Modular Normalisation of Classical Proofs

My PhD in half an hour

Benjamin Ralph

Inria Saclay, Team Parsifal

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My PhD in one slide

Ideology: Normalisation = Decomposition + Cut Elimination

I look at this paradigm in the context of classical logic.

	Part I	**Part II**
Logic	Propositional	First-Order
Inference Rule	Merge Contraction	Herbrand Expander
Main Theorem	Cycle Removal	Expansion Proofs Link
Publications	CSL 2017	LFCS 2018
Next Step	Develop Merges	Indicated Subs.

Open Deduction

The presentation of open deduction is slightly different to any before, trying to balance the clarity of, for example [Gun09] with the generality of [GGP10].

$$\dfrac{\phi}{\left(\dfrac{\psi}{\xi}\right)} \equiv \left(\dfrac{\phi_1}{\left(\dfrac{\psi_1}{\xi_1}\right)} \star \dfrac{\phi_2}{\left(\dfrac{\psi_2}{\xi_2}\right)} \right) \equiv \left(\dfrac{\left(\dfrac{\phi_1}{\psi_1}\right)}{\xi_1} \star \dfrac{\left(\dfrac{\phi_2}{\psi_2}\right)}{\xi_2} \right) \equiv \dfrac{\left(\dfrac{\phi}{\psi}\right)}{\xi}$$

Proof Systems

The basic proof systems for propositional classical logic are then introduced: the two most important being SKS (the system with cut) and KS (the cut-free system).

SKS

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}\!\uparrow \frac{a}{a \wedge a} \qquad \mathsf{aw}\!\uparrow \frac{a}{\mathsf{t}}$$

KS

$$\mathsf{ai}\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}\!\downarrow \frac{a \vee a}{a} \qquad \mathsf{aw}\!\downarrow \frac{\mathsf{f}}{a}$$

$$\mathsf{s} \frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m} \frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

The experiments method

To end the first chapter, we show the *experiments method* for propositional logic.

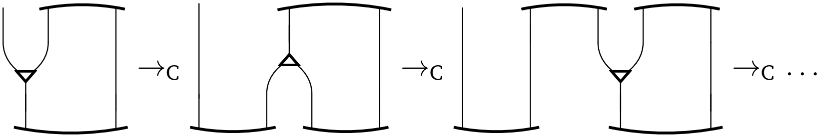
$$\begin{array}{c} \phi \, \Pi \, \mathsf{SKS} \\ A \end{array} \quad \longrightarrow \quad \begin{array}{c} TT_\phi \, \Pi \, \mathsf{KS} \\ \boxed{\begin{array}{c} (a_j)^{m_{(1,1)}} \wedge (\bar{a}_k)^{m_{(1,2)}} \\ \phi_1 \, \| \, \mathsf{SKS} \setminus \{\mathsf{ai}\uparrow\} \\ A \end{array} \vee \ldots \vee \begin{array}{c} (a_j)^{m_{(2^n,1)}} \wedge (\bar{a}_k)^{m_{(2^n,2)}} \\ \phi_{2^n} \, \| \, \mathsf{SKS} \setminus \{\mathsf{ai}\uparrow\} \\ A \end{array}} \\ \| \, \{\mathsf{ac}\downarrow, \mathsf{m}\} \\ A \end{array}$$

Decomposition and Cycles

The second chapter opens with a presentation of the contraction rewriting system, that will decompose a proof, if it terminates.

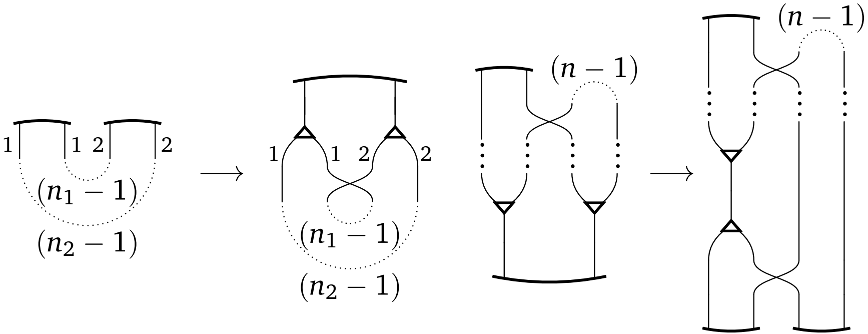
$$
\begin{array}{c}
A \\
\Vert\,\mathsf{SKS} \\
B
\end{array}
\quad \longrightarrow^{*}_{\mathsf{c}} \quad
\begin{array}{l}
A \\
\Vert\,\{\mathsf{ac}{\uparrow}\} \\
A' \\
\Vert\,\mathsf{SKS}\setminus\{\mathsf{ac}{\downarrow},\mathsf{ac}{\uparrow}\} \\
B' \\
\Vert\,\{\mathsf{ac}{\downarrow}\} \\
B
\end{array}
$$

The obstacles to termination are identified as *cycles*.



More on Cycles

Some important lemmas to do with breaking down more complicated cycles into simpler ones are stated and proved.



Merge Contractions

Definition

The $\vee$-*merge set* $M_\vee(A, B)$ of two formulae A and B is the minimum set that satisfies the following conditions:

- **M1** For any A and B, $A \vee B \in M_\vee(A, B)$.
- **M2** For any atom or unit a, $a \in M_\vee(a, a)$.
- **M3** For any A, $A \in M_\vee(A, \mathsf{f}), A \in M_\vee(\mathsf{f}, A), A \in M_\wedge(A, \mathsf{t})$ and $A \in M_\wedge(\mathsf{t}, A)$.
- **M4** For $\alpha \in \{\vee, \wedge\}$, if $C_1 \in M_\vee(A_1, B_1)$ and $C_2 \in M_\vee(A_2, B_2)$, then $C_1 \; \alpha \; C_2 \in M_\vee(A_1 \; \alpha \; A_2, B_1 \; \alpha \; B_2)$.

Definition

$\mathsf{mc}\!\downarrow \dfrac{A \vee B}{C}$ is a *merge contraction* if $C \in M_\vee(A, B)$.

Context Contractions

Another way to think about merges (pointed out by Willem) is as *context contractions*:

Definition

If $K\{\ \}$ is a context with n-holes, and $A_1, \ldots, A_n, B_1, \ldots, B_n$ are formulas, then

$$\mathsf{cc}\!\downarrow \frac{K\{A_1\}\ldots\{A_n\} \vee K\{B_1\}\ldots\{B_n\}}{K\{A_1 \vee B_1\}\ldots\{A_n \vee B_n\}}$$

is an instance of *context contraction*.

Theorem

$\mathsf{mc}\!\downarrow \dfrac{A \vee B}{C}$ *is a valid instance of a merge contraction iff* $\mathsf{cc}\!\downarrow \dfrac{A \vee B}{C}$ *is a valid instance of a context contraction (ish).*

SKSm

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \quad \text{atomic identity} \qquad\qquad \mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \quad \text{atomic cut}$$

$$\mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a} \quad \text{atomic weakening} \qquad\qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}} \quad \text{atomic coweakening}$$

$$\mathsf{mc}{\downarrow}\,\frac{A \vee B}{C} \quad \text{merge contraction} \qquad\qquad \mathsf{mc}{\uparrow}\,\frac{C}{A \wedge B} \quad \text{merge cocontraction}$$

$$\mathsf{s}{\downarrow}\,\frac{(A \vee B) \wedge (C \vee D)}{(A \wedge C) \vee (B \vee D)} \quad \text{switch (down)} \qquad\qquad \mathsf{s}{\uparrow}\,\frac{(A \wedge B) \wedge (C \vee D)}{(A \wedge C) \vee (B \wedge D)} \quad \text{switch (up)}$$

Translations between SKS and SKSm

We can define translations between SKS and SKSm:

$$
\begin{array}{ccccc}
\mathsf{SKSm} & & \mathsf{SKS} & & \mathsf{SKSm} \\[1em]
\mathsf{mc}{\downarrow}\,\dfrac{(a \wedge b) \vee (a \wedge c)}{a \wedge (b \vee c)}
& \begin{array}{c} \xleftarrow{\nu_{\max}} \\[1em] \xrightarrow{\mu} \end{array}
& \mathsf{m}\,\dfrac{(a \wedge b) \vee (a \wedge c)}{\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \wedge (b \vee c)}
& \begin{array}{c} \xrightarrow{\nu_{\min}} \\[1em] \xleftarrow{\mu} \end{array}
& \mathsf{mc}{\downarrow}\,\dfrac{(a \wedge b) \vee (a \wedge c)}{\mathsf{mc}{\downarrow}\,\dfrac{a \vee a}{a} \wedge (b \vee c)}
\end{array}
$$

Cycle Removal with Merge Contractions

With the material about merge contractions in place, we are able to prove cycle removal. The strategy is based around permuting *critical medials* as merge contractions down through the proof.

$$\mathsf{m}\,\frac{(A\{a\} \wedge B) \vee (C \wedge D\{\bar{a}\})}{(A\{a\} \vee C) \wedge (B \vee D\{\bar{a}\})}$$

The proof of termination is lengthy, but a lot of it is just checking various fairly straightforward cases. This method was developed over the last two years in collaboration with Alessio and Andrea.

Open Deduction and First-Order Logic

Chapter 3 introduces the open deduction proof theory of first-order logic. A lot of it is material formerly introduced by Kai Brünnler, in the Calculus of Structures formalism, but organised in a new way.

SKS

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}}$$

KS

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a}$$

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$

SKSq

$$\mathsf{u}{\uparrow}\,\frac{\exists x A \wedge \forall x B}{\exists x (A \wedge B)} \qquad \mathsf{m}_2{\uparrow}\,\frac{\exists x (A \wedge B)}{\exists x A \wedge \exists x B}$$

$$\mathsf{n}{\uparrow}\,\frac{\forall x A}{[t \Rightarrow x]A} \qquad \mathsf{m}_1{\uparrow}\,\frac{\forall x (A \wedge B)}{\forall x A \wedge \forall x B}$$

KSq

$$\mathsf{n}{\downarrow}\,\frac{[t \Rightarrow x]A}{\exists x A} \qquad \mathsf{m}_1{\downarrow}\,\frac{\exists x A \vee \exists x B}{\exists x [A \vee B]}$$

$$\mathsf{u}{\downarrow}\,\frac{\forall x [A \vee B]}{\forall x A \vee \exists x B} \qquad \mathsf{m}_2{\downarrow}\,\frac{\forall x A \vee \forall x B}{\forall x [A \vee B]}$$

Quantifier Cuts

An old result of Kai shows that we can separate ‘quantifier cuts’ from atomic cuts:

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SKSq} \\ A \end{array} \quad \longrightarrow \quad \begin{array}{c} \Vert\, \mathsf{KSq} \cup \{\mathsf{ai}\!\uparrow\} \\ A' \\ \Vert\, \{\mathsf{qi}\!\uparrow\} \\ A \end{array}$$

$$\mathsf{qi}\!\uparrow \frac{\forall x A \wedge \exists x \bar{A}}{\mathsf{f}}$$

Herbrand Proofs

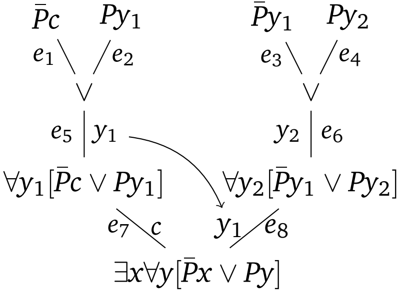
Chapter 4 develops the theory that links deep inference proof theory to two strains of proof theory deriving from Herbrand's Theorem.

First, the more classical approach is developed, defining a class of proofs, *Herbrand Proofs*, in a proof system called $\mathsf{KS}\mathfrak{h}1$ and proving a form of Herbrand's Theorem by showing they are complete for FOL.

$$
\begin{array}{c} \phi \,\Vert\, \mathsf{KSq} \\ A \end{array}
\quad\longrightarrow\quad
\begin{array}{c}
\Vert\, \mathsf{KS} \\
\forall\vec{x}[\vec{t} \Rightarrow \vec{y}]B \\
\Vert\, \{\mathsf{n}\!\downarrow\} \\
Q\{B\} \\
\Vert\, \mathsf{RP}_{\downarrow} \\
A' \\
\Vert\, \{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

Expansion Proofs

The second approach is based on expansion proofs.



KSh2

A proof system, KSh2 is defined, such that a class of proofs within it, in *Herbrand Normal Form* (HNF), intended to correspond to expansion proofs.

$$
\begin{array}{c}
Up(\phi) \, \|\, \mathsf{KS} \\
\forall \vec{x} H_\phi(A) \\
\| \, \{\exists \mathsf{w}\!\downarrow\} \\
\forall \vec{x} H_\phi^+(A) \\
Lo(\phi) \, \| \, \{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{h}\!\downarrow\} \\
A
\end{array}
$$

It is shown that there are translations between Herbrand Proofs and proofs in HNF.

KSh2

A proof system, KSh2 is defined, such that a class of proofs within it, in *Herbrand Normal Form* (HNF), intended to correspond to expansion proofs.

$$
\begin{array}{c}
Up(\phi) \,\| \,\mathsf{KS} \\
\forall \vec{x} H_{\phi}(A) \\
\| \, \{\exists \mathsf{w}\downarrow\} \\
\forall \vec{x} H^{+}_{\phi}(A) \\
Lo(\phi) \,\| \, \{\mathsf{r1}\downarrow, \mathsf{r2}\downarrow, \mathsf{h}\downarrow\} \\
A
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{c}
\| \,\mathsf{KS} \\
\forall \vec{x}[\vec{t} \Rightarrow \vec{y}]B \\
\| \, \{\mathsf{n}\downarrow\} \\
Q\{B\} \\
\| \, \mathsf{RP}_{\downarrow} \\
A' \\
\| \, \{\mathsf{qc}\downarrow\} \\
A
\end{array}
$$

It is shown that there are translations between Herbrand Proofs and proofs in HNF.

Translations between Expansion Proofs and HNF proofs

Next, it is shown that there are translations back and forward between expansion proofs and proofs in HNF. This material contains but also expands upon the material in [Ral18]. The two improvements are:

1. The translation from expansion proofs to proofs in HNF in [Ral18] requires an arbitrary total order on the nodes of the expansion proofs. In the thesis, this is improved to just requiring an order on the universal nodes.
2. The translation is extended to expansion proofs with cut and, respectively, HNF proofs with cut. This is essential if we are to use these translations as part of a cut elimination theorem.

Cut Elimination for SKSq

Finally, we state and prove a cut elimination theorem for SKSq:

$$\begin{array}{c} \phi \,\Vert\, \mathsf{SKSq} \\ A \end{array} \xrightarrow{1} \begin{array}{c} \phi_1 \,\Vert\, \mathsf{KSq}\cup\{\mathsf{ai}\!\uparrow\} \\ A \wedge B \\ \Vert\, \{\mathsf{qi}\!\uparrow\} \\ A \end{array} \xrightarrow{2} \begin{array}{c} \phi_2 \,\Vert\, \mathsf{KSh1}\cup\{\mathsf{ai}\!\uparrow\} \\ A \wedge B \\ \Vert\, \{\mathsf{qi}\!\uparrow\} \\ A \end{array} \xrightarrow{3} \begin{array}{c} \phi_3 \,\Vert\, \mathsf{KSh1} \\ A \wedge B \\ \Vert\, \{\mathsf{qi}\!\uparrow\} \\ A \end{array}$$

$$\xrightarrow{4} \begin{array}{c} \phi_4 \,\Vert\, \mathsf{KSh2} \\ A \wedge B \\ \Vert\, \{\mathsf{qi}\!\uparrow\} \\ A \end{array} \xrightarrow{5} \begin{array}{c} EC_{\phi_4} \,\Vert\, EPC \\ A \end{array} \xrightarrow{6} \begin{array}{c} E_{\phi_5} \,\Vert\, EP \\ A \end{array}$$

$$\xrightarrow{7} \begin{array}{c} \phi_5 \,\Vert\, \mathsf{KSh2} \\ A \end{array} \xrightarrow{8} \begin{array}{c} \phi_6 \,\Vert\, \mathsf{KSh1} \\ A \end{array} \xrightarrow{9} \begin{array}{c} \psi \,\Vert\, \mathsf{KSq} \\ A \end{array}$$