Understanding First-Order Cut Elimination and Herbrand's Theorem using Deep Inference

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In the study of a logic, the structure of its proofs greatly influences how easily meta-theorems can be established and what insights we can draw from the process of proving them. The deep-inference formalism allows for very fine-grained inference steps and proof transformations, and its introduction has produced important results and innovations in various logics. In this work, I will look at classical first-order logic, and at its foundational results, Herbrand's Theorem and Gentzen's Hauptsatz, through this lens.

First-Order Logic At the centre of the proof theory of classical first-order logic (FOL) lie two theorems that unpack the constructive content of first-order proofs in different ways: Herbrand's Theorem and Gentzen's *Hauptsatz*, or cut elimination theorem. Both now have applications and uses well beyond the aim for which they were formulated, as part of Hilbert's Program to prove the consistency of arithmetic, and many links between the two theorems have been studied.

As with much of proof theory, the setting of this study has been in Gentzen-style proof systems, especially the sequent calculus, which were designed specifically with the consistency proof in mind. However, despite the focus in proof theory having shifted to other concerns such as semantics and proof complexity, the proof theoretical setting for the study of these theorems has largely not changed. In fact, one of the techniques used for profitable study of these theorems is the ϵ-calculus [10], another product of Hilbert's Program.

Deep Inference The deep-inference formalism of open deduction differs from the sequent calculus in that composition of proofs is allowed with the same connectives that are used for the composition of formulae [5]. Thus in classical propositional logic, two proofs $\begin{array}{c}A\\ \phi\,\|\\ B\end{array}$ and $\begin{array}{c}C\\ \psi\,\|\\ D\end{array}$ can be composed not only with conjunction, as is possible in the sequent calculus, but also with disjunction:

$$\begin{array}{c}A\\ \phi\,\|\\ B\end{array} \wedge \begin{array}{c}C\\ \psi\,\|\\ D\end{array} = \begin{array}{c}A\wedge C\\ \phi\wedge\psi\,\|\\ B\wedge D\end{array}, \qquad \begin{array}{c}A\\ \phi\,\|\\ B\end{array} \vee \begin{array}{c}C\\ \psi\,\|\\ D\end{array} = \begin{array}{c}A\vee C\\ \phi\vee\psi\,\|\\ B\vee D\end{array}$$

This freedom of composition has enabled many proof-theoretic innovations: the reduction of cut to atomic form by a local procedure of polynomial-time complexity [3], and the development of a quasi-polynomial cut elimination procedure for propositional logic using a geometric invariant of proofs known as the *atomic flow* [1,6]. In FOPL, we also allow quantifiers to be applied to proofs, not only formulae:

$$\exists x \left[\begin{array}{c}A\\ \phi\,\|\\ B\end{array}\right] = \begin{array}{c}\exists x A\\ \exists x\phi\,\|\\ \exists x B\end{array}, \qquad \forall x \left[\begin{array}{c}A\\ \phi\,\|\\ B\end{array}\right] = \begin{array}{c}\forall x A\\ \forall x\phi\,\|\\ \forall x B\end{array}$$

Although there has been work on deep inference for FOL, including a direct cut-elimination procedure [2], there has been no work as of yet that uses the recent advances in the field to investigate the proof theory of FOL.

I will argue that deep inference can offer a different and perhaps superior proof theoretical setting for the investigation of FOL. In fact, there is reason to believe that using deep inference we can gain access to complexity results previously proven with the ϵ-calculus [10], hopefully in simpler fashion.

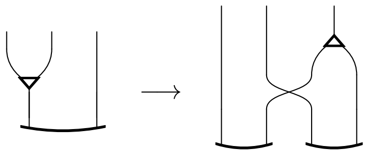
Cut elimination and Herbrand's Theorem in Deep Inference. As well as providing a cut elimination theorem, Brünnler's work on first-order deep inference [2] includes a statement and proof of Herbrand's Theorem. His approach mirrors that of Buss [4], who presents Herbrand's

Theorem for the sequent calculus as a proof decomposition, and proves it using cut elimination. However, the proof theoretical features of deep inference mean that we can understand the two theorems and their relationship differently, seeing a certain proof reduction as common to the proof of both, and understanding the theorems as unpicking two proof compression mechanisms (contraction of existential witnesses for Herbrand's Theorem and repeated use of lemmata for cut elimination) that become orthogonal in a way not permissible in the sequent calculus.

In particular, we observe that the reduction of an existential contraction above a cut is central to the proof of both:

$$\mathsf{cut}\,\dfrac{\boxed{\mathsf{cont}\,\dfrac{\exists xA \vee \exists xA}{\exists xA}} \wedge \forall x\bar{A}}{\mathsf{f}} \quad\longrightarrow\quad \dfrac{(\exists xA \vee \exists xA) \wedge \boxed{\mathsf{co{-}cont}\,\dfrac{\forall xA}{\forall x\bar{A} \wedge \forall x\bar{A}}}}{\boxed{\mathsf{cut}\,\dfrac{\exists xA \wedge \forall x\bar{A}}{\mathsf{f}}} \vee \boxed{\mathsf{cut}\,\dfrac{\exists xA \wedge \forall x\bar{A}}{\mathsf{f}}}}$$

This reduction is an example of a pattern that is central to much deep inference proof theory [5], the shape of which can be seen clearly in the atomic flows below:



While the pattern on the left is one simply instantiated in the sequent calculus, the pattern on the right contains features not expressible in the sequent calculus without *ad hoc* syntactic additions. Specifically, the fully two-dimensional open deduction formalism allows for more combinatorial possibilities than the tree-like structure of sequent calculus proofs, and this is exploited by the shape on the right.

This pattern also occurs in the work of Heijltjes and McKinley [7,8] on extending Miller's expansion trees [9] with an eliminable cut, reinforcing the view that deep inference is well-suited for a study of Herbrand's Theorem. Linking the work of Heijltjes and McKinley to deep inference should prove helpful and productive in both directions, enabling the translation of techniques and results between these fields of study.

In the talk, I will give the outline of a new cut elimination procedure for FOL in open deduction, as well as a decomposition-style presentation of Herbrand's Theorem called a *Herbrand Stratification* that is proved not as a corollary of cut elimination, but in tandem with it. In doing so, I hope that a valuable dialogue on these topics will be started between proof theorists working with deep inference and those in the Hilbert-Gentzen tradition.

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