i)} > 0,$$

$$m_{ij} = \frac{H(a_{ij} - \rho_i)a_{ij}}{\sum_{j=1}^{N} H(a_{ij} - \rho_i)} \geq 0 \quad (i \neq j).$$

[2] Hegselmann, R., & Krause, U. (2002). Journal of Artificial Societies and Social Simulation, 5(3).

[3] Mariano, S., Morărescu, I., Postoyan, R., & Zaccarian, L. (2020). IEEE Control Systems Letters, 4(3), 644–649.

The model and analysis

2. Sufficient conditions for convergence and for consensus

Theorem[4]

Let $\rho_i(t) = \rho$ be some constant which we call the *universal threshold*. Let $\boldsymbol{V}(0) \in \mathbb{R}^{N\times D}$ be any initial condition. Let $\boldsymbol{V}(t)$ evolve according to the system described above.

1. The opinions converge to some steady state:

$$\exists \boldsymbol{V}^\infty \in \mathbb{R}^{N\times D} : \lim_{t\to\infty} \boldsymbol{V}(t) = \boldsymbol{V}^\infty.$$

2. If $\rho < \rho_* := \frac{1}{(1+4\mu R_0^2)^{1/2}}$ where $R_0 := \max_i \|\mathbf{v}_i(0)\|$, then the opinions converge to average consensus:

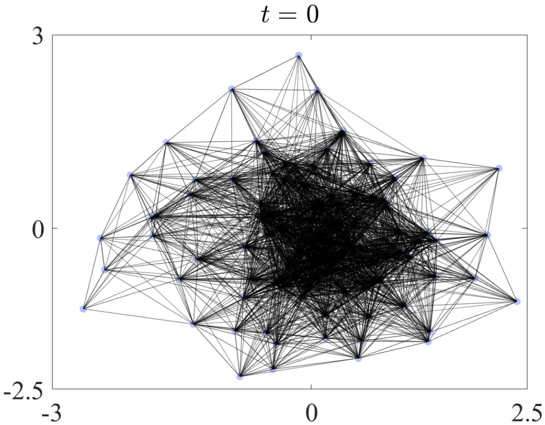
$$\forall i : \lim_{t\to\infty} \mathbf{v}_i(t) = \mathbf{v}_* := \frac{1}{N}\sum_{i=1}^{N} \mathbf{v}_i(0).$$

The dynamics can be interpreted as cooperation-seeking.[5]

[4] The proof builds on a result by Lorenz, J. (2005). Physica A: 355(1), 217–223.
[5] Rand, D. G., Arbesman, S., & Christakis, N. A. (2011). Proceedings of the National Academy of Sciences, 108(48), 19193–19198.

The model and analysis

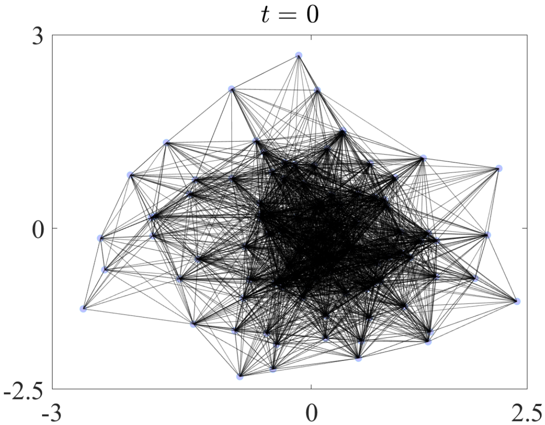
3. Long vs. short memory



With $\mu = 10$, agents 'hold grudges' and find it hard to listen to others.

The model and analysis

3. Long vs. short memory



With $\mu = 2$, agents 'forgive and forget' and manage to find consensus.

Numerical experiments
1. Methodology

Either $\rho_i(t) = \rho$, or more generally,

$$\rho_i(t) = \rho + (1-\rho)\left(1 - e^{-\alpha\|\mathbf{v}_i(t)\|}\right) \in [0,1),$$

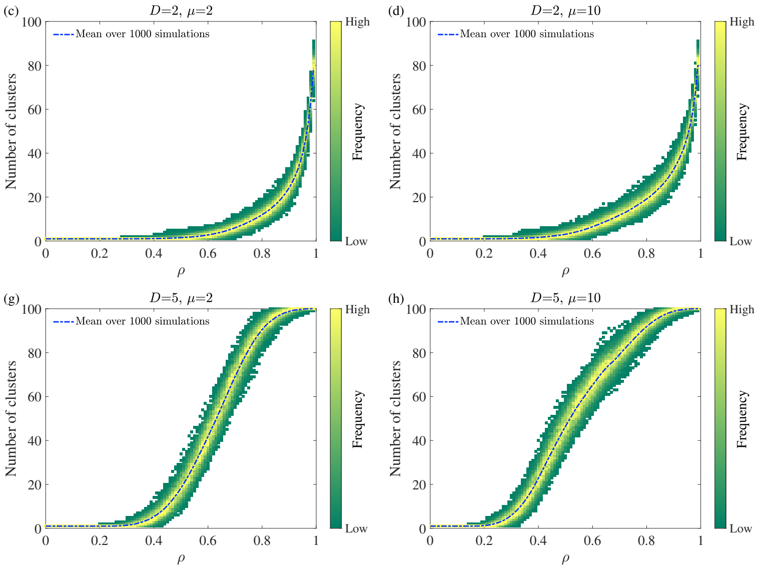
where $\alpha \geq 0$ is called the reinforcement rate. When $\alpha > 0$, the more extreme someone's opinion, the more stubborn they are.[6]

Parameter	**Values used**
D (Dimensions of opinion space)	1, 2, 3, 5.
μ (Memory capacity of population)	2, 10.
ρ (Universal/baseline threshold)	0, 0.01, 0.02, . . . , 0.99.
α (Reinforcement rate)	0, 0.1, 0.2, 0.4, 0.8.

1000 different initial configurations of 100 agents, for each D.

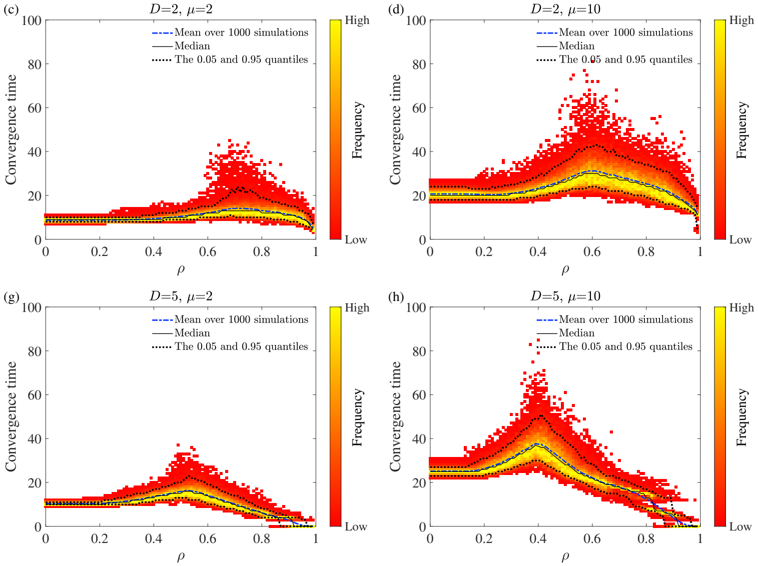
[6] Tian, Y., Jia, P., Mirtabatabaei, A., Wang, L., Friedkin, N.E., & Bullo, F. (2021). IEEE Transactions on Automatic Control, 67(2), 574–588.

Numerical experiments
2. Cluster formation: consensus vs. segregation ($\alpha = 0$)



Numerical experiments

3. Convergence time $(\alpha = 0)$



Numerical experiments

4. Extremisation ($\alpha > 0$)

Define the extremisation measure:

$$E := \left\| \frac{1}{N} \sum_{i=1}^{N} \mathbf{v}_i^{\infty} \right\| - \left\| \frac{1}{N} \sum_{i=1}^{N} \mathbf{v}_i(0) \right\|.$$

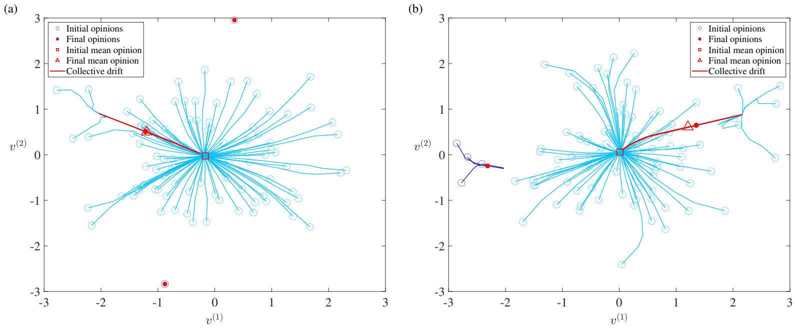
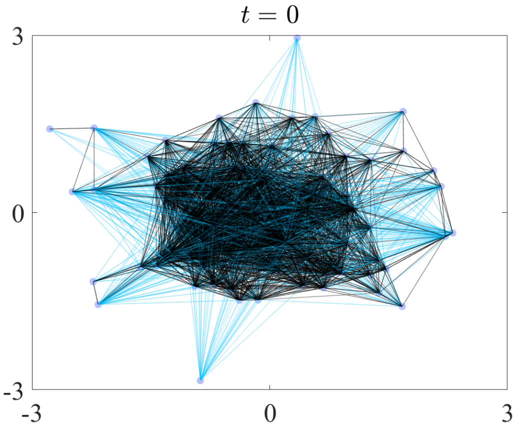


Figure: Two examples of collective drift and extremisation in 2D. Parameters: $\mu = 2, \rho = 0, \alpha = 0.4$.

Numerical experiments

4. Extremisation ($\alpha > 0$)

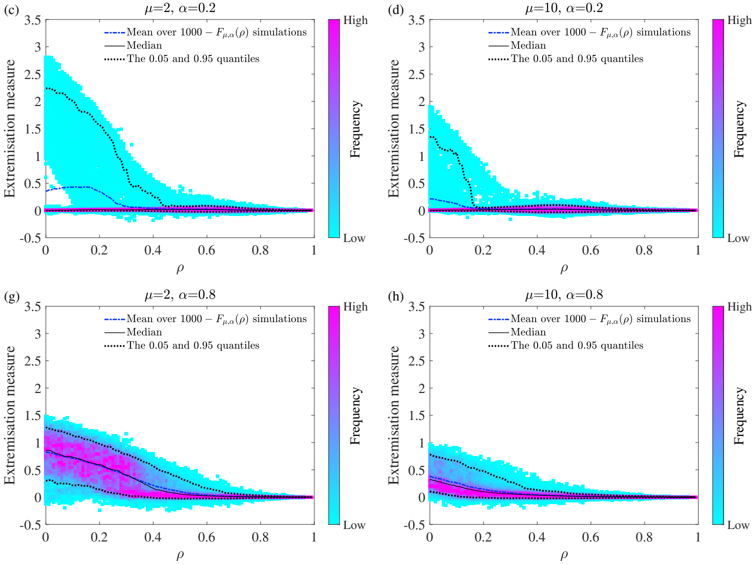


This process is reminiscent of group polarisation.[7]

[7]Sunstein, C.R. (1999).

Numerical experiments

4. Extremisation ($\alpha > 0$)



Numerical experiments

5. Failure to converge: collective oscillations

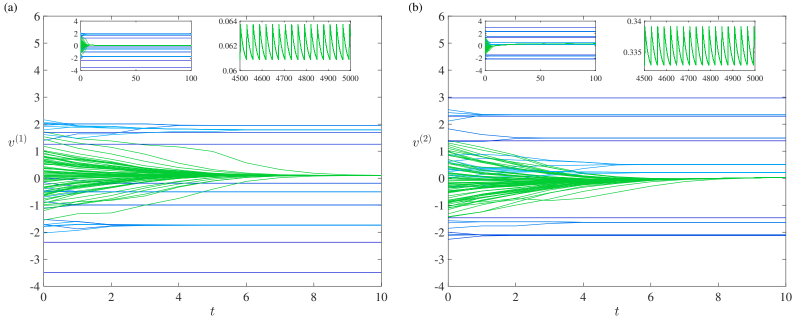


Figure: An example of collective oscillations in 2D.
Parameters: $\mu = 2, \rho = 0, \alpha = 0.8$.

We proved that periodic solutions exist, by explicitly constructing an N-body system where two extreme agents remain stationary while $N - 2$ agents collectively oscillate with period 2.

Conclusions

Novel model of opinion formation capable of mimicking socio-psychological phenomena such as emergent co-operation and group polarisation.

Extends existing theoretical findings and support experimental ones.

Sufficiently low universal threshold guarantees consensus.

A population which takes a longer history of itself into account is less susceptible to extremism.

Heterogeneous networks permit oscillatory opinion clusters.

Expect an even richer range of phenomena to emerge from more sophisticated initial opinion distributions.

Potential extensions could include: hierarchical populations; repulsive interactions; and stochastic fluctuations.

References

Cucker, F., & Smale, S. (2007). *Emergent behavior in flocks.* IEEE Transactions on Automatic Control, 52(5), 852–862.

Hegselmann, R., & Krause, U. (2002). *Opinion dynamics and bounded confidence models, analysis, and simulation.* Journal of Artificial Societies and Social Simulation, 5(3).

Lorenz, J. (2005). *A stabilization theorem for dynamics of continuous opinions.* Physica A: Statistical Mechanics and its Applications, 355 (1), 217–223.

Mariano, S., Morârescu, I., Postoyan, R., & Zaccarian, L. (2020). *A hybrid model of opinion dynamics with memory-based connectivity.* IEEE Control Systems Letters, 4(3), 644–649.

Rand, D.G., Arbesman, S., & Christakis, N.A. (2011). *Dynamic social networks promote cooperation in experiments with humans.* Proceedings of the National Academy of Sciences, 108 (48), 19193–19198.

Sunstein, C.R. (1999). *The law of group polarization.* University of Chicago Law School, John M. Olin Law & Economics Working Paper, 91.

Tian, Y., Jia, P., Mirtabatabaei, A., Wang, L., Friedkin, N.E., & Bullo, F. (2021). *Social power evolution in influence networks with stubborn individuals.* IEEE Transactions on Automatic Control, 67(2), 574-588.