Asymptotic and hyperasymptotic expansions of solutions to ODEs

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Abstract

In this project we look at the general theory of asymptotic power series and how they play a crucial role in solving ordinary differential equations (ODEs). In particular, we prove a theorem about a class of ODEs that have a solution with an asymptotic power series defined in some sector. Later we extend to hyperasymptotic power series and look at an example with a Riccati equation which involves the Airy function's asymptotics.

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Chapter 1

Introduction

When solving ordinary differential equations (ODEs) it is common to have the solutions in terms of series expansions. Sometimes these expansions converge but there are cases where they diverge.
This may seem useless to quote Niels Abel, "divergent series are the invention of the devil and it is a shame to base on them any demonstration whatsoever." However these divergent series can be useful, for instance they can approximate solutions to an ODE in a certain sector. Also, such series can be truncated to a finite number of terms and approximate certain quantities with less computation than their respective convergent series. These divergent series have uses both on the practical and theoretical level.
The main part of this project involves a theorem (Theorem 3.1.1) that looks at a general class of ODEs which admit an asymptotic power series. Then there exists an actual solution with this series in some sector. The proof of this theorem is outlined in this project in detail and a different approach is taken compare to the work of Wasow (1965). Later we extend to hyperasymptotic power series and original work is done by look at an example with a Riccati equation that uses the Airy function's asymptotics.

Chapter overviews

For Chapter 2 we talk about asymptotic power series in general, by defining what they are and exploring some of their properties. We will see what operations can be done on them through various theorems.
For Chapter 3 this is the core of the project, where we prove Theorem 3.1.1.
For Chapter 4, we look beyond standard asymptotic power series by looking at hyperasymptotic power series. We look at an original piece of work on a Riccati equation that can be solved in terms of the Airy equation.

Chapter 2

Theory of Asymptotic Power Series

2.1 Definition of an asymptotic power series

Definition 2.1.1. *Let the function $f(x)$ be defined in a point-set S of the complex x-plane having $x = 0$ as an accumulation point. The power series*

$$\sum_{r=0}^{m} a_r x^r$$

is said to represent $f(x)$ asymptotically as $x \to 0$ in S, if

$$x^{-m}\left[f(x) - \sum_{r=0}^{m} a_r x^r\right] \to 0,\ \forall m \geq 0 \text{ as } x \to 0 \text{ in } S. \tag{2.1}$$

Then

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r,\ x \in S,\ x \to 0.$$

An alternative definition is

$$f(x) = \sum_{r=0}^{m} a_r x^r + o(x^m),\ \forall m \geq 0 \text{ as } x \to 0 \text{ in } S, \tag{2.2}$$

then

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r,\ x \in S,\ x \to 0.$$

Following this the coefficients are defined as

$$\lim_{x\to 0} f(x) = a_0, \tag{2.3}$$

$$\lim_{x\to 0} x^{-m}\left[f(x) - \sum_{r=0}^{m-1} a_r x^r\right] = a_m,\ \forall m > 0, x \in S. \tag{2.4}$$

Unlike convergent power series where a series converges for any fixed value of x in the limit $m \to \infty$, asymptotic power series do not have to converge and generally diverge. They are useful when truncated to a finite number of terms (fixed m) for a value of x towards a limit L ($x \to L$). This provides benefits in approximating a function or a solution to an equation. Typically the best approximation is given by one where the relative error is the smallest from the actual function. It can often require less computation to achieve the same error bound than a convergent series.

2.2 Examples

There are numerous examples of asymptotic power series expansions, each having very useful features in many applications. Below are a few notable ones [Olver (1997)].

- Exponential Integral $\mathrm{Ei}(x)$

  $$\mathrm{Ei}(x) = \int_{-\infty}^{x} e^t t^{-1}\, dt \sim e^x \sum_{r=0}^{\infty} r! x^{-(r+1)},\ x < 0, x \in \mathbb{R}.$$

  The function has many applications including time-dependent heat transfer and solutions to the 1D neutron transport equation.

- Error function and its complement, $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$

  $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,\ \mathrm{erfc}(x) = 1 - \mathrm{erf}(x) \sim \frac{1}{\sqrt{\pi} x e^{x^2}} \left[1 + \sum_{r=1}^{\infty} (-1)^r \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2x^2)^r} \right], x \in \mathbb{C}.$$

  The functions are useful in probability and thermodynamics.

- Logarithmic Integral $\mathrm{Li}(x)$

  $$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln t} \sim x \sum_{r=0}^{\infty} \frac{r!}{(\ln x)^{r+1}},\ x \neq 1, x \in \mathbb{R},$$

  where the function itself is a very good approximation for the prime-counting function.

2.3 Properties

Now we shall look at some properties of asymptotic power series, including their representation of functions, linear combinations of different series and their analytical properties. First, we define a region on the complex plane that we will use for subsequent work in this project unless defined otherwise.

Definition 2.3.1. *The sector S_α^β is a set of points on the complex plane z, where $z = \{z \mid z = re^{i\theta}, 0 < r < \infty, \alpha \leq \theta \leq \beta$, where $r, \alpha, \beta \in \mathbb{R}\}$. The sector S_α^β has boundary rays at α and β. (See Figure 2.1 for an example)*

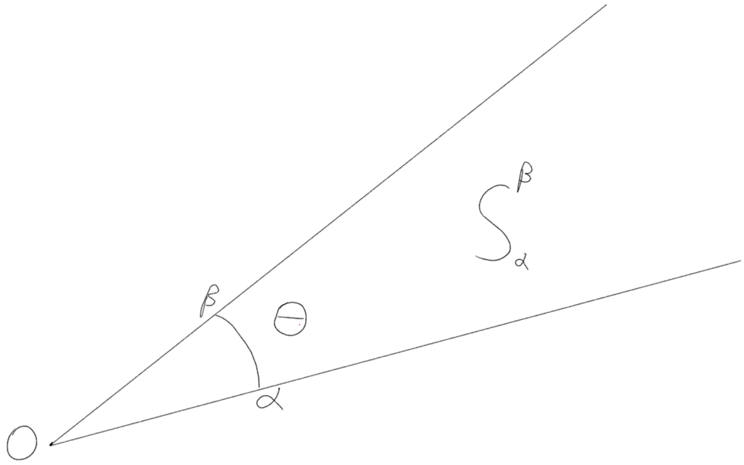

Figure 2.1: An example S_α^β with vertex at O

2.3.1 Algebraic properties

Theorem 2.3.1. *A function $f(x)$ can have at most one asymptotic series representation*

$$\sum_{r=0}^{\infty} a_r x^r,$$

as $x \to 0$ in a given set S_α^β.

Proof. From equations (2.3) and (2.4), we see that each coefficient a_r, $\forall r \geq 0$ has a limit representation which therefore defines a_r uniquely.

□

However, the converse of this theorem is not true, i.e. a power series can **never** determine a **unique** function which is asymptotic to it. For example, we look at the simple power series whose coefficients are all zero and lies on the positive x-axis. Let two functions be $f(x) \equiv 0$ and $g(x) = e^{-\frac{1}{x^3}}$, we see that as $x \to 0$ both are asymptotic to 0 in the sector $S_{-\epsilon}^{\epsilon}$ where $0 < \epsilon << 1$, but are not the same function. Also if we look on the negative x-axis as $x \to 0$ they each have radically different asymptotic behaviour.

Asymptotic power series can also be term-wise added and multiplied by constants like formal power series.

Theorem 2.3.2. *For $x \in S_\alpha^\beta, x \to 0$, if $m(x) = \alpha f(x) + \beta g(x)$ with constants α, β, where*

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r, \quad g(x) \sim \sum_{r=0}^{\infty} b_r x^r,$$

then

$$m(x) \sim \sum_{r=0}^{\infty} (\alpha a_r + \beta b_r) x^r.$$

Proof. We define two new functions with the constants α and β

$$\alpha f(x) \sim \sum_{r=0}^{\infty} \alpha a_r x^r, \quad \beta g(x) \sim \sum_{r=0}^{\infty} \beta b_r x^r \text{ for } x \in S_\alpha^\beta, \quad x \to 0.$$

Adding these functions together and their asymptotic power series expansions, the required result is obtained. □

2.3.2 Analytic (differential) properties

Now we consider only analytic or holomorphic functions on S_α^β, as we explore integration and differentiation in the complex plane.

Theorem 2.3.3. *If $f(x)$ is holomorphic in the whole annular neighbourhood $0 < |x| \leq x_0 < \infty$ of $x = 0$ and if*

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r, \quad x \to 0, \quad \forall \arg x,$$

then the series converges to $f(x)$ in that neighbourhood.

Proof. From equation (2.3), we consider the origin as a removable singularity, i.e. we define $f(0) = a_0$, so the extended function $f(x)$ is now holomorphic at $x = 0$. Hence there exists a convergent power series in $|x| \leq x_0$. Now, we use Theorem 2.3.1 so the series is identically the same as

$$\sum_{r=0}^{\infty} a_r x^r.$$

□

The idea that an asymptotic series for $f(x)$ converges in a sector S_α^β, does not mean the series converges to $f(x)$ anywhere. Unless S_α^β is a whole annular neighbourhood of $x = 0$. For example, the function $e^{-\frac{1}{x}}$ is not zero anywhere yet its asymptotic expansion converges to zero, as $x \to 0$ in the right-hand plane.

We now prove an important theorem about the term-wise integration of such series but we will see later term-wise differentiation does not work entirely in the same way.

Theorem 2.3.4. *If $f(x)$ is holomorphic in a sector S_α^β then*

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r, \quad x \in S_\alpha^\beta$$

implies

$$\int_0^x f(t)\, dt \sim \sum_{r=0}^{\infty} \frac{a_r}{r+1} x^{r+1},$$

through a path of integration that lies in S_α^β.

Proof. From the definition of asymptotic series

$$f(x) = \sum_{r=0}^{m} a_r x^r + E(x, m)x^m, \tag{2.5}$$

where

$$\lim_{x \to 0} E(x, m) = 0 \ , \ x \in S_\alpha^\beta.$$

We know that $\lim_{x\to 0} f(x)$, $x \in S_\alpha^\beta$ exists, so the integral from 0 to x of the function in equation (2.5) also exists and is independent of the path in S_α^β. We take this path to be a straight line segment. As the function $E(x, m)$ is holomorphic and tends to zero at the origin, $E(x, m)$ is then uniformly continuous on the closure of S_α^β, provided $E(x, m)$ is defined to be 0 at the origin. This implies that the right-hand side of the integral

$$\int_0^x E(t, m)t^m \, dt = x^{m+1} \int_0^1 E(\tau x, m)\tau^m \, d\tau \ , \ t = \tau x \ , \ 0 \le \tau \le 1$$

tends to zero uniformly in S_α^β as $x \to 0$. Finally we integrate (2.5) from 0 to x and we obtain the required result. □

As mentioned earlier, there is not a completely analogous property for term-wise differentiation. In general for functions with asymptotic power series that are restricted to a single ray or fixed argument, then term-wise differentiation of the series does not work. For instance, let $f(x) = e^{-1/x}\cos(e^{1/x})$. For $x > 0$, we have $f(x) \sim 0$ as $x \to 0$. However

$$f'(x) = x^{-2}[e^{-1/x}\cos(e^{1/x}) + \sin(e^{1/x})]$$

does not have an asymptotic power series on the positive real axis, because

$$\lim_{x\to 0} f'(x)$$

does not exist.

We now consider a theorem that uses the term-wise differentiation of asymptotic power series not restricted to a single ray.

Theorem 2.3.5. *If $f(x)$ is holomorphic in a sector S defined by the inequalities $0 < |x| \le x_0 < \infty$, $\theta_1 \le \arg x \le \theta_2$, with $\theta_2 > \theta_1$ and if $f(x)$ has an asymptotic expansion,*

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r, \quad x \in S,$$

then

$$f'(x) \sim \sum_{r=0}^{\infty} r a_r x^{r-1}$$

in every proper sub-sector $S^* : \theta_1 < \theta_1^* \le \arg x \le \theta_2^* < \theta_2$.

Proof. By assumption the equation (2.5) is valid in S. We then differentiate

$$f'(x) = \sum_{r=0}^{m} r a_r x^{r-1} + m x^{m-1} E(x, m) + x^m E'(x, m). \tag{2.6}$$

Now we denote by α, as a positive number so small that the circle C_x of radius $|x|\alpha$ about x lies in S, $\forall x \in S^*$. Also let $M(x,m)$ be the maximum of $|E(x,m)|$ on C_x. Then, since $E(x,m)$ is holomorphic on and inside the circle, we use Cauchy's integral formula to bound $E'(x,m)$

$$E(x,m) = \frac{1}{2\pi i}\oint_{C_x} \frac{E(y,m)}{y-x}dy,\ E'(x,m) = \frac{1}{2\pi i}\oint_{C_x} \frac{E(y,m)}{(y-x)^2}dy,$$

$$|E'(x,m)| \leq \frac{1}{2\pi}\left|\oint_{C_x} \frac{E(y,m)}{(y-x)^2}dy\right| = \frac{M(x,m)2\pi|x|\alpha}{2\pi(|x|\alpha)^2},$$

$$\left|\frac{dE(x,m)}{dx}\right| \leq \frac{M(x,m)}{|x|\alpha}. \tag{2.7}$$

Hence using (2.7) and equation (2.6)

$$\left|f'(x) - \sum_{r=0}^{m} r a_r x^{r-1}\right| \leq |x|^{m-1}\left[m|E(x,m)| + \frac{M(x,m)}{\alpha}\right],$$

where this expression tends to zero as $x \to 0$ in S^*, obtaining the required result. □

2.3.3 Existence properties

Now we link asymptotic series to formal power series with some interesting theorems that will later help out in the next chapter.

Theorem 2.3.6. *If $f(x)$ is holomorphic in the sector S_α^β and if all of the limits below exist*

$$f_r = \lim_{\substack{x\to 0\\ x\in S}} f^{(r)}(x)\ ,\ r = 0,1,\ldots, \tag{2.8}$$

then

$$f(x) \sim \sum_{r=0}^{\infty} \frac{f_r}{r!}x^r. \tag{2.9}$$

Note, if only a finite number m of limits exist of the form (2.8), then $f(x)$ has an asymptotic expansion of m terms. For example, $f(x) = e^x + x^3\log(x)$ has the following asymptotic expansion to four terms

$$e^x + x^3\log(x) = 1 + x + \frac{x^2}{2} + E(x,2)x^2.$$

Another important theorem discusses the use of formal power series and functions, similar to converse Theorem 2.3.1.

Theorem 2.3.7. *Corresponding to every formal power series*

$$\sum_{r=0}^{\infty} a_r x^r$$

with every sector S_α^β, $\exists$ a function $f(x)$ that is holomorphic in S_α^β for $|x| \leq x_0$ (x_0 is any real constant), such that

$$f(x) \sim \sum_{r=0}^{\infty} a_r x^r,\ x \in S_\alpha^\beta.$$

Chapter 3

Main Asymptotic Existence Theorem

3.1 Theorem statement

The following theorem stated below (Theorem 3.1.1) looks at a general class of ODEs, where Theorem 3.1.1 says the ODE admits an asymptotic power series then there exists a solution to the ODE with this series in some sector. The proof is based on the work of Wasow (1965), but will include more detail and a different approach is taken.

Theorem 3.1.1. *Let S be an open sector of the complex x-plane with a vertex at the origin O and a positive central angle not exceeding $\frac{\pi}{q+1}$, where $q \in \mathbb{Z}^+$. Let $f(x,z)$ be an N-dimensional vector function of x and an N-dimensional vector z with the following properties.*

(a) $f(x,z)$ is a polynomial in the components z_j for z, where $j = 1, 2, ..., N$. The coefficients are also holomorphic in x in the region

$$0 < x_0 \leq |x| < \infty,\ x \in S \text{ and } x_0 \in \mathbb{R}. \tag{3.1}$$

(b) The coefficients of the polynomial $f(x,z)$ have an asymptotic series expansion in powers of x^{-1}, as $x \to \infty$, in S.

(c) Let $f_j(x,z)$ denotes the components of $f(x,z)$ and the Jacobian matrix J ($N \times N$ matrix) have components

$$J_{jk} = \lim_{\substack{x \to \infty \\ x \in S}} \left. \frac{\partial f_j}{\partial z_k} \right|_{z=0}, \tag{3.2}$$

for $j, k = 1, 2, \ldots, N$. Then all of J's eigenvalues $\lambda_j, j = 1, 2, \ldots, N$ are non-zero.

(d) The differential equation

$$x^{-q} y' = f(x,y) \tag{3.3}$$

is formally satisfied by a power series of the form

$$y = \sum_{r=1}^{\infty} y_r x^{-r}. \tag{3.4}$$

Then there exists, for sufficiently large x in S, a solution for $y = \phi(x)$ in the equation (3.3) such that in every proper subsector of S,

$$\phi(x) \sim \sum_{r=1}^{\infty} y_r x^{-r} \ , \ x \to \infty. \tag{3.5}$$

3.2 Proof

We will only consider the case in which the eigenvalues of J (3.2) are distinct.

3.2.1 Transformation for the differential equation

We break down f defined Theorem 3.1.1 into its constant, linear and non-linear parts

$$a(x) := f(x,0) = \begin{bmatrix} f_1(x,0) \\ f_2(x,0) \\ \vdots \\ f_N(x,0) \end{bmatrix} \ , \ A(x) := \begin{bmatrix} \frac{\partial f_1}{\partial z_1}\Big|_{z=0} & \frac{\partial f_1}{\partial z_2}\Big|_{z=0} & \cdots & \frac{\partial f_1}{\partial z_n}\Big|_{z=0} \\ \frac{\partial f_2}{\partial z_1}\Big|_{z=0} & \frac{\partial f_2}{\partial z_2}\Big|_{z=0} & \cdots & \frac{\partial f_2}{\partial z_n}\Big|_{z=0} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1}\Big|_{z=0} & \frac{\partial f_n}{\partial z_2}\Big|_{z=0} & \cdots & \frac{\partial f_n}{\partial z_n}\Big|_{z=0} \end{bmatrix}, \tag{3.6}$$

$$g(x,z) := f(x,z) - a(x) - A(x)z \text{ (g contains all non-linear terms).} \tag{3.7}$$

The differential equation (3.3) becomes

$$x^{-q}y' = a(x) + A(x)y + g(x,y). \tag{3.8}$$

From assumption (b) of Theorem 3.1.1, the coefficients of $f(x,z)$ have asymptotic series expansions, in particular the linear coefficients are expressed as

$$A(x) \sim \sum_{r=0}^{\infty} A_r x^{-r} \ , \ x \to \infty \ , \ x \in S.$$

As $x \to \infty$ for $x \in S$, the nature of the solution for (3.8) is driven by the leading coefficient of $A(x)$, A_0. Without loss of generality we take A_0 to be diagonal due to the following reasoning.
Let an invertible linear transformation be applied to the differential equation (3.8) with $y = K\tilde{y}$. We first define the diagonal $N \times N$ matrix containing all the eigenvalues in (3.2) as $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ and K as the $N \times N$ matrix containing the corresponding eigenvectors $v_1, v_2, ..., v_n$.
Applying these ideas below, as $x \to \infty$ for $x \in S$

$$\begin{aligned} x^{-q}K\tilde{y}' &= a(x) + \left(A_0 + A_1 x^{-1} + \cdots\right) K\tilde{y} + g(x, K\tilde{y}) \\ \Rightarrow x^{-q}\tilde{y}' &= K^{-1}a(x) + K^{-1}A_0 K\tilde{y} + K^{-1}g(x, K\tilde{y}). \end{aligned}$$

Let $\tilde{a}(x) := K^{-1}a(x)$, $\tilde{A_0} := K^{-1}A_0 K = \Lambda$ and $\tilde{g}(x, K\tilde{y}) := K^{-1}g(x, K\tilde{y})$. This means we have a differential equation after applying this linear transformation, which has the same form as the original differential equation (3.8), just with A_0 replaced by its diagonalisation $\tilde{A_0}$.

Now using Theorem 2.3.7, $\exists$ a vector function $\phi(x)$ which is holomorphic for $|x| \geq x_0$ in S such that

$$\phi(x) \sim \sum_{r=1}^{\infty} y_r x^{-r} \ , \ x \to \infty \text{ in } S. \tag{3.9}$$

By constructing $\phi(x)$ for a larger sector than S and applying Theorem 2.3.5, we deduce equation (3.9) is term-wise differentiable in S.
Substituting

$$u = y - \phi(x), \quad y = u + \phi(x), \tag{3.10}$$

in the differential equation (3.8) gives

$$x^{-q}u' = a(x) + A(x)u + A(x)\phi(x) - x^{-q}\phi(x) + g(x, u + \phi(x)). \tag{3.11}$$

Since the series (3.4) solves the differential equation (3.8) formally, we use this series with equation (3.9) to define $b(x)$

$$b(x) := a(x) + A(x)\phi(x) + g(x, \phi(x)) - x^{-q}\phi(x),$$

where

$$b(x) \sim 0 \ , \ x \to \infty \ , \ x \in S. \tag{3.12}$$

Then equation (3.11) is written as

$$x^{-q}u' = b(x) + A(x)u + g(x, u + \phi(x)) - g(x, \phi(x)). \tag{3.13}$$

Consider the Taylor series expansion of $g(x, u + \phi(x))$

$$g(x, u + \phi(x)) - g(x, \phi(x)) = A^*(x)u + h(x, u), \tag{3.14}$$

where $h(x, u)$ is a non-linear polynomial in component of u_j from $j = 1, ..., N$ and its coefficients admits an asymptotic power series in x^{-1}, as $x \to \infty$ in S.
We denote the components of $g(x, u + \phi(x))$ by $g_j(x, u + \phi(x))$ for $j = 1, \ldots, N$. Then we define $A^*(x)$ as a $N \times N$ matrix with components

$$A^*(x)_{jk} = \left. \frac{\partial g_j(x, u + \phi(x))}{\partial u_k} \right|_{u=0}, \tag{3.15}$$

for $j, k = 1, 2, \ldots, N$. Looking at the asymptotic series for $\phi(x)$ in equation (3.9) we see that

$$\lim_{x \to \infty} \phi(x) = 0, \quad x \in S,$$

and as $g(x, z)$ contains only non-linear terms of z, then $\left. \frac{\partial g_j(x,z)}{\partial u_k} \right|_{z=0} = 0, \forall j = 1, \ldots, N$. Similarly in equation (3.15)

$$\lim_{x \to \infty} \left(\left. \frac{\partial g_j(x, u + \phi(x))}{\partial u_k} \right|_{u=0} \right) = 0, \quad x \in S,$$

so

$$\lim_{x \to \infty} A^*(x) = 0, \, x \in S.$$

Now looking back at the transformed differential equation (3.13), we substitute the relevant expressions from equation (3.14) and define a $N \times N$ matrix $B(x) := A(x) + A^*(x)$, to obtain

$$\lim_{x\to\infty} B(x) = \Lambda,\ x \in S \tag{3.16}$$

and

$$x^{-q}u' = b(x) + B(x)u + h(x,u). \tag{3.17}$$

Theorem 3.1.1 will be proved by showing that this transformed differential equation (3.17) has a solution where $u(x) \sim 0$, as $x \to \infty, x \in S$.

3.2.2 Integral form of the differential equation

For the transformed differential equation (3.17), we will manipulate it to show $u(x) \sim 0$ and therefore show that $y \sim \phi(x), x \to \infty$. This will require some bounds and estimates on functions in equation (3.17) and having it in the integral form makes it easier to work with.
First, we express equation (3.17) as

$$x^{-q}u' = \Lambda u + p(x,u), \tag{3.18}$$

where

$$p(x,u) = b(x) + (B(x) - \Lambda)u + h(x,u). \tag{3.19}$$

Here for large x and small u, $p(x,u) << u$, since $b(x) \sim 0$, $B(x) - \Lambda \to 0$ for $x \to \infty$ and $h(x,u)$ is a non-linear polynomial in u that tends to zero much more quickly than u. Also, note that if $p(x,u) \equiv 0$, $u \equiv 0$ is a solution.

Variation of parameters method

We apply a standard method for solving linear ODEs called variation of parameters [Lakshmikantham and Deo (1998)]. From equation (3.18) we first solve the homogeneous equation

$$x^{-q}u' = \Lambda u\ ,\ \frac{u'}{u} = \Lambda x^q$$
$$\Rightarrow \ln|u| = \frac{\Lambda x^{q+1}}{q+1} + D,\ D \text{ is a constant}$$
$$\Rightarrow u = k\exp\left[\frac{x^{q+1}}{q+1}\Lambda\right],\ M(x) := \exp\left[\frac{x^{q+1}}{q+1}\Lambda\right],$$

$$u_H := kM(x),\ \text{ where } k \in \mathbb{R} \text{ is a constant.} \tag{3.20}$$

Now we substitute $u(x) = M(x)C(x)$ into the full equation (3.18) solving the inhomogeneous equation and use the fact $M'(x) = \Lambda x^q M(x)$ to find $C(x)$. So,

$$x^{-q}[M(x)C(x)]' = x^{-q}M'(x)C(x) + x^{-q}M(x)C'(x) = \Lambda M(x)C(x) + p(x,u).$$

Now we rearrange for $C'(x)$ and integrate with respect to t from a fixed point a to x

$$C'(x) = x^q M^{-1}(x)p(x,u),$$
$$C(x) = \int_a^x M^{-1}(t)t^q p(t,u(t))\,dt,$$

then the solution to the inhomogeneous equation is

$$u_{IH} := \int_a^x M(x)M^{-1}(t)t^q p(t, u(t))\, dt. \tag{3.21}$$

As $u(x)$ is a N dimensional vector function, we have N scalar integrals. Instead of taking the same path of integration for all of them, we shall choose individual paths $\gamma_j(x), j = 1, \ldots, N$ which all end at x. These paths will be defined in the next Section 3.2.3 and the set of all of them are denoted by $\Gamma(x)$.
Then, we put this all together and substitute the constant $k = 0$. Since when we construct the estimates for $u(x)$ later in this section, it is a difference of successive estimates so $kM(x)$ in u_H (3.20) vanishes. So, we obtain

$$u(x) = u_H + u_{IH} = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1}\Lambda\right] t^q p(t, u(t))\, dt. \tag{3.22}$$

Integral operator form

Now we introduce the idea of a non-linear integral operator $\mathcal{P}$, where $\mathcal{P}$ is the integral in equation (3.22) acting on u. So we have

$$u(x) = \mathcal{P}(u(x)) \text{ where } \mathcal{P}y(x) = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1}\Lambda\right] t^q p(t, y(t))\, dt. \tag{3.23}$$

Our ultimate aim is to show equation (3.23) has a solution asymptotic to zero by using the method of successive approximations, similar to finding a root for a function using the Newton-Raphson method [Verbeke and Cools (1995)]. We define a sequence of vector functions, $u_r(x)$ for $r = 0, 1, \ldots,$

$$u_0 \equiv 0,\ u_{r+1} = \mathcal{P}u_r,\ r \geq 0. \tag{3.24}$$

Also we look at the convergence of this sequence, which will be established by estimating the differences between successive approximations and it should tend to zero,

$$u_{r+1} - u_r = \mathcal{P}u_r - \mathcal{P}u_{r-1}. \tag{3.25}$$

In the subsequent sections we will establish a suitable set for the paths of integration, $\Gamma(x)$ and some inequalities to help us show the existence and convergence for the equation (3.23).

3.2.3 Paths of integration

Looking at the integral (3.22), in particular the exponential function, we choose paths of integration $\gamma_j(x)$, $j = 1, 2, \ldots, N$ such that along the paths the exponential function is bounded. To construct this, we describe the plane of an auxiliary variable τ from the t-plane as

$$\tau = t^{q+1}. \tag{3.26}$$

Then we let the image of x under this mapping be

$$\xi = x^{q+1}. \tag{3.27}$$

The image of the sector S of the t-plane is a sector Ω in the τ-plane, with the central angle $< \pi$, matching the assumption that the central angle of S does not exceed $\frac{\pi}{q+1}$, where $q \in \mathbb{Z}^+$ in Theorem 3.1.1. We consider $2N$ rays in the τ-plane along which $Re(\tau\lambda_j) = 0$, for the eigenvalues $\lambda_j, j = 1, 2, ..., N$ for the matrix Λ defined in Section 3.2.1. Without loss of generality, none of these rays lie on the boundary of Ω.
We consider two classes of the eigenvalues λ_j, $j = 1, 2, ..., N$.

1.
$$Re(\tau\lambda_j) < 0 \text{ in } \Omega, \forall j = 1, ..., j_1, \text{ where } 0 \leq j_1 \leq N \tag{3.28}$$

2.
$$Re(\tau\lambda_k) > 0 \text{ in } \Omega, \forall j_1 < k \leq N \text{ or exactly one } \lambda_k \text{ where } Re(\tau\lambda_k) = 0 \text{ in } \Omega \tag{3.29}$$

Let ξ_1 be a point on the bisector of Ω such that $|\xi_1| > x_0^{q+1}$, where x_0 is defined as indicated in part a) of the Theorem 3.1.1 statement (3.1). Also, let Ω^* denote the closed sector with its vertex at ξ_1 and boundary rays parallel to Ω. Therefore, $\Omega^* \subset \Omega$ and $|\tau| > x_0^{q+1}$, $\tau \in \Omega$. The paths are the following in this region, where $\xi \in \Omega^*$:

1. For eigenvalues in the first class (3.28), we let $\delta_j(\xi)$ be the directed segment from ξ_1 to ξ, where $Re(\tau\lambda_j)$ decreases along this segment. The analogous path in the t-plane is $\gamma_j(x)$ for $j \leq j_1$.

2. For eigenvalues in the second class (3.29), $\forall j_1 < k \leq N$ we choose a ray l_k from the origin into Ω where $Re(\tau\lambda_k) > 0$. Let $\delta_k(\xi)$ be an infinite half-line in Ω^* that starts from infinity to ξ, parallel to l_k. Like the first path, $Re(\tau\lambda_k)$ decreases along $\delta_k(\xi)$. Again, the analogous path in the t-plane is $\gamma_k(x)$ for $k > j_1$.

See Figure 3.1 and 3.2 below for an illustration of the paths in the t and τ planes.

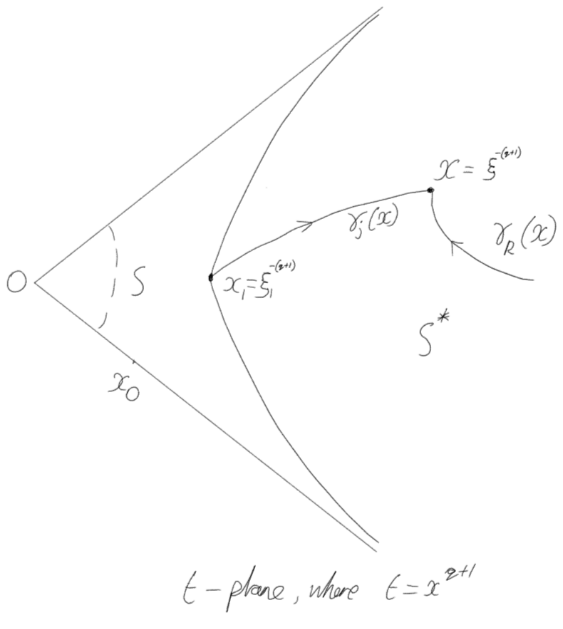

Figure 3.1: Paths in t-plane

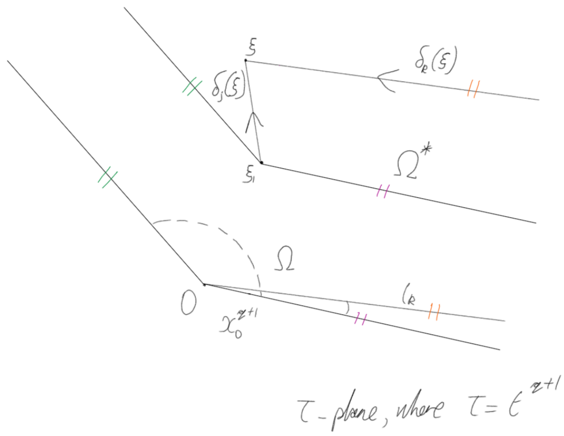

Figure 3.2: Paths in τ-plane

Note that, sector S^* in the t-plane is analogous to Ω^*. So, $S^* \subset S$ and is bounded by two curves meeting at $x_1 = \xi_1^{\frac{1}{q+1}}$ with boundary rays of S as asymptotes (see Figure 3.1). This implies that S^* contains all the points at a sufficiently large distance from the origin of any closed subsector of S. We see that S^* is a proper subsector of S and recall that Theorem 3.1.1 will be proved on proper subsectors of S.

3.2.4 Inequalities

These inequalities play a crucial role in the final stages of the proof of Theorem 3.1.1.

Lemma 3.2.1. *Let*

$$\lambda_0 = \min_{j=1,\dots,N} |\lambda_j|. \tag{3.30}$$

Then, $\exists$ a positive constant μ, independent of λ_0, j and ξ_1, such that

$$Re\left[\frac{(x^{q+1} - t^{q+1})\lambda_j}{q+1}\right] \leq -|x^{q+1} - t^{q+1}|\lambda_0\mu(q+1) \tag{3.31}$$

for $t \in \gamma_j(x)$, $x \in S^*$ (from Section 3.2.3) and where q is a positive integer.

Proof. Let $\xi = x^{q+1}$ and $\tau = t^{q+1}$ be paths as constructed in Section 3.2.3 and images for Lemma 3.2.1. By these constructions, we see that $(\tau - \xi)\lambda_j$ lies in a closed proper subsector of the right half-plane ($Re > 0$). So for its opposite sign, $(\xi - \tau)\lambda_j$ lies in the left half-plane ($Re < 0$). Then

$$0 < \cos\left[\arg(\tau - \xi)\lambda_j\right] < 1,$$
$$-1 < \cos\left[\arg(\xi - \tau)\lambda_j\right] \leq -\mu < 0,$$
$$Re\left[\frac{(\xi-\tau)\lambda_j}{q+1}\right] = \frac{|\xi - \tau||\lambda_j|}{q+1}\cos\left[\arg(\xi-\tau)\lambda_j\right] \leq -\frac{|\xi-\tau|}{q+1}\lambda_0\mu \leq -|x^{q+1} - t^{q+1}|\lambda_0\mu(q+1)$$

as required.

∎

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Lemma 3.2.1 will help us prove the next crucial lemma, Lemma 3.2.2. First let us define a vector norm which will be used for subsequent calculations unless stated otherwise.

Definition 3.2.1. *Let v be a column vector with components $v_1, \ldots, v_n$. Then the norm $||v||$ is defined as*

$$||v|| = \max_{j \in \{1,\ldots,n\}} |v_j|.$$

Lemma 3.2.2. *Let $\chi(x)$ be a vector function holomorphic for $x \in S^*$ satisfying an inequality of the form*

$$||\chi(x)|| \leq c|x|^{-m}, \tag{3.32}$$

where m is a non-negative integer and c is a constant. Then

$$\psi(x) = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1}\Lambda\right] t^q \chi(t)\, dt \tag{3.33}$$

is holomorphic in S^* and satisfies the inequality

$$||\psi(x)|| \leq Kc|x|^{-m}, \tag{3.34}$$

where K is a constant independent of $\chi(t)$ but dependent on m.

Proof. First we consider the components ψ_j of ψ for $j = 1, 2, ..., N$. This means it is enough to show that

$$|\psi_j(x)| \leq Kc|x|^{-m}, \text{ for } j = 1, 2, ..., N. \tag{3.35}$$

By considering the mappings defined for t and x with equations (3.26) and (3.27) respectively, we change the variables of integration, $d\tau = (q+1)t^q dt$ and the path will be $\delta_j(\xi)$ as defined in Section 3.2.3. We also look at each element of the matrix Λ from this use λ_j for $\psi_j(x)$. This means the integral (3.33) for $\psi_j(x)$ becomes

$$\psi_j(x) = \frac{1}{q+1} \int_{\delta_j(\xi)} \exp\left[\frac{(\xi - \tau)\lambda_j}{q+1}\right] \chi_j(t)\, d\tau. \tag{3.36}$$

Next we take the modulus of (3.36) with Lemma 3.2.1's result (3.31) and the assumption established in this Lemma 3.2.2 (3.32) to obtain

$$|\psi_j(x)| \leq \frac{c}{q+1} \int_{\delta_j(\xi)} \exp\left[\frac{-|\xi - \tau|\lambda_0 \mu}{q+1}\right] |\tau|^{-\frac{m}{q+1}} |d\tau|. \tag{3.37}$$

Before proving the inequality (3.35) by using equation (3.37), we construct a strategy for proof. We require $\psi_j(x)$ to be holomorphic in $S^*, j = 1, \ldots, N$. From equation (3.36), we take out the terms not involving τ and then it is enough to show that

$$\int_{\delta_j(\xi)} \exp\left[\frac{-\tau\lambda_j}{q+1}\right] \chi_j(t)\, d\tau \tag{3.38}$$

are holomorphic functions of ξ in Ω^* and therefore $S^*, \forall j = 1, ..., N$. Note that $\chi_j(t)$ is well defined and bounded $\forall j = 1, \ldots, N$ by the assumption in the lemma (3.32). We consider the two eigenvalue cases we defined in Section 3.2.3 and show equation (3.38) is holomorphic in both cases.

For the first case (3.29), recall that the path of $\delta_j(\xi)$ is a finite line segment from ξ_1 to ξ and $Re(\tau\lambda_j) < 0$ in Ω, $\forall j = 1, \dots, j_1 \leq N$ (See figure 3.2). Therefore, the exponential in the integrand (3.38) is well defined along $\delta_j(\xi)$. Thus, the integrand is well-defined and bounded on Ω, Ω^* and therefore on S^*. So, $\psi_j(x)$ is holomorphic on S^* for $j = 1, \dots, j_1 \leq N$.

For the second case (3.28), recall that the $\delta_k(\xi)$ is a semi infinite line segment from ∞ to ξ and $Re(\tau\lambda_k) > 0$ in Ω, $\forall j_1 < k \leq N$ or exactly one λ_k where $Re(\tau\lambda_k) = 0$ in Ω (See figure 3.2). Then we apply the same idea as the first eigenvalue case, looking at the exponential in the integrand of (3.38). However this time, we see that the integrand becomes exponentially small for large values of τ and so the integrand is holomorphic on S^* as required.

This forms the strategy of the proof, where now we consider the classes defined in (3.28) and (3.29) separately. First, we look at (3.28) case.

We set the difference between the mappings to be some complex number $\rho e^{i\alpha} := \xi - \tau$, with $\rho = |\xi - \tau|$ and α as the directional angle of the path of $\delta_j(\xi)$. Substituting in ρ into equation (3.37) we obtain

$$|\psi_j(x)| \leq \frac{c}{q+1} \int_0^{|\xi-\xi_1|} \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] |\tau|^{-\frac{m}{q+1}} \, d\rho =: I. \tag{3.39}$$

We split the integral I into two with paths of equal length, where $I = I_1 + I_2$ as defined below

$$I_1 = \frac{c}{q+1} \int_0^{\frac{|\xi-\xi_1|}{2}} \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] |\tau|^{-\frac{m}{q+1}} \, d\rho, \tag{3.40}$$

$$I_2 = \frac{c}{q+1} \int_{\frac{|\xi-\xi_1|}{2}}^{|\xi-\xi_1|} \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] |\tau|^{-\frac{m}{q+1}} \, d\rho. \tag{3.41}$$

First look at equation (3.40) with $\rho \leq \frac{|\xi-\xi_1|}{2}$. We need to find a lower bound for $|\tau|$. Our claim is $|\tau| \geq \frac{|\xi|}{2}$. To show this we refer back to Section 3.2.3, where from the construction of the sector Ω in the τ-plane, we know it has a central angle $< \pi$ and ξ_1 is the vertex of the bisector of Ω, namely Ω^* (See Figure 3.2). This means $\arg \xi_1 < \frac{\pi}{2}$, so $|\xi - \xi_1| \leq |\xi|$ and we use the triangle inequality for $\xi, \tau \in \Omega^*$

$$\begin{aligned} |\xi| &\leq |\xi - \tau| + |\tau| = \rho + |\tau| \\ &\leq \frac{|\xi - \xi_1|}{2} + |\tau| \leq \frac{|\xi|}{2} + |\tau|. \end{aligned}$$

Therefore, $|\tau|^{-1} \leq 2|\xi|^{-1}$ as required. See Figure 3.3 for an illustration of this on the τ-plane.

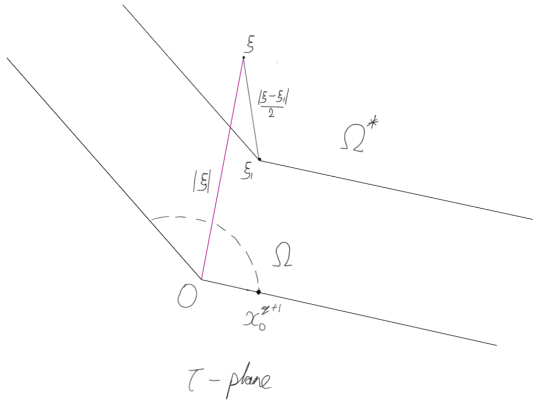

Figure 3.3: Lengths in τ-plane

We have enough to bound the integral (3.40)

$$I_1 \leq \frac{c}{q+1} 2^{\frac{m}{q+1}} |\xi|^{-\frac{m}{q+1}} \int_0^\infty \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] d\rho$$

$$= \frac{c}{q+1} 2^{\frac{m}{q+1}} |\xi|^{-\frac{m}{q+1}} \left[-\frac{q+1}{\lambda_0\mu} e^{-\frac{\rho\lambda_0\mu}{q+1}}\right]_{\rho=0}^{\rho=\infty}$$

$$= \frac{c 2^{\frac{m}{q+1}}}{\lambda_0\mu} |\xi|^{-\frac{m}{q+1}} = \frac{c 2^{\frac{m}{q+1}}}{\lambda_0\mu} |x|^{-m},$$

by using the mapping of ξ as defined in equation (3.27). Therefore, the inequality (3.35) is satisfied with a constant $K_1 = \frac{2^{\frac{m}{q+1}}}{\lambda_0\mu}$,

$$I_1 \leq K_1 c |x|^{-m} \tag{3.42}$$

as required.
Now, we look at I_2 (3.41), where $\rho > \frac{|\xi-\xi_1|}{2}$ and we see that $|\tau| \geq |\xi_1|$, by definition of ρ with $|\xi_1|$ as the starting point of the whole path of integration defined in Section 3.2.3 (Figure 3.2). Then, we can use a similar estimate to what we did for I_1 to obtain the inequality (3.42),

$$I_2 \leq \frac{c}{q+1} |\xi_1|^{-\frac{m}{q+1}} \int_{\frac{|\xi-\xi_1|}{2}}^\infty \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] d\rho = \frac{c}{q+1} |\xi_1|^{-\frac{m}{q+1}} \left[-\frac{q+1}{\lambda_0\mu} e^{-\frac{\rho\lambda_0\mu}{q+1}}\right]_{\rho=\frac{|\xi-\xi_1|}{2}}^{\rho=\infty},$$

$$I_2 \leq \frac{c}{\lambda_0\mu} |\xi_1|^{-\frac{m}{q+1}} \exp\left[\frac{-|\xi-\xi_1|\lambda_0\mu}{2(q+1)}\right]. \tag{3.43}$$

To show that the right hand side of the inequality is less than or equal to $Kc|x|^{-m}$ for a given K, we look at two cases within inequality (3.43).

1. $|\xi - \xi_1| \leq \frac{|\xi|}{2}$

2. $|\xi-\xi_1| > \frac{|\xi|}{2}$

In the first case we use the triangle inequality to obtain an upper bound on ξ_1,

$$|\xi| \leq |\xi-\xi_1| + |\xi_1| \leq \frac{|\xi|}{2} + |\xi_1|$$
$$\Rightarrow \frac{|\xi|}{2} \leq |\xi_1|, \, |\xi_1| \leq 2|\xi|^{-1}.$$

Also the exponential function is a strictly increasing function with $\exp(0) = 1$, we can then use the inequality (3.43) to obtain

$$I_2 \leq \frac{c2^{\frac{m}{q+1}}}{\lambda_0\mu}|\xi|^{-\frac{m}{q+1}} = K_2|x|^{-m}. \tag{3.44}$$

Like the inequality (3.42) for I_1, I_2 satisfies the inequality (3.35) as required with $K_2 = K_1$, the same constant as I_1.

In the second case we multiply the inequality (3.43) by $|\xi|^{\frac{m}{q+1}}$ and use the fact the exponential function is a strictly increasing function, $e^{-|\xi-\xi_1|} < e^{-\frac{|\xi|}{2}}$. Hence

$$|\xi|^{\frac{m}{q+1}} I_2 \leq \frac{c}{\lambda_0\mu}|\xi_1|^{-\frac{m}{q+1}} f(|\xi|), \tag{3.45}$$

where

$$f(X) = X^{\frac{m}{q+1}} \exp\left[\frac{-X\lambda_0\mu}{4(q+1)}\right], \, X = |\xi| \geq 0.$$

We find the maximum of $f(X)$ through differentiation,

$$f^{'}(X) = \exp\left[\frac{-X\lambda_0\mu}{4(q+1)}\right]\left[\frac{m}{q+1}X^{\frac{m}{q+1}-1} - \frac{\lambda_0\mu}{4(q+1)}X^{\frac{m}{q+1}}\right] = 0$$
$$\Rightarrow \frac{m}{q+1}X^{-1} = \frac{\lambda_0\mu}{4(q+1)}, \, X = |\xi| = \frac{4m}{\lambda_0\mu}.$$

Substituting this maximum value X into the inequality (3.45) and rearranging to get an upper bound of I_2 in terms of x using its mapping defined in equation (3.27),

$$I_2 \leq \frac{c}{\lambda_0\mu}|\xi_1|e^{-\frac{m}{q+1}}\left(\frac{4m}{\lambda_0\mu}\right)^{-\frac{m}{q+1}}|\xi|^{-\frac{m}{q+1}} = K_3|x|^{-m}. \tag{3.46}$$

This inequality (3.46) satisfies the inequality (3.35) with the constant $K_3 = \frac{c}{\lambda_0\mu}|\xi_1|e^{-\frac{m}{q+1}}\left(\frac{4m}{\lambda_0\mu}\right)^{-\frac{m}{q+1}}$ as required. Adding these quantities together for the two cases for I_2, we have an inequality for I_2 that satisfies the inequality (3.35).

We then conclude by combining the inequality for I_1 (3.42) with the two inequalities found for I_2 (3.44) and (3.46) to obtain the inequality for I,

$$I \leq Kc|x|^{-m}$$

where a constant K is a combination of the constants K_1, K_2 and K_3 as required.

Having proved Lemma 3.2.2 for the first class (3.29), we now consider Lemma 3.2.2 for the second class (3.29). First of all, we have to show the existence of a positive real number p that is independent of k (where k is the eigenvalue index for $Re(\tau\lambda_k) > 0$ defined in (3.29)), such that on the path $\delta_k(\xi)$

$$|\tau| \geq p|\xi|, \quad \tau \in \delta_k(\xi). \tag{3.47}$$

To show this, let P be the point where its perpendicular line from the origin meets the path $\delta_k(\xi)$. We shall consider two cases for P.

1. $P \notin \delta_k(\xi)$
2. $P \in \delta_k(\xi)$

For the first case, the path from the origin to ξ is the shortest path to $\delta_k(\xi)$ (see Figure 3.4), which means that $|\tau| \geq |\xi|$ for $\tau \in \delta_k(\xi)$. So the inequality (3.47) is true for $p = 1$.

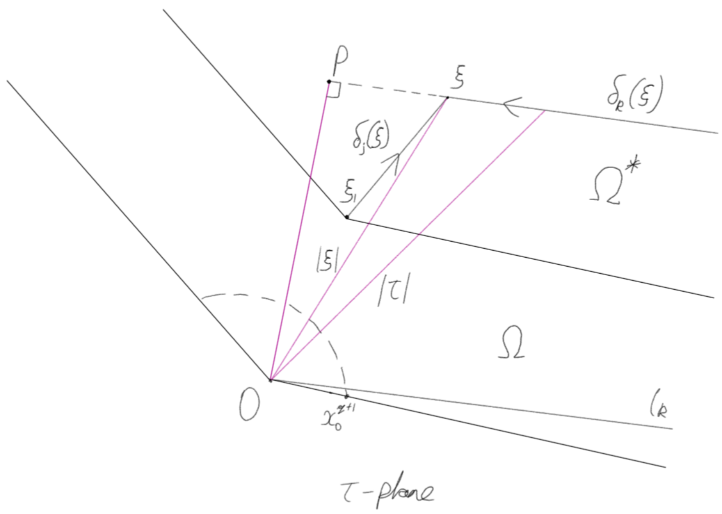

Figure 3.4: Case when $P \notin \delta_k(\xi)$

For the second case where $P \in \delta_k(\xi)$ (see Figure 3.5) we see that the angle $O\xi P$ is greater than or equal to the smaller of the positive angles l_k forms with the boundary rays of Ω. Let the minimum of these angles, which are both acute be called β. Then, as OP is the shortest distance from the origin to P like the first case, however now as $P \in \delta_k(\xi)$ we obtain

$$|\tau| \geq OP \geq |\xi| \sin \beta.$$

So inequality (3.47) is true for the second case when $p = \sin \beta > 0$ and therefore is true for all of the eigenvalues in the second class defined in (3.29).
With this, we have enough to tackle equation (3.37) in the same fashion as we did for the eigenvalues in the first class (3.28). We define $\rho = |\xi - \tau|$ again and using the lower bound for τ involving ξ only from inequality (3.47), we use the same method as we did in estimating integral I (3.39). However we do not have to split the path into two equal paths, because our lower bound of τ (3.47) is for the

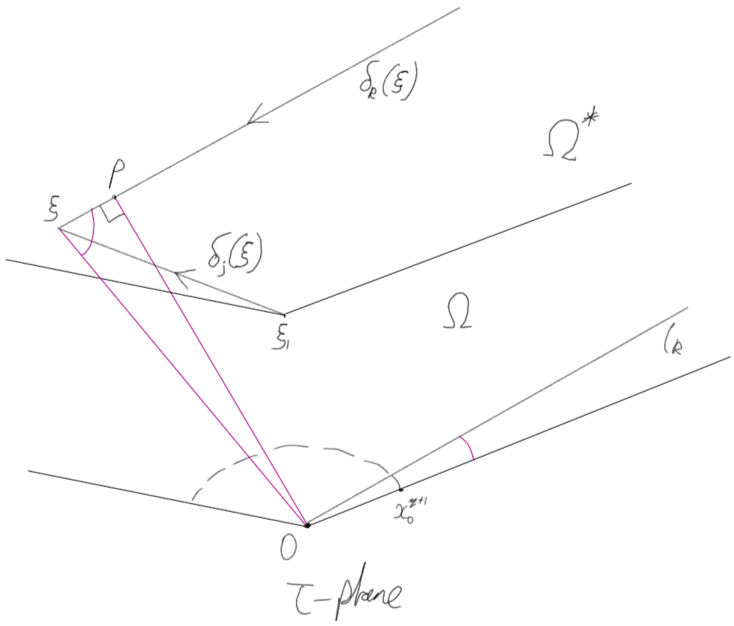

Figure 3.5: Case when $P \in \delta_k(\xi)$

whole path. Then substituting this lower bound into equation (3.39)

$$|\psi_j(x)| \leq \frac{c}{q+1} p^{-\frac{m}{q+1}} |\xi|^{-\frac{m}{q+1}} \int_0^\infty \exp\left[\frac{-\rho\lambda_0\mu}{q+1}\right] d\rho$$

$$= \frac{c}{q+1} p^{-\frac{m}{q+1}} |\xi|^{-\frac{m}{q+1}} \left[-\frac{q+1}{\lambda_0\mu} e^{-\frac{\rho\lambda_0\mu}{q+1}}\right]_{\rho=0}^{\rho=\infty}$$

$$= \frac{cp^{-\frac{m}{q+1}}}{\lambda_0\mu} |\xi|^{-\frac{m}{q+1}} = \frac{cp^{-\frac{m}{q+1}}}{\lambda_0\mu} |x|^{-m},$$

where $p, \mu \in \mathbb{R}^+, m \in 0 \cup \mathbb{Z}^+, q \in \mathbb{Z}^+$ (q is related to the central angle defined in Theorem 3.1.1), λ_0 is the minimum eigenvalue of Λ and τ, ξ are the mappings for t and x respectively as defined in equations (3.26) and (3.27). Thus, inequality (3.35) is satisfied $\forall j, 1 \leq j \leq N$. This concludes the proof of Lemma 3.2.2. $\square$

From Lemma 3.2.2, there are two remarks relating the the constant K in equation (3.34).

Remark 3.2.1. *While K depends on ξ_1, it does not need to be increased if $|\xi_1|$ is increased.*

Remark 3.2.2. *If $m = 0$, the constant K satisfies the inequality*

$$K \leq K_1\lambda_0^{-1},$$

where K_1 is a constant that is independent of λ_0. In addition, K_1 only depends on q, the directions of the boundary rays of S^*, a lower bound on $|\xi_1|$ and μ. The last constant μ depends only on the angles of the eigenvalues λ_j.

3.2.5 Estimating and solving the integral equation

Now we have all the major ingredients needed to tackle the final part of the proof. This section consists of applying the knowledge of the inequalities and path of integration explained in the last two sections with the goal of showing the existence of $u(x)$ for successive estimates from equations (3.24) and (3.25), showing its holomorphism and convergence. Hence, $u(x)$ will solve the integral equation (3.22).

Let $S^{'}$ be a closed subsector of sector S, the open sector we considered in Theorem 3.1.1. The construction of $S^* \subset S$ in Section 3.2.3 will be repeated with S' in place of S, therefore obtaining a region $S^{*'}$. By taking $S^{'}$ sufficiently close to S, we have achieved that the set of paths $\Gamma(x)$ lie in $S^{*'} \subset S'$ when $x \in S^{*'}$.

By the definition of $b(x)$ in equation (3.12), we take a positive integer m and a constant d which depends on a positive integer m such that

$$||b(x)|| \leq d|x|^{-m}, \quad x \in S^{*'}. \tag{3.48}$$

Next we bound $u_1(x)$ using equations (3.19), (3.24) and (3.25),

$$p(x, u_0) = p(x, 0) = b(x) + 0(B(x) - \Lambda) + h(x, 0) = b(x),$$

then

$$u_1(x) = \mathcal{P}u_0(x) = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1}\Lambda\right] t^q b(t)\, dt.$$

Hence, using (3.48) and the result from Lemma 2.2.2 (3.35) with $\psi(x) = u_1(x)$ and $c = d$ for the constant

$$||u_1(x)|| \leq Kd|x|^{-m}, \quad x \in S^{*'}. \tag{3.49}$$

Now we look at $p(x, u)$ as defined in equation (3.19) for successive estimates and establish an inequality that will be useful later on. Let $z^{(1)}$ and $z^{(2)}$ be vectors such that

$$||z^{(i)}|| \leq z_0 \text{ where } z_0 \text{ is a constant, } i = 1, 2 \tag{3.50}$$

so

$$||(p(x, z^{(2)}) - p(x, z^{(1)})|| = ||(B(x) - \Lambda)(z^{(2)} - z^{(1)}) + h(x, z^{(2)}) - h(x, z^{(1)})||. \tag{3.51}$$

We estimate the first term in the inequality (3.51) using the fact from equation (3.16) $B(x) \to \Lambda$. So,

$$||(B(x) - \Lambda)(z^{(2)} - z^{(1)}) \leq R_1||z^{(2)} - z^{(1)}||, \text{ where } R_1 \text{ is a constant.} \tag{3.52}$$

Looking at the second term in the inequality (3.51)

$$||h(x, z^{(2)}) - h(x, z^{(1)})||$$

we know from equation (3.14) that $h(x, z)$ is a non-linear polynomial in the components of z_j from $j = 1, \ldots, N$ with coefficients that admits an asymptotic power series in x^{-1}, as $x \to \infty$ in S. So only the constant coefficients will remain. Consider the 2D case for $h(x, z)$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \; z^{(i)} = \begin{bmatrix} z_1^{(i)} \\ z_2^{(i)} \end{bmatrix} \text{ for } i = 1, 2$$

and

$$h(x,z) = \begin{bmatrix} \sum_{j+k\geq 2}^{2N} a_{1jk} z_1^j z_2^k \\ \sum_{j+k\geq 2}^{2N} a_{2jk} z_1^j z_2^k \end{bmatrix},$$

where a_{1jk}, a_{2jk} are the coefficients that admit the asymptotic power series in x^{-1} for $j,k = 1,\dots,N$. Then we have

$$h(x,z^{(2)}) - h(x,z^{(1)}) = \begin{bmatrix} \sum_{j+k\geq 2}^{2N} a_{1jk} \left((z_1^{(2)})^j (z_2^{(2)})^k - (z_1^{(1)})^j (z_2^{(1)})^k\right) \\ \sum_{j+k\geq 2}^{2N} a_{2jk} \left((z_1^{(2)})^j (z_2^{(2)})^k - (z_1^{(1)})^j (z_2^{(1)})^k\right) \end{bmatrix}. \tag{3.53}$$

We want to estimate equation (3.53) in the same way as the inequality (3.52). Each component of equation (3.53) will have the same form so we focus on the first component. We see that

$$(z_1^{(2)})^j (z_2^{(2)})^k - (z_1^{(1)})^j (z_2^{(1)})^k = (z_1^{(2)})^j \left[(z_2^{(2)})^k - (z_2^{(1)})^k\right] + (z_2^{(1)})^k \left[(z_1^{(2)})^j - (z_1^{(1)})^j\right]$$

$$= (z_1^{(2)})^j \left[(z_2^{(2)})^{k-1} + \ldots + (z_2^{(1)})^{k-1}\right](z_2^{(2)} - z_2^{(1)}) + (z_2^{(2)})^k \left[(z_1^{(2)})^{j-1} + \ldots + (z_1^{(1)})^{j-1}\right](z_1^{(2)} - z_1^{(1)}).$$

Now we use the triangle inequality and the bound for $z^{(1)}$ and $z^{(2)}$ (3.50) to obtain

$$||(z_1^{(2)})^j (z_2^{(2)})^k - (z_1^{(1)})^j (z_2^{(1)})^k|| \leq k z_0^{j+k-1} |z_2^{(2)} - z_2^{(1)}| + j z_0^{j+k-1} |z_2^{(2)} - z_2^{(1)}|,$$

$$|(z_1^{(2)})^j (z_2^{(2)})^k - (z_1^{(1)})^j (z_2^{(1)})^k| \leq \max(j,k) z_0^{j+k-1} |z_2^{(2)} - z_2^{(1)}| + |z_1^{(2)} - z_1^{(1)}|. \tag{3.54}$$

Taking the norm and use of triangle inequality for equation (3.53), we obtain

$$||h(x,z^{(2)}) - h(x,z^{(1)})|| \leq \begin{bmatrix} \sum_{j+k\geq 2}^{2N} a_{1jk} \max(j,k) z_0^{j+k-1} \left(|(z_2^{(2)} - z_2^{(1)}) + (z_1^{(2)} - z_1^{(1)})|\right) \\ \sum_{j+k\geq 2}^{2N} a_{2jk} \max(j,k) z_0^{j+k-1} \left(|(z_2^{(2)} - z_2^{(1)}) + (z_1^{(2)} - z_1^{(1)})|\right) \end{bmatrix},$$

$$\Rightarrow ||h(x,z^{(2)}) - h(x,z^{(1)})|| \leq R_2 ||z^{(2)} - z^{(1)}||, \tag{3.55}$$

where R_2 is a constant involving the coefficients a_1jk, a_2jk, z_0 and $\max(j,k)$. This can be generalised further for higher dimensional cases.
Next use the inequalities (3.52) and (3.55) to obtain the inequality we require

$$||p(x,z^{(2)} - p(x,z^{(1)})|| = ||(B(x) - \Lambda)(z^{(2)} - z^{(1)}) + h(x,z^{(2)} - h(x,z^{(1)})|| \leq \gamma ||z^{(2)} - z^{(1)}||, \tag{3.56}$$

where $\gamma \in \mathbb{R}$ is a constant. The constant γ can be taken as small as we like by making $|\xi_1|$ (from Section 3.2.3) sufficiently large and z_0 (3.50) is sufficiently small.

Recall from Remark 3.2.2 that increasing $|\xi_1|$ does not affect the constant K in the inequality (3.49), then we assume that

$$\gamma < K^{-1}. \tag{3.57}$$

Also by increasing x_0 from the theorem 3.1.1 (3.1) but fixing γ and K, we construct the inequality

$$\frac{dK}{1-\gamma K} |x|^{-m} \leq z_0, \text{ for } x \in S^{*'}. \tag{3.58}$$

With these conditions we will now prove the following two inequalities involving u for $x \in S^{*'}$

$$||u_{r+1} - u_r|| \leq \gamma^r K^{r+1} d |x|^{-m}, \ r = 0, 1, ..., \tag{3.59}$$

$$||u_{r+1}|| \leq \frac{dK}{1-\gamma K} |x|^{-m}, \ r = 0, 1, ... \ . \tag{3.60}$$

Proof by induction for equation (3.59)

For $r = 0$

$$||u_1 - u_0|| = ||u_1|| \leq \gamma^0 K d|x|^{-m} = K d|x|^{-m},$$

which matches equation (3.49) as required.
Assume true $\forall r \leq j-1$, then for $r = j-1$ we have

$$||u_j - u_{j-1}|| = ||\mathcal{P}u_{j-1} - \mathcal{P}u_{j-2}|| \leq \gamma^{j-1} K^j d|x|^{-m}. \tag{3.61}$$

Now for $r = j$

$$u_{j+1} - u_j = \mathcal{P}u_j - \mathcal{P}_{j-1} = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1}\Lambda\right] t^q (p(t, u_j(t)) - p(t, u_{j-1}(t)))\, dt.$$

Therefore, using equations (3.56) and (3.61)

$$||p(t, u_j(t)) - p(t, u_{j-1}(t))|| \leq \gamma ||u_j - u_{j-1}|| = \gamma^j K^j d|x|^{-m}.$$

Using the result from Lemma 3.2.2 (3.34) for $\psi(x) = u_{j+1} - u_j$ with its constant $c = \gamma^j K^j d$

$$||u_{j+1} - u_j|| \leq \gamma^j K^{j+1} d|x|^{-m}$$

as required.
We have proved the inequality (3.59) is true for $r = j$, thus by induction it is true $\forall r \in \mathbb{N}$ with the constant $c = \gamma^r K^r d$ for each iteration of inequality (3.34).
To show inequality (3.60), we look at the norm of u_{r+1} as a series of the differences in u and use the triangle inequality and the geometric series formula

$$||u_{r+1}|| = \left|\left|\sum_{i=0}^{r} (u_{i+1} - u_i)\right|\right| \leq \sum_{i=0}^{r} ||u_{i+1} - u_i||,$$

$$||u_{r+1}|| \leq dK|x|^{-m} \sum_{i=0}^{r} \gamma^i K^i \leq \frac{dK}{1 - \gamma K}|x|^{-m}, \tag{3.62}$$

since $\gamma K < 1$ from inequality (3.57). Next we apply this logic to all $r \in \mathbb{N}$ and we have then shown inequality (3.60).
Now, we need to show that $u(x) = \lim_{r\to\infty} u_r(x)$ uniformly. Using the inequality (3.59), we see $u_r(x)$ is dominated by a convergent geometric series as $r \to \infty$, for $x \in S^{*'}$. For uniform convergence we need to use the Weierstrass M-test, which will be stated and proved below.

Weierstrass M-Test

Lemma 3.2.3. *Suppose that f_n is a sequence of real or complex valued functions defined on a set A and $\exists$ a sequence of non-negative numbers M_n such that*

- $|f_n(x)| \leq M_n$, $\forall n \geq 1$ *and* $x \in A$
- $\sum_{n=1}^{\infty} M_n$ *converges*

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely and uniformly on A.

Proof. First, consider the sequence of partial sums

$$Y_n(x) = \sum_{i=1}^{n} f_i(x).$$

We know that $\sum_{n=1}^{\infty} M_n$ converges and $M_n \geq 0$, $\forall n \geq 1$, then use Cauchy's convergence test

$$\forall \epsilon > 0, \exists N > 0 \text{ s.t } \forall m > n > N, \sum_{i=n+1}^{m} M_i \leq \epsilon.$$

Next we apply this thinking to the sequence of functions, $\forall x \in A$, $\forall m > n > N$

$$|Y_m(x) - Y_n(x)| = \left| \sum_{i=n+1}^{m} f_i(x) \right| \leq \sum_{i=n+1}^{m} |f_i(x)| \leq \sum_{i=n+1}^{m} M_i < \epsilon, \tag{3.63}$$

where we use the first assumption in Lemma 3.2.3 and the triangle inequality. We have just shown that $Y_n(x)$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$. Recall that all Cauchy sequences are convergent [Cohen (1987)], so this sequence converges to a number $Y(x)$. Then $\forall n > N$,

$$|Y_n(x) - Y(x)| = |Y_n(x) - \lim_{m\to\infty} Y_m(x)| = \lim_{m\to\infty} |Y_n(x) - Y_m(x)| < \epsilon.$$

As N does not depend on x, the sequence of partial sums $Y_n(x)$ converges uniformly to $Y(x)$ for $x \to \infty$, meaning by definition $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly as required. □

Now we apply this to the function $u(x)$ using inequality (3.59) with r in place of n from the Weierstrass M-Test lemma 3.2.3

$$f_r(x) = u_{r+1} - u_r \ , \ M_r = \gamma^r K^{r+1} d|x|^{-m}.$$

As shown in equation (3.62), $|f_r(x)| \leq M_r$, $\forall r \geq 0$ and $x \in S^{*'}$. Also, $\sum_{i=0}^{\infty} M_i$ is convergent from the inequality (3.57) involving γ and K. So

$$u(x) = \lim_{r\to\infty} u_r(x) = \lim_{r\to\infty} \sum_{i=0}^{r-1} (u_{i+1} - u_i) \tag{3.64}$$

exists and converges uniformly from Weierstrass M-test and is therefore holomorphic for $x \in S^{*'}$. We have now shown that $u(x)$ solves the integral equation $u = \mathcal{P}u$.
Recall from equations (3.22) and (3.23) that

$$u(x) = \int_{\Gamma(x)} \exp\left[\frac{x^{q+1} - t^{q+1}}{q+1} \Lambda\right] t^q p(t, u(t))\, dt,$$

$$u(x) = \mathcal{P}(u(x)),$$

where $\mathcal{P}$ is the integral operator acting on $u(x)$ in equation (3.23). Since $\lim_{r\to\infty} u_r(x) = u(x)$ uniformly then by the bounded convergence integral theorem [Lewin (1987)],

$$\mathcal{P}u_r(x) \to \mathcal{P}u(x) \iff \lim_{r\to\infty} \mathcal{P}u_r = \mathcal{P} \lim_{r\to\infty} u_r.$$

Concluding the proof by using inequality (3.60), we see $u(x) \sim 0$ as $x \to \infty$, since m is arbitrary. The region S^* depends on the choice of $x_1 = \xi_1^{\frac{1}{q+1}}$ in the paths of integration (Figure 3.2) which is dependent on the choice of m. However, we do not need to consider this as $u(x)$ is independent of m and therefore exists in a region that does not depend on m. So, by the definition of $u(x)$ established in equation (3.10) and definition of $\phi(x)$ in equation (3.9)

$$u(x) = u = y - \phi(x) \sim 0,$$
$$y = \phi(x) \sim \sum_{r=1}^{\infty} y_r x^{-r} \text{ as } x \to \infty,\ x \in S^* \subset S.$$

This completes the proof of Theorem 3.1.1.

Chapter 4

Hyperasymptotics

The types of asymptotic power series we have looked at so far are sometimes not good enough to capture all the information about a function. Behaviour of these asymptotics cannot explain certain phenomena, for example Stokes phenomenon for non-linear ODEs.
There has been research into a different type of asymptotic power series involving exponential terms which are called hyperasymptotic power series. These achieve better accuracy and representation of some functions compared to regular asymptotic power series.
In this chapter, we take the known asymptotics of the Airy function and use them in a particular type of non-linear ODE called the Riccati equation. The calculations for this are original work.

4.1 Airy equation

There is a famous ODE called the Airy equation

$$\frac{d^2y}{dz^2} = zy, \tag{4.1}$$

where the solution is an function called the Airy function $y(z)$ (see Figure 4.1). Historically, the function was first developed when British astronomer and physicist George Biddell Airy was studying the intensity near an optical directional caustic such as a rainbow [Airy et al. (1838)].

There exists two independent solutions [Olver (1997)] with asymptotic expansions where $\zeta = \frac{2}{3}z^{3/2}$

$$y_1(z) = \mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}}\sum_{r=0}^{\infty}(-1)^r u_r \zeta^{-r}, \ -\pi < \arg z < \pi, \tag{4.2}$$

$$y_2(z) = e^{-\pi i/6}\mathrm{Ai}(ze^{-2\pi i/3}) \sim \frac{e^{\zeta}}{2\sqrt{\pi}z^{1/4}}\sum_{r=0}^{\infty} u_r \zeta^{-r}, \ -\frac{\pi}{3} < \arg z < \frac{5\pi}{3}, \tag{4.3}$$

where $u_0 = 1$ and $u_r = \frac{\Gamma(r+\frac{1}{6})\Gamma(r+\frac{5}{6})}{2\pi r! 2^r}$ for $r \geq 1$.
Looking at the positive real axis ($\arg z = 0$), we see that $y_1(z)$ (4.2) has recessive behaviour, due the negative exponential term in the asymptotic expansion, as $\zeta \to \infty$. In contrast, $y_2(z)$ (4.3) has dominant behaviour as $\zeta \to \infty$.

4.1.1 Stokes phenomenon for Airy function

Stokes phenomenon is where for a function with multiple asymptotic power series expansions that are defined across different sectors, there is different asymptotic behaviour across these sectors. As these series approach boundaries of such sectors, moving from one series to another causes a sudden jump in asymptotic behaviour. For example, an exponentially small term can contribute only in one sector and is not present in another sector. The places where this change of behaviour occur are called Stokes lines.

We look at the Airy function's asymptotic behaviour on the negative real axis $\arg z = \pi$ compared to the positive real axis $\arg z = 0$. We need a solution $y_3(z)$ for the Airy equation (4.1) in the sector $\frac{2\pi}{3} < \arg z < \frac{4\pi}{3}$. As the Airy equation is a second order ODE with two independent solutions $y_1(z)$ and $y_2(z)$, $y_3(z)$ will then be linear combination of these independent solutions.

$$y_3(z) = Ay_1(ze^{-4\pi i/3}) + Ky_2(z) \tag{4.4}$$

where A and $K \in \mathbb{C}$ are constants. Due to the Stokes phenomenon, the constant K will change in different sectors for $\arg z$.

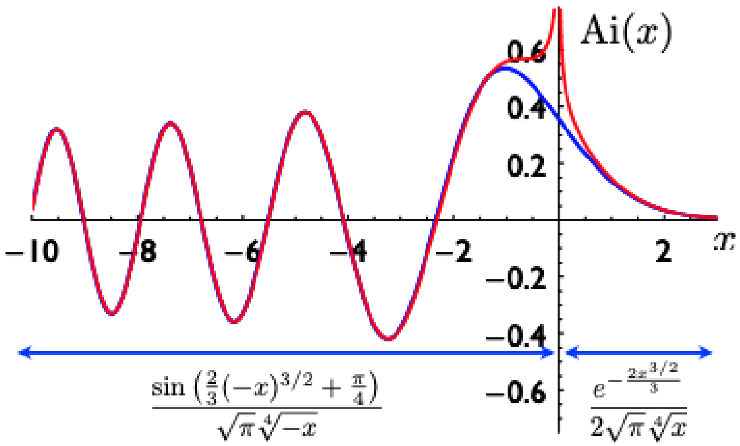

Figure 4.1: Airy function

4.2 Riccati equation form of Airy equation

Riccati equations are a class of non-linear second order ODEs which can be converted to linear first order ODEs [Olde Daalhuis (2005)]. The following is original work on a particular Riccati equation. We consider a Riccati equation for a function $v(z)$

$$v'(z) + v^2(z) - z = 0. \tag{4.5}$$

Let

$$v(z) = \frac{y'(z)}{y(z)}, \tag{4.6}$$

therefore its derivative is

$$v'(z) = \frac{y''(z)y(z) - y'(z)^2}{y(z)^2}. \tag{4.7}$$

Substituting (4.6) and (4.7) into the Riccati equation (4.5)

$$\frac{y''(z)y(z) - y'(z)^2}{y(z)^2} + \frac{y'(z)^2}{y(z)^2} - z = 0$$
$$\Rightarrow y''(z) = zy(z),$$

which is the Airy equation (4.1).

For now, suppose we have two independent solutions to equation (4.1) y_1 and y_2 such that $y = \alpha y_1 + \beta y_2$ where $|\alpha|, |\beta| \neq 0, \in \mathbb{C}$. Then we have

$$v(z, C) = \frac{\alpha y_1'(z) + \beta y_2'(z)}{\alpha y_1(z) + \beta y_2(z)} = \frac{y_1'(z) + C y_2'(z)}{y_1(z) + C y_2(z)} \text{ where } C = \frac{\beta}{\alpha} \in \mathbb{C}. \tag{4.8}$$

With some manipulation a series expansion with C is formed

$$v(z, C) = (y_1' + C y_2')((y_1 + C y_2)^{-1} = \left(\frac{y_1'}{y_1} + C\frac{y_2'}{y_1}\right)\left(1 + C\frac{y_2}{y_1}\right)^{-1}$$
$$= \left(\frac{y_1'}{y_1} + C\frac{y_2'}{y_1}\right)\left(\sum_{n=0}^{\infty}(-C\frac{y_2}{y_1})^n\right)$$
$$= \sum_{n=0}^{\infty}\left[\left(-C\frac{y_2}{y_1}\right)^n \frac{y_1'}{y_1} + (-1)^n C^{n+1}\left(\frac{y_2}{y_1}\right)^n \frac{y_2'}{y_1}\right].$$

We re-index $n = n - 1$ for the second term in the series

$$\frac{y_1^{'}}{y_1} + \sum_{n=1}^{\infty} C^n \left(-\frac{y_2}{y_1}\right)^n \left[\frac{y_1'}{y_1} - \frac{y_2'}{y_1}\right]$$
$$= \frac{y_1'}{y_1} + \sum_{n=1}^{\infty} C^n (-1)^n \left(\frac{y_2^{n-1}}{y_1^{n+1}}\right)[y_2 y_1' - y_1 y_2'],$$

then we have the form of the solution for v (4.8)

$$v(z, C) = \frac{y_1'(z)}{y_1(z)} + \sum_{n=1}^{\infty} C^n (-1)^n \left(\frac{y_2(z)^{n-1}}{y_1(z)^{n+1}}\right)\mathcal{W}(y_2(z), y_1(z)) = \sum_{n=1}^{\infty} C^n v_n(z) \tag{4.9}$$

where

$$v_0(z) = \frac{y_1'(z)}{y_1(z)}, \tag{4.10}$$

$$v_n(z) = (-1)^n \frac{y_2(z)^{n-1}}{y_1(z)^{n+1}}\mathcal{W}(y_2(z), y_1(z)) = \frac{\mathcal{W}(y_2(z), y_1(z))}{y_1(z)y_2(z)}\left(-\frac{y_2(z)}{y_1(z)}\right)^n, \quad \forall n \geq 1. \tag{4.11}$$

The solution (4.9) is for a constant $C \in \mathbb{C}$ and the Wronskian is defined as $\mathcal{W}(y_2(z), y_1(z)) = \det\begin{pmatrix} y_2(z) & y_1(z) \\ y_2'(z) & y_1'(z)\end{pmatrix}$, which is also a constant. Note that the constant C can change across different sectors, this is an example of Stokes phenomenon for non-linear ODEs.

4.2.1 Asymptotic expansions for Riccati equation

Now we substitute $v(z, C)$ (4.9) into the Riccati equation (4.5) to obtain

$$(v_0'(z) + Cv_1'(z) + C^2v_2'(z) + \ldots) + (v_0(z) + Cv_1(z) + C^2v_2(z) + \ldots)^2 - z = 0,$$

$$(v_0' + Cv_1' + C^2v_2' + \ldots) + (v_0^2 + 2Cv_0v_1 + 2C^2v_0v_2 + C^2v_1^2 + 2C^3v_1v_2 + C^4v_2^2 + \ldots) - z = 0. \quad (4.12)$$

Looking at $O(1)$ terms we have

$$v_0'(z) + v_0^2(z) - z = 0 \quad (4.13)$$

we see (4.13) is the same as the starting Riccati equation (4.5) we had in Section 4.2. It follows that the solution is $v_0(z) = \frac{y_1'(z)}{y_1(z)}$, which is consistent with the change in variables in (4.6).
Now looking at $O(C)$ terms we have

$$v_1'(z) + 2v_0(z)v_1(z) = 0. \quad (4.14)$$

Looking at $O(C^2)$ terms we have

$$v_2'(z) + 2v_0(z)v_2(z) + v_1(z)^2 = 0. \quad (4.15)$$

For general $O(C^n), n \geq 1$ terms we have

$$v_n'(z) + \sum_{m=0}^{n} v_m(z)v_{n-m}(z) = 0 \quad n = 1, 2, \ldots. \quad (4.16)$$

We now substitute the asymptotic expansions of $y_1(z)$ and $y_2(z)$ from (4.2) and (4.3) to obtain the asymptotic expansions for $v_n(z), \forall n \geq 0, n \in \mathbb{N}$ as $\zeta = \frac{2}{3}z^{3/2} \to \infty$.
For $v_0(z)$ (4.10) we have

$$\ln y_1 \sim -\zeta - \ln A - \frac{1}{4}\ln z + \sum_{r=0}^{\infty} b_r\zeta^{-r}, \quad (4.17)$$

where $\ln A$ is a real constant and $\sum_{r=0}^{\infty} \frac{b_r}{\zeta^r} = \ln(1 + \sum_{r=1}^{\infty} \frac{a_r}{\zeta^r})$ with a_r and b_r defined in terms of u_r from in (4.2) and (4.3) for $r = 0, 1 \ldots$.
We differentiate (4.17)

$$\frac{y_1'(z)}{y_1(z)} \sim -\frac{1}{4z} + z^{1/2}\left[-1 - \sum_{r=1}^{\infty} rb_r\zeta^{-(r+1)}\right]$$

and let

$$W(\zeta) = \sum_{r=0}^{\infty} c_r\zeta^{-r} := -1 - \sum_{r=1}^{\infty} rb_r\zeta^{-(r+1)}, \quad (4.18)$$

where c_r are coefficients to be determined for $r = 0, 1, \ldots$. So we know the asymptotic expansion of $v_0(z)$ from (4.10)

$$v_0(z) = \frac{y_1'(z)}{y_1(z)} \sim z^{1/2}W(\zeta) + \frac{1}{4z} \quad \text{for } \zeta = \frac{2}{3}z^{3/2} \to \infty, \quad (4.19)$$

where $W(\zeta)$ is defined as in (4.18).
We now perform a similar calculation for $v_n(z), \forall n \geq 1$ by looking at the expression in (4.11)

$$\frac{\mathcal{W}(y_2(z), y_1(z))}{y_1 y_2} \sim B z^{1/2} \sum_{r=0}^{\infty} d_r \zeta^{-r},$$
$$\left(-\frac{y_2(z)}{y_1(z)}\right)^n \sim e^{2n\zeta} \sum_{r=0}^{\infty} f_r$$

where $B \in \mathbb{R}$ is a constant in terms of π and $\mathcal{W}(y_2(z), y_1(z))$. The coefficients d_r and f_r are part of their asymptotic series expansions. So for $v_n(z), n \geq 1$

$$v_n(z) \sim z^{1/2} e^{2n\zeta} \sum_{r=0}^{\infty} \gamma_r \zeta^{-r}, \forall n \geq 1, \tag{4.20}$$

where the coefficients γ_r are coefficients to be determined for $r = 0, 1 \ldots$ and $\zeta = \frac{2}{3} z^{3/2} \to \infty$.

Focusing on $v_0(z)$, we substitute its asymptotic expansion (4.19) into the ODE for $v_0(z)$ (4.13). We see that

$$v_0'(z) \sim \frac{W(\zeta)}{2z^{1/2}} + z^{1/2} \frac{d\zeta}{dz} W'(\zeta) - \frac{1}{4z^2} = \frac{W(\zeta)}{2z^{1/2}} + zW'(\zeta) - \frac{1}{4z^2}$$

and

$$v_0^2 \sim zW(\zeta)^2 + \frac{W(\zeta)}{2z^{1/2}} + \frac{1}{16z^2}.$$

For (4.13)

$$0 = v_0'(z) + v_0^2(z) - z \sim \frac{W(\zeta)}{z^{1/2}} + zW'(\zeta) + zW(\zeta)^2 - \frac{3}{16z^2} - z.$$

Dividing by z and getting all powers of z in terms of ζ

$$W'(\zeta) + W(\zeta)^2 + \frac{W(\zeta)}{z^{3/2}} - \frac{3}{16z^3} - 1 \sim 0,$$
$$W'(\zeta) + W(\zeta)^2 + \frac{2W(\zeta)}{3\zeta} - \frac{1}{12\zeta^2} - 1 \sim 0 \quad \text{for } \zeta \to \infty, \tag{4.21}$$

where $W(\zeta)$ is defined as in (4.18).
To find the coefficients c_r of the asymptotic power series for $W(\zeta)$, we substitute (4.18) into (4.21)

$$-(c_1\zeta^{-2} + c_2\zeta^{-3} + \ldots) + (c_0 + c_1\zeta^{-1} + \ldots)^2 + \frac{2}{3}(c_0\zeta^{-1} + c_1\zeta^{-2} + \ldots) - \frac{\zeta^{-2}}{12} - 1 = 0,$$
$$-(c_1\zeta^{-2} + c_2\zeta^{-3} + \ldots) + (c_0^2 + 2c_0c_1\zeta^{-1} + c_1^2\zeta^{-2} \ldots) + \frac{2}{3}(c_0\zeta^{-1} + c_1\zeta^{-2} + \ldots) - \frac{\zeta^{-2}}{12} - 1 = 0. \tag{4.22}$$

We then look at the equations at $O(\zeta^{-n}), n = 0, 1, \ldots$ like we did for (4.12) and find the coefficients c_r in (4.18) via recurrence relations.

Similarly for $v_n(z), n \geq 1$, we can substitute its asymptotic expansion (4.20) into the ODE (4.16). This will form another ODE similar to (4.21) giving a series of divergent series called a transseries. To find coefficients of γ_r in (4.20), we would conduct a similar method as we did for (4.22).

Chapter 5

Conclusion

In this project, we have looked at the general theory of asymptotic power series and their properties in Chapter 2. Then in Chapter 3, we proved Theorem 3.1.1 that looks at a general class of ODEs where there exists a solution which has an asymptotic power series representation in some sector. In Chapter 4 we looked at hyperasymptotics, specifically an original worked example involving the asymptotics of the Airy function with a Riccati equation (4.5). We started to calculate the series expansion explicitly (4.22) by the link to the Airy function. However, it is possible to generate the series expansion by substituting the general form (4.9) and obtain recurrence relations for the coefficients. This would generate a hyperasymptotic power series containing exponentially small terms. These exponentially small terms contain critical information that determines the arbitrary constant C in (4.9).

Future work can involve looking at higher-order ODEs or ODEs where they cannot be solved in terms of linear ODEs and explore what transseries they form. Also, derive their respective hyperasymptotic power series and maybe prove such series exist for these ODEs.

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