Decomposing First-Order Proofs

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Open Deduction

$$A \atop {\Phi \, \| \atop B} \qquad\qquad C \atop {\Psi \, \| \atop D}$$

Open Deduction

1. Inference Rule σ:

$$\begin{array}{c} A \\ \Phi \| \\ B \\ \sigma \, \overline{} \\ C \\ \Psi \| \\ D \end{array}$$

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2. Binary Connective $\star$:

$$\begin{array}{c} A \\ \Phi \Big\| \\ B \end{array} \star \begin{array}{c} C \\ \Big\| \Psi \\ D \end{array} = \begin{array}{c} A \star C \\ \Phi\star\Psi \Big\| \\ B \star D \end{array}$$

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3. Quantifier Qx:

$$Qx \left[\begin{array}{c} A \\ \phi \| \\ B \end{array} \right] = \begin{array}{c} QxA \\ Qx\phi \| \\ QxB \end{array}$$

SKS

Structural rules:

$$\mathsf{ai}{\downarrow}\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{ac}{\downarrow}\frac{a \vee a}{a} \qquad \mathsf{aw}{\downarrow}\frac{\mathsf{f}}{a}$$

identity *contraction* *weakening*

$$\mathsf{ai}{\uparrow}\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{ac}{\uparrow}\frac{a}{a \wedge a} \qquad \mathsf{aw}{\uparrow}\frac{a}{\mathsf{t}}$$

cut *cocontraction* *coweakening*

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$$\mathsf{s}\,\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

switch *medial*

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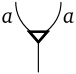
contraction



$$\mathsf{aw}\!\downarrow \frac{\mathsf{f}}{a}$$

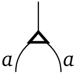
weakening

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}}$$

cut

$$\mathsf{ac}\!\uparrow \frac{a}{a \wedge a}$$

cocontraction



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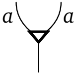

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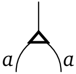

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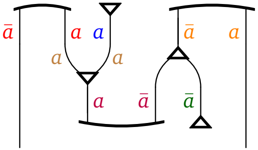
coweakening

Atomic Flow Example

$$\dfrac{\dfrac{\dfrac{\mathsf{t}}{\bar a \vee a} \vee \dfrac{\mathsf{f}}{a}}{\bar a \vee \dfrac{a \vee a}{a}} \wedge \dfrac{\mathsf{t}}{\dfrac{\bar a}{\bar a \wedge \dfrac{\bar a}{\mathsf{t}}} \vee a}}{\mathsf{s}^2\,\dfrac{[\bar a \vee a] \wedge [a \vee \bar a]}{\bar a \vee \mathsf{ai}{\uparrow}\,\dfrac{a \wedge \bar a}{\mathsf{f}} \vee a}}$$



SKSh

Structural Rules:

$$\mathsf{n}\!\downarrow \frac{A[\tau/x]}{\exists x A} \qquad \mathsf{qi}\!\downarrow \frac{\mathsf{t}}{\forall x A \vee \exists x \bar{A}} \qquad \mathsf{qc}\!\downarrow \frac{\exists x A \vee \exists x A}{\exists x A} \qquad \mathsf{m_2}\!\downarrow \frac{\forall x A \vee \forall x B}{\forall x [A \vee B]}$$

$$\mathsf{n}\!\uparrow \frac{\forall x A}{A[\tau/x]} \qquad \mathsf{qi}\!\uparrow \frac{\exists x A \wedge \forall x \bar{A}}{\mathsf{f}} \qquad \mathsf{qc}\!\uparrow \frac{\forall x A}{\forall x A \wedge \forall x A} \qquad \mathsf{m_2}\!\uparrow \frac{\exists x (A \wedge B)}{\exists x A \wedge \exists x B}$$

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Retract rules (B is free for x):

$$\mathsf{r1}{\downarrow}\frac{\forall x A \vee B}{\forall x[A \vee B]} \qquad \mathsf{r2}{\downarrow}\frac{\forall x A \wedge B}{\forall x(A \wedge B)} \qquad \mathsf{r3}{\downarrow}\frac{\exists x A \vee B}{\exists x[A \vee B]} \qquad \mathsf{r4}{\downarrow}\frac{\exists x A \wedge B}{\exists x(A \wedge B)}$$

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Example

Drinker's Formula ($\exists x \forall y \left[\bar{P}x \vee Py \right]$):

$$\text{qc}\downarrow \dfrac{\text{qi}\uparrow \dfrac{\text{t}}{\text{n}\downarrow \dfrac{\forall x \left[\text{w}\downarrow \dfrac{Px}{\bar{P}a \vee Px} \right]}{\exists x \forall x \left[\bar{P}x \vee Py \right]} \vee \exists x \left[\text{r1}\downarrow \dfrac{\text{w}\downarrow \dfrac{\bar{P}x}{\bar{P}x \vee \forall y Py}}{\forall y \left[\bar{P}x \vee Py \right]} \right]}}{\exists x \forall y \left[\bar{P}x \vee Py \right]}$$

Cut Elimination in Propositional Logic

Theorem

If there is an SKS *proof* $\overset{\Psi \Vert}{A}$, *then there is an* KS *proof* $\overset{\Psi \Vert}{A}$.

a fortiori: ai$\uparrow$ *is admissible for* SKS.

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1. Splitting (Sequent Calculus style) [Brünnler 2003]

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1. Splitting (Sequent Calculus style) [Brünnler 2003]
2. Quasipolynomial-time procedure [Bruscoli et al. 2010]

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Proof.

1. Splitting (Sequent Calculus style) [Brünnler 2003]
2. Quasipolynomial-time procedure [Bruscoli et al. 2010]
3. Decomposition + Splitting

□

Decomposition

Where does complexity come from?

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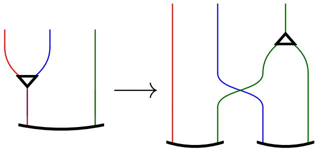

Decomposition

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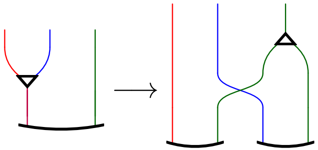

Normalisation separates into:

Decomposition

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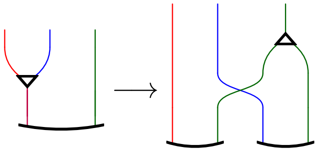

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1. A *decomposition* phase.

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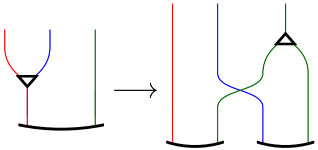

Normalisation separates into:

1. A *decomposition* phase.
2. An *elimination* phase.

First-Order Logic

What new technology do we have in FOL?

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$$\text{qc}\!\downarrow \frac{\text{n}\!\downarrow \dfrac{Pa}{\exists x P x} \vee \text{n}\!\downarrow \dfrac{Pb}{\exists x P x}}{\exists x P x}$$

First-Order Logic

What new technology do we have in FOL?

$$\text{qc}{\downarrow}\,\dfrac{\text{n}{\downarrow}\,\dfrac{Pa}{\exists x Px} \vee \text{n}{\downarrow}\,\dfrac{Pb}{\exists x Px}}{\exists x Px}$$

$$\text{i}{\uparrow}\,\dfrac{\forall x A \wedge \exists x \bar{A}}{\mathsf{f}}$$

Herbrand's Theorem

There exist two irrational numbers a and b such that a^b is rational.

FO: $$\exists a, b \in \mathbb{R}(\overline{\mathbb{Q}}(a) \wedge \overline{\mathbb{Q}}(b) \wedge \mathbb{Q}(a^b))$$

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$$\swarrow \qquad \searrow$$

Prop: $\overline{\mathbb{Q}}(\sqrt{2}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(\sqrt{2}^{\sqrt{2}}) \quad \vee \quad \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(2)$

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Prop: $\overline{\mathbb{Q}}(\sqrt{2}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(\sqrt{2}^{\sqrt{2}}) \quad \vee \quad \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}}) \wedge \overline{\mathbb{Q}}(\sqrt{2}) \wedge \mathbb{Q}(2)$

$$\downarrow \qquad\qquad \downarrow$$

Taut: $\mathbb{Q}(\sqrt{2}^{\sqrt{2}}) \quad \vee \quad \overline{\mathbb{Q}}(\sqrt{2}^{\sqrt{2}})$

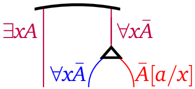
Herbrand Expanders

$$\mathsf{qc}{\downarrow}\ \frac{\exists x A \vee \boxed{\mathsf{n}{\downarrow}\ \dfrac{A[a/x]}{\exists x A}}}{\exists x A} \quad\longrightarrow\quad \mathsf{h}{\downarrow}\ \frac{\exists x A \vee A[a/x]}{\exists x A}$$

$$\mathsf{qc}{\uparrow}\ \frac{\forall x A}{\forall x A \wedge \boxed{\mathsf{n}{\uparrow}\ \dfrac{\forall x A}{A[a/x]}}} \quad\longrightarrow\quad \mathsf{h}{\uparrow}\ \frac{\forall x A}{\forall x A \wedge A[a/x]}$$

Herbrand Stratification

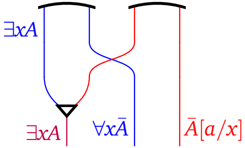
$$\mathsf{qi}\!\downarrow \frac{\mathsf{t}}{\exists xA \vee \mathsf{h}\!\uparrow \dfrac{\forall x\bar{A}}{\forall x\bar{A} \wedge \bar{A}[a/x]}}$$



$$\downarrow$$

$$\downarrow$$

$$\mathsf{qi}\!\downarrow \frac{\mathsf{qi}\!\downarrow \dfrac{\mathsf{t}}{\exists xA \vee \forall x\bar{A}} \wedge \mathsf{i}\!\downarrow \dfrac{\mathsf{t}}{A[a/x] \vee \bar{A}[a/x]}}{\mathsf{h}\!\downarrow \dfrac{\exists xA \vee A[a/x]}{\exists xA} \vee \left(\forall x\bar{A} \wedge \bar{A}[a/x]\right)}$$



Decomposition Theorem

Theorem

Every proof in SKSh *can be transformed into a proof of the following form:*

$$\begin{array}{c} \Phi \, \overset{}{\|}\, \mathsf{SKSh} \\ A \end{array} \quad \longrightarrow \quad \begin{array}{c} \Phi' \, \|\, \mathsf{SKSh} \setminus \{\mathsf{n}{\downarrow}, \mathsf{n}{\uparrow}\} \\ B \\ \| \, \{\mathsf{h}{\downarrow}\} \\ A \end{array}$$

Loops

Loops threaten the possibility that the decomposition procedure will not terminate.

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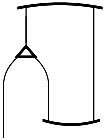
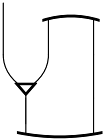
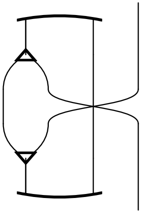
$L_{1\downarrow}$



$L_{1\uparrow}$



L_2



Escaping the loops

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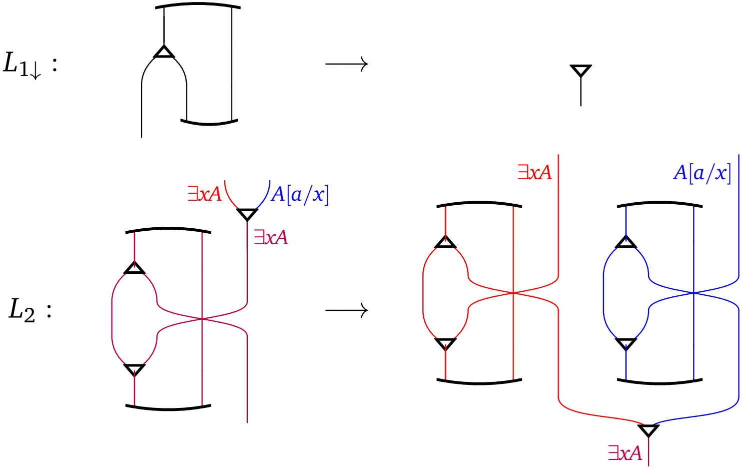
The h$\downarrow$ and h$\uparrow$ rules cannot occur *inside* the loop.

Escaping the loops

The h↓ and h↑ rules cannot occur *inside* the loop.



Proving Decomposition

Theorem

Every proof in SKSh *can be transformed into a proof of the following form:*

$$\begin{array}{c} \Vert \\ A \end{array} \quad \longrightarrow \quad \begin{array}{l} \Vert\,\mathsf{SKSh}\setminus\{\mathsf{n}\!\downarrow,\mathsf{n}\!\uparrow\} \\ B \\ \Vert\,\{\mathsf{h}\!\downarrow\} \\ A \end{array}$$

Proving Decomposition

Theorem

Every proof in SKSh *can be transformed into a proof of the following form:*

$$\begin{array}{c} \Pi \\ A \end{array} \quad \longrightarrow \quad \begin{array}{l} \Pi \, \mathsf{SKSh} \setminus \{\mathsf{n}{\downarrow}, \mathsf{n}{\uparrow}\} \\ B \\ \| \, \{\mathsf{h}{\downarrow}\} \\ A \end{array}$$

Proof.

We first eliminate any $L_{1\downarrow}$ or $L_{1\uparrow}$ loops in the proof. We then convert all $\mathsf{n}{\downarrow}$ ($\mathsf{n}{\uparrow}$) rules to $\mathsf{h}{\downarrow}$ ($\mathsf{h}{\uparrow}$) rules, and then push them down (up) the proof. Taking care to pass through any L_2 loops, the procedure terminates with the proof in the appropriate form. □

A different view of CE and Herbrand's Theorem

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Old Picture

$$\text{Proof} \xrightarrow{CE} \text{Cut-free Proof} \xrightarrow{HT} \text{Herbrand-Expanded Proof}$$

A different view of CE and Herbrand's Theorem

Old Picture

Proof $\xrightarrow{CE}$ Cut-free Proof $\xrightarrow{HT}$ Herbrand-Expanded Proof

New Picture

Herbrand Expanded Proof

$\nearrow$ $\searrow$

Proof Normalised Proof

$\searrow$ $\nearrow$

Cut-free Proof

Further Work

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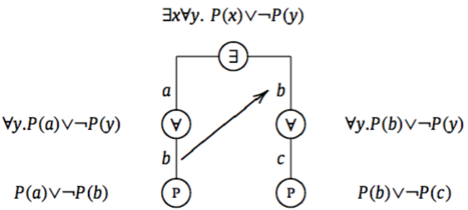
- Study further the complexity of the Herbrand Stratification procedure.
- Compare with results proved using ϵ-calculus [7].
- Compare with other similar systems, for example Heijltjes' Expansion Trees with cut. [6]



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