Lecture 2 (Edinburgh): BPRE and properites of random walk

V.A. Vatutin (Steklov Mathematical Institute, Moscow)

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1 Branching processes in random environment

In this lecture fundamental properties of branching processes in a random environment are developed. In such a process individuals reproduce independently of each other according to random offspring distributions which vary from one generation to the other. To give a formal definition let $\mathcal{P} = \{\pi\}$ be the space of probability measures on $\mathbb{N}_0 = \{0, 1, ...,\}$:

$$\pi = (\pi(0), \pi(1), ...), \sum_{i=0}^{\infty} \pi(i) = 1, \pi(i) \geq 0.$$

Equipped with the metric of total variation $\mathcal{P}$ becomes a Polish space.

With each π we assiciate the respective probability generating function

$$f(s) = \sum_{i=0}^{\infty} \pi\{i\} s^i \tag{1}$$

and let $\mathbf{F}$ be the set of such generating functions.

Let Π be a random variable (probability distribution) taking values in $\mathcal{P}$, or, what is the same, a random probability generating function F. Then, an infinite sequence $\bar{\Pi} = (\Pi_0, \Pi_1, \ldots)$ of i.i.d. copies of Π (or, what is the same, an infinite sequence $\bar{F} = (F_0, F_1, \ldots)$ of i.i.d. copies of F) is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z(0), Z(1), \ldots$ is called a *branching process in the random environment (BPRE)* $\bar{\Pi}$, if $Z(0)$ is independent of $\bar{\Pi}$ and given $\bar{\Pi}$ the process $Z = (Z(0), Z(1), \ldots)$ is a Markov chain with

$$\mathcal{L}\left(Z(n+1) \mid Z(n) = z_n, \bar{\Pi} = (\pi_0, \pi_1, \ldots)\right) = \mathcal{L}\left(\xi_{n1} + \cdots + \xi_{nz_n}\right) \tag{2}$$

for every n, $z_n \in \mathbb{N}_0$ and $\pi_0, \pi_1, \ldots \in \mathcal{P}$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution π_n.

In the language of branching processes Z_n is the nth generation size of the population and Π_n is the distribution of the number of children of an individual at generation n. Thus, letting

$$f_n(s) = \sum_{i=0}^{\infty} \Pi_n \{i\} s^i \tag{3}$$

we get the offspring generating function of the individuals of the (n)th generation. Given the environment the evolution of the standard BPRE is described by the relations

$$Z(0) = 1,\ \mathbf{E}\left[s^{Z(n)} | \, f_0, ..., f_{n-1}; Z(0), Z(1), ..., Z(n-1)\right] = (f_{n-1}(s))^{Z(n-1)}.$$

From here it is not difficult to see that

$$\begin{aligned} \mathbf{E}_\pi s^{Z(n)} &:= \mathbf{E}\left[s^{Z(n)} | \, f_0, ..., f_{n-1}\right] \\ &= \mathbf{E}_\pi (f_{n-1}(s))^{Z(n-1)} = f_0(f_1(...(f_{n-1}(s))...)) := f_{0,n}(s). \end{aligned}$$

In view of the equality

$$Z(n) = \xi_{n-1,1} + \cdots + \xi_{n-1,Z_{n-1}} \tag{4}$$

and the Wald identity we get for the standard BPRE

$$\begin{aligned} a(n) := \mathbf{E}_\pi Z(n) = \mathbf{E}_\pi[\xi_{n-1,1} + \cdots + \xi_{n-1,Z_{n-1}}] \\ = \mathbf{E}_\pi Z(n-1) \mathbf{E}_\pi \xi_{n-1,1} = f'_{n-1}(1) \mathbf{E}_\pi Z(n-1) \\ = \prod_{i=0}^{n-1} f'_i(1). \end{aligned}$$

As it turns out the properties of Z are first of all determined by its associated random walk $S = (S_0, S_1, \ldots)$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1}$, $n \geq 1$ defined as

$$X_n := \log \sum_{y=0}^{\infty} y \ \Pi_{n-1}(\{y\}) = \log f'_{n-1}(1),$$

which are i.i.d. copies of the logarithmic mean offspring number

$$X := \log \sum_{y=0}^{\infty} y \ \Pi(\{y\}) = \log f'(1).$$

We assume that X is a.s. finite. In view of (2) the conditional expectation of $Z(n)$ given the environment $\bar{\Pi}$

$$a(n) := \mathbf{E}_\pi[Z(n) \mid Z(0) = 1]$$

can be expressed by means of S as

$$a(n) = e^{S_n} \quad \mathbf{P}\text{–}a.s.$$

According to fluctuation theory of random walks (compare Feller, Volume 2, Chapter XII) one may distinguish three different types of branching processes in random environment (BPRE). First, S can be a random walk with positive drift, which means that $\lim_{n\to\infty} S_n = \infty$ a.s. Second, S can have negative drift, i.e., $\lim_{n\to\infty} S_n = -\infty$ a.s. Finally, S may be an oscillating random walk meaning that $\limsup_{n\to\infty} S_n = \infty$ a.s. and at the same time $\liminf_{n\to\infty} S_n = -\infty$ a.s. There exists one more, degenerate possibility $S_n = 0$ with probability 1. However, in general, we **do not** require that the expectation of X exists.

We say that a BPRE is

subcritical if

$$\lim_{n\to\infty} S_n = -\infty \text{ a.s.};$$

degenerate critical if $X = 0$ with probability 1;

nondegenerate critical if

$$\limsup_{n\to\infty} S_n = +\infty, \ \liminf_{n\to\infty} S_n = -\infty \text{ a.s.}$$

and

supercritical if

$$\lim_{n\to\infty} S_n = +\infty \text{ a.s.}$$

Clearly, if $\mathbf{E}X$ exists then the process is subcritical if $\mathbf{E}X < 0$ is critical if $\mathbf{E}X = 0$ and is supercritical if $\mathbf{E}X > 0$ (by LLN and CLT).

Lemma 1 *If the process is either subcritical or critical then*

$$\lim_{n\to\infty} \mathbf{P}_\pi \left(Z(n) = 0\right) := \lim_{n\to\infty} \mathbf{P}\left(Z(n) = 0 | \bar{\Pi}\right) = 1 \ a.s.$$

Proof. If $\mathbf{P}\left(X \neq 0\right) > 0$ we have

$$\mathbf{P}_\pi \left(Z(n) > 0\right) \leq \mathbf{E}_\pi Z(n) = e^{S_n}$$

and

$$\mathbf{P}_\pi \left(Z(n) > 0\right) \leq \min_{0\leq k\leq n} \mathbf{P}_\pi \left(Z(k) > 0\right) \leq e^{\min_{0\leq k\leq n} S_k} \to 0 \text{ a.s.}$$

The case $\mathbf{P}\left(X = 0\right) = 1$ needs additional arguments.

In this case we have for

$$q_\pi(n) = \mathbf{P}_\pi \left(Z(n) = 0\right) = f_0(f_1(...f_{n-1}(0)...))$$

and

$$q_\pi = q_\pi(\infty)$$

that is

$$q_\pi = f_0(q_{T\pi})$$

where T means the shift transformation $T\pi = (\pi_1, \pi_2, ...)$.

The event $\{q_\pi = 1\}$ is shift invariant that is,

$$\{\omega : q_\pi = 1\} = \{\omega : q_{T\pi} = 1\}$$

with probability 1. Hence, by ergodic theorem

$$\mathbf{P}\,(q_\pi = 1) = 1 \text{ or } 0.$$

Let now $\log f_i'(1) = 0$ with probability 1 and assume the contrary, that is that $q_\pi < 1$ with probability 1. Observe that in this case

$$1 - q_\pi = 1 - f_0(q_{T\pi}) \le 1 - q_{T\pi}.$$

Denote

$$0 \le h_\pi = -\log\left(1 - q_\pi\right) < \infty \text{ a.s.}$$

and

$$r_\pi = -\log \frac{1 - f_0(q_{T\pi})}{1 - q_{T\pi}} = -\log \frac{1 - q_\pi}{1 - q_{T\pi}} \ge 0.$$

Then

$$h_\pi = r_\pi + h_{T\pi}$$

or

$$h_\pi = r_\pi + r_{T\pi} + ... + r_{T^n\pi} + h_{T^{n+1}\pi}.$$

By ergodic theorem

$$\begin{aligned} \frac{1}{n+1} h_\pi &\ge \frac{1}{n+1} h_\pi - \frac{1}{n+1} h_{T^{n+1}\pi} \\ &= \frac{1}{n+1} \sum_{i=0}^{n} r_{T^i\pi} \to \mathbf{E} r_\pi \ge 0 \end{aligned}$$

Clearly,

$$\lim_{n\to\infty} \frac{1}{n+1} h_\pi = 0.$$

This implies $\mathbf{E} r_\pi = 0$ and $r_\pi = 0$ with probability 1 leading to

$$q_{T\pi} = f_0(q_{T\pi}) \text{ a.s.}$$

that is $q_{T\pi} = 1$.

Now we would like to recall a natural correspondence between the branching processes in random environment and the simple random walk in random environment. The construction below is a natural extension of the construction given by Harris in [13] for the ordinary inhomogeneous Galton-Watson processes.

Let $G_r, r = 0, 1, 2, ...$, be a discrete-time simple random walk in random environment on the real line (see, for instance, [14] or [16]) specified by a tuple of independent and identically distributed nonnegative random vectors (p_n, q_n), $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$, with $p_n + q_n = 1$, $p_n q_n > 0$ with probability 1. In the other words, if a walking particle is at point G_r at moment r then at the next step it moves to the state $G_r + 1$ with probability p_{G_r} and to the state $G_r - 1$ with probability q_{G_r}. Assume now that $G_0 = 0$ and let $\theta = \min\{r : G_r = -1\}$. Denote by G_r^* the random walk stopped at moment $\theta : G_r^* = G_{r\wedge\theta}, r \geq 0$. Suppose that $\mathbf{P}(\theta < \infty) = 1$, that is our random walk in random environment is either recurrent or drifts to $-\infty$ with probability 1 (this is true, for instance, if $\mathbf{E}\log(p_n/q_n) \leq 0$, see [16]). Let a realization $\Pi_{r.env.} = \{p_n, q_n, n \in \mathbb{Z}\}$ of the environment for our random walk in random environment *be fixed* and let

$$Y(n) = \#\left\{r \in \mathbb{N}_0 : G_r^* = n, G_{r+1}^* = n - 1\right\} \tag{5}$$

be the number of jumps of the stopped random walk from level n to level $n-1$. It is not difficult to understand that if $\mathbf{P}(\theta < \infty) = 1$ then the number of visits of the random walk $\{G_r^*\}_{r\geq 0}$ to a state $n \in \mathbb{N}_0$, i.e., the local time $\ell(n)$ of G_r^* at n, is

$$\ell(n) := \#\{r \in \mathbb{N}_0 : G_r^* = n\} = Y(n+1) + Y(n) \tag{6}$$

for almost all realizations $\Pi_{r.env.}$. According to the definition of the stopped random walk $Y(0) = 1$, while the assumptions above show that the random variable $Y(1) = \#\left\{r \in \mathbb{N}_0 : G_r^* = 1, G_{r+1}^* = 0\right\}$ has a geometric distribution with

$$\mathbf{P}(Y(1) = j \mid \Pi_{r.env.}) = \mathbf{P}\left(\eta_{(0)} = j \mid \Pi_{r.env.}\right) = q_0 p_0^j$$

a.s. and, moreover,

$$Y(n) = \eta_{1,n-1} + ... + \eta_{n-1,Y(n-1)},\ n \geq 1, \tag{7}$$

where $\eta_{in} \stackrel{d}{=} \eta_{(n)}, i = 1, 2, ...,$ are independent identically distributed random variables with

$$\mathbf{E}\left[s^{\eta_{(n)}} \mid \Pi_{r.env.}\right] = \frac{q_n}{1 - p_n s}. \tag{8}$$

Consider now a branching process in random environment the support of the measure $\mathbb{P}$ of which is concentrated on the environments with reproduction generating functions of the form

$$f_n(s) = \frac{q_n}{1 - p_n s} = \mathbf{E}_\pi\left[s^{\xi_{(n)}}\right],\ n \in \mathbb{N}_0, \tag{9}$$

where the pairs (p_n, q_n) are distributed the same as above for the case of random walk in random environment. Thus,

$$Z(n) = \xi_{1,n-1} + ... + \xi_{n-1,Y(n-1)},\ n \geq 1, \tag{10}$$

Clearly, $\mathbb{E}X = \mathbb{E}\log f_n'(1) = \mathbb{E}\log(p_n/q_n)$ (if exists). Comparing (7) and (8) with (10) and (9) we see that these relations specify one and the same stochastic process. Thus, one can define on a common probability space the random walk $\{G_r^*\}_{r\geq 0}$ and the respective branching process $Z(n), n = 0, 1, ...,$ in random environment in such a way that $\{Z(n)\}_{n\geq 0} = \{Y(n)\}_{n\geq 0}$ with probability 1. Hence, for this particular case of possible reproduction laws we can reformulate all results of our future lectures established for branching processes in random environment into the respective results for stopped random walks in random environment $\{G_r^*\}_{r\geq 0}$ and vice versa. In particular, for almost all realizations of the environment we have for $k \in \mathbb{N}$

$$\{\ell(k) = j\} = \{Z(k+1) + Z(k) = j\}\,, \{Z(k) > 0\} = \left\{\max_{r\geq 0} G_r^* > k\right\}. \tag{11}$$

These relations allow us to make conclusions about the distribution of various characteristics of the local lime of the first excursion of the random walk in random environment. In this course of lectures we will use the described correspondence to study, in particular, the distribution of $\ell(\cdot)$ at the levels corresponding to the sequential moments of minima of the random walk

$$S_0 = 0,\ S_n = \log(p_0/q_0) + \log(p_1/q_1) + \ldots + \log(p_{n-1}/q_{n-1}),\ n \geq 1,$$

given $\{\max_{r\geq 0} G_r^* > n\}$

Our results will cover the so-called Sinai case $\mathbf{E}\log(p_n/q_n) = 0$ (see the classical paper [14] related to this subject) and the natural generalization of the Sinai case to the case when the random variables $\log(p_n/q_n)$ have no expectation (for instance, for *any* random walk in random environment whose distribution of $\log(p_n/q_n)$ is symmetric).

2 Random walk and Spitzer's condition

From now on we consider the NONDEGENERATE CRITICAL processes only and let $S = \{S_n\}_{n\geq 0}$ be it's associated random walk. It happens that the associated random walk plays a crucial role in our subsequent arguments and for this reason we first study properties of such random walk under rather weak assumptions. Let

$$\gamma_0 := 0, \quad \gamma_{j+1} := \min(n > \gamma_j : S_n < S_{\gamma_j}),\ \ j \geq 0,$$

and

$$\Gamma_0 := 0, \quad \Gamma_{j+1} := \min(n > \Gamma_j : S_n > S_{\Gamma_j}),\ j \geq 0,$$

be the strict descending and ascending ladder moments of $\{S_n\}_{n\geq 0}$. Introduce the renewal functions (PICTURE!!!)

$$V(x) := \sum_{j=0}^{\infty} \mathbf{P}(S_{\gamma_j} \geq -x), \quad x > 0, \quad V(0) = 1, \quad V(x) = 0, \quad x < 0,$$

and

$$U(x) := \sum_{j=0}^{\infty} \mathbf{P}(S_{\Gamma_j} < x), \quad x > 0, \quad U(0) = 1, \quad U(x) = 0, \quad x < 0.$$

We suppose that the random walk satisfies the **Spitzer-Doney condition**:

Condition A1. There exists $0 < \rho < 1$ such that

$$\frac{1}{n}\sum_{m=1}^{n}\mathbf{P}(S_m > 0) \to \rho, \quad n \to \infty,$$

or, what is the same

$$\mathbf{P}(S_n > 0) \to \rho, \quad n \to \infty.$$

Set

$$\begin{aligned}\Gamma &: = \min\{n \geq 1 : S_n \geq 0\},\ \tilde{\Gamma} := \min\{n \geq 1 : S_n \leq 0\},\\ \gamma &: = \min\{n \geq 1 : S_n < 0\}.\end{aligned}$$

Under condition A1 S_n is oscillating and, therefore,

$$\mathbf{P}(\tilde{\Gamma} < \infty) = \mathbf{P}(\gamma < \infty) = 1.$$

Set

$$D = \sum_{j=1}^{\infty} j^{-1}\mathbf{P}(S_j = 0).$$

Clearly, $D = 0$ if the distribution of X is absolute continuous.

Lemma 2 *For any oscillating random walk*

$$\mathbf{E}U(-X)I\{-X > 0\} = e^{-D}, \mathbf{E}U(x - X)I\{x - X > 0\} = U(x),\ x > 0, \quad (12)$$

$$\mathbf{E}V(X) = V(0) = 1, \quad \mathbf{E}V(x + X) = V(x),\ x \geq 0. \quad (13)$$

Proof. Set

$$a := \mathbf{P}(S_1 = 0) + \sum_{j=1}^{\infty}\mathbf{P}(0 < S_1, \ldots, 0 < S_j, S_{j+1} = 0).$$

Then

$$\begin{aligned}1 = \mathbf{P}(\tilde{\Gamma} < \infty) &= \mathbf{P}(\tilde{\Gamma} = 1) + \sum_{n=1}^{\infty}\mathbf{P}(\tilde{\Gamma} = n + 1)\\ &= \mathbf{P}(S_1 \leq 0) + \sum_{n=1}^{\infty}\mathbf{P}(\tilde{\Gamma} > n, S_{n+1} \leq 0)\\ &= \mathbf{P}(S_1 < 0) + \sum_{n=1}^{\infty}\mathbf{P}(\tilde{\Gamma} > n, S_{n+1} < 0) + a\\ &= \mathbf{P}(0 < -X) + \sum_{n=1}^{\infty}\mathbf{P}(\tilde{\Gamma} > n, S_n < -X) + a\\ &\qquad = \mathbf{E}U(-X)I\{-X > 0\} + a.\end{aligned}$$

In view of the equality $a = 1 - e^{-D}$ (see Feller, Ch. XII, §10) the first equality follows from (12). Further, for $x > 0$

$$\begin{aligned}
U(x) &= 1 + \mathbf{P}(0 < S_1 < x) + \sum_{j=1}^{\infty} \mathbf{P}(0 < S_1, \ldots, 0 < S_j, 0 < S_{j+1} < x) \\
&= 1 + \mathbf{P}(S_1 < x) + \sum_{j=1}^{\infty} \mathbf{P}(0 < S_1, \ldots, 0 < S_j, S_{j+1} < x) \\
-\mathbf{P}(S_1 &< 0) - \sum_{j=1}^{\infty} \mathbf{P}(0 < S_1, \ldots, 0 < S_j, S_{j+1} < 0) - a \\
&= 1 + \mathbf{P}(0 < x - X) + \sum_{j=1}^{\infty} \mathbf{P}(0 < S_1, \ldots, 0 < S_j, S_j < x - X) \\
-\mathbf{P}(0 &< -X) - \sum_{j=1}^{\infty} \mathbf{P}(0 < S_1, \ldots, 0 < S_j, S_j < -X) - a \\
&= 1 + \mathbf{E}U(x - X)I\{x - X > 0\} - \mathbf{E}U(-X)I\{-X > 0\} - a \\
&= \mathbf{E}U(x - X)I\{x - X > 0\}. \qquad (14)
\end{aligned}$$

The remaining part can be proved using $1 = \mathbf{P}(\Gamma < \infty)$.

Let

$$M_n := \max_{0 \le k \le n} S_k, \quad L_n := \min_{0 \le k \le n} S_k = S_{\tau(n)},$$
$$m_n(x) := \mathbf{P}(L_n \ge -x), \qquad \widetilde{m}_n(x) := \mathbf{P}(M_n < x).$$

and let $\tau(n)$ be the left-most point of minimum of $S_j, 0 \le j \le n$.

Lemma 3 *If Spitzer's condition is valid then*

$$\lim_{n \to \infty} \mathbf{P}\left(\frac{\tau(n)}{n} \le x\right) = \frac{\sin \rho\pi}{\pi} \int_0^x \frac{dy}{y^{\rho}(1-y)^{1-\rho}}, 0 \le x \le 1.$$

Proof. Omitted.

Lemma 4 *For $r \in (0,1)$ and $Re\lambda \le 0$*

$$\sum_{n=0}^{\infty} r^n \mathbf{E}\left[e^{\lambda S_n}; \Gamma > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n < 0\right]\right\},$$

$$\sum_{n=0}^{\infty} r^n \mathbf{E}\left[e^{\lambda S_n}; \gamma > n\right] = \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{E}\left[e^{\lambda S_n}; S_n \ge 0\right]\right\}.$$

In particular,

$$v(r) = \sum_{n=0}^{\infty} r^n \mathbf{P}(\Gamma > n) = \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{P}(S_n < 0)\right\}$$

and

$$u(r)=\sum_{n=0}^{\infty}r^{n}\mathbf{P}\left(\gamma>n\right)=\exp\left\{\sum_{n=1}^{\infty}\frac{r^{n}}{n}\mathbf{P}\left(S_{n}\geq0\right)\right\}.$$

Lemma 5 *(Spitzer identity) For $r\in(0,1)$ and $Re\lambda\leq0$*

$$\sum_{n=0}^{\infty}r^{n}\mathbf{E}e^{\lambda M_{n}}=\exp\left\{\sum_{n=1}^{\infty}\frac{r^{n}}{n}\mathbf{E}\left[e^{\lambda S_{n}};S_{n}\geq0\right]\right\}$$

and

$$\sum_{n=0}^{\infty}r^{n}\mathbf{E}e^{-\lambda L_{n}}=\exp\left\{\sum_{n=1}^{\infty}\frac{r^{n}}{n}\mathbf{E}\left[e^{-\lambda S_{n}};S_{n}\leq0\right]\right\}.$$

A function $l(x)$ is said to be a slowly varying at infinity if

$$\lim_{x\to\infty}\frac{l(tx)}{l(x)}=1$$

for any fixed $t>0$.

Examples: $\ln x$.

From now on we assume that X has absolute continuous distribution.

Lemma 6 *Under condition A1 there exist slowly varying functions $l_1(n)$ and $l_2(n)$, with $l_1(n)\,l_2(n)\sim\pi/\sin\pi\rho,\ n\to\infty$, and constants $c_1>0$, $c_2>0$ such that for any $n\geq1$ and all $x\in[0,\infty)$*

$$m_{n}(x)\leq c_{1}V(x)/\left(n^{1-\rho}l_{1}(n)\right),\qquad\widetilde{m}_{n}(x)\leq c_{2}U(x)/\left(n^{\rho}l_{2}(n)\right)$$

and for any fixed $x\in(0,\infty)$ as $n\to\infty$

$$m_{n}(x)\sim V(x)/\left(n^{1-\rho}l_{1}(n)\right),\qquad\widetilde{m}_{n}(x)\sim U(x)/\left(n^{\rho}l_{2}(n)\right).\tag{15}$$

Besides,

$$m_{n}(0)=\mathbf{P}(\gamma>n)\sim1/\left(n^{1-\rho}l_{1}(n)\right),\qquad\mathbf{P}(\Gamma>n)\sim1/\left(n^{\rho}l_{2}(n)\right).\tag{16}$$

Proof. By Lemma 5

$$\begin{aligned}v'(r)&=\sum_{n=0}^{\infty}nr^{n-1}\mathbf{P}\left(\Gamma>n\right)=\left(\sum_{n=1}^{\infty}r^{n-1}\mathbf{P}\left(S_{n}<0\right)\right)\exp\left\{\sum_{n=1}^{\infty}\frac{r^{n}}{n}\mathbf{P}\left(S_{n}<0\right)\right\}\\&=\left(\sum_{n=1}^{\infty}r^{n-1}\mathbf{P}\left(S_{n}<0\right)\right)v(r),\end{aligned}$$

and, by Spitzer-Doney condition

$$\frac{(1-r)v'(r)}{v(r)}\to1-\rho$$

implying (see Feller, Volume II)

$$v(r) \sim \frac{\Gamma(1-\rho)}{(1-r)^{1-\rho} l_2(1/(1-r))}$$

or

$$\sum_{k=1}^{n} \mathbf{P}(\Gamma > k) \sim \frac{\Gamma(1-\rho)}{l_2(n)} n^{1-\rho}$$

implying by Tauberian theorem (see Feller, Volume II)

$$\mathbf{P}(\Gamma > n) \sim \frac{1}{n^{\rho} l_2(n)}.$$

Similarly,

$$u(r) \sim \frac{\Gamma(1-\rho)}{(1-r)^{\rho} l_1(1/(1-r))}$$

and

$$m_n(0) = \mathbf{P}(\gamma > n) \sim \frac{1}{n^{1-\rho} l_1(n)}.$$

Further,

$$\begin{aligned} v(r)u(r) &= \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{P}(S_n < 0)\right\} \exp\left\{\sum_{n=1}^{\infty} \frac{r^n}{n} \mathbf{P}(S_n \geq 0)\right\} \\ &= \frac{1}{1-r} \sim \frac{\Gamma(1-\rho)}{(1-r)^{1-\rho} l_2(1/(1-r))} \times \frac{\Gamma(\rho)}{(1-r)^{\rho} l_1(1/(1-r))}. \end{aligned}$$

Hence

$$l_2(n) l_1(n) \sim \Gamma(\rho) \Gamma(1-\rho) = \frac{\pi}{\sin \pi\rho}.$$

The rest is omitted.

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