Towards a Combinatorial Proof Theory

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Hilbert's 24th Problem

David Hilbert, 1900:

"Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof."

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"Two proofs are the same iff they have identical combinatorial proofs."

Ben Ralph:
"Two structural proof systems are the same iff they have identical homomorphism classes."

Structure

Formulae without Syntax

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$\mathfrak{G}$: Formulae $\longrightarrow$ Cographs

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Proofs without Syntax

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Proofs without Syntax

$\mathfrak{G}$: MLL Proofs $\longrightarrow$ Critically Chorded R&B Graphs

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Proof Systems without Syntax

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- Formula Equivalence: $\equiv \, = (\equiv_C \cup \equiv_A)^*$, where:

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- What candidates for $f : \mathcal{F} \to S$ with $f(A) = f(B)$ iff. $A \equiv B$?

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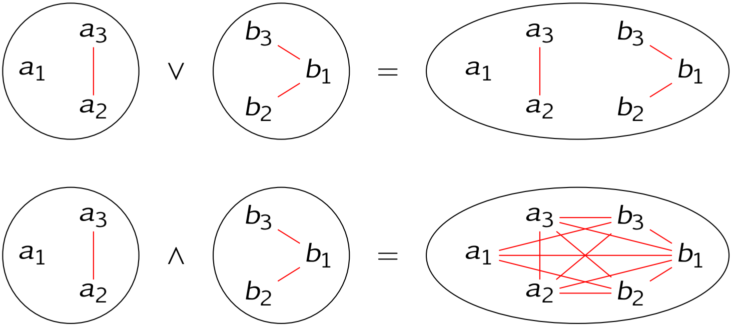

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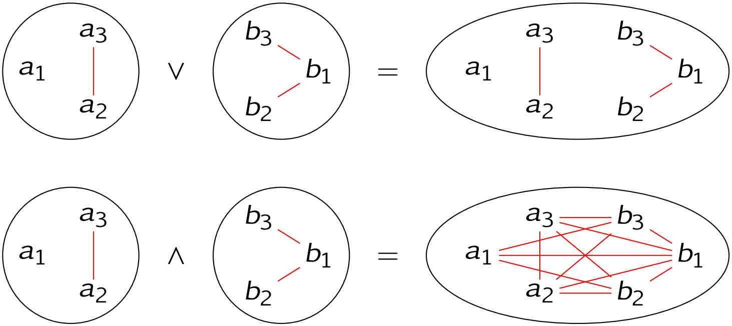

- Thus we have $\mathfrak{G}\colon \mathcal{F} \to \mathcal{G}_{\mathcal{F}}$, with $\mathfrak{G}(A) = \mathfrak{G}(B)$ iff $A \equiv B$.

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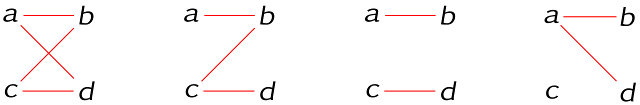

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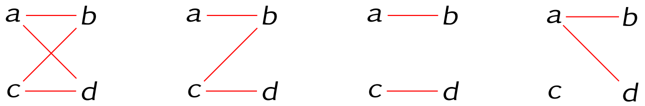

$(a \vee c) \wedge (b \vee d)$? $(a \wedge b) \vee (c \wedge d)$ $(a \wedge (b \vee d)) \vee c$

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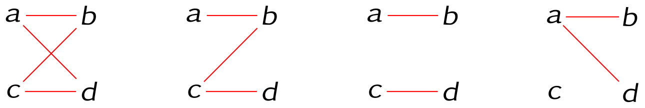

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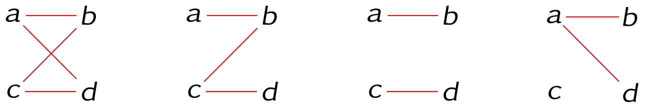

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- The second graph, the P_4 *graph*, is the only one not in $\mathcal{G}_{\mathcal{F}}$.
- A *cograph* is a graph that contains no P_4 subgraph.

Theorem (e.g. Duffin (1965))

$\mathcal{G}_{\mathcal{F}}$ is exactly the set of cographs.

Structure

Formulae without Syntax

$\mathfrak{G}$: Formulae $\longrightarrow$ Cographs

Proofs without Syntax

$\mathfrak{G}$: MLL Proofs $\longrightarrow$ Critically Chorded R&B Graphs

$\mathfrak{G}$: Classical Proofs $\longrightarrow$ Combinatorial Proofs

Proof Systems without Syntax

$\mathfrak{G}$: Proof Systems $\longrightarrow$ Homomorphism Classes

Multiplicative Linear Logic

$$\mathsf{MLL} = \boxed{\mathsf{ax}\,\frac{}{\vdash a, \bar{a}} \qquad \wedge_R\,\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \wedge B} \qquad \vee_R\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a,\bar{a}}\,\mathsf{ax} \quad \dfrac{}{\vdash b,\bar{b}}\,\mathsf{ax}}{\vdash (a \wedge b), \bar{a}, \bar{b}}\,\wedge_R}{\vdash (a \wedge b), (\bar{a} \vee \bar{b})}\,\vee_R \quad \dfrac{\dfrac{\dfrac{}{\vdash a,\bar{a}}\,\mathsf{ax} \quad \dfrac{}{\vdash b,\bar{b}}\,\mathsf{ax}}{\vdash (a \wedge b), \bar{a}, \bar{b}}\,\wedge_R}{\vdash (a \wedge b), (\bar{a} \vee \bar{b})}\,\vee_R}{\dfrac{\vdash (a \wedge b), (a \wedge b), ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}{\vdash (a \wedge b) \vee (a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}\,2\vee_R}\,\wedge_R$$

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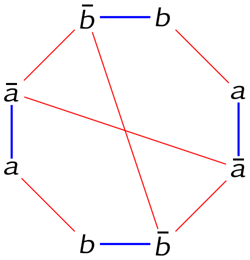
Are there graph theoretic criteria to determine which cographs correspond to theorems of MLL?

R&B Graphs and MLL Proofs

Theorem (Retore (1999))

There is a MLL *proof of* A *with* $\mathfrak{G}(A) = (V, R)$ *iff there is a critically chorded R&B cograph* $(V; R, B)$.



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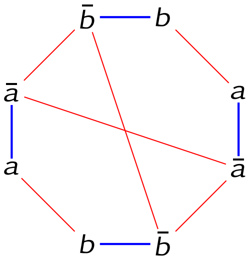

$$
{\scriptstyle 2\vee_R}\dfrac{{\scriptstyle \wedge_R}\dfrac{{\scriptstyle \vee_R}\dfrac{{\scriptstyle \wedge_R}\dfrac{{\scriptstyle \text{ax}}\dfrac{}{\vdash a, \bar{a}} \quad {\scriptstyle \text{ax}}\dfrac{}{\vdash b, \bar{b}}}{\vdash (a \wedge b), \bar{a}, \bar{b}}}{\vdash (a \wedge b), (\bar{a} \vee \bar{b})} \qquad {\scriptstyle \vee_R}\dfrac{{\scriptstyle \wedge_R}\dfrac{{\scriptstyle \text{ax}}\dfrac{}{\vdash a, \bar{a}} \quad {\scriptstyle \text{ax}}\dfrac{}{\vdash b, \bar{b}}}{\vdash (a \wedge b), \bar{a}, \bar{b}}}{\vdash (a \wedge b), (\bar{a} \vee \bar{b})}}{\vdash (a \wedge b), (a \wedge b), ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}}{\vdash (a \wedge b) \vee (a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}
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Classical Logic

$$\mathsf{LK}_R = \mathsf{MLL} + \boxed{\;\mathsf{c_R}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \mathsf{w_R}\,\frac{\vdash \Gamma}{\vdash \Gamma, A}\;}$$

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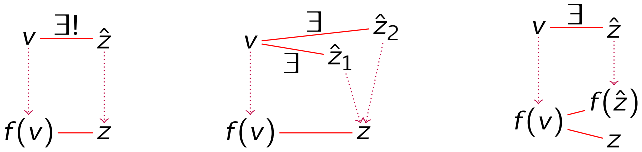
Are there graph theoretic criteria to determine which cographs correspond to theorems of classical logic?

Fibrations and Skew Fibrations



Let $f\colon G \to H$ be a graph homomorphism such that for every $v \in V_G, z \in V_H$ with $f(v)z \in E_H$, there is some $v\hat{z} \in E_G$. We consider 3 conditions on $\hat{z}$.

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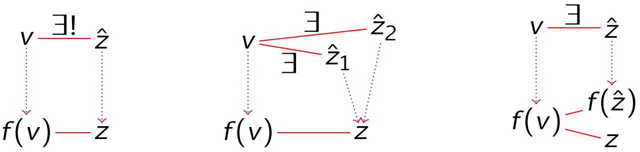

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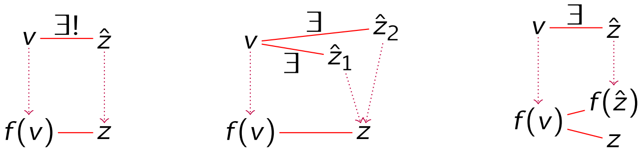

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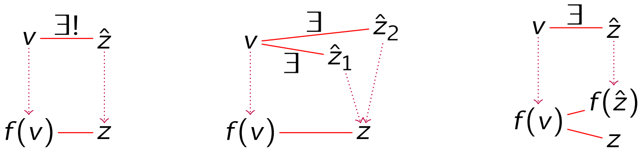

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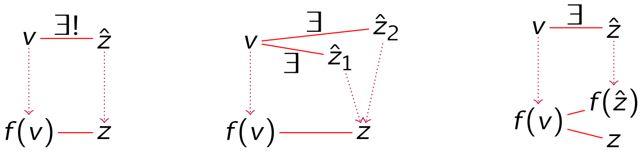

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For f to be a *fibration*, $\hat{z}$ must satisfy **(F1)** and **(F2)**, for a *weak fibration* it must satisy **(F1)** and for a *skew fibration* it must satisfy only **(SF1)**.

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A combinatorial proof of a formula A is a critically chorded cograph $G_{R\&B}$ together with a skew fibration $f\colon G_R \to \mathfrak{G}(A)$.

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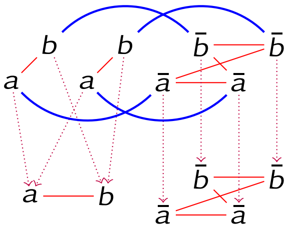

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Critically chorded R&B graphs are invariant of MLL sequent calculus proofs. Does the skew fibration correspond to a contraction-weakening extension of the proof?

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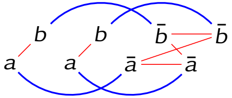

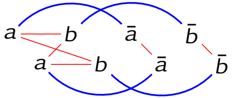

$$\text{ax}\dfrac{}{\vdash a,\bar a}\quad \text{ax}\dfrac{}{\vdash b,\bar b}\quad \text{ax}\dfrac{}{\vdash a,\bar a}\quad \text{ax}\dfrac{}{\vdash b,\bar b}$$

$$\wedge_R\dfrac{\dfrac{\vee_R\dfrac{\wedge_R\dfrac{\vdash a,\bar a \quad \vdash b,\bar b}{\vdash (a\wedge b),\bar a,\bar b}}{\vdash (a\wedge b),(\bar a\vee\bar b)} \qquad \vee_R\dfrac{\wedge_R\dfrac{\vdash a,\bar a \quad \vdash b,\bar b}{\vdash (a\wedge b),\bar a,\bar b}}{\vdash (a\wedge b),(\bar a\vee\bar b)}}{}}{\vdash (a\wedge b),(a\wedge b),((\bar a\vee\bar b)\wedge(\bar a\vee\bar b))}$$

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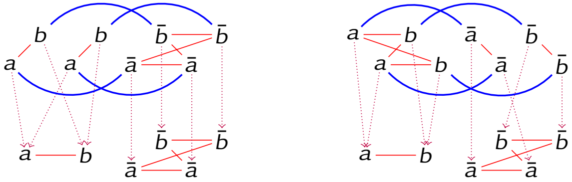

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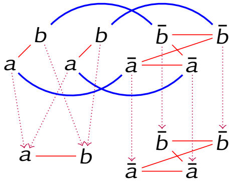

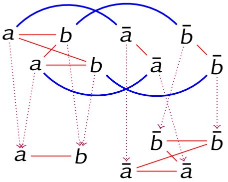

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Structure

Formulae without Syntax

$\circlearrowleft$:	Formulae	$\longrightarrow$	Cographs

Proofs without Syntax

$\circlearrowleft$:	MLL Proofs	$\longrightarrow$	Critically Chorded R&B Graphs
$\circlearrowleft$:	Classical Proofs	$\longrightarrow$	Combinatorial Proofs

Proof Systems without Syntax

$\circlearrowleft$:	Proof Systems	$\longrightarrow$	Homomorphism Classes

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Formulae without Syntax

₲: Formulae ⟶ Cographs

Proofs without Syntax

₲: MLL Proofs ⟶ Critically Chorded R&B Graphs

₲: Decomposed Proofs ⟶ Combinatorial Proofs

Proof Systems without Syntax

₲: Proof Systems ⟶ Homomorphism Classes

Open Deduction

We use a different proof formalism to correspond to skew fibrations, *open deduction*.

1. Inference Rule $\sigma \in \mathsf{S}$:

$$\begin{array}{c} A \\ \Phi \| \mathsf{S} \\ \sigma \dfrac{B}{C} \\ \Psi \| \mathsf{S} \\ D \end{array}$$

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2. Binary Connective $\star \in \{\wedge, \vee\}$:

$$\begin{array}{c} A \\ \Phi \Vert \mathsf{S} \\ B \end{array} \star \begin{array}{c} C \\ \Psi \Vert \mathsf{S} \\ D \end{array} = \begin{array}{c} A \star C \\ \Phi \star \Psi \Vert \mathsf{S} \\ B \star D \end{array}$$

KS

MLL

$$\mathsf{s}\,\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \mathsf{ai}{\downarrow}\,\frac{}{a \vee \bar{a}}$$

KS

MLL

$$\mathsf{s}\frac{A \wedge (B \vee C)}{(A \wedge B) \vee C}$$
switch

$$\mathsf{ai}{\downarrow}\frac{}{a \vee \bar{a}}$$
atomic identity

Structural rules

$$\mathsf{c}{\downarrow}\frac{A \vee A}{A}$$
contraction

$$\mathsf{ac}{\downarrow}\frac{a \vee a}{a}$$
atomic contraction

$$\mathsf{m}\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)}$$
medial

$$\mathsf{w}{\downarrow}\frac{A}{A \vee B}$$
weakening

KS

MLL

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switch — *atomic identity*

Structural rules

$$\mathsf{c}{\downarrow}\,\frac{A \vee A}{A} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \mathsf{w}{\downarrow}\,\frac{A}{A \vee B}$$

contraction — *atomic contraction* — *medial* — *weakening*

$$\mathsf{KSg} = \mathsf{MLL} \cup \{\mathsf{c}{\downarrow}, \mathsf{w}{\downarrow}\} \qquad \mathsf{KS} = \mathsf{MLL} \cup \{\mathsf{ac}{\downarrow}, \mathsf{m}, \mathsf{w}{\downarrow}\}$$

Examples

Two proofs of $(a \wedge b) \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))$, the first in KSg, the second in KS.

$$\mathsf{2s}\,\dfrac{\mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\,\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})} \vee \mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\,\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})}}{\mathsf{c}{\downarrow}\,\dfrac{(a \wedge b) \vee (a \wedge b)}{a \wedge b} \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}$$

$$\mathsf{2s}\,\dfrac{\mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\,\dfrac{}{a \vee \bar{a}}}{(a \vee a) \wedge (\bar{a} \wedge \bar{a})} \vee \mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\,\dfrac{}{b \vee \bar{b}} \wedge \mathsf{ai}{\downarrow}\,\dfrac{}{b \vee \bar{b}}}{(b \vee b) \wedge (\bar{b} \wedge \bar{b})}}{\left(\mathsf{ac}{\downarrow}\,\dfrac{a \vee a}{a} \wedge \mathsf{ac}{\downarrow}\,\dfrac{b \vee b}{b}\right) \vee \mathsf{m}\,\dfrac{(\bar{a} \wedge \bar{a}) \vee (\bar{b} \wedge \bar{b})}{(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})}}$$

Decomposed Proofs

Derivations in open deduction are called *decomposed* if they are stratified into different sections of inference rules.

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$$\begin{array}{c} \| \mathsf{MLL} \\ A \\ \| \{\mathsf{c}\!\downarrow, \mathsf{w}\!\downarrow\} \\ B \end{array} \quad \text{or} \quad \begin{array}{c} \| \mathsf{MLL} \\ A \\ \| \{\mathsf{ac}\!\downarrow, \mathsf{m}, \mathsf{w}\!\downarrow\} \\ B \end{array}$$

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Theorem

All proofs in KS/KSg *can be decomposed.*

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Theorem

All proofs in KS/KSg *can be decomposed.*

Theorem (Strassburger (2007,2017))

A formula has a decomposed proof in KS/KSg *iff it has a combinatorial proof. The top section corresponds to a critically chorded R&B Graph, and the bottom section corresponds to the skew fibration.*

Decomposed Proof Example

We can now extend ₲ to a class of classical proofs - decomposed proofs.

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$$
\mathsf{2s}\,\dfrac{
\mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})}
\vee
\mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})}
}{
\mathsf{c}{\downarrow}\dfrac{(a \wedge b) \vee (a \wedge b)}{a \wedge b} \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))
}
$$

$$
\mathsf{2s}\,\dfrac{
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}{
\left(\mathsf{ac}{\downarrow}\dfrac{a \vee a}{a} \wedge \mathsf{ac}{\downarrow}\dfrac{b \vee b}{b}\right) \vee \mathsf{m}\,\dfrac{(\bar{a} \wedge \bar{a}) \vee (\bar{b} \wedge \bar{b})}{(\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b})}
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$$

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$$\mathsf{2s}\dfrac{\mathsf{2s}\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})} \vee \mathsf{2s}\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a \vee \bar{a}} \wedge \mathsf{ai}{\downarrow}\dfrac{}{b \vee \bar{b}}}{(a \wedge b) \vee (\bar{a} \vee \bar{b})}}{\mathsf{c}{\downarrow}\dfrac{(a \wedge b) \vee (a \wedge b)}{a \wedge b} \vee ((\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{b}))}$$

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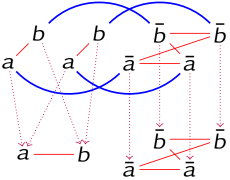

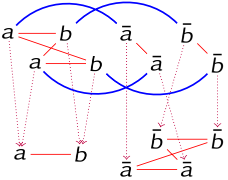

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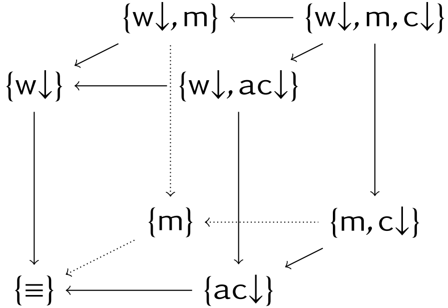
₲:	Structural Proof Systems	$\longrightarrow$	Homomorphism Classes	

Structural Proof Systems

We can think of proof systems as a lattice, ordered by derivability:

$\langle \mathcal{S}_{\downarrow}, \preceq \rangle =$



Homomorphism Classes

Critically chorded R&B cographs:

$$\mathfrak{G}\begin{pmatrix}\phi \parallel \mathsf{MLL} \\ A\end{pmatrix} = \mathfrak{G}(A)_{R\&B} \xrightarrow{Iso} \mathfrak{G}(A)$$

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Combinatorial Proofs:

$$\mathfrak{G}\begin{pmatrix} \| \text{MLL} \\ B \\ \| \{\text{w}\downarrow, \text{c}\downarrow\} \\ A \end{pmatrix} = \mathfrak{G}(B)_{R\&B} \xrightarrow{SkFib} \mathfrak{G}(A)$$

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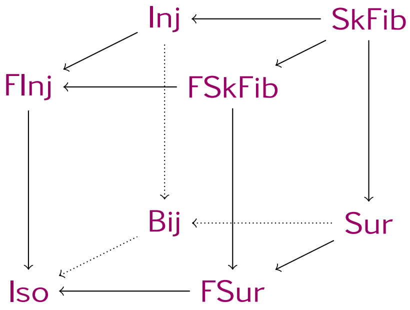
What about other homomorphism classes?

$$\mathfrak{G}(B)_{R\&B} \xrightarrow{H} \mathfrak{G}(A)$$

Homomorphism Class Lattice

We can also consider a lattice of homomorphism classes, ordered by set inclusion:

$\langle 2^{\mathsf{SkFib}}, \subseteq \rangle =$



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We now have two lattices:

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$$\mathfrak{G} \colon \langle \mathcal{S}_{\downarrow}, \preceq \rangle \rightarrow \langle 2^{\mathsf{SkFib}}, \subseteq \rangle$$

where

$$\mathfrak{G}\begin{pmatrix} \Vert \mathsf{MLL} \\ B \\ \Vert \mathsf{S} \\ A \end{pmatrix} = \mathfrak{G}(B)_{R\&B} \xrightarrow{\mathfrak{G}(\mathsf{S})} \mathfrak{G}(A)$$

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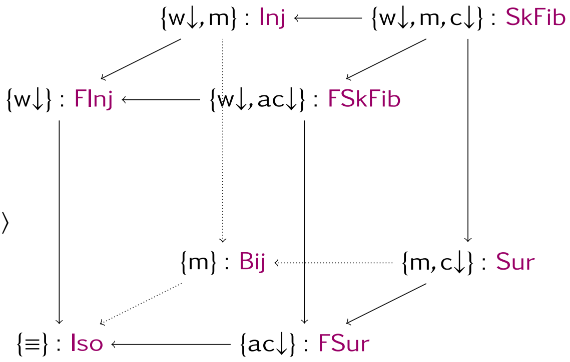
We can consider homomorphism classes as invariants of proof systems.

Corresponding Lattices

$$\mathfrak{G}(\langle \mathcal{S}_{\downarrow}, \preceq \rangle) = \langle 2^{\mathsf{SkFib}}, \subseteq \rangle$$



Slogans

Slogans

No Weakening = Surjectivity

Slogans

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Deep Inference	=	Skew Fibrations

Previously Known Results

- **Affine Logic** - Hughes (2006)
  $\mathfrak{G}(\{\mathsf{w}\!\downarrow\}) = \mathsf{FInj}, \quad \mathfrak{G}(\{\mathsf{w}\!\downarrow, \mathsf{m}\}) = \mathsf{Inj}$

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- **Relevance Logic** - Acclavio, Strassburger (2019)
  $\mathfrak{G}(\{\mathsf{m}, \mathsf{c}{\downarrow}\}) = \mathfrak{G}(\{\mathsf{m}, \mathsf{c}{\downarrow}\}) = \mathsf{FSur}, \mathfrak{G}(\{\mathsf{ac}{\downarrow}\}) = \mathsf{Sur}$

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- **Relevance Logic** - Acclavio, Strassburger (2019)
  $\mathfrak{G}(\{m, c\downarrow\}) = \mathfrak{G}(\{m, c\downarrow\}) = \mathsf{FSur}, \mathfrak{G}(\{ac\downarrow\}) = \mathsf{Sur}$
- **Logic of Bijections** - Hughes (2006), Strassburger (2007)
  $\mathfrak{G}(\{m\}) = \mathsf{Bij}$

A Logic of Fibrations

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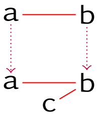
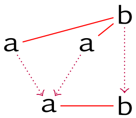
$$\mathsf{w}{\downarrow}\,\frac{a}{a \vee c} \wedge b$$



$$\mathsf{ac}{\downarrow}\,\frac{a \vee a}{a} \wedge b$$



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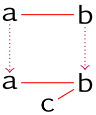
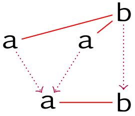


$$\mathsf{ac}{\downarrow}\frac{a \vee a}{a} \wedge b$$



In both cases, it is precisely the *deepness* of the rules that breaks the fibration.

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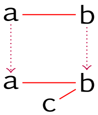

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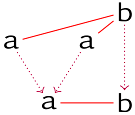
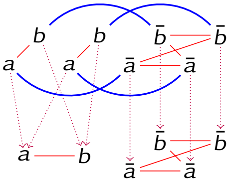


In both cases, it is precisely the *deepness* of the rules that breaks the fibration.
Therefore, the proof system corresponding to fibrations has *shallow* contraction and weakening:

$$\mathfrak{G}(\{\mathsf{sw}{\downarrow}, \mathsf{sc}{\downarrow}\}) = \mathsf{Fib}, \quad \mathfrak{G}(\{\mathsf{sw}{\downarrow}, \mathsf{c}{\downarrow}\}) = \mathsf{WFib}$$

Examples

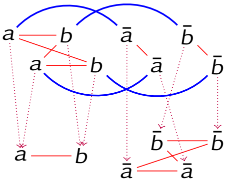
Fibration



$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a,\bar a}\,\mathsf{ax}\quad \dfrac{}{\vdash b,\bar b}\,\mathsf{ax}}{\vdash (a\wedge b),\bar a,\bar b}\,\wedge_R}{\vdash (a\wedge b),(\bar a\vee\bar b)}\,\vee_R \qquad \dfrac{\dfrac{\dfrac{}{\vdash a,\bar a}\,\mathsf{ax}\quad \dfrac{}{\vdash b,\bar b}\,\mathsf{ax}}{\vdash (a\wedge b),\bar a,\bar b}\,\wedge_R}{\vdash (a\wedge b),(\bar a\vee\bar b)}\,\vee_R}{\dfrac{\vdash (a\wedge b),(a\wedge b),((\bar a\vee\bar b)\wedge(\bar a\vee\bar b))}{\dfrac{\vdash (a\wedge b),((\bar a\vee\bar b)\wedge(\bar a\vee\bar b))}{\vdash (a\wedge b)\vee((\bar a\vee\bar b)\wedge(\bar a\vee\bar b))}\,\vee_R}\,\mathsf{c}{\downarrow}_R}\,\wedge_R$$

$$\mathsf{2s}\,\dfrac{\mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a\vee\bar a}\wedge\mathsf{ai}{\downarrow}\dfrac{}{b\vee\bar b}}{(a\wedge b)\vee(\bar a\vee\bar b)}\ \vee\ \mathsf{2s}\,\dfrac{\mathsf{ai}{\downarrow}\dfrac{}{a\vee\bar a}\wedge\mathsf{ai}{\downarrow}\dfrac{}{b\vee\bar b}}{(a\wedge b)\vee(\bar a\vee\bar b)}}{\mathsf{c}{\downarrow}\dfrac{(a\wedge b)\vee(a\wedge b)}{a\wedge b}\vee((\bar a\vee\bar b)\wedge(\bar a\vee\bar b))}$$

Skew Fibration



$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash a,\bar a}\,\mathsf{ax}\quad \dfrac{}{\vdash a,\bar a}\,\mathsf{ax}}{\vdash a,a,(\bar a\wedge\bar a)}\,\wedge_R}{\vdash (a\vee a),(\bar a\wedge\bar a)}\,\vee_R \qquad \dfrac{\dfrac{\dfrac{}{\vdash b,\bar b}\,\mathsf{ax}\quad \dfrac{}{\vdash b,\bar b}\,\mathsf{ax}}{\vdash b,b,(\bar b\wedge\bar b)}\,\wedge_R}{\vdash (b\vee b),(\bar b\wedge\bar b)}\,\vee_R}{\dfrac{\vdash (a\vee a)\wedge(b\vee b),(\bar a\wedge\bar a),(\bar b\wedge\bar b)}{?}}\,\wedge_R$$

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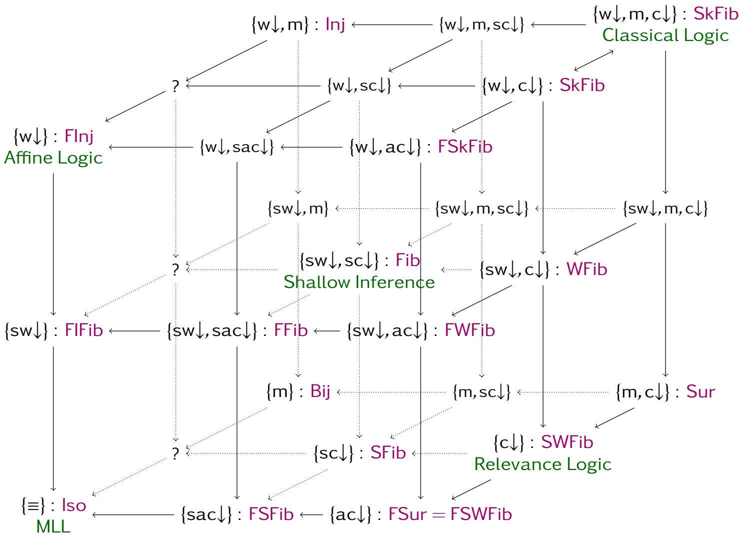
Proof Systems and Homomorphism Classes



Closing Remarks

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What if we replace MLL with different logics? IMLL? BV?