Lecture 4 (Edinburgh): BPRE and Queueing systems

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1 Galton-Watson processes with immigration

The Galton-Watson process with immigration: is specified by

$$f(s) = \mathbf{E}s^{\xi},\ g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}\left(\eta = k\right) s^{k},$$

and

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)}, \qquad \eta^{(n)} \overset{d}{=} \eta, \text{ and iid.}$$

We have

$$\begin{aligned} \Phi(n+1,s) &= \mathbf{E}\left[s^{Y(n+1)}|Y(0)=0\right] \\ &= \mathbf{E}\left[s^{\xi_1^{(n)}+...+\xi_{Y(n)}^{(n)}+\eta^{(n)}}|Y(0)=0\right] \\ &= g(s)\Phi(n,f(s)) = ... = \prod_{k=0}^{n+1} g(f_k(s)). \end{aligned}$$

Theorem 1 *If $g'(1) < \infty$ and $A = f'(1) < 1$ then there exists the limit*

$$\Phi(s) = \mathbf{E}s^{Y} = \lim_{n\to\infty} \Phi(n,s) = \prod_{k=0}^{\infty} g(f_k(s)) > 0.$$

Proof. Indeed,

$$1 - g(f_k(s)) \le g'(1)\left(1 - f_k(s)\right) \le g'(1)A^k\left(1 - s\right).$$

Hence

$$\sum_{k=0}^{\infty} \left(1 - g(f_k(s))\right) < \infty$$

which shows that $1 \ge \prod_{k=0}^{\infty} g(f_k(s)) > 0$ for all $s \in [0,1]$ finishing the proof.

Theorem 2 *If* $g^{'}(1) = b < \infty$ *and* $f'(1) = 1$, $B = f"(1) \in (0, \infty)$ *then for* $\theta = 2b/B$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{Y(n)}{Bn} \leq x\right) = F(x) = \frac{1}{\Gamma(\theta)} \int_0^x y^{\theta-1} e^{-y} dy. \qquad (1)$$

1.1 $M^{[X]}|G|1$ systems with permanent customers and $FIFO-$ discipline

(OPTIMIZATION OF A DISK SPACE).

Consider a queueing system with nonordinary Poisson flow of customers with PGF $h(s)$ and the intensity Λ. Assume that there is 1 permanent customer in the queue. The service time of the permanent customer is distributed according to $G_p(x)$ while the distribution of the service time of non-permanent custiomers is $G(x)$. Initially only the permanent customer is in the queue and its service starts. When the service is ended the premanent customer joins the queue consisting of the customers coming during the its service time and becomes the last one in the queue. The service dicipline is FIFO - first-in-first-out.

Let $Y(n)$ be the number of nonpermanent customers in the queue just after the moment when the nth service of the permanent customer is finished. Then

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)}$$

where $\xi_i^{(n)}-$ is the number of customers arriving during the service time of the $i-$th nonpermanent customer being in the queue at the end of the $(n-1)$-th service of the permananet customer and $\eta^{(n)}$ the number of customers arriving during the $n-$th service of the permanent customer.

Thus, at these moments we have a Galton-Watson branching process with immigration. Its ingredients are specified by the Poisson flow of intensity Λ.

Let $\mu(u)$ be the number of batches of customers arriving within the interval $[0, u]$. Then its probability generating function is

$$\mathbf{E}s^{\mu(u)} = \sum_{k=0}^{\infty} \mathbf{P}(\mu(u) = k)s^k = e^{\Lambda u(s-1)}.$$

Thus, the offspring probability generating function $f(s)$ for the number of new customers arriving during the service time l of a nonpermanent customer is

$$\begin{aligned} f(s) &= \mathbf{E}s^{\xi} = \int_0^{\infty} \mathbf{E}\left[s^{\xi}|l = u\right] dG(u) \\ &= \int_0^{\infty} \mathbf{E}\left[s^{M(u)}\right] dG(u) = \int_0^{\infty} e^{\Lambda u(h(s)-1)} dG(u) \end{aligned}$$

and the offspring probability generating function $g(s)$ for the number of new customers arriving during the service time l_p of the permanent customer is

$$\begin{aligned} g(s) &= \mathbf{E}s^{\eta} = \int_0^{\infty} \mathbf{E}\left[s^{\eta} | l_p = u\right] dG_p(u) \\ &= \int_0^{\infty} e^{\Lambda u(h(s)-1)} dG_p(u). \end{aligned}$$

And if $g'(1) = \Lambda h'(1) \int_0^{\infty} u dG_p(u) < \infty$ and $A = f'(1) = \Lambda h'(1) \int_0^{\infty} u dG(u) < 1$ we have a stationary distribution for the size of queue at the moments of the end of the service of the permanent customer.

2 The Galton-Watson process with immigration at zero:

$$f(s) = \mathbf{E}s^{\xi},\ g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}\left(\eta = k\right) s^k.$$

We have

$$Y(n+1) = \xi_1^{(n)} + ... + \xi_{Y(n)}^{(n)} + \eta^{(n)} I\left\{Y(n) = 0\right\}.$$

$$\xi_i^{(n)} \overset{d}{=} \xi,\ \eta^{(n)} \overset{d}{=} \eta \text{ and iid.}$$

If

$$\Pi(n,s) = \mathbf{E}s^{Y(n)}$$

then

$$\begin{aligned} \Pi(n+1,s) &= \Pi\left(n, f(s)\right) - \Pi(n,0) + \Pi(n,0)g(s) \\ &= \Pi\left(n, f(s)\right) - (1-g(s))\Pi(n,0) \\ &= \Pi\left(0, f_{n+1}(s)\right) - \sum_{k=0}^{n} (1 - g(f_k(s))\Pi(n-k,0). \end{aligned}$$

In particular, if $Y(0) = 0$ then

$$\Pi(n+1,0) = 1 - \sum_{k=0}^{n} (1 - g(f_k(0))\Pi(n-k,0).$$

If $A < 1$ and

$$g^{'}(1) = b,\ g(0) > 0,$$

then we have a stationary distribution for the process $Y(n)$ as $n \to \infty$.

Indeed, it is known that if a Markov chain is irreducible and nonperiodic then either

1) for any pair of states $p_{ij}^{(n)} \to 0, n \to \infty$, and, therefore, there exists no stationary distribution;

or

2) all the states are ergodic, that is,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0$$

and in this case $\{\pi_j\}$ is a stationary distribution and no other stationary distributions exists.

In our case take $p_{00}^{(n)} = \Pi(n,0) = P\left(Y(n) = 0\right)$. Assuming that there is NO stationary distribution we get by dominated convergence theorem a contradiction:

$$\lim_{n\to\infty} \Pi(n+1,0) = 0 = 1 - \lim_{n\to\infty} \sum_{k=0}^{n} (1 - g(f_k(0))\Pi(n-k,0) = 1$$

since the series

$$\sum_{k=0}^{\infty} (1 - g(f_k(0)) \leq b \sum_{k=0}^{\infty} (1 - f_k(0)) \leq b \sum_{k=0}^{\infty} A^k < \infty.$$

Thus, we have a stationary distribution

$$\Pi\left(s\right) = \mathbf{E}s^Y = \lim_{n\to\infty} \mathbf{E}s^{Y(n)}$$

where

$$\Pi\left(s\right) = \Pi\left(f(s)\right) - \pi_0\left(1 - g(s)\right)$$

or

$$\Pi\left(s\right) = 1 - \pi_0 \sum_{k=0}^{\infty} \left(1 - g(f_k\left(s\right))\right).$$

From here

$$\pi_0 = 1 - \pi_0 \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)$$

leading to

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)}.$$

Hence

$$\Pi\left(s\right) = 1 - \frac{\sum_{k=0}^{\infty} \left(1 - g(f_k\left(s\right))\right)}{1 + \sum_{k=0}^{\infty} \left(1 - g(f_k\left(0\right))\right)}.$$

Introduce the following classes of functions: $K_1 = K\left(b_1, b_2\right) = \{g(s) = Es^{\eta}\}$ of probability generating functions (PGF):

$$0 < b \leq g'(1) = E\eta = b; g(0) > 0; E\eta^2 \leq b_2 < \infty$$

and $K_2 = K_2\left(C_1, C_2\right) = \left\{f(s) = Es^{\xi}\right\}$ of PGF specified by B_1, B_2:

$$A = E\xi, 0 < C_1 \leq f"(1) = E\xi(\xi - 1) = B, E\xi^3 \leq C_2 < \infty$$

Let $g(m,s), m = 1, 2, ...$ be a sequence of PGF belonging to class $K_1 = K\left(b_1, b_2\right)$ and $f(m,s), m = 1, 2, ...$be a sequence of PGF belonging to class $K_2 = K_2\left(C_1, C_2\right)$..

Theorem 3 *If $A_m = f'(m,1) < 1$ and $B_m = f''(m,1) < \infty$ and the functions $g(m,s), f(m,s), m = 1,2,..$vary within the classes K_1 and K_2 in such a way that as $m \to \infty$*

$$b_m \to b, \ A_m \nearrow 1, \lim_{m\to\infty} B_m = B$$

and if $Y_m(n)$ is the branching process with immigration at zeroand reproduction functions $(g(m,s), \ f(m,s))$ with $Y_m \overset{d}{=} lim_{n\to\infty} Y(\infty)$ then we have (under heavy trafic!)

$$\lim_{m\to\infty} \mathbf{P}\left(\frac{\ln Y_m}{\ln \frac{1}{1-A_m}} \leq x\right) = x, \ x \in (0,1].$$

2.1 Queueing systems with batch service

$M^{[X]}|G|1$

Λ- the intensity of the input Poisson flow. The customers arrive in batches of random size. The size of the $i-$th group is $\eta^{(i)}$

$$g(s) = \mathbf{E}s^{\eta} = \sum_{k=1}^{\infty} \mathbf{P}\left(\eta = k\right) s^k.$$

The first customer $\to$ to the server

$\nu(1)$- the number of customers coming during the service time of the $first$ customer.

$\nu(2)$- the number of customers coming during the service time of all first $\nu(1)$ customers.

$\nu(j)$- the number of customers coming during the service time of all $\nu(j-1)$ customers.

If NO customers arrive during the service time of a group of customers then we wait for the new batch and take all of them. We have

$$\nu(n+1) = \xi_1^{(n)} + ... + \xi_{\nu(n)}^{(n)} + \eta^{(n)} I\left\{\nu(n) = 0\right\}.$$

$$\xi_i^{(n)} \overset{d}{=} \xi, \text{ and iid.}$$

This is a BRANCHING PROCESS WITH IMMIGRATION AT ZERO. Clearly,

$$\begin{aligned} \mathbf{E}s^{\xi} &= \sum_{j=0}^{\infty} \mathbf{P}\left(\xi = j\right) s^j = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\Lambda u} \frac{(\Lambda u)^k}{k!} g^k(s) dG(u) \\ &= \int_0^{\infty} e^{-\Lambda u(1-g(s))} dG(u) = f(s). \end{aligned}$$

Direct calculations show that

$$A = \mathbf{E}\xi = f^{'}(1) = \Lambda g^{'}(1) \int_0^{\infty} u dG(u) = \Lambda g^{'}(1) L$$

where L is the expected service time of a customer. Hence we can apply the previous theorem to study the queueing system under heavy traffic when $A = \Lambda g^{'}(1) L \nearrow 1$.

3 Continuous time Markov processes

Only nonformal definition: if there are i particles at some moment then each of them has exponential remaining life-length with parameter, say, ρ, and dying produces children in accordance with the pgf

$$f(s) = \sum_{k=0}^{\infty} \mathbb{P}(\xi = k) s^k = \sum_{k=0}^{\infty} p_k s^k, 0 \le s \le 1,$$

independently of other individuals.

From here for $f^{(\rho)}(s) = \rho(f(s) - s)$ and $i = 1$ we have for

$$F(t, s) = E\left[s^{Z(t)} | Z(0) = 1\right]$$

the following equations

$$\frac{\partial F(t; s)}{\partial t} = f^{(\rho)}(s) \frac{\partial F(t; s)}{\partial s}, \ F(0, s) = s,$$

and

$$\begin{aligned} \frac{\partial F(t; s)}{\partial t} &= \rho(f(F(t, s)) - F(t, s)) = f^{(\rho)}(F(t, s)), \\ F(0, s) &= s. \end{aligned} \tag{2}$$

3.1 Classification

Let

$$A(t) = EZ(t).$$

Then

$$A(t) = e^{at}, a = \rho(f'(1) - 1).$$

A continuous time Markov branching process is called supercritical, critical, subcritical if, respectively $f'(1) > 1, = 1, < 1$.

3.1.1 Criterion

A Markov process does not explode if and only if for any $\varepsilon \in (0, 1)$

$$\int_{1-\varepsilon}^{1} \frac{du}{1 - f(u)} = \infty.$$

4 Branching processes counted by random characteristics (branching processes with final product)

We consider continuous time Markov branching process with exponential life-time distribution with parameter ρ and the reproduction function $f(s)$.

Now we suppose that at the end of life any particle produces along with random number ξ of children a final product $\chi \geq 0$ which is not changed later on and denote by $\varphi^{\chi}(s,\lambda)$ the joint probability generating function of the vector (ξ,χ) specified by

$$\varphi^{\chi}(s,\lambda) = \mathbf{E}s^{\xi}e^{-\lambda\chi}.$$

χ is called a random characteristics or the final product. It is assumed that the size of the final product of a particle IS INDEPENDENT of her life-length.

Examples. $\chi = I\{\xi = k\}, \chi = I\{\xi \geq k\}$ and so on.

Let

$$Z^{\chi}(t) = \sum_{D} \chi_D$$

where the summation is taken over all particles D which died up to the moment t.

We deduce integral and diferential equations for the probability generating fucntion of the pair $(Z(t), Z^{\chi}(t))$ assuming that the final product of a particle IS INDEPENDENT of her life-length. We have by the total probability formula for

$$\Phi(t,s,\lambda) = \mathbf{E}\left[s^{Z(t)}e^{-\lambda Z^{\chi}(t)} | (Z(0), Z^{\chi}(0)) = (1,0)\right]$$

and $G(t) = 1 - e^{-\rho t}$:

$$\Phi(t,s,\lambda) = s\left(1 - G(t)\right) + \int_0^t \varphi^{\chi}(\Phi(t-u,s,\lambda),\lambda)dG(u).$$

Hence

$$\frac{\partial \Phi(t,s,\lambda)}{\partial t} = \rho\left(\varphi^{\chi}(\Phi(t,s,\lambda),\lambda) - \Phi(t,s,\lambda)\right),\ \Phi(0,s,\lambda) = s.$$

In particular, for

$$\Phi(t,\lambda) := \mathbf{E}\left[e^{-\lambda Z^{\chi}(t)} | (Z(0), Z^{\chi}(0)) = (1,0)\right] = \Phi(t,1,\lambda)$$

we get

$$\Phi(t,\lambda) = (1 - G(t)) + \int_0^t \varphi^{\chi}(\Phi(t-u,\lambda),\lambda)dG(u) \quad (3)$$

and

$$\frac{\partial \Phi(t,\lambda)}{\partial t} = \rho\left(\varphi^{\chi}(\Phi(t,\lambda),\lambda) - \Phi(t,\lambda)\right)$$

with

$$\Phi\left(0,\lambda\right)=1.$$

Thus, if

$$A^{\chi}(t)=\mathbf{E}Z^{\chi}(t),$$

then denoting by l the lifelength of the initial particle we get from (3) by differentiating with respect to λ and setting $\lambda=0$:

$$\begin{aligned} A^{\chi}(t) &= \mathbf{E}\xi\int_0^t A^{\chi}(t-u)dG(u)+\int_0^t \mathbf{E}[\chi|l=u]dG(u) \\ &= \text{(by independence of } \chi \text{ of the lifelength)} \\ &= \mathbf{E}\xi\int_0^t A^{\chi}(t-u)dG(u)+\mathbf{E}\chi G(t) \end{aligned}$$

giving

$$A^{\chi}(t)=\frac{\mathbf{E}\chi}{\mathbf{E}\xi-1}e^{(\mathbf{E}\xi-1)t}-\frac{\mathbf{E}\chi}{\mathbf{E}\xi-1}$$

if $\mathbf{E}\xi\neq 1$ and

$$A^{\chi}(t)=t\mathbf{E}\chi$$

if $\mathbf{E}\xi=1$.

Passing to the limit as $t\to\infty$ we get for

$$\begin{aligned} \Phi\left(\lambda\right) &:= Ee^{-\lambda Z^{\chi}(\infty)}=\lim_{t\to\infty}\Phi\left(t,\lambda\right) \\ &= \lim_{t\to\infty}\mathbf{E}\left[e^{-\lambda Z^{\chi}(t)}|(Z(0),Z^{\chi}(0))=(1,0)\right] \end{aligned}$$

(since $Z^{\chi}(t)$ is nondecreasing this limit always exists) that

$$\Phi\left(\lambda\right)=\varphi^{\chi}(\Phi\left(\lambda\right),\lambda).$$

This is a reflection of the relation

$$Z^{\chi}(t)\stackrel{d}{=}\left[\chi_0+Z_1^{\chi}(t-l_0)+...+Z_{\xi}^{\chi}(t-l_0)\right]I\left\{l_0\leq t\right\}$$

and, therefore,

$$Z^{\chi}(\infty)\stackrel{d}{=}\chi_0+Z_1^{\chi}(\infty)+...+Z_{\xi}^{\chi}(\infty).$$

In particular, for the total number of particles born in the process ($\chi=1$) we get

$$\varphi^{\chi}(s,\lambda)=\mathbf{E}s^{\xi}e^{-\lambda\chi}=e^{-\lambda}\mathbf{E}s^{\xi}=e^{-\lambda}f(s)$$

and

$$\Phi\left(\lambda\right)=e^{-\lambda}f(\Phi\left(\lambda\right)).$$

For instance, for the case

$$f(s)=\frac{1}{2-s} \tag{4}$$

we get

$$\Phi\left(\lambda\right)=1-\sqrt{1-e^{-\lambda}}\text{ or }(=1-\sqrt{1-s}).$$

5 Branching processes and Queueing system with SIRO (service in random order) discipline

System with one server and the infinite capacity queue.

Consider a queueing system in which initially there are $n+1$ customers in the queue one of them is marked and the server is idle (free). The subsequent customer is selected for service from the queue at random. Let π_i be the service time of the $i-$th customer being served:

$$\mathbf{P}(\pi_i \leq x) = G(x),$$

and let ξ_i be the number of new customers arriving to the system during the service time π_i. Assume that the pairs $(\xi_i, \pi_i), i = 1, 2, ...$ are iid (for instance this is valid for any $M|G|1$ system). Denote by T_n^π the waiting time for the start of the service of the marked customer. Clearly, if the marked customer is served as the $(N+1)-$th customer then

$$T_n^\pi = \pi_1 + \pi_2 + ... + \pi_N.$$

We consider this from a more general point of view: $\pi_i \to \chi_i$ that is, a final product χ_i is produced at moment $\pi_1 + \pi_2 + ... + \pi_i$ and the final products are accumulated in the process. For instance, if $\chi_i = 1$ then $T_n^\chi = N$ if $\chi_i = \xi_i - 1$ then $T_n^\chi + n$ is the length of the queue when the service of the marked customer starts and so on.

The associated branching process is described as follows. The process starts by $n+1$ individuals, each of them (say, D) is treated as a customer. The start of splitting of the individual D is the start of the service of the customer D . The number of children of D is the number of new customers arriving during the service time of D. The end of the splitting is the moment of the end of service of D when it produces a final product χ_D. The life-length distributions of particles are exponential with parameter 1. Thus, each particle presenting in the process at moment t has one and the same probability to produce the final product first:

$$T_n^\chi = \chi_1 + \chi_2 + ... + \chi_N.$$

Using the construction above one can show the validity of the following statement.

Theorem 4 *The queueing system above and the associated branching process can be specified on a common probability space in such a way that*

$$T_n^\chi = Z_1^\chi(\tau) + ... + Z_n^\chi(\tau) \text{ a.s.}$$

where τ and the random variables $Z^\chi(t), i = 1, 2, ..., n$ are independent, $\mathbf{P}(\tau \leq x) = 1 - e^{-x}$ and

$$Z_i^\chi(\tau) \stackrel{d}{=} Z^\chi(\tau).$$

In particular,

$$\mathbf{E}e^{-\lambda T_n^\chi} = \int_0^\infty e^{-t}\Phi^n(t,\lambda)dt$$

and

$$\mathbf{E}\left(T_n\right)^p = \int_0^\infty e^{-t}\mathbf{E}\left(Z_1^\chi(t)+...+Z_n^\chi(t)\right)^p dt.$$

Thus, if the characteristics χ is independent of the life-time then

$$\begin{aligned}
\mathbf{E}T_n &= n\int_0^\infty e^{-t}\mathbf{E}Z^\chi(t)dt = \frac{n\mathbf{E}\chi}{\mathbf{E}\xi-1}\int_0^\infty e^{-t}(e^{(\mathbf{E}\xi-1)t}-1)dt \\
&= \frac{n\mathbf{E}\chi}{\mathbf{E}\xi-1}\int_0^\infty \left(e^{(\mathbf{E}\xi-2)t}-e^{-t}\right)dt \\
&= \frac{n\mathbf{E}\chi}{\mathbf{E}\xi-1}\left(\frac{1}{2-\mathbf{E}\xi}-1\right) = \frac{n\mathbf{E}\chi}{2-\mathbf{E}\xi}.
\end{aligned}$$

Hence $\mathbf{E}\xi < 2$ gives finite expectation for T_n (even for $n=1$).
One can show that

$$\begin{aligned}
\mathbf{E}T_n^2 &= n\left(\frac{\mathbf{E}\chi^2}{2-\mathbf{E}\xi}+\frac{2\mathbf{E}\chi\mathbf{E}\xi\chi}{\left(2-\mathbf{E}\xi\right)^2}+\frac{2\left(\mathbf{E}\chi\right)^2\left(\mathbf{E}\xi^2-2\right)}{\left(3-2\mathbf{E}\xi\right)\left(2-\mathbf{E}\xi\right)^2}\right) \\
&\quad +\frac{n^2\mathbf{E}\chi^2}{\left(3-2\mathbf{E}\xi\right)\left(2-\mathbf{E}\xi\right)}.
\end{aligned}$$

Remark. It is interesting to understand when $\mathbf{P}\left(T_n<\infty\right)=1$. Clearly,

$$\begin{aligned}
\mathbf{P}\left(T_n<\infty\right) &= 1 \Longleftrightarrow \mathbf{P}\left(Z^\chi(\tau)<\infty\right)=1 \\
&\Longleftrightarrow \mathbf{P}\left(Z^\chi(t)<\infty\right)=1
\end{aligned}$$

for almost all t and hence for all $t>0$ and this, in turn, means that, under reasonable assumption on χ (say, $0<c_1\le\chi\le c_2<\infty$ for some constants $c_1,c_2>0$) that

$$\mathbf{P}\left(Z^\chi(t)<\infty\right)=1 \Longleftrightarrow \mathbf{P}\left(Z(t)<\infty\right)=1.$$

Therefore, by the non-explosion criterion for ordinary Markov processes the following statement is valid (under reasonable assumption on χ) :

Theorem 5 $\mathbf{P}\left(T_n<\infty\right)=1$ *if and only if*

$$\int_0^1 \frac{du}{1-f(1-u)} = \infty.$$

For $M|G|1$ system with ordinary Poisson input and intensity 1 we have

$$f(s) = \mathbf{E}s^\xi = \int_0^\infty e^{(s-1)x}dG(x)$$

and, therefore

$$f(1-u) = \int_0^\infty e^{-ux} dG(x).$$

An unusual phenomena: Consider an $M|G|1$ system having the following ingredients:

The flow of customers is Poisson with intensity, say Λ, and the service time distribution is $G(x)$. The service intensity of customers is 1. Consider two such systems with underlying distributions $G_i(x), i = 1, 2$. Combine the two flows of customers into one, that is assume that the customers have the service time distributed as

$$\frac{1}{2}\left(G_1(x) + G_2(x)\right)$$

and with *any* service intensity $c > 0$. Then there are two distribution functions $G_i(x), i = 1, 2$, such that the waiting time of a customer under the stationary regime in the new system is infinite while for each separate system they are finite (see Grishechkin, TPA, V.21, 1986 for more details).

Let $m = \mathbf{E}\xi - 1$.

Theorem 6 *As* $n \to \infty$

$$\frac{T_n^\chi}{n\mathbf{E}\chi} \xrightarrow{d} \zeta$$

where the distribution function of the random variable ζ *is*

$$F_m(x) = 1 - (1+mx)^{-1/m}, \ 0 \le x \le x_m,$$

where

$$x_m = -\frac{1}{m}, \ m < 0, \ x_m = \infty, \ m \ge 0,$$

and

$$F_0(x) = 1 - e^{-x}.$$

Proof. We have

$$\begin{aligned}
\mathbf{P}\left(\frac{T_n^\chi}{n\mathbf{E}\chi} \le x\right) &= \mathbf{P}\left(\frac{Z_1^\chi(\tau) + ... + Z_n^\chi(\tau)}{n\mathbf{E}\chi} \le x\right) \\
&= \int_0^\infty e^{-t}\mathbf{P}\left(\frac{Z_1^\chi(t) + ... + Z_n^\chi(t)}{n\mathbf{E}\chi} \le x\right) dt.
\end{aligned}$$

Since

$$A^\chi(t) = \frac{\mathbf{E}\chi}{\mathbf{E}\xi - 1} e^{(\mathbf{E}\xi - 1)t} - \frac{\mathbf{E}\chi}{\mathbf{E}\xi - 1} = \frac{\mathbf{E}\chi}{m}\left(e^{mt} - 1\right),$$

we have by the law of large numbers

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{Z_1^\chi(t) + ... + Z_n^\chi(t)}{n\mathbf{E}\chi} \le x\right) = \mathbf{P}\left(\frac{1}{m}\left(e^{mt} - 1\right) \le x\right).$$

Now if $m < 0$ we get for $mx > -1$

$$\begin{aligned}\mathbf{P}\left(\frac{1}{m}\left(e^{mt}-1\right)\leq x\right) &= \mathbf{P}\left(e^{mt}\geq mx+1\right)\\ &= \mathbf{P}\left(t\leq \frac{1}{m}\ln\left(mx+1\right)\right)\end{aligned}$$

and by the dominated convergence theorem

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\infty e^{-t}\mathbf{P}\left(\frac{Z_1^\chi(t)+...+Z_n^\chi(t)}{n\mathbf{E}\chi}\leq x\right)dt\\ =&\int_0^\infty e^{-t}\mathbf{P}\left(\frac{1}{m}\left(e^{mt}-1\right)\leq x\right)dt\\ =&\int_0^\infty e^{-t}\mathbf{P}\left(t\leq\frac{1}{m}\ln\left(mx+1\right)\right)dt\\ =&\int_0^{\frac{1}{m}\ln(mx+1)} e^{-t}dt = 1-\left(1+mx\right)^{-1/m}.\end{aligned}$$

For $m = 0$

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\infty e^{-t}\mathbf{P}\left(\frac{Z_1^\chi(t)+...+Z_n^\chi(t)}{n\mathbf{E}\chi}\leq x\right)dt\\ =&\int_0^\infty e^{-t}\mathbf{P}\left(t\leq x\right)dt=\int_0^x e^{-t}dt=1-e^{-x}.\end{aligned}$$

The case $m > 0$ can be treated similarly.

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