Stable processes

submitted by

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Summary

We consider several first passage problems for stable processes, giving explicit formulas for hitting distributions, hitting probabilities and potentials of stable processes killed at first passage. Our principal tools are the Lamperti representation of positive self-similar Markov processes and the Wiener–Hopf factorisation of Lévy processes. As part of the proof apparatus, we introduce a new class of Lévy processes with explicit Wiener–Hopf factorisation, which appear repeatedly in Lamperti representations derived from stable processes. We also apply the Lamperti–Kiu representation of real self-similar Markov processes and obtain results on the exponential functional of Markov additive processes, in order to find the law of the first time at which a stable process reaches the origin.

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Three years is a long time, and there is no way that I could have got through all the highs and lows without my friends in Bath and London – a huge thank you to, in no particular order, Charlie, Will, Siân, Melina and Marion, and an apology to anybody I have left out. Finally, my parents and my sister, who took me in during the holidays and kept me going through the tough times – thank you.

Contents

Chapter 1

Introduction

In probability theory, we are often concerned with models representing the result of a series of small events or influences. It is a remarkable feature of such models that, even when very little is known about the precise nature of these small effects, we can often say a great deal about the overall result. Indeed, it is this 'universality at large scales' which accounts for much of the success of probabilistic methods in the natural and social sciences.

As an illustration, consider the following toy model. Suppose that we have a collection of real-valued random variables $X_1, X_2, \dots,$ which are independent and all have the same distribution, and should model small-scale events. Denote by S_n the sum of the first n of these, that is, $S_n = X_1 + \cdots + X_n$. We are then interested in the behaviour of S_n when n is very large. In particular, suppose there exist sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ such that the rescaled, recentered sums

$$a_n^{-1} S_n - nb_n$$

converge in distribution to a random variable U. What distributions can U possess?

It emerges that only very few distributions can arise in this manner; they are called the normal distribution and stable distributions, and they occupy a central position in probability theory.

On the other hand, if we wish to capture in the limit not only the final sum S_n, but also the whole path $(S_1, S_2, \dots, S_n)$, we require the theory of stochastic processes. A stochastic process X is a collection of random variables $(X_t)_{t\geq 0}$, with each X_t representing the position of X at time t. The natural processes which arise as limits in this context are Brownian motion and stable processes. In fact, the distribution of a Brownian motion at any time is a normal distribution, while the distribution of a stable process at any time is a stable distribution.

Brownian motion has been extensively studied for many decades, and there is a huge body of research on a great many aspects of the process. This is due in part to the wide variety of mathematical tools which may be applied to its study, such as analysis via its generator, potential theory and connections to differential equations, excursion theory, martingale theory and stochastic calculus. Stable processes have proven more difficult to analyse by these means, and have therefore historically taken second place to Brownian motion, in spite of their central importance in the theory of stochastic processes. The contribution of this thesis is to shed light on certain aspects of stable processes via the 'Lamperti representation', which has in recent years shown itself to be a very useful tool in the analysis of stable processes.

We will now go into more detail on the topic of Brownian motion and stable processes, highlighting their relationship to one another and their key properties. We begin by presenting a mathematical definition of Brownian motion. A stochastic process W is said to be a Brownian motion if it satisfies the following conditions.

(si) Stationary increments. Fix two times $s < t$. Then the distribution of the increment $W_t - W_s$ depends only on the value $t - s$. This means that, if you start at time s and wait a certain time, the change in position of the 'particle' depends only on how long you waited, and not on the value of s.

(ii) Independent increments. Fix a collection of times $0 \leq t_1 < \cdots < t_n$. Then the increments $W_{t_n} - W_{t_{n-1}}, \dots, W_{t_2} - W_{t_1}$ are independent of one another. This property essentially guarantees that sections of the path of W that do not overlap in time may be made independent of each other by recentering them in space.

(2-ss) Self-similarity. Fix any $c > 0$ and define $\tilde{W}_t = cW_{tc^{-2}}$ for each $t \geq 0$. Then the random process $\tilde{W}$ has the same distribution as the process W. This is known as *self-similarity* or the *scaling property* with parameter 2, and the result is that one may 'zoom in' on a Brownian motion by rescaling space and time appropriately, and retain the characteristic features of Brownian motion.

One of the most important features of Brownian motion, which follows from the above three, is that almost every path of Brownian motion is continuous, that is, has no jumps. This may be seen in Figure 1-1, which shows a sample path of Brownian motion.[1]

Brownian motion is an example of a *Lévy process*. A random process X is called a Lévy process if it satisfies the properties **(si)** and **(ii)** above, written with X in place of W. Lévy processes form a wide class of random models, and are more flexible than Brownian motion since the path of a Lévy process X need

[1] All simulations in this thesis were performed by the author using Python and the `numpy` and `matplotlib` libraries. Stable distributions were simulated using the algorithm of Chambers et al. [19].

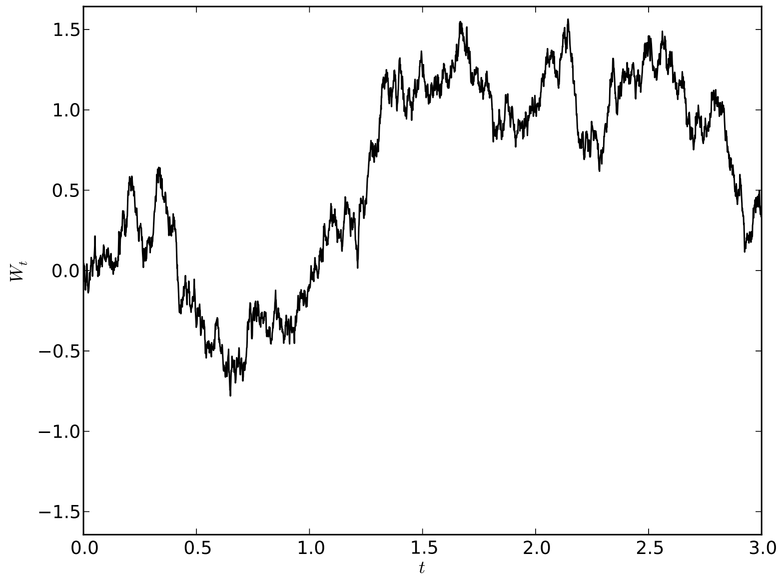

Figure 1-1: The graph of a Brownian motion in one dimension. The path is very rough, but does not contain any jumps.

not be continuous; that is, it may contain jumps. Remarkably, many features of these processes can be derived from nothing more than the stationarity and self-similarity of their increments. However, the property **(2-ss)** is a rather useful one, and we would like to find some sort of replacement for it. This leads us to *stable processes*, which are the main focus of this thesis.

A stable process is a Lévy process X which additionally satisfies the following property, for some choice of α such that $0 < \alpha < 2$.

(α-ss) Fix any $c > 0$ and define $\tilde{X}_t = cX_{tc^{-\alpha}}$ for each $t \geq 0$. Then the random process $\tilde{X}$ has the same distribution as the process X.

This *scaling property with parameter* α is very similar to that of Brownian motion, except that the spatial scale at which one 'zooms in' is different.

When studying stable processes, we have access to the rich literature on general Lévy processes, but do not restrict ourselves to continuous paths as in the case of Brownian motion. In fact, any stable process has *infinite* jump activity, in that it is guaranteed to have an infinite number of (mostly very small) jumps in any finite time period. Figure 1-2 depicts sample paths of stable processes for several choices of α.

The reader may notice the presence of very large jumps in Figure 1-2, particularly when α is small. This is related to the so-called 'heavy tailed' property of stable processes. In particular, for any t, the expectation $\mathrm{E}|X_t|^\beta$ is finite when

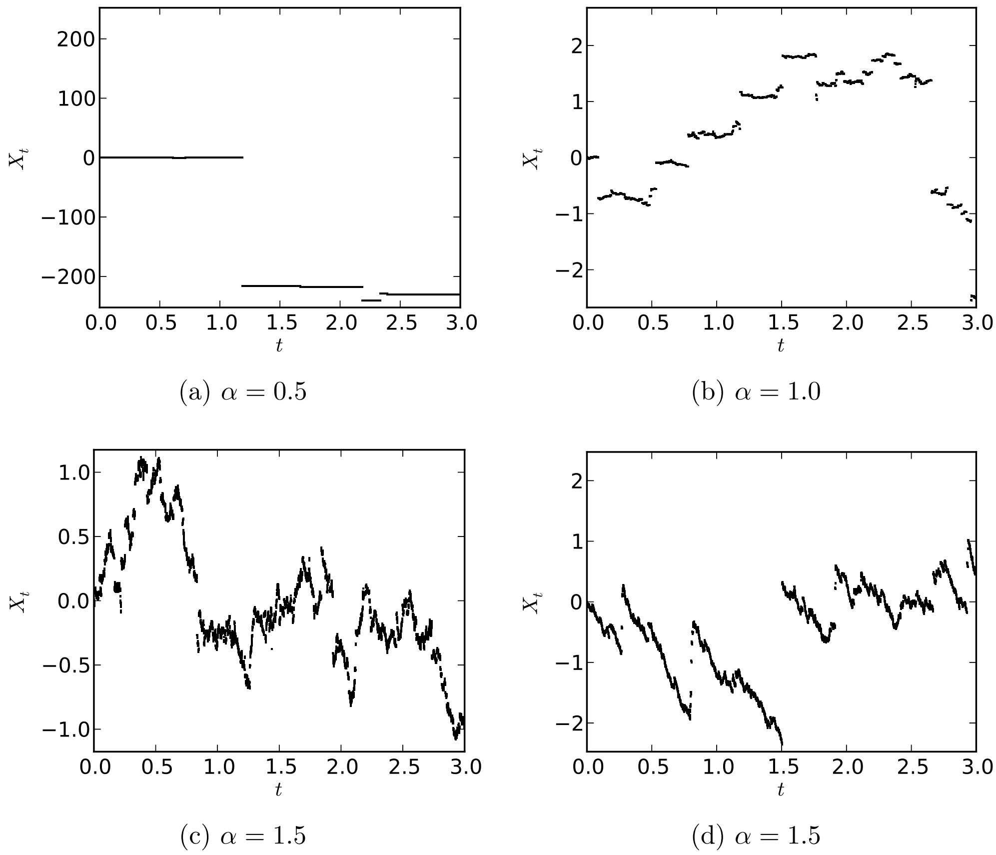

Figure 1-2: Graphs of stable processes for varying choices of α. Plots (a) through (c) show symmetric stable processes, while (d) shows a stable process which only jumps upwards. Note the scale in plot (a).

$0 \leq \beta < \alpha$, but when $\beta \geq \alpha$, it is infinite; in particular, when $\alpha \leq 1$, even the expected value of X_t is not well-defined. The upshot of this is that a stable process may exhibit large-magnitude, low-intensity jumps which are very rare but whose size forces the expectation to be infinite.

Taken together, these properties have led many to incorporate stable processes when modelling situations which feature rare but significant events. Furthermore, perhaps due to the scaling property already discussed, distributions associated with stable processes have appeared in many models in the physical sciences, including in classical and quantum physics, signal processing and biology; see chapter 1 of the book of Zolotarev [83] for several examples.

1.1 Outline

This thesis is dedicated to the computation of explicit identities for spatial and temporal aspects of stable processes. As part of the apparatus of proof, we also define and study a new class of Lévy processes of independent interest.

We now give a detailed summary of the main body of this thesis.

Chapter 2. Preliminaries. Here we review definitions and results which will be of use. Firstly, we give more detailed definitions of Lévy processes and stable processes, and review some of their key properties.

We then define positive self-similar Markov processes (pssMps), which loosely speaking are Markov processes with positive values, satisfying a property similar to (**α-ss**) above. If Y is a pssMp, then by a remarkable transformation due to Lamperti [58], Y may be represented by a Lévy process ξ; such a process ξ is sometimes called a Lamperti–Lévy process. We therefore give details of the Lamperti representation, and describe explicitly several known examples of Lamperti–Lévy processes which are obtained by starting with the stable process X, transforming its path to obtain a pssMp Y, and then looking at its Lamperti representation ξ.

We then return to the theory of Lévy processes, giving first some results related to the so-called Wiener–Hopf factorisation, and then reviewing a certain 'hypergeometric class' of Lévy processes. We shall bring these tools to bear on the new examples of Lamperti–Lévy processes that we encounter in the remainder of the thesis.

Chapter 3. The extended hypergeometric class. Our approach in this thesis will be roughly as follows. We intend to study stable processes via the theory of self-similar Markov processes, in the following way. Suppose we wish to compute the distribution of some function $f(X)$ of the stable process X. Our technique is to construct a suitable positive self-similar Markov process Y out of X in some way, and then to consider the Lamperti representation ξ of Y. If we have chosen our construction correctly, there will be some function g such that $f(X) = g(\xi)$, and so we only need to compute the distribution of $g(\xi)$; it frequently turns out to be considerably simpler to work with the Lamperti–Lévy process ξ in this way than to perform computations directly with the original stable process.

For this scheme to be effective, we will need some results about the Lamperti–Lévy processes we expect to encounter. It is already known that many of the processes derived from stable processes via the Lamperti transformation fall into the 'hypergeometric class' as defined in section 2.7. In the course of this thesis, we will obtain several Lamperti–Levy processes, derived from transformations of stable processes, which are indeed hypergeometric Lévy processes when $\alpha \leq 1$, but which fall outside this class when $\alpha > 1$. We therefore introduce an 'extended hypergeometric class', which will encompass these cases. We derive several results about this general class, and then consider two relevant examples.

Chapter 4. Hitting distributions and path censoring. The efforts of mathematicians over the years have produced many distributional identities for Brownian motion, some found by probabilistic arguments, and some via potential theory and connections with partial differential equations, among other methods. A large sample of quantities which have been computed explicitly may be found in the book of Borodin and Salminen [13]. To give a simple example which is close in spirit to our work, consider a Brownian motion started at the point 0, and an interval $[a, b]$ such that $a \leq 0 \leq b$. We define the exit time

$$\sigma = \inf\{t \geq 0 : W_t \notin [a, b]\},$$

and ask for the distribution of the random variable W_σ. Since W has continuous paths, it is sufficient to decide whether the process exits the interval $[a, b]$ by hitting the lower boundary a or the upper boundary b, and there is an elegant martingale method for settling this question; see [71, Proposition 7.3]. The problem becomes dramatically different if one replaces W with an α-stable process X, primarily because X may very well jump over the boundary: the law of X_σ is determined not solely by whether the exit is over the upper or lower boundary, but also by how far the process overshoots.

The law of X_σ was computed explicitly by Rogozin [73] in 1972, by means of solving a system of coupled integral equations. A similar approach was taken by Blumenthal et al. [11] in considering the following related hitting problem. Suppose that we define the hitting time

$$\tau = \inf\{t \geq 0 : X_t \in (a, b)\},$$

and ask for the distribution of X_τ when started from some point not in $[a, b]$.

For a Brownian motion, the absence of jumps means the problem is entirely trivial, but for a stable process it appears to be rather difficult. The question was resolved in [11] for X a symmetric stable process in one or more dimensions, but in the asymmetric case the problem has remained open for several decades. In chapter 4, we solve the hitting problem in one dimension, by means of looking at the Lamperti representation of a path transformation of X which we called 'path censoring'. We also compute the probability that X hits the point 0 before passing above the level 1, and characterise the distribution of the time that X spends in $(0, \infty)$ before hitting zero.

Chapter 5. Potentials of killed processes. Potentials of stochastic processes are closely related to hitting problems. Suppose that ζ is a random time, and define the potential measure of X killed at ζ:

$$U_\zeta(x, \mathrm{d}y) = \mathrm{E}_x \int_0^\zeta \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t,$$

where E_x indicates expectation when X starts at the point x. If σ and τ are the random exit and entrance times alluded to above, we note that both of the measures U_σ and U_τ can be connected to certain simpler potentials for Lévy processes obtained through the Lamperti transformation. In chapter 5, we compute U_σ and U_τ, along with closely related quantities, including potentials for the stable process reflected in its infimum.

Chapter 6. The hitting time of zero. The final chapter is dedicated to the only temporal identity in this thesis. Since the Lamperti representation involves a time-change, spatial identities of X are easily related to spatial identities of Lamperti–Lévy processes; but the presence of time makes matters more difficult. We consider the law of the hitting time of zero, which is related to the so-called exponential functional of a Lévy process. Indeed, the Lamperti transformation alone proves insufficient for our purposes, and we turn to the recent Lamperti–Kiu transformation, which yields a Markov additive process instead of a Lévy process. We present a number of applications, including to the process conditioned to avoid zero. Inspired by the methods in this chapter, we also suggest future work on conditionings of self-similar processes at zero.

1.2 Publication and collaboration details

Much of the work in this thesis is the result of collaboration.

Chapter 3 is joint work with Andreas E. Kyprianou (AEK) and Juan-Carlos Pardo (JCP), and a modified version is in preparation as a paper. Parts of sections 3.3 and 3.4 also appear in [49].

Chapter 4 is also joint work with AEK and JCP. This has been accepted for publication, and forms reference [56].

Chapter 5 is also joint work with AEK and JCP, and parts of it are included in [56], the remainder being prepared for submission separately.

Chapter 6 is joint work with Alexey Kuznetsov, AEK and JCP, and the majority of it has been submitted as [49].

Chapter 2

Preliminaries

The purpose of this chapter is to make this thesis as self-contained as is reasonably possible by introducing definitions and results which we work with in the rest of the text.

We set out to answer questions about the way in which stable processes hit sets. Since stable processes are self-similar Lévy processes, we begin by defining first Lévy processes, then stable processes, and then positive self-similar Markov processes. We then outline the connection between the first and third of these by discussing the Lamperti representation, and offer three important special cases, each derived from the stable process. We then commence our study of the fluctuations of Lévy processes by introducing the Wiener–Hopf factorisation, which will be our main probabilistic and analytic tool. The Wiener–Hopf factorisation facilitates both decomposition and synthesis of Lévy processes, and we explore useful classes of subordinators as well as the theory of philanthropy, in order to exploit this. Finally, we review a class of 'hypergeometric Lévy processes', which will encompass many of the processes we meet in the remaining chapters.

2.1 Lévy processes

Lévy processes have two roles in this work. On the one hand, our basic object of study is the stable process, which is a Lévy process satisfying a certain self-similarity property; on the other hand, a major component of our analysis is the Lamperti representation, which is a bijection between Lévy processes and certain self-similar processes. We therefore begin by defining Lévy processes and reviewing some fundamental results about their structure. For more details, we refer the reader to the books of Bertoin [6], Doney [31], Kyprianou [51] and Sato [74].

A *Lévy process* is a stochastic process issued from the origin with stationary and independent increments and càdlàg paths, with state space $\mathbb{R} \cup \{\partial\}$. The state ∂

is a cemetery state in which the process is absorbed.

If $X := (X_t)_{t\geq 0}$ is a one-dimensional Lévy process with law P, then the classical Lévy–Khintchine formula states that for all $t \geq 0$, the characteristic exponent Ψ, given by $e^{-t\Psi(\theta)} = \mathrm{E}(e^{\mathrm{i}\theta X_t})$ for $\theta \in \mathbb{R}$, satisfies

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left[1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta l(x)\right] \Pi(\mathrm{d}x) + q,$$

where $a \in \mathbb{R}$ is known as the *centre*, $\sigma \geq 0$ is the *Brownian coefficient*, and Π is a measure (the *Lévy measure*) with no atom at 0 such that $\int_{\mathbb{R}} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$.

The value $q \geq 0$ is the *killing rate* of the Lévy process. When $q = 0$, the process never reaches ∂, and is said to be *unkilled*. When $q > 0$, the process X remains in the set $\mathbb{R}$ until an exponentially distributed time, of rate q and independent of the path of X, is reached; at that point, X is sent to ∂. Such a process is referred to as a *killed* Lévy process.

The function l is known as a *cutoff function*, and should satisfy

$$\begin{aligned} l(x)/x &= 1 + o(x), && x \to 0, \\ l(x)/x &= O(1/x), && |x| \to \infty. \end{aligned}$$

An extremely common choice of cutoff function is $l(x) = x\mathbb{1}_{[-1,1]}(x)$, and we will use this unless otherwise specified. Alternatives include $l(x) = x/(1 + x^2)$ and $l(x) = \sin x$; see also [74, §8]. The choice of l affects only the value of a, and not any other parameters in the Lévy–Khintchine representation.

The meaning of the terms in the Lévy–Khintchine representation is loosely as follows: the parameter σ is the volatility of a Brownian motion; the integral with respect to Π is the combination of a large-jump compound Poisson process and an L^2 limit of compensated small-jump compound Poisson processes; and the quantity a is a combination of deterministic linear drift and compensation terms. The precise statement of this discussion is known as the *Lévy–Itō decomposition*, and may be found in, for example, Sato [74, Chapter 4].

One may ask when the drifting and jumping behaviours of a Lévy process X can be separated, and this question leads to the notion of bounded variation. A càdlàg function $f\colon [0, t] \to \mathbb{R}$ is said to be of *bounded* (or *finite*) *variation* if

$$\sup_{s_1,\dots,s_n} \sum_{j=1}^{n} |f(s_j) - f(s_{j-1})| < \infty,$$

where the supremum is over all finite partitions of $[0, t]$. Then, it is known (see [74, Section 21]) that a Lévy process X has paths which are almost surely of bounded variation on every set $[0, t]$ if and only if $\sigma = 0$ and either $\Pi(\mathbb{R}) < \infty$, or $\int_{|x|\le 1}|x|\,\Pi(\mathrm{d}x) < \infty$. In this case, one may write

$$\Psi(\theta) = -\mathrm{i}\mathbf{d}\theta + \int_{\mathbb{R}} (1 - e^{\mathrm{i}\theta x})\,\Pi(\mathrm{d}x) + q, \qquad \theta \in \mathbb{R},$$

and $\mathbf{d} \in \mathbb{R}$ is known as the *drift* of the Lévy process X.

A particular case of bounded variation occurs when X has paths which are increasing almost surely. In this case, X is known as a *subordinator*, and instead of the characteristic exponent Ψ, it is more common to work with the *Laplace exponent* ψ, given by

$$\mathrm{E}e^{-\lambda X_t} = e^{-t\psi(\lambda)}. \tag{2.1}$$

Then, one has

$$\psi(\lambda) = \mathbf{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\,\Pi(\mathrm{d}x) + q,$$

and once again $\mathbf{d}$ is called the drift of the subordinator X. Subordinators occupy a special role in the theory of Lévy processes, as many more general theorems can be reduced to the study of carefully chosen subordinators.

Let us make a remark about sign convention. When we speak about a Laplace exponent, we will always mean a function defined as in (2.1) above. There will be several occasions on which we wish to discuss an analogous function, but with signs of both the exponents in (2.1) reversed. Such a function is also often known as a Laplace exponent in the literature, but we will make an effort to distinguish the two, calling the latter a Laplace$^+$ exponent. As to the sign on the characteristic exponent, we follow [51], but many authors refer to $-\Psi$ as the characteristic exponent.

2.2 Stable processes

A process X with law P is said to be a *(strictly) α-stable process*, or just a *stable process*, if it is a Lévy process which also satisfies the *scaling property*: under P, for every $c > 0$, the process $(cX_{tc^{-\alpha}})_{t\ge 0}$ has the same law as X. It is known that $\alpha \in (0, 2]$, and the case $\alpha = 2$ corresponds to Brownian motion, which we exclude. The Lévy-Khintchine representation of such a process is as follows: $\sigma = 0$, and the Lévy measure Π is absolutely continuous with density given by

$$c_+ x^{-(\alpha+1)} \mathbb{1}_{\{x>0\}} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{\{x<0\}}, \qquad x \in \mathbb{R},$$

where $c_+, c_- \ge 0$, and $c_+ = c_-$ when $\alpha = 1$. It holds that $a = (c_+ - c_-)/(\alpha - 1)$ when $\alpha \ne 1$; however, if $\alpha = 1$, we specify that $a = 0$, which is a restriction ensuring that the only 1-stable process we consider is the symmetric Cauchy process.

These choices mean that, up to a multiplicative constant $c > 0$, X has the characteristic exponent

$$\Psi(\theta) = \begin{cases} c|\theta|^\alpha(1 - \mathrm{i}\beta \tan \frac{\pi\alpha}{2} \operatorname{sgn}\theta) & \alpha \in (0,2) \setminus \{1\}, \\ c|\theta| & \alpha = 1, \end{cases} \qquad \theta \in \mathbb{R}, \tag{2.2}$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$ and $c = -(c_+ + c_-)\Gamma(-\alpha)\cos\bigl(\frac{\pi\alpha}{2}\bigr)$. For more details, see Sato [74, §14].

For consistency with the literature we appeal to, we shall always parameterise our stable process such that

$$c_+ = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \quad \text{and} \quad c_- = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)},$$

where $\rho = \mathrm{P}(X_t \geq 0) = \mathrm{P}(X_t > 0)$ is the positivity parameter, and $\hat\rho = 1 - \rho$. This corresponds to

$$\beta = \frac{\tan(\pi\alpha(\rho - 1/2))}{\tan(\pi\alpha/2)} \quad \text{and} \quad c = \cos(\pi\alpha(\rho - 1/2)).$$

We take the point of view that the class of stable processes, with this normalisation, is parameterised by α and ρ; the reader will note that all the quantities above can be written in terms of these parameters. We shall restrict ourselves a little further within this class by excluding the possibility of having only one-sided jumps. Together with our assumption about the case $\alpha = 1$, this gives us the following set of admissible parameters:

$$\begin{aligned} \mathcal{A}_{\mathrm{st}} = &\bigl\{(\alpha,\rho) : \alpha \in (0,1),\ \rho \in (0,1)\bigr\} \\ &\cup \bigl\{(\alpha,\rho) : \alpha \in (1,2),\ \rho \in (1-1/\alpha, 1/\alpha)\bigr\} \cup \bigl\{(\alpha,\rho) = (1,1/2)\bigr\}. \end{aligned}$$

For each $x \in \mathbb{R}$, we shall denote by P_x the law of $X + x$ under P.

Remark 2.1. We comment on the three 'gaps' in our collection of stable processes. The first is the Brownian case $\alpha = 2$. This is easily justified, for most of the hitting results we prove are trivial for Brownian motion, and the other results are already known, and may be found, for example, in [13].

The second gap is the case where X jumps only in one direction. Such a process is either a subordinator or the negative of a subordinator, if $\alpha \leq 1$, or a totally asymmetric Lévy process, that is, one with only negative or only positive jumps but which does not have monotone paths, if $\alpha > 1$. In this case our results can be proved fairly simply with standard techniques, and we will remark on this again at the appropriate points in the text; see Proposition 4.3 and Remarks 4.22, 4.23, 5.2, 5.4 and 6.9.

The third gap is the case of 1-stable processes which are not symmetric; such processes are the sum of a symmetric Cauchy process and a deterministic drift. The reason for this omission is our use of the Lamperti representation, and the paper of Caballero and Chaumont [14], which we rely on, omits this case also.

Remark 2.2. Let us also mention the following properties of X, which may be found, for example, in [6]: namely Proposition VIII.8 and the discussion immediately preceding it, and the remarks on page 34. When $\alpha \in (0, 1]$, points are polar for X, while when $\alpha \in (1, 2)$, every point is recurrent almost surely. On the other hand, when $\alpha \in (0, 1)$, the process X is transient a.s., while when $\alpha \in [1, 2)$ the process is recurrent almost surely, in the sense that, with probability 1, it returns to every compact set infinitely often. Combining these two observations allows us to make the following remark.

If we define

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

then the following holds under P_x, for any $x \neq 0$.

(i) If $\alpha \in (0, 1)$, $T_0 = \infty$ and X is transient almost surely.

(ii) If $\alpha = 1$, $T_0 = \infty$ and every neighbourhood of zero is an a.s. recurrent set for X.

(iii) If $\alpha \in (1, 2)$, $T_0 < \infty$ and X hits zero continuously, by means of an infinite number of jumps on either side.

2.3 Positive, self-similar Markov processes

A *positive self-similar Markov process* (*pssMp*) with *self-similarity index* $\alpha > 0$ is a standard Markov process $Y = (Y_t)_{t\geq 0}$ with filtration $(\mathscr{G}_t)_{t\geq 0}$ and probability laws $(\mathrm{P}_x)_{x>0}$, on $[0, \infty)$, which has 0 as an absorbing state and which satisfies the *scaling property*, that for every $x, c > 0$,

$$\text{the law of } (cY_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}. \tag{2.3}$$

Here, we mean "standard" in the sense of [10], which is to say, $(\mathscr{G}_t)_{t\geq 0}$ is a complete, right-continuous filtration, and Y has càdlàg paths and is strong Markov and quasi-left-continuous.

2.4 The Lamperti representation

In the seminal paper [58], Lamperti describes a bijective correspondence between pssMps and Lévy processes, which we now outline. It may be worth noting that we have presented a slightly different definition of pssMp from Lamperti; for the connection, see [80, §0].

Let

$$S(t) = \int_0^t (Y_u)^{-\alpha} \, \mathrm{d}u. \tag{2.4}$$

This process is continuous and strictly increasing until Y reaches zero. Let $(T(s))_{s\geq 0}$ be its inverse, and define

$$\xi_s = \log Y_{T(s)} \qquad s \geq 0.$$

Then $\xi := (\xi_s)_{s\geq 0}$ is a Lévy process started at $\log x$, possibly killed at an independent exponential time; the law of the Lévy process and the rate of killing do not depend on the value of x. The real-valued process ξ with probability laws $(\mathbb{P}_y)_{y\in\mathbb{R}}$ is called the *Lévy process associated to* Y, or the *Lamperti transform of* Y. We will sometimes refer to a Lévy process arising in this way from the Lamperti transform as a *Lamperti–Lévy process*.

An equivalent definition of S and T, in terms of ξ instead of Y, is given by taking

$$T(s) = \int_0^s \exp(\alpha \xi_u) \, \mathrm{d}u$$

and S as its inverse. (Note that here and elsewhere, all functions are considered to evaluate to zero when applied to the cemetery state ∂.) Then,

$$Y_t = \exp(\xi_{S(t)}) \tag{2.5}$$

for all $t \geq 0$, and this shows that the Lamperti transform is a bijection.

Let $T_0 = \inf\{t > 0 : Y_t = 0\}$ be the first hitting time of the absorbing state zero. Then the large-time behaviour of ξ can be described by the behaviour of Y at T_0, as follows:

(i) If $T_0 = \infty$ a.s., then ξ is unkilled and either oscillates or drifts to $+\infty$.

(ii) If $T_0 < \infty$ and $Y_{T_0-} = 0$ a.s., then ξ is unkilled and drifts to $-\infty$.

(iii) If $T_0 < \infty$ and $Y_{T_0-} > 0$ a.s., then ξ is killed.

It is proved in [58] that the events in (i)–(iii) above satisfy a zero-one law independently of x, and so the three possibilites above are an exhaustive classification of pssMps.

It is immediate that

$$T_0 = T(\infty) = \int_0^\infty \exp(\alpha \xi_u) \, \mathrm{d}u.$$

The latter quantity is known as the *exponential functional* of the Lévy process $\alpha\xi$, and we will denote it $I(\alpha\xi)$. More will be said about exponential functionals in chapters 3 and 6.

2.5 Three examples: killed and conditioned stable processes

In this section, we describe three basic examples of pssMps and their Lamperti representations. All three are obtained from the stable process: as we have seen, this process is self-similar, but unless it is a subordinator, it is certainly not $[0,\infty)$-valued; moreover, when $\alpha > 1$, the process may very well hit zero and exit again. Nonetheless, certain transformations of stable processes are indeed pssMps; following Caballero and Chaumont [14], where their Lamperti representations were first described, we call them *killed and conditioned stable processes*.

Our motivation is threefold: firstly, we provide interesting and immediate examples of the Lamperti representation; secondly, we introduce processes which we will use in the sequel; and thirdly, we motivate the introduction of the *hypergeometric class* of Lévy processes in section 2.7.

2.5.1 The stable process killed on exiting $[0,\infty)$

For our first case, we start with the stable process X, and define

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\},$$

the first hitting time of $(-\infty, 0)$ for the process X. Then, let

$$X_t^* = X_t \mathbb{1}_{\{t<\tau_0^-\}}.$$

We will denote the laws of this process by $(\mathrm{P}_x^*)_{x>0}$. It is clear that this process is a pssMp; we call it the *stable process killed on exiting* $[0,\infty)$. We will denote the Lévy process associated to X^* in the Lamperti representation by ξ^*, and its laws by $(\mathbb{P}_y^*)_{y\in\mathbb{R}}$.

It is shown in [14] that the characteristic exponent of ξ^* is given by the Lévy–Khintchine representation

$$\Psi^*(\theta) = \mathrm{i}a\theta + \int_{\mathbb{R}} \big[1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta(e^x - 1)\mathbb{1}_{\{|e^x-1|<1\}}\big]\pi^*(x)\,\mathrm{d}x + c_-/\alpha, \qquad \theta \in \mathbb{R}.$$

where a is as in section 2.2, and the Lévy density π^* is given by

$$\pi^*(x) = c_+ \frac{e^x}{(e^x-1)^{\alpha+1}}\mathbb{1}_{\{x>0\}} + c_- \frac{e^x}{(1-e^x)^{\alpha+1}}\mathbb{1}_{\{x<0\}}, \qquad x \in \mathbb{R}. \tag{2.6}$$

We draw attention to the unusual cutoff function $x \mapsto (e^x-1)\mathbb{1}_{\{|e^x-1|<1\}}$ in the Lévy–Khintchine representation; this is an artifact of the Lamperti transform's effect on the generator of X^*, and has the effect of keeping the centre of the Lévy–Khintchine representation equal to the familiar quantity a.

Note that the pssMp X^* reaches zero by a jump, and is therefore in the class (iii) of section 2.4. This is reflected in the killing rate $c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$ seen in the characteristic function Ψ^*.

2.5.2 The stable process conditioned to stay positive

It is known (see [21, 14]) that the function

$$h^{\uparrow}(x) = x^{\alpha\hat{\rho}} \mathbb{1}_{\{x>0\}} \qquad x \geq 0,$$

is invariant for the stable process killed on exiting $[0, \infty)$, in the sense that for every $x \geq 0$ and $t \geq 0$,

$$\mathrm{E}_x[h^{\uparrow}(X_t);\, t < \tau_0^-] = h^{\uparrow}(x).$$

It follows that, for each $x > 0$, the process $(h^{\uparrow}(X_t)/h^{\uparrow}(x))_{t\geq 0}$ is a martingale with mean one under the law $\mathrm{P}_x[\cdot;\, t < \tau_0^-]$. We may therefore define a collection of probability laws via the *Doob h-transform*, as follows:

$$\mathrm{P}_x^{\uparrow}(\Lambda) = \frac{1}{h^{\uparrow}(x)} \mathrm{E}_x\big[h^{\uparrow}(X_t)\mathbb{1}_{\Lambda};\, t < \tau_0^-\big], \qquad x > 0,\, t \geq 0,\, \Lambda \in \mathscr{F}_t, \tag{2.7}$$

where, here and in the sequel, $(\mathscr{F}_t)_{t\geq 0}$ is the standard filtration associated with the stable process X. The canonical process associated with these laws is called the *stable process conditioned to stay positive*, and it is a strong α-self-similar Markov process which remains in the set $(0, \infty)$ at all times and drifts to $+\infty$ almost surely. In particular, it is a pssMp with index α. We will denote it by $X^{\uparrow}$.

The Lamperti representation of $X^{\uparrow}$ is calculated in [14]. We will denote it $\xi^{\uparrow}$. It has characteristic exponent

$$\Psi^{\uparrow}(\theta) = \mathrm{i}a^{\uparrow}\theta + \int_{\mathbb{R}} \big[1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta(e^x - 1)\mathbb{1}_{\{|e^x-1|<1\}}\big]\pi^{\uparrow}(x)\,\mathrm{d}x,$$

where a formula for the centre is given by

$$a^{\uparrow} = a - c_+ \int_0^1 \frac{(1+x)^{\alpha\hat{\rho}} - 1}{x^{\alpha}}\,\mathrm{d}x - c_- \int_0^1 \frac{(1-x)^{\alpha\hat{\rho}} - 1}{x^{\alpha}}\,\mathrm{d}x$$

and the Lévy density is given by

$$\pi^{\uparrow}(x) = c_+ \frac{e^{(\alpha\hat{\rho}+1)x}}{(e^x-1)^{\alpha+1}} \mathbb{1}_{\{x>0\}} + c_- \frac{e^{(\alpha\hat{\rho}+1)x}}{(1-e^x)^{\alpha+1}} \mathbb{1}_{\{x<0\}}, \qquad x \in \mathbb{R}.$$

2.5.3 The stable process conditioned to hit zero continuously

Another conditioned stable process is given by a different choice of harmonic function. It is known (see [20, 14]) that the function

$$h^{\searrow}(x) = x^{\alpha\hat{\rho}-1}, \qquad x > 0,$$

is harmonic for the stable process killed on exiting $[0,\infty)$, and so as before we may define

$$\mathrm{P}^{\searrow}_x(\Lambda) = \frac{1}{h^{\searrow}(x)}\mathrm{E}_x\big[h^{\searrow}(X_t)\mathbb{1}_\Lambda;\, t < \tau_0^-\big], \qquad x > 0,\, t \geq 0,\, \Lambda \in \mathscr{F}_t.$$

The canonical process associated with these laws, which we will write $X^{\searrow}$, is called the *stable process conditioned to hit zero continuously*, and it is a pssMp with index α which in finite time reaches the point zero continuously, in the sense that

$$\mathrm{P}^{\searrow}_x(X^{\searrow}_{T_0-} = 0, T_0 < \infty) = 1.$$

The Lamperti representation of $X^{\searrow}$ is calculated in [14], and we denote it $\xi^{\searrow}$. The process $\xi^{\searrow}$ has characteristic exponent

$$\Psi^{\searrow}(\theta) = \mathrm{i}a^{\searrow}\theta + \int_{\mathbb{R}}\big[1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta(e^x - 1)\mathbb{1}_{\{|e^x-1|<1\}}\big]\pi^{\searrow}(x)\,\mathrm{d}x,$$

where a formula for the centre is given by

$$a^{\searrow} = a - c_+\int_0^1 \frac{(1+x)^{\alpha\hat{\rho}-1} - 1}{x^\alpha}\,\mathrm{d}x - c_-\int_0^1 \frac{(1-x)^{\alpha\hat{\rho}-1} - 1}{x^\alpha}\,\mathrm{d}x$$

and the Lévy density is given by

$$\pi^{\searrow}(x) = c_+\frac{e^{\alpha\hat{\rho}x}}{(e^x-1)^{\alpha+1}}\mathbb{1}_{\{x>0\}} + c_-\frac{e^{\alpha\hat{\rho}x}}{(1-e^x)^{\alpha+1}}\mathbb{1}_{\{x<0\}}, \qquad x \in \mathbb{R}.$$

2.6 Wiener–Hopf factorisation

In this section, we describe the Wiener–Hopf factorisation of a Lévy process and related topics. This factorisation decomposes the process into two so-called ladder height processes, which are subordinators (that is, increasing Lévy processes). The factorisation will prove extremely useful; however, in order to apply it, one needs to recognise which subordinators come into play through their Laplace exponents. We therefore consider a class of subordinators which will appear frequently when describing ladder height processes. We look also at two transformations of subordinators, and discuss so-called special and complete subordinators. We then turn the situation on its head, and ask when one may build a Lévy process out of two desired ladder height processes. Finally, we describe a class of processes with semi-explicit Wiener–Hopf factorisation, known as the 'meromorphic class'.

We begin with a short sketch of the Wiener–Hopf factorisation. For further details, see [51, Chapter 6] and [6, §VI.2].

The Wiener–Hopf factorisation describes the characteristic exponent of a Lévy process in terms of the Laplace exponents of two subordinators. Recall that a *subordinator* is an increasing Lévy process, possibly killed at an independent exponentially distributed time and sent to the cemetery state ∂ (which may be identified with $+\infty$). If H is a subordinator with expectation operator $\mathbb{E}$, we define its *Laplace exponent* ψ by the equation

$$\mathbb{E}\big[\exp(-\lambda H_1)\big] = \exp(-\psi(\lambda)), \qquad \lambda \geq 0.$$

Indeed, the function ψ may even be analytically extended to the positive half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$. Similarly, let ξ be a (possibly killed) Lévy process, again with expectation $\mathbb{E}$, and denote its characteristic exponent by Ψ, so that

$$\mathbb{E}\big[\exp(\mathrm{i}\theta\xi_1)\big] = \exp(-\Psi(\theta)), \qquad \theta \in \mathbb{R}.$$

The *Wiener–Hopf factorisation*[1] of ξ consists of the decomposition

$$k\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}, \tag{2.8}$$

where $k > 0$ is a constant which may, without loss of generality, be taken equal to one, and the functions κ and $\hat{\kappa}$ are the Laplace exponents of certain subordinators which we denote H and $\hat{H}$.

Any decomposition of the form (2.8) is unique, up to the constant k, provided that the functions κ and $\hat{\kappa}$ are Laplace exponents of subordinators. The exponents κ and $\hat{\kappa}$ are termed the *Wiener–Hopf factors* of ξ.

The subordinator H can be identified in law as an appropriate time change of the running maximum process $\bar{\xi} := \big(\bar{\xi}_t\big)_{t\geq 0}$, where $\bar{\xi}_t = \sup\{\xi_s,\, s \leq t\}$. In particular, the range of H and $\bar{\xi}$ are the same. Similarly, $\hat{H}$ is equal in law to an appropriate time-change of $-\underline{\xi} := \big(-\underline{\xi}_t\big)_{t\geq 0}$, with $\underline{\xi}_t = \inf\{\xi_s,\, s \leq t\}$, and they

[1] The name 'Wiener–Hopf factorisation' encompasses several decompositions of stochastic processes. The factorisation we present is purely spatial. However, to give a result which also involves time, it is also known that, if $\mathbf{e}_p$ is an independent exponentially distributed random variable of rate p, one may decompose $\mathbb{E}\big[e^{\mathrm{i}\vartheta\mathbf{e}_p + \mathrm{i}\theta\xi_{\mathbf{e}_p}}\big]$ as the product of two characteristic functions of bivariate (space-time) infinitely divisible distributions; see [51, Theorem 6.16(i)]. One may also view the Wiener–Hopf factorisation as the decomposition of the path of a Lévy process into a Poisson point process of marked excursions from the maximum.

have the same range. Intuitively speaking, H and $\hat{H}$ keep track of how ξ reaches its new maxima and minima, and they are therefore termed the *ascending* and *descending ladder height processes* associated to ξ.

Remark 2.3. Occasionally, in deference to cited works, we will find it more convenient to express the Wiener–Hopf factorisation in terms of the Laplace$^+$ exponent of ξ. If we write this as ϕ, that is $e^{\phi(z)} = \mathbb{E}e^{z\xi_1}$, then (2.8) reads

$$\phi(z) = -\kappa(-z)\kappa(z),$$

and this holds for all $z \in \mathbb{C}$ where the left-hand side is defined. In principle, the domain of definition need not be any larger than the imaginary axis, but in sections 2.7 and 3.1 we shall see examples where the domain is a vertical strip in the complex plane.

2.6.1 Lamperti-stable subordinators

A *Lamperti-stable subordinator* is characterised by parameters in the admissible set

$$\big\{(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}) : \mathtt{a} \in (0,1),\ \mathtt{b} \le 1 + \mathtt{a},\ q, \mathtt{c}, \mathtt{d} \ge 0\big\},$$

and it is defined as the (possibly killed) subordinator with killing rate q, drift $\mathtt{d}$, and Lévy density

$$\mathtt{c}\frac{e^{\mathtt{b}x}}{(e^x - 1)^{\mathtt{a}+1}}, \qquad x > 0. \tag{2.9}$$

It is simple to deduce from [16, Theorem 3.1] that the Laplace exponent of such a process is given, for $\lambda \ge 0$, by

$$\psi(\lambda) = q + \mathtt{d}\lambda - \mathtt{c}\Gamma(-\mathtt{a})\left(\frac{\Gamma(\lambda + 1 - \mathtt{b} + \mathtt{a})}{\Gamma(\lambda + 1 - \mathtt{b})} - \frac{\Gamma(1 - \mathtt{b} + \mathtt{a})}{\Gamma(1 - \mathtt{b})}\right). \tag{2.10}$$

Lamperti [58, pages 223–224] proved that, if one treats an α-stable subordinator as a pssMp and finds the Lévy process associated with it, this is precisely a Lamperti-stable subordinator with $\mathtt{a} = \alpha$, $\mathtt{b} = 1$ and $q = \mathtt{d} = 0$. This motivates the name *Lamperti-stable*.[2]

[2]Indeed, in [16], a more general class of *Lamperti-stable Lévy processes* is defined, and the processes ξ^*, $\xi^\uparrow$ and $\xi^\searrow$ of section 2.5 belong to this class. We do not discuss this further, since we will find it more convenient to identify ξ^*, $\xi^\uparrow$ and $\xi^\searrow$ as members of the class of hypergeometric Lévy processes.

2.6.2 Special and complete Bernstein functions and transformations of subordinators

The Lamperti-stable subordinators we have just encountered will not suffice to disentangle the Wiener–Hopf factors in chapters 3 and 4. We therefore introduce two transformations of subordinators in order to expand our repertoire of processes.

The first of these is the classical Esscher transformation. The second, the $\mathcal{T}_\beta$ transformation, is more recent, but we will see that, in the cases we are concerned with, it is closely connected to the Esscher transform. We refer the reader to [51, §3.3] and [54, §2] respectively for details. In connection with the second transformation, we introduce the notion of special and complete Bernstein functions and conjugation of subordinators, which may also be seen as a type of transformation.

The following result is classical.

Lemma 2.4. *Let H be a subordinator with Laplace exponent ψ, and let $\beta > 0$. Define the function*

$$\mathcal{E}_\beta\psi(\lambda) = \psi(\lambda+\beta) - \psi(\beta), \qquad \lambda \geq 0.$$

Then, $\mathcal{E}_\beta\psi$ is the Laplace exponent of a subordinator, known as the Esscher transform *of H (or of ψ).*

The Esscher transform of H has no killing and the same drift coefficient as H, and if the Lévy measure of H is Π, then its Esscher transform has Lévy measure $e^{-\beta x}\Pi(\mathrm{d}x)$.

Before giving details of the $\mathcal{T}_\beta$ transform, we review the notions of special and complete Bernstein functions and subordinators. A detailed account of the theory we are about to sketch may be found in Song and Vondraček [78], Schilling et al. [75] and Jacob [41].

A function $\psi\colon [0,\infty) \to \mathbb{R}$ is called a *Bernstein function* if it is the Laplace exponent of a subordinator. Consider a function ψ, and define $\psi^*\colon [0,\infty) \to \mathbb{R}$ by

$$\psi^*(\lambda) = \lambda/\psi(\lambda).$$

The function ψ is called a *special Bernstein function* if both ψ and ψ^* are Bernstein functions; the subordinator associated to ψ is then called a *special subordinator*. In this case, ψ and ψ^* are said to be *conjugate* to one another, as are their corresponding subordinators.

An alternative criterion for a subordinator to be special, which is often easier to verify, is given (see [78, Theorem 2.1]) as follows. The *renewal measure* of a subordinator H is the measure

$$V(A) = \mathbb{E} \int_0^\infty \mathbb{1}_{\{H_t \in A\}} \, \mathrm{d}t,$$

defined on the Borel sets of $[0, \infty)$. A subordinator H is a special subordinator if and only if the measure V possesses a density on $(0, \infty)$ which is integrable at zero and decreasing. (V may have an atom at zero, however.)

An important subclass of special Bernstein functions consists of *complete Bernstein functions.* We say that a function ψ is a complete Bernstein function if it admits the representation

$$\psi(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda x} \chi(x) \, \mathrm{d}x, \qquad \lambda > 0,$$

where χ is also a Bernstein function. If H is the subordinator associated to a complete Bernstein function ψ, then H is called a *complete subordinator.* There are several alternative definitions:

Lemma 2.5 ([41, Theorem 3.9.29] and [75, Chapter 6]). *The following conditions are equivalent:*

(i) ψ *is a complete Bernstein function;*

(ii) the conjugate function ψ^**, given by* $\psi^*(\lambda) = \lambda/\psi(\lambda)$*, is a complete Bernstein function;*

(iii) ψ *is a Bernstein function, and the Lévy measure* Υ *of its associated subordinator has a density* υ *which is* completely monotone*, in the sense that* υ *is infinitely differentiable and its derivatives satisfy* $(-1)^n \upsilon^{(n)} \geq 0$ *for every integer* $n \geq 0$*;*

(iv) ψ *is a Bernstein function, and the Lévy measure* Υ *of its associated subordinator has a density* υ *given by*

$$\upsilon(x) = \int_{(0,\infty)} e^{-xy} \, \gamma(\mathrm{d}y), \qquad x > 0,$$

where γ *is a measure on* $(0, \infty)$ *which satisfies*

$$\int_{(0,1)} y^{-1} \, \gamma(\mathrm{d}y) + \int_{(1,\infty)} y^{-2} \, \gamma(\mathrm{d}y) < \infty.$$

In particular, any complete Bernstein function is a special Bernstein function.

As a complement to this discussion, we give the following example.

Lemma 2.6. *Let* ψ *be given by*

$$\psi(\lambda) = \frac{\Gamma(\lambda + 1 - \mathtt{b} + \mathtt{a})}{\Gamma(\lambda + 1 - \mathtt{b})}, \qquad \lambda \geq 0,$$

where $\mathtt{a} \in (0,1)$ *and* $\mathtt{b} \leq 1$*; this is the Laplace exponent of a Lamperti-stable subordinator with parameters*

$$\left(\frac{\Gamma(1 - \mathtt{b} + \mathtt{a})}{\Gamma(1 - \mathtt{b})}, \mathtt{a}, \mathtt{b}, -\frac{1}{\Gamma(-\mathtt{a})}, 0 \right).$$

Then ψ *is a complete Bernstein function.*

Proof. In [55, Example 2] it is shown that the conjugate function ψ^* is a Bernstein function whose associated Lévy density is completely monotone. Using criteria (ii) and (iii) of Lemma 2.5, we see that this is sufficient for ψ to be a complete Bernstein function. □

We now give our second transformation of subordinators.

Proposition 2.7. *Let* H *be a subordinator with Laplace exponent* ψ*, and let* $\beta > 0$*. Define*

$$\mathcal{T}_\beta \psi(\lambda) = \frac{\lambda}{\lambda + \beta} \psi(\lambda + \beta), \qquad \lambda \geq 0. \tag{2.11}$$

Then $\mathcal{T}_\beta \psi$ *is the Laplace exponent of a subordinator with no killing and the same drift coefficient as* H*.*

Furthermore, if ψ *is a special (resp., complete) Bernstein function conjugate to* ψ^**, then* $\mathcal{T}_\beta \psi$ *is a special (resp., complete) Bernstein function conjugate to*

$$\mathcal{E}_\beta \psi^* + \psi^*(\beta). \tag{2.12}$$

Proof. The first assertion is proved in Gnedin [37, page 124] as the result of a path transformation, and directly, for spectrally negative Lévy processes (from which the case of subordinators is easily extracted) in Kyprianou and Patie [54, Lemma 2.1]. The killing rate and drift coefficient can be read off as $\mathcal{T}_\beta \psi(0)$ and $\lim_{\lambda \to \infty} \mathcal{T}_\beta \psi(\lambda)/\lambda$.

If ψ is a special Bernstein function, the second claim can be seen immediately by rewriting (2.11) as

$$\mathcal{T}_\beta \psi(\lambda) = \frac{\lambda}{\psi^*(\lambda + \beta)}$$

and observing that $\psi^*(\lambda + \beta) = \mathcal{E}_\beta \psi^*(\lambda) + \psi^*(\beta)$ for $\lambda \geq 0$.

If ψ is additionally a complete Bernstein function, we proceed via Lemma 2.5: it follows that ψ^* is also a complete Bernstein function, and hence its associated subordinator has a completely monotone Lévy density, say π_{ψ^*}. Using the representation (2.12), we see that that the subordinator conjugate to $\mathcal{T}_\beta\psi$ has Lévy density $x \mapsto e^{-\beta x}\pi_{\psi^*}(x)$. Since the product of two completely monotone functions is again completely monotone, and it follows that $\mathcal{T}_\beta\psi$ is a complete Bernstein function. □

2.6.3 Friendship and philanthropy

The theory we have so far encountered allows us to decompose a Lévy process into a pair of subordinators. An equally interesting question is whether one can *manufacture* a process out of two subordinators, thereby obtaining a Lévy process with a Wiener–Hopf factorisation chosen in advance. Vigon [79] calls this *le problème des amis*, the problem of friends, and provides an elegant solution, which we now present. The following results may be found in chapter 7 of [79]; the forthcoming [53] provides an English-language reference upon which the following presentation is based.

Let H and $\hat{H}$ be two subordinators with respective Laplace exponents κ and $\hat{\kappa}$. Suppose that κ has the following Lévy–Khintchine representation:

$$\kappa(\lambda) = \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \Upsilon(\mathrm{d}x) + p, \qquad \lambda \geq 0,$$

and define the symbols $\hat{\delta}$, $\hat{\Upsilon}$ and $\hat{p}$ similarly. One says that H and $\hat{H}$ are *friends* if the equation

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R},$$

holds, where Ψ is the characteristic exponent of a (possibly killed) Lévy process ξ. The central theorem is as follows.

Theorem 2.8 (*Theorème des amis*)**.** *Suppose that H and $\hat{H}$ are friends. Then Υ and $\hat{\Upsilon}$ are absolutely continuous and possess densities υ and $\hat{\upsilon}$, such that the Lévy measure of ξ, say Π, has tails*

$$\Pi(x,\infty) = \int_{(0,\infty)} \hat{\Upsilon}(u,\infty)\Upsilon(x+\mathrm{d}u) + \hat{\delta}\upsilon(x) + \hat{p}\Upsilon(x,\infty), \qquad (2.13)$$

$$\Pi(-\infty,-x) = \int_{(0,\infty)} \Upsilon(u,\infty)\hat{\Upsilon}(x+\mathrm{d}u) + \delta\hat{\upsilon}(x) + p\hat{\Upsilon}(x,\infty), \qquad (2.14)$$

for $x > 0$.

Conversely, if Υ and $\hat{\Upsilon}$ are absolutely continuous and possess densities υ and $\hat{\upsilon}$, such that the expressions on the right-hand sides of (2.13) and (2.14) are both decreasing in x, then H and $\hat{H}$ are friends.

Since the expressions in (2.13) and (2.14) are not particularly simple, another criterion for friendship is useful. One says that a subordinator H is a *philanthropist* if its Lévy measure admits a decreasing density. Vigon shows that a subordinator is a philanthropist if and only if it is the friend of a unkilled pure drift subordinator; but more important to us is the following result.

Proposition 2.9. *Any two philanthropists are friends.*

This simple criterion for friendship will prove very useful to us in chapter 3. It is worth noting that any complete subordinator is a philanthropist.

2.6.4 Meromorphic Lévy processes

In this section we introduce the meromorphic class of Lévy processes first defined by Kuznetsov, Kyprianou and Pardo [48]. This is a large class of processes whose Wiener–Hopf factors may be written in semi-explicit form. We shall apply the theory in chapter 3, but it is worth noting that, although we do not exploit the fact, every Lamperti–Lévy process in the remainder of the text falls within the meromorphic class.

An unkilled Lévy process ξ is said to be in the *meromorphic class* of Lévy processes if its Lévy measure Π is absolutely continuous with density given by

$$\Pi(\mathrm{d}x)/\mathrm{d}x = \begin{cases} \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x}, & x > 0, \\ \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}, & x < 0, \end{cases}$$

for some positive sequences $(a_n)_{n\geq 1}$, $(\hat{a}_n)_{n\geq 1}$ and positive, strictly increasing sequences $(\rho_n)_{n\geq 1}$ and $(\hat{\rho}_n)_{n\geq 1}$, such that $\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \hat{\rho}_n = +\infty$. The processes in this class are a subset of the *Lévy processes with completely monotone jump density* considered in Rogers [70], which have Lévy measure

$$\Pi(\mathrm{d}x)/\mathrm{d}x = \int_{\mathbb{R}} e^{-t|x|}\, \mu(\mathrm{d}x), \qquad x \in \mathbb{R},$$

for some measure μ. Such a process is in the meromorphic class precisely when the measure μ is purely atomic. The paper of Rogers gives results on the Wiener–Hopf factors of Lévy processes with completely monotone jump density; we shall shortly see similar results for processes in the meromorphic class.

The article [48] gives several equivalent definitions of the meromorphic class, one of which is as follows.

Proposition 2.10 ([48, Theorem 1(v)]). *A Lévy process ξ is in the meromorphic* *class if and only if for some $q > 0$ its Laplace$^+$ exponent ϕ admits the factorisation*

$$\phi(z) - q = -q \prod_{n\geq 1} \frac{1 - z/\zeta_n(q)}{1 - z/\rho_n} \prod_{n\geq 1} \frac{1 + z/\hat{\zeta}_n(q)}{1 + z/\hat{\rho}_n}, \qquad z \in \mathbb{C}, \tag{2.15}$$

holds, where the $\zeta_n(q)$ and $-\hat{\zeta}_n(q)$ are all the zeroes of $\phi(z) - q$ and the ρ_n and $-\hat{\rho}_n$ are its poles, with the sequences satisfying the interlacing condition

$$\cdots - \hat{\rho}_2 < -\hat{\zeta}_2(q) < -\hat{\rho}_1 < -\hat{\zeta}_1(q) < 0 < \zeta_1(q) < \rho_1 < \zeta_2(q) < \rho_2 < \cdots .$$

If (2.15) holds for some $q > 0$, then it holds for all $q > 0$.

The factorisation (2.15) looks like a Wiener–Hopf factorisation of the process ξ killed at rate q, and indeed it is; this is the content of [48, Theorem 2(i)], which comes from [44, Theorem 1]. For the case $q = 0$, the following result is useful.

Proposition 2.11 ([48, Corollary 2(i)]). *Suppose that ξ is a process in the meromorphic class which drifts to $+\infty$. Then its Laplace$^+$ exponent admits the Wiener–Hopf factorisation $\phi(z) = -\kappa(-z)\kappa(z)$, where*

$$\kappa(z) = z\mathbb{E}[\xi_1] \prod_{n\geq 1} \frac{1 + z/\zeta_{n+1}(0)}{1 + z/\rho_n},$$

$$\hat{\kappa}(z) = \prod_{n\geq 1} \frac{1 + z/\hat{\zeta}_n(0)}{1 + z/\hat{\rho}_n},$$

and $\zeta_n(0)$, $\hat{\zeta}_n(0)$, ρ_n and $\hat{\rho}_n$ are as in the previous proposition.

A similar result holds when ξ oscillates, with a factor z in both κ and $\hat{\kappa}$; see [48, Corollary 2(ii)].

2.7 The hypergeometric class

Although the connection between the characteristics of the processes ξ^*, $\xi^\uparrow$ and $\xi^\searrow$ is evident at a glance, the structural similarities run deeper. We demonstrate this by describing the *hypergeometric class*[3] of Lévy processes given in Kuznetsov and Pardo [46], to which each of these processes belongs.

The hypergeometric class will occur several times in the sequel: we will extend the range of parameters in chapter 3, where the original class will also appear

[3] This designation is somewhat ambiguous. A class of 'hypergeometric Lévy processes' was first defined in [57, §6.5], and expanded in [47, §3.2]. The class introduced by [46], which we consider in this section, is larger than that of [57] but smaller than that of [47]. However, since we use the class of [46] so frequently, we reserve the term for these processes.

in an auxiliary role; and the Lamperti representation of both the radial part of the stable process and the path-censored stable process will be of hypergeometric class when $\alpha \in (0,1]$; see sections 3.3 and 4.4.1.

To begin with, define the set of admissible parameters

$$\mathcal{A}_{\mathrm{HG}} = \{\beta \le 1,\, \gamma \in (0,1),\, \hat\beta \ge 0,\, \hat\gamma \in (0,1)\},$$

and consider the meromorphic function

$$\phi(z) = -\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)}\frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(\hat\beta+z)}, \qquad z \in \mathbb{C}. \tag{2.16}$$

It is then known that, providing the parameters are chosen from the set $\mathcal{A}_{\mathrm{HG}}$, the function ϕ is the Laplace$^+$ exponent of a Lévy process, say ξ, in the sense that $e^{\phi(z)} = \mathbb{E}[e^{z\xi_1}]$. First defining

$$\eta = 1 - \beta + \gamma + \hat\beta + \hat\gamma,$$

we have:

Proposition 2.12 ([46, Proposition 1])**.**

(i) The function ϕ is the Laplace$^+$ exponent of a Lévy process ξ, with Wiener–Hopf factorisation

$$\kappa(z) = \frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(1-\beta+z)}, \quad \hat\kappa(z) = \frac{\Gamma(\hat\beta+\hat\gamma+z)}{\Gamma(\hat\beta+z)}, \qquad z \ge 0.$$

(ii) The process ξ has absolutely continuous Lévy measure with density

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat\gamma)\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat\gamma; e^{-x}), & x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat\gamma)} e^{(\hat\beta+\hat\gamma)x} {}_2F_1(1+\hat\gamma, \eta; \eta-\gamma; e^{x}), & x < 0, \end{cases}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

(iii) When $\beta < 1$ and $\hat\beta > 0$, the process ξ is killed at rate

$$q = \frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)}\frac{\Gamma(\hat\beta+\hat\gamma)}{\Gamma(\hat\beta)}.$$

Otherwise, $q = 0$. When $\beta = 1$ and $\hat\beta > 0$ (resp., $\beta < 1$ and $\hat\beta = 0$), ξ drifts to $+\infty$ (resp., $-\infty$). When $\beta = 1$ and $\hat\beta = 0$ the process oscillates.

We call the process ξ a *Lévy process in the hypergeometric class*, or a *hypergeometric Lévy process*, with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$.

We remark that $-\xi$ is also a hypergeometric Lévy process, with parameters $(1-\hat{\beta}, \hat{\gamma}, 1-\beta, \gamma)$.

The Wiener–Hopf factors κ and $\hat{\kappa}$ are given in Proposition 2.12 in terms of Laplace exponents of subordinators; but it is simple to identify precisely the ladder height processes as members of the Lamperti-stable family of subordinators:

Corollary 2.13. *The ascending ladder height process H is a Lamperti-stable subordinator with parameters*

$$(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}) = \left(\frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)},\ \gamma,\ \beta,\ -\frac{1}{\Gamma(-\gamma)},\ 0 \right).$$

The ascending renewal measure $V(\mathrm{d}x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{H_t \in \mathrm{d}x\}}\,\mathrm{d}t$ *is*

$$V(\mathrm{d}x) = \frac{1}{\Gamma(\gamma)} e^{-(1-\beta)x}(1-e^{-x})^{\gamma-1}\,\mathrm{d}x.$$

The descending ladder height process $\hat{H}$ is a Lamperti-stable subordinator with parameters

$$(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}) = \left(\frac{\Gamma(\hat{\beta}+\hat{\gamma})}{\Gamma(\hat{\beta})},\ \hat{\gamma},\ 1-\hat{\beta},\ -\frac{1}{\Gamma(-\hat{\gamma})},\ 0 \right).$$

The descending renewal measure $\hat{V}(\mathrm{d}x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{\hat{H}_t \in \mathrm{d}x\}}\,\mathrm{d}t$ *is*

$$\hat{V}(\mathrm{d}x) = \frac{1}{\Gamma(\hat{\gamma})} e^{-\hat{\beta}x}(1-e^{-x})^{\hat{\gamma}-1}\,\mathrm{d}x.$$

Proof. Either substitute in (2.10), or see the proof of [46, Proposition 1] and compare with the construction of the 'general hypergeometric Lévy process' of [47]. The renewal measures may be verified via the Laplace transform identity

$$\int_{[0,\infty)} e^{-\lambda x} V(\mathrm{d}x) = 1/\kappa(\lambda). \qquad \square$$

We record for future use a result on the killed potential of hypergeometric Lévy processes. Let

$$S_0^- = \inf\{t \geq 0 : \xi_t < 0\},$$

and for $x, y > 0$, define

$$u(x,y)\,\mathrm{d}y = \mathbb{E}_x \int_0^{S_0^-} \mathbb{1}_{\{\xi_s \in \mathrm{d}y\}}\,\mathrm{d}s,$$

the potential density of ξ killed outside $[0,\infty)$, provided this exists.

Proposition 2.14. *Suppose that the killing rate $q = 0$, that is, either $\beta = 1$ or $\hat{\beta} = 0$. Then for $x, y > 0$,*

$$u(x,y) = \begin{cases} \frac{1}{\Gamma(\gamma)\Gamma(\hat{\gamma})} e^{-\hat{\beta}(x-y)} (1-e^{-(x-y)})^{\gamma+\hat{\gamma}-1} \\ \quad \times \int_0^{\frac{1-e^{-y}}{1-e^{-x}}} s^{\gamma-1}(1-s)^{(1-\beta+\hat{\beta})-1}(1-e^{-(x-y)}s)^{\beta-\hat{\beta}-\gamma-\hat{\gamma}}\, \mathrm{d}s, \\ \hfill 0 < y < x, \\ \frac{1}{\Gamma(\gamma)\Gamma(\hat{\gamma})} e^{-(1-\beta)(y-x)} (1-e^{-(y-x)})^{\gamma+\hat{\gamma}-1} \\ \quad \times \int_0^{\frac{1-e^{-x}}{1-e^{-y}}} s^{\hat{\gamma}-1}(1-s)^{(1-\beta+\hat{\beta})-1}(1-e^{-(y-x)}s)^{\beta-\hat{\beta}-\gamma-\hat{\gamma}}\, \mathrm{d}s, \\ \hfill x < y. \end{cases}$$

Proof. The proof proceeds via Silverstein's identity [6, Theorem VI.20]: since the renewal measures of ξ are absolutely continuous, we find that the density $u(x,\cdot)$ exists for each $x > 0$, and

$$u(x,y) = \begin{cases} \int_{x-y}^{x} \hat{v}(z)v(z+y-x)\,\mathrm{d}z, & 0 < y < x, \\ \int_0^x \hat{v}(z)v(z+y-x)\,\mathrm{d}z, & y > x, \end{cases}$$

where v and $\hat{v}$ are the ascending and descending renewal densities of ξ. Inserting the expressions from Corollary 2.13 and substituting in the integrals then yields the result. □

Clearly the formula simplifies considerably when $1 = \beta = \hat{\beta} + \gamma + \hat{\gamma}$, and when we apply this result to the hypergeometric process appearing in chapter 4, this will be the case.

The utility of the hypergeometric class for us is due primarily to the following result, whose proof is assembled from Kuznetsov and Pardo [46], Kyprianou et al. [57] and Chaumont et al. [24]. The following presentation is taken from [46].

Proposition 2.15 ([46, Theorem 1])**.** *The processes ξ^*, $\xi^\uparrow$ and $\xi^\searrow$ of section 2.5 are hypergeometric Lévy processes, with parameters as given in the following table.*

	β	γ	$\hat{\beta}$	$\hat{\gamma}$
ξ^*	$1-\alpha\hat{\rho}$	$\alpha\rho$	$1-\alpha\hat{\rho}$	$\alpha\hat{\rho}$
$\xi^\uparrow$	1	$\alpha\rho$	1	$\alpha\hat{\rho}$
$\xi^\searrow$	0	$\alpha\rho$	0	$\alpha\hat{\rho}$

In the next chapter, we will be interested in computing the law of exponential functionals of processes in the extended hypergeometric class, and our approach relies upon similar results for the hypergeometric class given in [46]. We therefore give a brief summary of these results.

We are interested in the *exponential functional* of the process ξ,

$$I(-\xi/\delta) = \int_0^\infty e^{-\xi_t/\delta}\,\mathrm{d}t,$$

for $\delta > 0$, which is an a.s. finite random variable provided that $\hat{\beta} > 0$. We characterise its law via the *Mellin transform*

$$\mathcal{M}(s) = \mathbb{E}[I(-\xi/\delta)^{s-1}],$$

defined for s in some subset of $\mathbb{C}$ to be determined.

[46] gives an expression for $\mathcal{M}$ in terms of the *double gamma function* of Alexeiewsky and Barnes, which may be expressed, for $z \in \mathbb{C}$ and $|\arg(\tau)| < \pi$, as the product

$$G(z,\tau) = \frac{z}{\tau} e^{az/\tau + bz^2/(2\tau)} \prod_{m\geq 0} \prod_{n\geq 0}{}' \left(1 + \frac{z}{m\tau + n}\right) e^{-z/(m\tau+n) + z^2/(2(m\tau+n)^2)},$$

where the prime on the second product indicates that the term corresponding to $m = n = 0$ is omitted, and the functions a and b of τ are as chosen in Barnes [3]. The function $G(z;\tau)$ is analytic with respect to z, and possesses certain quasiperiodicity properties, which we now list for future reference:

$$G(z+1;\tau) = \Gamma\left(\frac{z}{\tau}\right) G(z;\tau), \tag{2.17}$$

$$G(z+\tau;\tau) = (2\pi)^{\frac{\tau-1}{2}} \tau^{\frac{1}{2}-z} \Gamma(z) G(z;\tau). \tag{2.18}$$

The proof of these identities may be found in [3].

We may now state the result of [46].

Proposition 2.16 ([46, Theorem 2])**.** *Suppose that $\hat{\beta} > 0$. Then*

$$\mathcal{M}(s) = C\Gamma(s) \frac{G((1-\beta)\delta + s;\delta)}{G((1-\beta+\gamma)\delta + s;\delta)} \frac{G((\hat{\beta}+\hat{\gamma})\delta + 1 - s;\delta)}{G(\hat{\beta}\delta + 1 - s;\delta)},$$

for $\operatorname{Re} s \in (0, 1 + \hat{\beta}\delta)$*, where C is such that $\mathcal{M}(1) = 1$.*

The proof of Proposition 2.16 is based upon the following 'verification result', which will be fundamental to our approach in the next chapter and which we therefore record here.

Proposition 2.17 ([46, Proposition 2]). *Let ϕ be the Laplace$^+$ exponent of a Lévy process ξ, such that either ξ is killed or $\mathbb{E}\xi_1 > 0$. Denote by $\mathcal{M}$ the Mellin transform of the exponential functional $I(-\xi/\delta)$. Assume further that ξ satisfies the* Cramér condition *with Cramér number $-\theta$: there exist $z_0 < 0$ and $\theta \in (0, -z_0)$ such that ϕ is defined on $(z_0, 0)$, and $\phi(-\theta) = 0$. Suppose there exists a function f whose domain contains the vertical strip $\{s \in \mathbb{C} : \operatorname{Re} s \in (0, 1+\theta)\}$, and that f satisfies the following properties:*

(i) f is analytic and zero-free in the strip $\operatorname{Re} s \in (0, 1+\theta)$,

(ii) $f(1) = 1$, and

$$f(s+1) = -sf(s)/\phi(-s/\delta), \qquad s \in (0, \theta), \tag{2.19}$$

(iii) $|f(s)|^{-1} = o(\exp(2\pi |\operatorname{Im} s|))$ as $|\operatorname{Im} s| \to \infty$ uniformly in $\operatorname{Re} s \in (0, 1+\theta)$.

Then, $\mathcal{M}(s) = f(s)$ when $\operatorname{Re} s \in (0, 1+\theta)$.

Chapter 3

The extended hypergeometric class

In this chapter, we present a class of Lévy processes whose Laplace$^+$ exponent is identical to that of the hypergeometric Lévy processes, but where the range of admissible parameters is different; namely, instead of $\mathcal{A}_{\mathrm{HG}}$, we consider

$$\mathcal{A}_{\mathrm{EHG}} = \big\{\beta \in [1,2],\, \gamma, \hat{\gamma} \in (0,1),\, \hat{\beta} \in [-1,0];\, 1-\beta+\hat{\beta}+\gamma \geq 0,\, 1-\beta+\hat{\beta}+\hat{\gamma} \geq 0\big\}.$$

We will first prove that such processes exist, by means of synthesising them from their Wiener–Hopf factors. Then we will find an expression for the exponential functional of such a process, by connecting it to that of a related hypergeometric Lévy process. Finally, we will justify our interest in this class of processes by giving two examples which arise naturally from transformations of the symmetric stable process in conjunction with the Lamperti representation. As a sample application, we give a re-derivation of the law of the first time at which a symmetric stable process hits zero. A third example of an extended hypergeometric Lévy process will feature in chapter 4 in the form of the Lamperti transform of the 'path-censored stable process'; see Theorem 4.18.

3.1 Existence and Wiener–Hopf factorisation

For parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma}) \in \mathcal{A}_{\mathrm{EHG}}$, define

$$\phi(z) = -\frac{\Gamma(1-\beta+\gamma-z)}{\Gamma(1-\beta-z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}, \tag{3.1}$$

as in the hypergeometric case of (2.16), and also define the auxiliary parameter

$$\eta = 1-\beta+\gamma+\hat{\beta}+\hat{\gamma},$$

which again is the same as in the hypergeometric case. We begin with the following existence result.

Proposition 3.1. *There exists a Lévy process* ξ *such that* $\mathbb{E}(e^{z\xi_1}) = e^{\phi(z)}$. *Its Wiener–Hopf factorisation may be expressed as*

$$\kappa(z) = (-\hat{\beta} + z)\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(2-\beta+z)}, \qquad \hat{\kappa}(z) = (\beta - 1 + z)\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(1+\hat{\beta}+z)}.$$

The density of its Lévy measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}), & x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta-\gamma; e^{x}), & x < 0, \end{cases} \tag{3.2}$$

where ${}_2F_1$ *is the Gauss hypergeometric function.*

The behaviour at large times is as follows:

(a) ξ *drifts to* $+\infty$ *if* $\beta > 1$, $\hat{\beta} = 0$.

(b) ξ *drifts to* $-\infty$ *if* $\beta = 1$, $\hat{\beta} < 0$.

(c) ξ *oscillates if* $\beta = 1$, $\hat{\beta} = 0$. *In this case,* ξ *is a hypergeometric Lévy process.*

(d) X *is killed if* $\beta \in (1,2)$, $\hat{\beta} \in (-1,0)$. *The killing rate is* $\frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)}\frac{\Gamma(\hat{\beta}+\hat{\gamma})}{\Gamma(\hat{\beta})}$.

Proof. We remark that there is nothing to do in case (c) since such processes are analysed in [46]; however, the proof below also works in this case.

We first identify the proposed ascending and descending ladder processes. Once we have shown that ϕ really is the Laplace$^+$ exponent of a Lévy process, this will be the proof of the Wiener-Hopf factorisation.

Some simple algebraic manipulation shows that

$$\kappa = \left(\mathcal{T}_{-\hat{\beta}} \upsilon^*\right)^*,$$

where $\mathcal{T}_{-\hat{\beta}}$ is the transformation of section 2.6.2, and

$$\upsilon^*(z) = \frac{\Gamma(2-\beta+\hat{\beta}+z)}{\Gamma(1-\beta+\hat{\beta}+\gamma+z)},$$

provided that υ^* is a special Bernstein function. In fact, under the constraint that $1-\beta+\hat{\beta}+\gamma \geq 0$, υ^* is precisely the sort of function considered in Lemma 2.6, and is therefore even a complete Bernstein function, and in particular a special

Bernstein function. Furthermore, we may identify this υ^* as the Laplace exponent of a Lamperti-stable subordinator with parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(\frac{\Gamma(2-\beta+\hat{\beta})}{\Gamma(1-\beta+\hat{\beta}+\gamma)}, \, 1-\gamma, \, \beta-\hat{\beta}-\gamma, \, -\frac{1}{\Gamma(\gamma-1)}, \, 0\right),$$

with notation as in section 2.6.1.

For the descending factor, we see that

$$\hat{\kappa} = \big(\mathcal{T}_{\beta-1}\hat{\upsilon}^*\big)^*,$$

with

$$\hat{\upsilon}^*(z) = \frac{\Gamma(2-\beta+\hat{\beta}+z)}{\Gamma(1-\beta+\hat{\beta}+\hat{\gamma}+z)},$$

where, by the same reasoning as before, the function $\hat{\upsilon}^*$ is a complete Bernstein function provided that $1-\beta+\hat{\beta}+\hat{\gamma} \geq 0$. Again, $\hat{\upsilon}^*$ is the Laplace exponent of a Lamperti-stable subordinator with parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(\frac{\Gamma(2-\beta+\hat{\beta})}{\Gamma(1-\beta+\hat{\beta}+\hat{\gamma})}, \, 1-\hat{\gamma}, \, \beta-\hat{\beta}-\hat{\gamma}, \, -\frac{1}{\Gamma(\hat{\gamma}-1)}, \, 0\right).$$

Having identified the ladder height processes we are looking at, we use Vigon's theory of philanthropy to demonstrate the existence of a Lévy process with the given Wiener–Hopf factorisation; see section 2.6.3.

We now make some deductions, using Lemma 2.5 several times. Since υ^* is a complete Bernstein function, it follows from Proposition 2.7 that both $\mathcal{T}_{\hat{\beta}}\upsilon^*$ and its conjugate κ are also complete Bernstein functions. The subordinator corresponding to κ therefore has a completely monotone Lévy density, and thus in particular is a philanthropist. The same applies to $\hat{\kappa}$. Thus by Proposition 2.9, there exists a Lévy process ξ which, by construction, has the Wiener–Hopf factorisation we claim.

The claim about the large time behaviour follows from the Wiener-Hopf factorisation: $\kappa(0) = 0$ if and only if the range of ξ is a.s. unbounded above, and $\hat{\kappa}(0) = 0$ if and only if the range of ξ is a.s. unbounded below; so we need only examine the values of $\kappa(0)$, $\hat{\kappa}(0)$ in each of the four parameter regimes.

We now proceed to calculate the Lévy measure of ξ. A fairly simple way to do this is to make use of the theory of the meromorphic class of Lévy processes, which we discussed in section 2.6.4. We first show that ξ is in the meromorphic class. Initially suppose that

$$1-\beta+\hat{\beta}+\gamma > 0, \; 1-\beta+\hat{\beta}+\hat{\gamma} > 0; \tag{3.3}$$

we will relax this assumption later. Looking at the expression (3.1) for ϕ, we see that it has zeroes $(\zeta_n)_{n\geq 1}, (-\hat{\zeta}_n)_{n\geq 1}$ and (simple) poles $(\rho_n)_{n\geq 1}, (-\hat{\rho}_n)_{n\geq 1}$ given as follows:

$$\begin{aligned}
&\zeta_1 = -\hat{\beta}, && \zeta_n = n-\beta, && n\geq 2,\\
& && \rho_n = n-\beta+\gamma, && n\geq 1,\\
&\hat{\zeta}_1 = \beta-1, && \hat{\zeta}_n = \hat{\beta}+n-1, && n\geq 2,\\
& && \hat{\rho}_n = \hat{\beta}+\hat{\gamma}+n-1, && n\geq 1,
\end{aligned}$$

and that they satisfy the interlacing condition

$$\cdots < -\hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \cdots .$$

We now give an infinite product representation of the Wiener–Hopf factors κ, $\hat{\kappa}$. Using the Weierstrass representation [38, 8.322] of the gamma function, we have, for $z \geq 0$,

$$\begin{aligned}
\kappa(z) &= (-\hat{\beta}+z)\frac{\Gamma(1-\beta+\gamma+z)}{\Gamma(2-\beta+z)}\\
&= (-\hat{\beta}+z)e^{-\mathbf{C}(\gamma-1)}\frac{2-\beta+z}{1-\beta+\gamma+z}\prod_{k=1}^{\infty} e^{(\gamma-1)/k}\frac{1+\frac{2-\beta+z}{k}}{1+\frac{1-\beta+\gamma+z}{k}}\\
&= (-\hat{\beta})\left[\frac{2-\beta}{1-\beta+\gamma}e^{-\mathbf{C}(\gamma-1)}\prod_{k=1}^{\infty} e^{(\gamma-1)/k}\frac{1+\frac{2-\beta}{k}}{1+\frac{1-\beta+\gamma}{k}}\right]\left(1+\tfrac{z}{-\hat{\beta}}\right)\prod_{n=1}^{\infty}\frac{1+\frac{z}{n+1-\beta}}{1+\frac{z}{n-\beta+\gamma}}\\
&= (-\hat{\beta})\frac{\Gamma(1-\beta+\gamma)}{\Gamma(2-\beta)}\prod_{n=1}^{\infty}\frac{1+\frac{z}{\zeta_n}}{1+\frac{z}{\rho_n}},
\end{aligned}$$

where $\mathbf{C}$ is the Euler–Mascheroni constant. By a very similar calculation,

$$\hat{\kappa}(z) = (\beta-1)\frac{\Gamma(\hat{\beta}+\hat{\gamma})}{\Gamma(1+\hat{\beta})}\prod_{n=1}^{\infty}\frac{1+\frac{z}{\hat{\zeta}_n}}{1+\frac{z}{\hat{\rho}_n}}, \qquad z\geq 0.$$

Now, if $q > 0$, we may compare these product representations with Proposition 2.10 to see that ξ is in the meromorphic class. If $q = 0$ we proceed differently. Suppose that ξ drifts to $+\infty$, so that $\hat{\beta} = 0$ and the linear factor in $\kappa(z)$ is equal to z. Modifying the calculation above to avoid incorporating that factor into the product, we obtain

$$\kappa(z) = z\frac{\Gamma(1-\beta+\gamma)}{\Gamma(2-\beta)}\prod_{n=1}^{\infty}\frac{1+\frac{z}{\zeta_{n+1}}}{1+\frac{z}{\rho_n}}, \qquad \hat{\kappa}(z) = (\beta-1)\Gamma(\hat{\beta}+\hat{\gamma})\prod_{n=1}^{\infty}\frac{1+\frac{z}{\hat{\zeta}_n}}{1+\frac{z}{\hat{\rho}_n}}.$$

Apart from the normalisation, this is precisely the representation given in Proposition 2.11 of the Wiener–Hopf factors of a meromorphic Lévy process. Since

ξ has the same Wiener–Hopf factorisation as a meromorphic Lévy process, it is one. If ξ drifts to $-\infty$, the proof is essentially the same, but with the factor z appearing in $\hat{\kappa}$ instead of κ, and if ξ oscillates, one uses [48, Corollary 2(ii)], and a factor z appears in both κ and $\hat{\kappa}$.

We now calculate the Lévy density. By the definition of the meromorphic class, the Lévy measure of ξ has a density of the form

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x} + \mathbb{1}_{\{x<0\}} \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}, \tag{3.4}$$

where $(a_n)_{n\geq 1}$, $(\hat{a}_n)_{n\geq 1}$ are some positive coefficients, and the ρ_n and $\hat{\rho}_n$ are as above. Furthermore, from [48, equation (8)], we see that

$$a_n \rho_n = -\operatorname{Res}(\phi(z) : z = \rho_n),$$

with a similar expression also holding for $\hat{a}_n \hat{\rho}_n$. (This remark is made on page 1111 of [48].) From here it is simple to compute

$$a_n \rho_n = -\frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\Gamma(1-\gamma-n)} \frac{\Gamma(\eta+n-1)}{\Gamma(\eta-\hat{\gamma}+n-1)}, \qquad n \geq 1,$$

and a similar identity for $\hat{a}_n \hat{\rho}_n$. The expression (3.2) follows by substituting in (3.4) and using the series definition of the hypergeometric function.

Suppose now that (3.3) fails, and that we have, say, $1-\beta+\hat{\beta}+\hat{\gamma}=0$. Then $\zeta_1 = \rho_1$, which is to say the first zero-pole pair to the right of the origin is removed. It is clear that ξ still falls into the meromorphic class, and indeed, our expression for π remains valid: although the initial pole ρ_1 no longer exists, the corresponding coefficient $a_1 \rho_1$ vanishes as well. Similarly, we may allow $1-\beta+\hat{\beta}+\gamma=0$, in which case the zero-pole pair to the left of the origin is removed; or we may allow both expressions to be zero, in which case both pairs are removed. The proof carries through in all cases. □

We propose to call this the *extended hypergeometric class* of Lévy processes.

Remark 3.2. If ξ is a process in the extended hypergeometric class, with parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$, then the dual process $-\xi$ also lies in this class, and has parameters $(1-\hat{\beta}, \hat{\gamma}, 1-\beta, \gamma)$.

Remark 3.3. We remark here that, instead of adopting the parameters in $\mathcal{A}_{\text{EHG}}$ as we have, one could extend the parameter range $\mathcal{A}_{\text{HG}}$ by moving only β, or only $\hat{\beta}$. To be precise, both

$$\mathcal{A}^{\beta}_{\text{EHG}} = \left\{\beta \in [1,2],\, \gamma, \hat{\gamma} \in (0,1),\, \hat{\beta} \geq 0;\, 1-\beta+\hat{\beta}+\gamma \leq 0,\, 1-\beta+\hat{\beta}+\hat{\gamma} \geq 0\right\}$$

and

$$\mathcal{A}^{\hat{\beta}}_{\mathrm{EHG}} = \bigl\{\beta \le 1,\, \gamma, \hat{\gamma} \in (0,1),\, \hat{\beta} \in [-1,0];\, 1-\beta+\hat{\beta}+\gamma \ge 0,\, 1-\beta+\hat{\beta}+\hat{\gamma} \le 0\bigr\}$$

are suitable parameter regimes, and, as in the above proof, one may use the meromorphic class of Lévy processes to develop a similar theory for parameters in $\mathcal{A}^{\beta}_{\mathrm{EHG}}$ or $\mathcal{A}^{\hat{\beta}}_{\mathrm{EHG}}$; for instance, for parameters in $\mathcal{A}^{\beta}_{\mathrm{EHG}}$, one has the Wiener–Hopf factors

$$\kappa(z) = \frac{\Gamma(2-\beta+\gamma+z)}{\Gamma(2-\beta+z)}, \qquad \hat{\kappa}(z) = \frac{\beta-1+z}{\beta-1-\gamma+z}\frac{\Gamma(\hat{\beta}+\hat{\gamma}+z)}{\Gamma(\hat{\beta}+z)}.$$

However, we are not aware of any natural examples of processes in these classes.

3.2 The exponential functional

The exponential functional of a Lévy process has been studied extensively; the paper of Bertoin and Yor [9] gives a survey of the literature. In the context of self-similar Markov processes, the exponential functional appears in the entrance law of a pssMp started at zero (see, for example, Bertoin and Yor [8]), and Pardo [65] relates the exponential functional of a Lévy process to envelopes of its associated pssMp. We are interested in the exponential functional because it is related to the hitting time of points for the pssMp associated with it via the Lamperti representation; and we will use the results of this section, in a special case, for precisely this purpose in section 3.3.

Suppose that ξ is a Lévy process in the extended hypergeometric class with $\beta > 1$, which is to say either ξ is killed or it drifts to $+\infty$.

As in section 2.7, we seek to characterise the law of the *exponential functional* of the process,

$$I(-\xi/\delta) = \int_0^\infty e^{-\xi_t/\delta}\,\mathrm{d}t,$$

for $\delta > 0$, which is an a.s. finite random variable under the conditions we have just outlined.

Define the *Mellin transform* of this quantity,

$$\mathcal{M}(s) = \mathbb{E}\bigl[I(-\xi/\delta)^{s-1}\bigr].$$

As we explained in section 2.7, if ξ is a hypergeometric Lévy process, the function $\mathcal{M}$ can be given in terms of gamma functions and double gamma functions. Our goal in this section is the following result, which does the same for the extended hypergeometric class.

Proposition 3.4. *Suppose that ξ is a Lévy process in the extended hypergeometric class with $\beta > 1$. Define $\theta = \delta(\beta - 1)$.*

Then, the Mellin transform $\mathcal{M}$ of $I(-\xi/\delta)$ is given, for $\mathrm{Re}(s) \in (0, 1+\theta)$, *by*

$$\mathcal{M}(s) = c\widetilde{\mathcal{M}}(s)\frac{\Gamma(\delta(1-\beta+\gamma)+s)}{\Gamma(-\delta\hat{\beta}+s)}\frac{\Gamma(\delta(\beta-1)+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)}, \tag{3.5}$$

where $\widetilde{\mathcal{M}}$ is the Mellin transform of $I(-\zeta/\delta)$, and ζ is an auxiliary Lévy process in the hypergeometric class, with parameters $(\beta-1, \gamma, \hat{\beta}+1, \hat{\gamma})$. The constant c is such that $\mathcal{M}(1) = 1$.

Proof. The process ξ/δ has Laplace$^+$ exponent ϕ_δ given by $\phi_\delta(z) = \phi(z/\delta)$. The relationship with ζ comes from the following calculation:

$$\begin{aligned}\phi_\delta(z) &= \frac{-\hat{\beta}-z/\delta}{1-\beta+\gamma-z/\delta}\frac{\beta-1+z/\delta}{\hat{\beta}+\hat{\gamma}+z/\delta}\frac{\Gamma(2-\beta+\gamma-z/\delta)}{\Gamma(2-\beta-z/\delta)}\frac{\Gamma(1+\hat{\beta}+\hat{\gamma}+z/\delta)}{\Gamma(1+\hat{\beta}+z/\delta)} \\ &= \frac{-\hat{\beta}-z/\delta}{1-\beta+\gamma-z/\delta}\frac{\beta-1+z/\delta}{\hat{\beta}+\hat{\gamma}+z/\delta}\tilde{\phi}_\delta(z),\end{aligned} \tag{3.6}$$

where $\tilde{\phi}_\delta$ is the Laplace$^+$ exponent of a Lévy process ζ/δ, with ζ as in the statement of the theorem.

Denote the right-hand side of (3.5) by $f(s)$. The proof now proceeds via the 'verification result' which we quoted in Proposition 2.17.

Recall that a Lévy process with Laplace$^+$ exponent ϕ is said to satisfy the *Cramér condition* with Cramér number $-\theta$ if there exists $z_0 < 0$ and $\theta \in (0, -z_0)$ such that $\phi(z)$ is defined for all $z \in (z_0, 0)$ and $\phi(-\theta) = 0$.

Inspecting the Laplace$^+$ exponent ϕ_δ reveals that ξ/δ satisfies the Cramér condition with Cramér number $-\theta$, where $\theta = \delta(\beta-1)$.

Let $\tilde{\theta} = \delta(\hat{\beta}+1)$. Then ζ/δ satisfies the Cramér condition with Cramér number $-\tilde{\theta}$. It follows from [69, Lemma 2] that $\widetilde{\mathcal{M}}$ is analytic for $\mathrm{Re}\, s \in (0, 1+\tilde{\theta})$, and since $I(-\zeta/\delta)^{s-1}$ is a positive random variable, $\widetilde{\mathcal{M}}$ also has no zeroes in this set. The constraints in the parameter set $\mathcal{A}_{\mathrm{EHG}}$ then ensure that $\tilde{\theta} \geq \theta$; this, together with inspecting the right-hand side of (3.5) and comparing again with the conditions in $\mathcal{A}_{\mathrm{EHG}}$, demonstrates that $f(s)$ is analytic and zero-free in the strip $\mathrm{Re}(s) \in (0, 1+\theta)$.

We must then check the functional equation

$$f(s+1) = -sf(s)/\phi_\delta(-s),$$

for $s \in (0,\theta)$. Apply (3.6) to write

$$
\begin{aligned}
-\frac{s}{\phi_\delta(-s)} &= -\frac{s}{\tilde{\phi}_\delta(-s)} \frac{1-\beta+\gamma+s/\delta}{-\hat{\beta}+s/\delta} \frac{\hat{\beta}+\hat{\gamma}-s/\delta}{\beta-1-s/\delta} \\
&= \frac{\widetilde{\mathcal{M}}(s+1)}{\widetilde{\mathcal{M}}(s)} \frac{\delta(1-\beta+\gamma)+s}{-\delta\hat{\beta}+s} \frac{\delta(\hat{\beta}+\hat{\gamma})-s}{\delta(\beta-1)-s} \\
&= \frac{\widetilde{\mathcal{M}}(s+1)}{\widetilde{\mathcal{M}}(s)} \frac{\Gamma(-\delta\hat{\beta}+s)}{\Gamma(-\delta\hat{\beta}+s+1)} \frac{\Gamma(\delta(1-\beta+\gamma)+s+1)}{\Gamma(\delta(1-\beta+\gamma)+s)} \\
&\quad \times \frac{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})-s)} \frac{\Gamma(\delta(\beta-1)-s)}{\Gamma(\delta(\beta-1)+1-s)},
\end{aligned}
$$

making use of the same functional equation for the Mellin transform $\widetilde{\mathcal{M}}$. It is then clear that the right-hand side is equal to $f(s+1)/f(s)$.

Finally, it remains to check that $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$, uniformly for $\mathrm{Re}(s) \in (0, 1+\theta)$ as $|\mathrm{Im}(s)| \to \infty$.

We know from [46, proof of Theorem 2] that

$$\log\big(|\widetilde{\mathcal{M}}(s)|^{-1}\big) = \frac{\pi}{2}(1-\gamma+\hat{\gamma})|\mathrm{Im}\, s| + o(\mathrm{Im}\, s),$$

uniformly in the (wider) strip $\mathrm{Re}\, s \in (0, 1+\tilde{\theta})$. Recall Stirling's asymptotic formula:

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\log(2\pi) + O(z^{-1}), \qquad z \to \infty, \tag{3.7}$$

which is uniform in $|\arg z| < \pi - \omega$ for any choice of $\omega > 0$; see [62, Chapter 8, §4]. From this we derive the simpler

$$\log\Gamma(z) = z\log z - z + O(\log z),$$

and when applied to the remaining terms in our expression for f this gives

$$\log\left|\frac{\Gamma(\delta(1-\beta+\gamma)+s)}{\Gamma(-\delta\hat{\beta}+s)} \frac{\Gamma(\delta(\beta-1)+1-s)}{\Gamma(\delta(\hat{\beta}+\hat{\gamma})+1-s)}\right|^{-1} = O(\log s) = o(\mathrm{Im}\, s).$$

That is, we have

$$\log|f(s)|^{-1} = \frac{\pi}{2}(1-\gamma+\hat{\gamma})|\mathrm{Im}\, s| + o(\mathrm{Im}\, s).$$

We make the following (standard) calculation, recalling that $1-\gamma+\hat{\gamma} \in (0,2)$:

$$\begin{aligned}\frac{|f(s)|^{-1}}{\exp(\pi|\operatorname{Im} s|)} &= \exp\left[\frac{\log|f(s)|^{-1} - \pi|\operatorname{Im} s|}{|\operatorname{Im} s|}|\operatorname{Im} s|\right] \\ &= \exp\left[\left(\frac{\log|f(s)|^{-1} - \frac{\pi}{2}(1-\gamma+\hat{\gamma})|\operatorname{Im} s|}{|\operatorname{Im} s|}\right.\right. \\ &\qquad\qquad \left.\left. - \pi\bigl(1-\tfrac{1}{2}(1-\gamma+\hat{\gamma})\bigr)\right)|\operatorname{Im} s|\right],\end{aligned}$$

and by picking $|\operatorname{Im} s|$ large in order to make the term in parentheses negative, we obtain $|f(s)|^{-1} = o(\exp(\pi|\operatorname{Im} s|))$, uniformly in the strip $\operatorname{Re}(s) \in (0, 1+\theta)$, as required.

Hence, $f(s) = \mathcal{M}(s)$ when $\operatorname{Re}(s) \in (0, 1+\theta)$. □

Let us add that the domain $\operatorname{Re}(s) \in (0, 1+\theta)$ is delineated by poles of $\mathcal{M}$ at the points 0 and $1+\theta$, and so in general $\mathcal{M}$ is not defined in any strictly larger strip. However, in the next section we shall see an example where $\hat{\beta} = 0$, causing a pole–zero pair in $\mathcal{M}$ to cancel and allowing the domain of $\mathcal{M}$ to be expanded.

We further remark on the powerful effect that the Cramér condition has on the exponential functional. Let us denote by $\mathcal{M}_{\mathrm{HG}}$ the Mellin transform which appeared in Proposition 2.16 for the hypergeometric class. Both our function $\mathcal{M}$ and the function $\mathcal{M}_{\mathrm{HG}}$ satisfy the functional equation (2.19) and the asymptotic growth which are necessary to be candidate Mellin transforms, but when we are working in the extended hypergeometric class, the function $\mathcal{M}_{\mathrm{HG}}$ has a pole at $1+\delta\hat{\beta} < 1$, and hence is certainly not analytic in the whole strip $0 < \operatorname{Re} s < 1+\delta(\beta-1)$.

3.3 The radial part of the symmetric stable process

Let X be the stable process as defined in section 2.2, and assume further that it is symmetric; that is, $\rho = 1/2$. Recall the definition of the stopping time

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

which may be infinite. Then define the process $R = (R_t)_{t\geq 0}$ by

$$R_t = |X_t| \mathbb{1}_{\{t<T_0\}}.$$

It is not difficult to see that this process, equipped with probability laws $(\mathrm{P}_x)_{x>0}$, is a pssMp. Denote its Lamperti representation by ξ^R.

Caballero et al. [17] considered the Lévy process ξ^R associated to the radial part of a d-dimensional stable process, and found the Wiener–Hopf factorisation of ξ^R when $\alpha < d$; it emerges that the process is a hypergeometric Lévy process. However, the methods in [17] did not allow the authors to deal with the case $\alpha \geq d = 1$.

In [17], Caballero et al. also showed that when $\alpha < d = 1$, the process ξ^R could be decomposed into a compound Poisson process and a Lévy process of infinite jump activity. By observing that this decomposition holds also for $\alpha \geq d = 1$, we give a new derivation of the characteristic exponent of ξ^R in one dimension. The Wiener–Hopf factorisation of ξ^R is then clear, since it is a hypergeometric Lévy process when $\alpha \leq 1$. Furthermore, since we also have the extended hypergeometric class at our disposal, we give the Wiener-Hopf factorisation of ξ^R when $\alpha > d = 1$, which is a new result.

Note that, when X is symmetric, the parameters c_+ and c_- coincide and have the common value

$$c_\pm = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)} = \frac{2^{\alpha-1}\alpha\Gamma(1/2+\alpha/2)}{\Gamma(1-\alpha/2)\Gamma(1/2)}.$$

We mention the equivalent value on the right-hand side since this expression is used in [17]. The proof of this equality is via the doubling formula [38, 8.335.1] for the gamma function.

We prove a short lemma, and then begin to expose the structure of ξ^R. Define the stopping time

$$\tau = \inf\{t \geq 0 : X_t < 0\},$$

which is the killing time of X^*, the stable process killed upon passing below the level 0.

Lemma 3.5. *For any $x > 0$, the joint law of $(R_\tau, R_{\tau-})$ under P_x is equal to that of $(xR_\tau, xR_{\tau-})$ under P_1.*

Proof. The proof is an application of scaling; we give it only for the distribution of R_τ, but the proof for the joint law is essentially identical.

Fix $c > 0$, and define the rescaled process $(\tilde{R}_t)_{t\geq 0}$ by $\tilde{R}_t = cR_{c^{-\alpha}t}$. Let $\tilde{\tau} = \inf\{t \geq 0 : \tilde{R}_t < 0\}$. Then,

$$c^\alpha \tau = \inf\{c^\alpha t : t \geq 0,\ R_t < 0\} = \inf\{t \geq 0 : cR_{c^{-\alpha}t} < 0\} = \tilde{\tau}.$$

This implies that for every $c, x > 0$, the measures $\mathrm{P}_x(R_\tau \in \cdot)$ and $\mathrm{P}_{cx}(c^{-1}R_\tau \in \cdot)$ are equal. The claim follows by setting $c = 1/x$. □

Proposition 3.6 (Structure of ξ^R). *The Lévy process ξ^R is the sum of two independent Lévy processes, $\xi^{R,\mathrm{L}}$ and $\xi^{R,\mathrm{C}}$, such that* *(i) The Lévy process $\xi^{R,\mathrm{L}}$ has characteristic exponent*

$$\Psi^*(\theta) - c_\pm/\alpha, \qquad \theta \in \mathbb{R}, \tag{3.8}$$

where Ψ^* is the characteristic exponent of the process ξ^*, which was defined in section 2.5 to be the Lamperti transform of X^*, the stable process killed upon first passage below zero. That is, $\xi^{R,\mathrm{L}}$ is formed by removing the independent killing from ξ^*.

(ii) The process $\xi^{R,\mathrm{C}}$ is a compound Poisson process whose jumps occur at rate $c_\pm/\alpha$, whose Lévy density is

$$\pi^{R,\mathrm{C}}(y) = c_\pm \frac{e^y}{(1+e^y)^{\alpha+1}}, \qquad y \in \mathbb{R}. \tag{3.9}$$

Proof. In Proposition 1 of [17], the authors decompose the Lévy measure of ξ^R into that of $\xi^{R,\mathrm{L}}$ and the measure with density $\pi^{R,\mathrm{C}}$. They do so under the assumption $\alpha \in (0,1)$; however, this is not used in the proof, which works for any $\alpha \in (0,2)$. Since the law of a compound Poisson process is determined entirely by its Lévy measure, the proposition will be proved once we show that ξ^R is the sum of a process with the law of $\xi^{R,\mathrm{L}}$ and a compound Poisson process.

It is clear that the path section $(R_t)_{t<\tau}$ agrees with $(X^*_t)_{t<\tau}$; however, rather than being killed at time τ, the process R jumps to a positive state. Recall now that the effect of the Lamperti transform on the time τ is to turn it into an exponential time of rate $c_\pm/\alpha$ which is independent of $(\xi^R_s)_{s<S(\tau)}$. This immediately yields the decomposition of ξ^R into the sum of $\xi^{R,\mathrm{L}} := (\xi^{R,\mathrm{L}}_s)_{s\geq 0}$ and $\xi^{R,\mathrm{C}} := (\xi^{R,\mathrm{C}}_s)_{s\geq 0}$, where $\xi^{R,\mathrm{C}}$ is a process which jumps at the times of a Poisson process with rate $c_\pm/\alpha$, but whose jumps may depend on the position of ξ^R prior to this jump.

What remains is to be shown is that the values of the jumps of $\xi^{R,\mathrm{C}}$ are also independent of $\xi^{R,\mathrm{L}}$. By the strong Markov property, it is sufficient to show that the *first* jump of $\xi^{R,\mathrm{C}}$ is independent of the previous path of $\xi^{R,\mathrm{L}}$, or equivalently, of ξ^R. Now, using only the independence of the jump times of $\xi^{R,\mathrm{L}}$ and $\xi^{R,\mathrm{C}}$, we can compute

$$\begin{aligned}
\Delta R_\tau := R_\tau - R_{\tau-} &= \exp(\xi^{R,\mathrm{L}}_{S(\tau)} + \xi^{R,\mathrm{C}}_{S(\tau)}) - \exp(\xi^{R,\mathrm{L}}_{S(\tau)-} + \xi^{R,\mathrm{C}}_{S(\tau)-}) \\
&= \exp(\xi^R_{S(\tau)-})\big[\exp(\Delta\xi^{R,\mathrm{C}}_{S(\tau)}) - 1\big] \\
&= R_{\tau-}\big[\exp(\Delta\xi^{R,\mathrm{C}}_{S(\tau)}) - 1\big],
\end{aligned}$$

where S is the Lamperti time change for R, and $\Delta\xi^{R,\mathrm{C}}_s = \xi^{R,\mathrm{C}}_s - \xi^{R,\mathrm{C}}_{s-}$. Now,

$$\exp(\Delta\xi^{R,\mathrm{C}}_{S(\tau)}) = 1 + \frac{\Delta R_\tau}{R_{\tau-}} = \frac{R_\tau}{R_{\tau-}}. \tag{3.10}$$

Hence, it is sufficient to show that $\frac{R_\tau}{R_{\tau-}}$ is independent of $(R_t, t < \tau)$. The proof of this is identical to that of Theorem 4(iii) in Chaumont et al. [25], which we reproduce here for clarity.

First, observe that one consequence of Lemma 3.5 is that, for g a Borel function and $x > 0$,

$$\mathrm{E}_x\left[g\left(\frac{R_\tau}{R_{\tau-}}\right)\right] = \mathrm{E}_1\left[g\left(\frac{R_\tau}{R_{\tau-}}\right)\right].$$

Now, fix $n \in \mathbb{N}$, f and g Borel functions and $s_1 < s_2 < \cdots < s_n = t$. Then, using the Markov property and the above equality,

$$\begin{aligned}\mathrm{E}_1\left[f(R_{s_1},\dots,R_t)g\left(\frac{R_\tau}{R_{\tau-}}\right)\mathbb{1}_{\{t<\tau\}}\right] &= \mathrm{E}_1\left[f(R_{s_1},\dots,R_t)\mathbb{1}_{\{t<\tau\}}\mathrm{E}_{R_t}\left[g\left(\frac{R_\tau}{R_{\tau-}}\right)\right]\right] \\ &= \mathrm{E}_1\left[f(R_{s_1},\dots,X_t)\mathbb{1}_{\{t<\tau\}}\right]\mathrm{E}_1\left[g\left(\frac{R_\tau}{R_{\tau-}}\right)\right].\end{aligned}$$

We have now shown that $\xi^{R,\mathrm{L}}$ and $\xi^{R,\mathrm{C}}$ are independent, and as discussed above, this completes the proof. □

Remark 3.7. In [17, Proposition 1], it is proved that

$$\pi^R = \pi^{R,\mathrm{L}} + \pi^{R,\mathrm{C}}, \tag{3.11}$$

where π^R and $\pi^{R,\mathrm{L}}$ are the Lévy densities of ξ^R and $\xi^{R,\mathrm{L}}$ respectively, and $\pi^{R,\mathrm{C}}$ is the function given in (3.9). When $\alpha < 1$, which is a necessary and sufficient condition for all processes involved to be of bounded variation, this suffices to prove Proposition 3.6. However, when $\alpha \geq 1$, although (3.11) is sufficient to prove that ξ^R is the sum of $\xi^{R,\mathrm{C}}$ and a process with the same Lévy measure as $\xi^{R,\mathrm{L}}$, it is not clear how to ensure that the centre in the Lévy–Khintchine representation of $\xi^{R,\mathrm{L}}$ is correct. Hence, we give the more robust proof above.

Theorem 3.8 (Characteristic exponent)**.** *The characteristic exponent of the Lévy process $2\xi^R$ is given by*

$$\Psi^R(2\theta) = 2^\alpha \frac{\Gamma(\alpha/2 - \mathrm{i}\theta)}{\Gamma(-\mathrm{i}\theta)} \frac{\Gamma(1/2 + \mathrm{i}\theta)}{\Gamma((1-\alpha)/2 + \mathrm{i}\theta)}, \qquad \theta \in \mathbb{R}, \tag{3.12}$$

where Ψ^R is the characteristic exponent of ξ^R.

Proof. Using Proposition 2.15 to obtain an expression for the characteristic exponent Ψ^* and substituting in the formula (3.8), we have, for the characteristic exponent of $\xi^{R,\mathrm{L}}$:

$$\Psi^{R,\mathrm{L}}(\theta) = \frac{\Gamma(\alpha - \mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(\alpha/2 - \mathrm{i}\theta)\Gamma(1-\alpha/2+\mathrm{i}\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)}.$$

We calculate the characteristic exponent of $\xi^{R,\mathrm{C}}$ by hand, using the beta integral [38, 8.380.3]:

$$\begin{aligned}
\Psi^{R,\mathrm{C}}(\theta) &= c_\pm \int_{-\infty}^{\infty} (1 - e^{\mathrm{i}\theta y}) \pi^{R,\mathrm{C}}(y) \mathrm{d}y \\
&= c_\pm \int_0^\infty \left(\frac{1}{(t+1)^{\alpha+1}} - \frac{t^{\mathrm{i}\theta}}{(t+1)^{\alpha+1}} \right) \mathrm{d}t \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha/2)\Gamma(1-\alpha/2)} \left[\frac{1}{\alpha} - \frac{\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha+1)} \right].
\end{aligned}$$

Summing these, then using product–sum identities and [38, 8.334.2–3],

$$\begin{aligned}
\Psi^R(\theta) &= \Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta) \left[\frac{\sin\big(\pi(\alpha/2 - \mathrm{i}\theta)\big)}{\pi} - \frac{\sin\big(\pi\alpha/2\big)}{\pi} \right] \\
&= \frac{2}{\pi} \Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta) \cos\frac{\pi(\alpha-\mathrm{i}\theta)}{2} \sin\frac{-\mathrm{i}\theta\pi}{2} \\
&= 2\pi \frac{\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma((\alpha-\mathrm{i}\theta)/2+1/2)} \frac{\Gamma(1+\mathrm{i}\theta)}{\Gamma((1+\mathrm{i}\theta)/2+1/2)} \frac{1}{\Gamma(-\mathrm{i}\theta/2)\Gamma((1-\alpha+\mathrm{i}\theta)/2)}.
\end{aligned}$$

Now, applying the doubling formula

$$\frac{\Gamma(2x)}{\Gamma(x+1/2)} = \frac{\Gamma(x)2^{2x-1}}{\sqrt{\pi}},$$

twice, we obtain the expression in the theorem. □

We now identify the Wiener–Hopf factorisation of ξ^R, which will depend on the value of α. However, note the factor 2^α in (3.12). In the context of the Wiener–Hopf factorisation, we could ignore this factor by picking an appropriate normalisation of local time; however, another approach is as follows.

Let us write $R' = \frac{1}{2}R$, and denote by $\xi^{R'}$ the Lamperti transform of R'. Then the scaling of space on the level of the self-similar process is converted by the Lamperti transform into a scaling of time, so that $\xi^R_s = \log 2 + \xi^{R'}_{s2^\alpha}$. In particular, if we write $\Psi^{R'}$ for the characteristic exponent of $\xi^{R'}$, it follows that $\Psi^{R'} = 2^{-\alpha}\Psi^R$. This allows us to disregard the inconvenient constant factor in (3.12), if we work with $\xi^{R'}$ instead of ξ^R.

The following corollary is now simple when we bear in mind the hypergeometric class of Lévy processes introduced in section 2.7. We emphasise that this Wiener–Hopf factorisation was derived by different methods in [17] for $\alpha < 1$, though not $\alpha = 1$.

Corollary 3.9 (Wiener–Hopf factorisation, $\alpha \in (0,1]$)**.** *The Wiener–Hopf fac-*

torisation of $2\xi^{R'}$ *when* $\alpha \in (0,1]$ *is given by*

$$\Psi^{R'}(2\theta) = \frac{\Gamma(\alpha/2 - \mathrm{i}\theta)}{\Gamma(-\mathrm{i}\theta)} \times \frac{\Gamma(1/2 + \mathrm{i}\theta)}{\Gamma((1-\alpha)/2 + \mathrm{i}\theta)}, \qquad \theta \in \mathbb{R},$$

and $2\xi^{R'}$ *is a Lévy process of the hypergeometric class with parameters*

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha/2, (1-\alpha)/2, \alpha/2).$$

Proof. It suffices to compare the characteristic exponent with that of a hypergeometric Lévy process. □

When $\alpha > 1$, the process $\xi^{R'}$ is not a hypergeometric Lévy process; however, it is in the extended hypergeometric class, and we therefore have the following result, which is new.

Theorem 3.10 (Wiener–Hopf factorisation, $\alpha \in (1,2)$)**.** *The Wiener–Hopf factorisation of* $2\xi^{R'}$ *when* $\alpha \in (1,2)$ *is given by*

$$\Psi^{R'}(2\theta) = \left(\frac{\alpha-1}{2} - \mathrm{i}\theta\right)\frac{\Gamma(\alpha/2 - \mathrm{i}\theta)}{\Gamma(1 - \mathrm{i}\theta)} \times \mathrm{i}\theta \frac{\Gamma(1/2 + \mathrm{i}\theta)}{\Gamma((3-\alpha)/2 + \mathrm{i}\theta)}, \tag{3.13}$$

and $2\xi^{R'}$ *is a Lévy process in the extended hypergeometric class, with parameters*

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1, \alpha/2, (1-\alpha)/2, \alpha/2).$$

Proof. Simply use Theorem 3.8; using the formula $x\Gamma(x) = \Gamma(x+1)$ yields (3.13). That this is indeed the Wiener-Hopf factorisation follows once we recognise $2\xi^{R'}$ as a process in the extended hypergeometric class, and apply Proposition 3.1. □

As an illustration of the utility of the extended hypergeometric class, we will now derive an expression for the Mellin transform of the exponential functional for the dual process $-\xi^{R'}$. This quantity is linked by the Lamperti representation to the hitting time of zero for X; see section 2.4. In particular, if

$$T_0 = \inf\{t \ge 0 : X_t = 0\},$$

we have that

$$T_0 = \int_0^\infty e^{-\alpha\xi_t^R}\, \mathrm{d}t = \int_0^\infty e^{-\alpha\xi^{R'}_{2^\alpha t}}\, \mathrm{d}t = 2^{-\alpha}\int_0^\infty e^{-\alpha\xi_s^{R'}}\, \mathrm{d}s = 2^{-\alpha} I(-\alpha\xi^{R'}). \tag{3.14}$$

Since $-2\xi^{R'}$ is an extended hypergeometric Lévy process with parameters $\left(\frac{\alpha+1}{2}, \frac{\alpha}{2}, 0, \frac{\alpha}{2}\right)$, which drifts to $+\infty$, we may apply the theory just developed to compute the Mellin transform of the right-hand side of (3.14). Denote this by

$\mathcal{M}$; that is,

$$\mathcal{M}(s) = \mathbb{E}\big[I(-\alpha\xi^{R'})^{s-1}\big],$$

for some range of $s \in \mathbb{C}$ to be determined.

Proposition 3.11. *For* $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathcal{M}(s) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\tfrac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}. \tag{3.15}$$

Proof. Let ζ be a hypergeometric Lévy process with parameters $\big(\frac{\alpha-1}{2}, \frac{\alpha}{2}, 1, \frac{\alpha}{2}\big)$, and denote by $\widetilde{\mathcal{M}}$ the Mellin transform of the exponential functional $I(-\alpha/2\cdot\zeta)$, which is known to be defined for $\operatorname{Re} s \in (0, 1 + 2/\alpha)$.

We can then use Proposition 3.4 to make the following calculation, provided that $\operatorname{Re} s \in (0, 2 - 1/\alpha)$. Here G is again the double gamma function, as in section 2.7, and we use [46, equation (19)] in the third line and the identity $x\Gamma(x) = \Gamma(x+1)$ in the final line. For normalisation constants C (and C') to be determined, we have

$$\begin{aligned}
\mathcal{M}(s) &= C\widetilde{\mathcal{M}}(s)\frac{\Gamma(\frac{1}{\alpha}+s)}{\Gamma(s)}\frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\
&= C\frac{G\big(\frac{3}{\alpha}-1+s;\frac{2}{\alpha}\big)}{G\big(\frac{3}{\alpha}+s;\frac{2}{\alpha}\big)}\frac{G\big(\frac{2}{\alpha}+2-s;\frac{2}{\alpha}\big)}{G\big(\frac{2}{\alpha}+1-s;\frac{2}{\alpha}\big)}\Gamma(s)\frac{\Gamma(\frac{1}{\alpha}+s)}{\Gamma(s)}\frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\
&= C\frac{\Gamma\big(1+\frac{\alpha}{2}-\frac{\alpha}{2}s\big)}{\Gamma\big(\frac{3-\alpha}{2}+\frac{\alpha}{2}s\big)}\Gamma(\tfrac{1}{\alpha}+s)\frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\
&= C'\frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})}\Gamma(\tfrac{1}{\alpha}-1+s)\frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)}.
\end{aligned}$$

The condition $\mathcal{M}(1) = 1$ means that we can calculate

$$C' = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})}, \tag{3.16}$$

and this gives the Mellin transform explicitly, for $\operatorname{Re} s \in (0, 2 - 1/\alpha)$.

We now expand the domain of $\mathcal{M}$. Note that, in contrast to the general case of Proposition 3.4, the right-hand side of (3.15) is well-defined in the domain $\operatorname{Re} s \in (-1/\alpha, 2-1/\alpha)$, and is indeed analytic in this region. (The reason for this difference is the cancellation of a simple pole and zero at the point 0.) Theorem 2 of [59] shows that, if the Mellin transform of a probability measure is analytic in a neighbourhood of the point $1 \in \mathbb{C}$, then it is analytic in a strip $\operatorname{Re} s \in (a, b)$, where $-\infty \le a < b \le \infty$; and futhermore, the function has singularities at a and b, if they are finite. It then follows that the right-hand side of (3.15) must actually be equal to $\mathcal{M}$ in all of $\operatorname{Re} s \in (-1/\alpha, 2-1/\alpha)$, and this completes the proof. □

Since $\mathcal{M}$ turns out to consist of gamma functions, one might hope, in retrospect, to prove (3.15) without the need for the full power of the extended hypergeometric family. Indeed, this is the approach taken in [49]. We reproduce that proof below.

Alternate proof of Proposition 3.11. Write the Laplace$^+$ exponent of $-2\xi^{R'}$ as ϕ, that is,

$$\phi(z) = -\frac{\Gamma(1/2-z)}{\Gamma((1-\alpha)/2-z)}\frac{\Gamma(\alpha/2+z)}{\Gamma(z)},$$

and write $f(s)$ for the right-hand side of (3.15). We begin by noting that $-\alpha\xi^{R'}$ satisfies the Cramér condition with Cramér number $-(1-1/\alpha)$; the verification result of Proposition 2.17 therefore allows us to prove the claim in the region $\operatorname{Re} s \in (0, 2-1/\alpha)$ once we verify some conditions which f should satisfy on this domain.

It is straightforward to check that $f(s)$ is analytic and zero-free in the strip $\operatorname{Re} s \in (0, 2-1/\alpha)$, and that it satisfies (2.19) with $\delta = 2/\alpha$.

Finally, we need to investigate the asymptotics of $f(s)$ as $|\operatorname{Im} s| \to \infty$. To do this, we will use the fact that

$$\lim_{|y|\to\infty} \frac{|\Gamma(x+\mathrm{i}y)|}{|y|^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}} = \sqrt{2\pi}, \tag{3.17}$$

This can be derived from the Stirling's asymptotic formula (3.7), which we recall holds uniformly as $z \to \infty$ and $|\arg(z)| < \pi - \omega$, for fixed $\omega > 0$; hence the convergence in (3.17) is also uniform in x belonging to a compact subset of $\mathbb{R}$. We calculate, giving C' the same value as in (3.16) and assuming $|\operatorname{Im} s| > 1$,

$$\begin{aligned}
|f(s)|^{-1} &\sim \frac{C'}{\sqrt{2\pi}}\left(\frac{\alpha}{2}\right)^{-\frac{1}{2}-\alpha+\alpha\operatorname{Re} s} |\operatorname{Im} s|^{1-\alpha+(\alpha-3)\operatorname{Re} s}\exp\left(\frac{\pi}{2}|\operatorname{Im} s|\right) \\
&\le \frac{C'}{\sqrt{2\pi}}\left(\frac{\alpha}{2}\right)^{-(\frac{1}{2}+\alpha)} \exp\left(\frac{\pi}{2}|\operatorname{Im} s|\right) \\
&= o(\exp(\pi|\operatorname{Im} s|))
\end{aligned}$$

where all asymptotics are uniform in $\{s \in \mathbb{C} : \operatorname{Re} s \in (0, 2-1/\alpha),\ |\operatorname{Im} s| > 1\}$ as $|\operatorname{Im} s| \to \infty$. This gives the asymptotic growth that we require.

The conditions of [46, Propositon 2] are therefore satisfied, and it follows that the formula in the proposition holds for $\operatorname{Re} s \in (0, 2-1/\alpha)$. Since $f(s)$ is analytic in the wider strip $\operatorname{Re} s \in (-1/\alpha, 2-1/\alpha)$, we conclude, just as in the previous proof of this proposition, that (3.15) holds in this larger domain as well. □

Thanks to (3.14), Proposition 3.11 yields an expression for the Mellin transform $\mathrm{E}_1[T_0^{s-1}]$ of the hitting time of zero for the symmetric stable process, which agrees with the expression of Cordero [29, equation (1.36)] and the representation of Yano et al. [81, Theorem 85.3]. We will reiterate and extend this result in chapter 6.

3.4 The radial part of the symmetric stable process conditioned to avoid zero

In [25, §4.2], Chaumont, Pantí and Rivero discuss a harmonic transform of a stable process with $\alpha > 1$ which results in *conditioning to avoid zero*. The results quoted in that paper are a special case of the notion of conditioning a Lévy process to avoid zero, which is explored in Pantí [63, 64].

In these works, the authors define

$$h^{\updownarrow}(x) = K(\alpha)(1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1}, \qquad x \in \mathbb{R}, \tag{3.18}$$

where the constant

$$K(\alpha) = \frac{\Gamma(2-\alpha)\sin(\pi\alpha/2)}{c\pi(\alpha-1)(1+\beta^2\tan^2(\pi\alpha/2))}$$

is defined in terms of

$$c = -(c_+ + c_-)\Gamma(-\alpha)\cos(\pi\alpha/2) = \cos(\pi\alpha(\rho - 1/2))$$

and

$$\beta = \frac{c_+ - c_-}{c_+ + c_-} = \frac{\tan(\pi\alpha(\rho-1/2))}{\tan(\pi\alpha/2)},$$

which are the quantities appearing in (2.2). If we write the function $h^{\updownarrow}$ in terms of the (α, ρ) parameterisation which we prefer, we obtain

$$h^{\updownarrow}(x) = \begin{cases} -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\hat{\rho})}{\pi}x^{\alpha-1}, & x > 0, \\ -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\rho)}{\pi}x^{\alpha-1}, & x < 0. \end{cases}$$

In [63], Pantí proves the following proposition for all Lévy processes, and $x \in \mathbb{R}$, with a suitable definition of $h^{\updownarrow}$. Here we quote only the result for stable processes and $x \neq 0$. Hereafter, $(\mathscr{F}_t)_{t\geq 0}$ is the standard filtration associated with X, and n refers to the excursion measure of the stable process away from zero, normalised (see [63, (7)] and [34, (4.11)]) such that

$$n(1 - e^{-q\zeta}) = 1/u^q(0),$$

where ζ is the excursion length and u^q is the q-potential density of the stable process.

Proposition 3.12 ([63, Theorem 2, Theorem 6]). *Let X be a stable process, and $h^\updownarrow$ the function in (3.18).*

(i) The function $h^\updownarrow$ is invariant for the stable process killed on hitting 0, that is,

$$\mathrm{E}_x[h^\updownarrow(X_t), t < T_0] = h^\updownarrow(x), \qquad t > 0,\ x \neq 0. \tag{3.19}$$

Therefore, we may define a family of measures $\mathrm{P}_x^\updownarrow$ by

$$\mathrm{P}_x^\updownarrow(\Lambda) = \frac{1}{h^\updownarrow(x)} \mathrm{E}_x[h^\updownarrow(X_t)\mathbb{1}_\Lambda, t < T_0], \qquad x \neq 0,\ \Lambda \in \mathscr{F}_t,$$

for any $t \geq 0$.

(ii) The function $h^\updownarrow$ can be represented as

$$h^\updownarrow(x) = \lim_{q\downarrow 0} \frac{\mathrm{P}_x(T_0 > \mathbf{e}_q)}{n(\zeta > \mathbf{e}_q)}, \qquad x \neq 0,$$

where $\mathbf{e}_q$ is an independent exponentially distributed random variable with parameter q. Furthermore, for any stopping time T and event $\Lambda \in \mathscr{F}_T$, and any $x \neq 0$,

$$\lim_{q\downarrow 0} \mathrm{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathrm{P}_x^\updownarrow(\Lambda).$$

This justifies the name 'the stable process conditioned to avoid zero' for the canonical process associated with the measures $(\mathrm{P}_x^\updownarrow)_{x\neq 0}$. We will denote this process by $X^\updownarrow$.

We now consider a symmetric stable process X with $\alpha > 1$, as in the last section. (When $\alpha \leq 1$, the process never hits zero, so we neglect this case.) There, we computed the Lamperti transform $\xi^{R'}$ of the pssMp

$$R'_t = \frac{1}{2}|X_t|\mathbb{1}_{\{t<T_0\}}, \qquad t \geq 0,$$

and gave its characteristic exponent $\Psi^{R'}$ in Theorem 3.10.

Consider now the process

$$R_t^\updownarrow = \frac{1}{2}|X_t^\updownarrow|, \qquad t \geq 0.$$

This is also a pssMp, and we may consider its Lamperti transform, which we will denote by $\xi^{\updownarrow}$. In [25], Chaumont et al. compute the characteristics of the so-called Lamperti–Kiu representation of $X^{\updownarrow}$. We shall discuss this representation in detail in chapter 6; for now, we merely wish to remark that the characteristic exponent, $\Psi^{\updownarrow}$, of $\xi^{\updownarrow}$ could be computed from this information. However, this is not necessary, since the harmonic transform in Proposition 3.12(i) gives us the following straightforward relationship between characteristic exponents:

$$\Psi^{\updownarrow}(\theta) = \Psi^{R'}(\theta - \mathrm{i}(\alpha - 1)).$$

This allows us to calculate

$$\Psi^{\updownarrow}(\theta) = \frac{\Gamma(1/2 - \mathrm{i}\theta/2)}{\Gamma((1-\alpha)/2 - \mathrm{i}\theta/2)} \frac{\Gamma(\alpha/2 + \mathrm{i}\theta/2)}{\Gamma(\mathrm{i}\theta/2)}, \qquad \theta \in \mathbb{R}.$$

It is immediately apparent that $2\xi^{\updownarrow}$ is the dual Lévy process to $2\xi^{R'}$, and in particular, that it is an extended hypergeometric Lévy process with parameters

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = ((\alpha + 1)/2, \alpha/2, 0, \alpha/2).$$

It follows from duality that R' is a time-reversal of $R^{\updownarrow}$, in the sense of [22, §2]: roughly speaking, if one fixes $x > 0$, starts the process $R^{\updownarrow}$ at zero (as in [15], say) and runs it backward from its last moment below some level y, where $y > x$, simultaneously conditioning on the position of the left limit at this time taking value x, then one obtains the law of R' under P_x.

We remark that this relationship is already known for Brownian motion, where $R^{\updownarrow}$ is a Bessel process of dimension 3. However, it seems unlikely that any such time-reversal property will hold for a general Lévy process conditioned to avoid zero.

Chapter 4

Hitting distributions and path censoring

After Brownian motion, stable processes are often considered an exemplary family of processes for which many aspects of the general theory of Lévy processes can be illustrated in closed form. First passage problems, which are relatively straightforward to handle in the case of Brownian motion, become much harder in the setting of a general Lévy process on account of the inclusion of jumps. A collection of articles through the 1960s and early 1970s, appealing largely to potential analytic methods for general Markov processes, were relatively successful in handling a number of first passage problems, in particular for symmetric stable processes in one or more dimensions. See, for example, [11, 35, 36, 67, 72] to name but a few.

However, following this cluster of activity, several decades have passed since new results on these problems have appeared. The last few years have seen a number of new, explicit first passage identities for one-dimensional stable processes, thanks to a better understanding of the intimate relationship between the aforesaid processes and positive self-similar Markov processes. See, for example, [14, 17, 24, 46, 57].

In this chapter we return to the work of Blumenthal et al. [11], published in 1961, which gave the law of the position of first entry of a symmetric stable process into the unit ball. Specifically, we are interested in establishing the same law, but now for all the one-dimensional stable processes which fall within the parameter regime $\mathcal{A}_{\mathrm{st}}$ given in section 2.2; we remark that Port [67, §3.1, Remark 3] found this law for processes with one-sided jumps, which justifies our exclusion of these processes in this work. Our method is modern in the sense that we appeal to the relationship of stable processes with certain positive self-similar Markov processes. However, there are two notable additional innovations. First, we make use of a type of path censoring. Second, we are able to describe in explicit analytical detail a non-trivial Wiener–Hopf factorisation of an auxiliary

Lévy process from which the desired solution can be sourced. Moreover, as a consequence of this approach, we are able to deliver in explicit form a number of additional, related identities for stable processes.

4.1 Main results

We now state the main results of this chapter. Let the process X with laws $(\mathrm{P}_x)_{x\in\mathbb{R}}$ be the stable process with scaling parameter α and positivity parameter ρ, as defined in section 2.2. We introduce the first hitting time of the interval $(-1,1)$,

$$\tau_{-1}^1 = \inf\{t \ge 0 : X_t \in (-1,1)\}.$$

Note that, for $x \notin \{-1,1\}$, $\mathrm{P}_x\bigl(X_{\tau_{-1}^1} \in (-1,1)\bigr) = 1$, so long as X is not spectrally one-sided; and taking the parameter set $\mathcal{A}_{\mathrm{st}}$ ensures this. However, in Proposition 4.3, we will consider a spectrally negative stable process, for which $X_{\tau_{-1}^1}$ may take the value -1 with positive probability.

Theorem 4.1. *Let $x > 1$. Then, when $\alpha \in (0,1]$,*

$$\begin{aligned}
&\mathrm{P}_x\bigl(X_{\tau_{-1}^1} \in \mathrm{d}y,\, \tau_{-1}^1 < \infty\bigr)/\mathrm{d}y \\
&\quad = \frac{\sin(\pi\alpha\hat\rho)}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat\rho}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat\rho}(x-y)^{-1},
\end{aligned}$$

for $y \in (-1,1)$. When $\alpha \in (1,2)$,

$$\begin{aligned}
&\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y)/\mathrm{d}y \\
&\quad = \frac{\sin(\pi\alpha\hat\rho)}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat\rho}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat\rho}(x-y)^{-1} \\
&\qquad - (\alpha-1)\frac{\sin(\pi\alpha\hat\rho)}{\pi}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat\rho}\int_1^x (t-1)^{\alpha\hat\rho-1}(t+1)^{\alpha\rho-1}\,\mathrm{d}t,
\end{aligned}$$

for $y \in (-1,1)$.

When X is symmetric, Theorem 4.1 reduces immediately to Theorems B and C of [11]. Moreover, the following hitting probability can be obtained.

Corollary 4.2. *When $\alpha \in (0,1)$, for $x > 1$,*

$$\mathrm{P}_x(\tau_{-1}^1 = \infty) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha)}\int_0^{\frac{x-1}{x+1}} t^{\alpha\hat\rho-1}(1-t)^{-\alpha}\,\mathrm{d}t.$$

This extends Corollary 2 of [11], as can be seen by differentiating and using the doubling formula [38, 8.335.2] for the gamma function.

The spectrally one-sided case can be found as the limit of Theorem 4.1, as we now explain. The first part of the coming proposition is due to Port [67], but we re-state it for the sake of clarity.

Proposition 4.3. *Let $\alpha \in (1,2)$, and suppose that X is spectrally negative, that is, $\rho = 1/\alpha$. Then, the hitting distribution of $[-1,1]$ is given by*

$$\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y) = \frac{\sin \pi(\alpha-1)}{\pi}(x-1)^{\alpha-1}(1-y)^{1-\alpha}(x-y)^{-1}\mathrm{d}y \\ + \frac{\sin \pi(\alpha-1)}{\pi}\int_0^{\frac{x-1}{x+1}} t^{\alpha-2}(1-t)^{1-\alpha}\,\mathrm{d}t\,\delta_{-1}(\mathrm{d}y),$$

for $x > 1$, $y \in [-1,1]$, where δ_{-1} is the unit point mass at -1. Furthermore, the measures on $[-1,1]$ given in Theorem 4.1 converge weakly, as $\rho \to 1/\alpha$, to the limit above.

A further result concerns the first passage of X into the half-line $(1,\infty)$ before hitting zero. Let

$$\tau_1^+ = \inf\{t \geq 0 : X_t > 1\} \quad \text{and} \quad T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

Recall that when $\alpha \in (0,1]$, $\mathrm{P}_x(T_0 = \infty) = 1$, while when $\alpha \in (1,2)$, it holds that $\mathrm{P}_x(T_0 < \infty) = 1$, for $x \neq 0$. In the latter case, we can obtain a hitting probability as follows.

Theorem 4.4. *Let $\alpha \in (1,2)$. When $0 < x < 1$,*

$$\mathrm{P}_x(T_0 < \tau_1^+) = (\alpha-1)x^{\alpha-1}\int_1^{1/x}(t-1)^{\alpha\rho-1}t^{\alpha\hat{\rho}-1}\,\mathrm{d}t.$$

When $x < 0$,

$$\mathrm{P}_x(T_0 < \tau_1^+) = (\alpha-1)(-x)^{\alpha-1}\int_1^{1-1/x}(t-1)^{\alpha\hat{\rho}-1}t^{\alpha\rho-1}\,\mathrm{d}t.$$

It is not difficult to push Theorem 4.4 a little further to give the law of the position of first entry into $(1,\infty)$ on the event $\{\tau_1^+ < T_0\}$. Indeed, by the Markov property, for $x < 1$,

$$\mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y,\, \tau_1^+ < \tau_0) = \mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y) - \mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y, T_0 < \tau_1^+) \\ = \mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y) - \mathrm{P}_x(T_0 < \tau_1^+)\mathrm{P}_0(X_{\tau_1^+} \in \mathrm{d}y). \quad (4.1)$$

Moreover, Rogozin [73] found that, for $x < 1$ and $y > 1$,

$$\mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y) = \frac{\sin(\pi\alpha\rho)}{\pi}(1-x)^{\alpha\rho}(y-1)^{-\alpha\rho}(y-x)^{-1}\,\mathrm{d}y. \quad (4.2)$$

Hence substituting (4.2) together with the hitting probability from Theorem 4.4 into (4.1) yields the following corollary.

Corollary 4.5. *Let* $\alpha \in (1,2)$*. Then, when* $0 < x < 1$*,*

$$\begin{aligned}
&\mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y,\, \tau_1^+ < T_0)/\mathrm{d}u \\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}(1-x)^{\alpha\rho}(y-1)^{-\alpha\rho}(y-x)^{-1} \\
&\qquad - (\alpha-1)\frac{\sin(\pi\alpha\rho)}{\pi}x^{\alpha-1}(y-1)^{-\alpha\rho}y^{-1}\int_1^{1/x}(t-1)^{\alpha\rho-1}t^{\alpha\hat{\rho}-1}\,\mathrm{d}t,
\end{aligned}$$

for $y > 1$*. When* $x < 0$*,*

$$\begin{aligned}
&\mathrm{P}_x(X_{\tau_1^+} \in \mathrm{d}y,\, \tau_1^+ < T_0)/\mathrm{d}y \\
&\quad = \frac{\sin(\pi\alpha\rho)}{\pi}(1-x)^{\alpha\rho}(y-1)^{-\alpha\rho}(y-x)^{-1} \\
&\qquad - (\alpha-1)\frac{\sin(\pi\alpha\rho)}{\pi}(-x)^{\alpha-1}(y-1)^{-\alpha\rho}y^{-1}\int_1^{1-1/x}(t-1)^{\alpha\hat{\rho}-1}t^{\alpha\rho-1}\,\mathrm{d}t,
\end{aligned}$$

for $y > 1$*.*

Finally, the path-censored stable process also yields a result on the occupation time of $(0,\infty)$ before hitting zero for the stable process: let

$$\mathcal{I} = \int_0^{T_0} \mathbb{1}_{\{X_t > 0\}}\,\mathrm{d}t,$$

a random variable which is a.s. finite when $\alpha > 1$, and denote by $\mathcal{M}$ the Mellin transform of this quantity:

$$\mathcal{M}(s) = \mathrm{E}_1[\mathcal{I}^{s-1}],$$

for an appropriate range of s to be determined. Then we have the following, which is a simple application of the ideas developed in chapter 3. In this proposition, G is the double gamma function we first encountered in section 2.7.

Proposition 4.6. *When* $\alpha > 1$*, the Mellin transform of the random variable* $\mathcal{I}$ *is given, for* $\mathrm{Re}(s) \in (\rho - 1/\alpha, 2 - 1/\alpha)$*, by*

$$\begin{aligned}
\mathcal{M}(s) = c\,&\frac{G(2/\alpha - 1 + s; 1/\alpha)}{G(2/\alpha - \rho + s; 1/\alpha)}\frac{G(1/\alpha + \rho + 1 - s; 1/\alpha)}{G(1/\alpha + 1 - s; 1/\alpha)} \\
&\times \frac{\Gamma(1/\alpha - \rho + s)}{\Gamma(\rho + 1 - s)}\Gamma(2 - 1/\alpha - s),
\end{aligned}$$

where c *is a normalising constant such that* $\mathcal{M}(1) = 1$*. When* X *is in the class*

$\mathcal{C}_{k,l}$ introduced by Doney [30], which is to say

$$\rho + k = l/\alpha,$$

for $k, l \in \mathbb{Z}$, equivalent expressions may be found solely in terms of gamma functions and elementary functions. For example, when $k, l \geq 0$, one has

$$\begin{aligned}\mathcal{M}(s) = c(-1)^l(2\pi)^{l(1/\alpha-1)}(1/\alpha)^{l(1-2/\alpha)} \frac{\Gamma(\frac{1-l}{\alpha}+k+s)}{\Gamma(\frac{l}{\alpha}+1-k-s)} \frac{\Gamma(2-1/\alpha-s)}{\Gamma(2-l-\alpha-\alpha s)} \\ \times \prod_{j=1}^{l} \Gamma(j/\alpha+1-s)\Gamma(2/\alpha-(j/\alpha+1-s)) \prod_{i=0}^{k-1} \frac{\sin(\pi\alpha(s+i))}{\pi},\end{aligned}$$

and when $k < 0$, $l \geq 0$,

$$\begin{aligned}\mathcal{M}(s) &= c(-1)^l(2\pi)^{l(1/\alpha-1)}(1/\alpha)^{l(1-2/\alpha)} \\ &\times \frac{\Gamma(\frac{1-l}{\alpha}+k+s)\Gamma(2-1/\alpha-s)\Gamma(l+1+\alpha-\alpha s)\Gamma(2-l+\alpha k+\alpha s)}{\Gamma(\frac{l}{\alpha}+1-k-s)} \\ &\times \prod_{j=1}^{l} \Gamma(j/\alpha+1-s)\Gamma(2/\alpha-(j/\alpha+1-s)) \prod_{i=2}^{-k-1} \frac{\pi}{\sin(\pi\alpha(s-i))}.\end{aligned}$$

Similar expressions may be obtained when $k \geq 0$, $l < 0$ and $k, l < 0$.

We conclude this section by giving an overview of the rest of the chapter. In section 4.2, we explain the operation which gives us the path-censored stable process Y, that is to say the stable process with the negative components of its path removed. We show that Y is a positive self-similar Markov process, and can therefore be written as the exponential of a time-changed Lévy process, say ξ. We show that the Lévy process ξ can be decomposed into the sum of a compound Poisson process and a Lamperti-stable process of the sort considered in section 2.5. Section 4.3 is devoted to finding the distribution of the jumps of this compound Poisson component, which we then use in section 4.4 to compute in explicit detail the Wiener–Hopf factorisation of ξ. Finally, we make us of the explicit nature of the Wiener–Hopf factorisation in section 4.5 to prove Theorem 4.1. There we also prove Theorem 4.4 via a connection with the stable process conditioned to stay positive.

4.2 The path-censored process and its Lamperti transform

We now describe the construction of the path-censored stable process that will lie at the heart of our analysis, show that it is a pssMp and discuss its Lamperti transform.

As in the previous section, X, with probability laws $(\mathrm{P}_x)_{x\in\mathbb{R}}$, denotes the stable process defined in section 2.2. Define the occupation time of $(0,\infty)$ up to time t,

$$A_t = \int_0^t \mathbb{1}_{\{X_s>0\}}\,\mathrm{d}s,$$

and let $\gamma(t) = \inf\{s \geq 0 : A_s > t\}$ be its right-continuous inverse. Define a process $(\check{Y}_t)_{t\geq 0}$ by setting $\check{Y}_t = X_{\gamma(t)}$, for $t \geq 0$. This is the process formed by erasing the negative components of X and joining up the gaps.

Recalling that $(\mathscr{F}_t)_{t\geq 0}$ is the augmented natural filtration of X, write $\mathscr{G}_t = \mathscr{F}_{\gamma(t)}$, for $t \geq 0$.

Proposition 4.7. *The process $\check{Y}$ is strong Markov with respect to the filtration $(\mathscr{G}_t)_{t\geq 0}$ and satisfies the scaling property with self-similarity index α.*

Proof. The fact that the strong Markov property holds is due to general facts about time-changed Markov processes; see Rogers and Williams [71, III.21] or Sharpe [76, Theorem 65.9]. We now consider the scaling property. For $c > 0$, define the rescaled process $(\tilde{X}_t, t \geq 0)$ by $\tilde{X}_t = cX_{c^{-\alpha}t}$, and, correspondingly, let $\tilde{\gamma}$ be the right-inverse of $\int_0^{\cdot} \mathbb{1}_{\{\tilde{X}_s>0\}}\,\mathrm{d}s$. By writing the equation $A_{\gamma(c^{-\alpha}t)} = c^{-\alpha}t$, a short calculation shows that

$$c^{\alpha}\gamma(c^{-\alpha}t) = \tilde{\gamma}(t).$$

The scaling property of $\check{Y}$ now follows:

$$\begin{aligned} \text{under } \mathrm{P}_x,\ \left(c\check{Y}_{c^{-\alpha}t}\right)_{t\geq 0} &= \left(cX_{\gamma(c^{-\alpha}t)}\right)_{t\geq 0} \\ &= \left(\tilde{X}_{\tilde{\gamma}(t)}\right)_{t\geq 0} \\ &\stackrel{d}{=} \left(X_{\gamma(t)}\right)_{t\geq 0}, \text{ under } \mathrm{P}_{cx} \\ &= \check{Y}. \end{aligned}$$

This completes the proof. □

We now make zero into an absorbing state. Define the stopping time

$$\tau_0 = \inf\{t > 0 : \check{Y}_t = 0\}$$

and the process

$$Y_t = \check{Y}_t \mathbb{1}_{\{t<\tau_0\}}, \qquad t \geq 0,$$

so that $Y := (Y_t)_{t\geq 0}$ is $\check{Y}$ absorbed at zero. We call the process Y with probability laws $(\mathrm{P}_x)_{x>0}$ the *path-censored stable process.*

Proposition 4.8. *The process Y is a pssMp with respect to the filtration $(\mathscr{G}_t)_{t\geq 0}$.*

Proof. The scaling property follows from Proposition 4.7, and zero is evidently an absorbing state. It remains to show that Y is a standard process, and the only point which may be in doubt here is quasi-left-continuity. This, however, follows from the Feller property, which in turn follows from scaling and the Feller property of X; see Kallenberg [42, Proposition 25.20]. □

Remark 4.9. The definition of Y via time-change and stopping at zero bears some resemblance to a number of other constructions:

(a) Bertoin's construction [5, §3.1] of the Lévy process conditioned to stay positive. The key difference here is that, when a negative excursion is encountered, instead of simply erasing it, [5] patches the last jump from negative to positive onto the final value of the previous positive excursion.

(b) Bogdan, Burdzy and Chen's 'censored stable process', censored in the domain $D = (0, \infty)$; see [12], in particular Theorem 2.1 and the preceding discussion. Here the authors suppress any jumps of a symmetric stable process X by which the process attempts to escape the domain, and kill the process if it reaches the boundary continuously.

Both processes (a) and (b) are also pssMps with index α. These processes, together with the process Y just described, therefore represent three choices of how to restart a stable process in a self-similar way after it leaves the positive half-line. We illustrate this in Figure 4-1.

Now, let $\xi = (\xi_s)_{s\geq 0}$ be the Lamperti transform of Y. That is,

$$\xi_s = \log Y_{T(s)}, \qquad s \geq 0, \tag{4.3}$$

where T is a time-change. As in section 2.4, we will write $\mathbb{P}_y$ for the law of ξ started at $y \in \mathbb{R}$; note that $\mathbb{P}_y$ corresponds to $\mathrm{P}_{\exp(y)}$.

Let us give a broad overview of the behaviour of Y and ξ. Referring to Remark 2.2, we first observe the following. For $\alpha \in (0, 1]$, points are polar for X, which implies that $\tau_0 = \infty$ a.s.; hence, in this case, $Y = \check{Y}$. Meanwhile, for $\alpha \in (1, 2)$, every point is recurrent, so $\tau_0 < \infty$ almost surely. However, the process X makes infinitely many jumps across zero before hitting it. Therefore, in this case Y approaches zero continuously.

Since zero is an instantaneous point for the stable process X, and X also enters the set $(0, \infty)$ immediately when started at zero, it is clear that $\check{Y}$ exits zero continuously, and is the unique *recurrent extension* of Y, in the spirit of [68] and [33], which has this property.

Using the behaviour of Y at τ_0 and the space transformation (4.3), we may summarise the large time behaviour of ξ as follows.

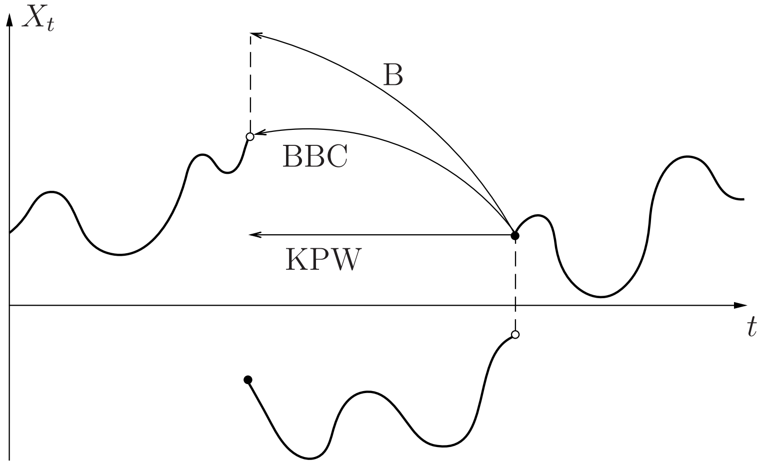

Figure 4-1: The construction of three related processes from X, the stable process: 'B' is (a), the stable process conditioned to stay positive [5]; 'BBC' is (b), the censored stable process [12]; and 'KPW' is the process Y in this chapter.

(i) If $\alpha \in (0,1)$, $\tau_0 = \infty$ and X (and hence Y) is transient almost surely. Therefore, ξ is unkilled and drifts to $+\infty$.

(ii) If $\alpha = 1$, $\tau_0 = \infty$ and every neighbourhood of zero is an a.s. recurrent set for X, and hence also for Y. Therefore, ξ is unkilled and oscillates.

(iii) If $\alpha \in (1,2)$, $\tau_0 < \infty$ and Y hits zero continuously. Therefore, ξ is unkilled and drifts to $-\infty$.

Furthermore, we have the following result.

Proposition 4.10. *The Lévy process ξ is the sum of two independent Lévy processes ξ^{L} and ξ^{C}, which are characterised as follows:*

(i) The Lévy process ξ^{L} has characteristic exponent

$$\Psi^{\mathrm{L}}(\theta) = \Psi^*(\theta) - c_-/\alpha, \qquad \theta \in \mathbb{R},$$

where Ψ^* is the characteristic exponent of the process ξ^* defined in section 2.5. That is, ξ^{L} is formed by removing the independent killing from ξ^*.

(ii) The process ξ^{C} is a compound Poisson process whose jumps occur at rate c_-/α.

Proof. The proof is identical to that of Proposition 3.6. □

Remark 4.11. Let us consider the effect of the Lamperti transform on each of the pssMps in Remark 4.9.

(a) For the process conditioned to stay positive, the associated Lévy process is the process $\xi^{\uparrow}$ which we discussed in section 2.5.

(b) As regards the censored stable process in $(0, \infty)$, we can reason as in the above proposition to deduce that its Lamperti transform is simply the process ξ^{L} which we have just defined.

4.3 Jump distribution of the compound Poisson component

In this section, we express the jump distribution of ξ^{C} in terms of known quantitites, and hence derive its characteristic function and density.

We define two stopping times,

$$\tau = \inf\{t \geq 0 : X_t < 0\} \quad \text{and} \quad \sigma = \inf\{t \geq \tau : X_t > 0\},$$

which are, respectively, the hitting and return times of $(-\infty, 0)$ and $(0, \infty)$ for X. Note that, due to the time-change γ, $Y_\tau = X_\sigma$, while $Y_{\tau-} = X_{\tau-}$.

We begin with two lemmas; the first we have more or less seen before.

Lemma 4.12. *The joint law of* $(X_\tau, X_{\tau-}, X_\sigma)$ *under* P_x *is equal to that of* $(xX_\tau, xX_{\tau-}, xX_\sigma)$ *under* P_1.

Proof. The proof is a minor alteration of that of Lemma 3.5. □

Let $\hat{X}$ be an independent copy of the dual process $-X$, and denote its probability laws by $(\hat{\mathrm{P}}_x)_{x\in\mathbb{R}}$. Define also the stopping time

$$\hat{\tau} = \inf\{t \geq 0 : \hat{X}_t < 0\}$$

for $\hat{X}$. Denote by $\Delta\xi^{\mathrm{C}}$ a random variable whose law is the same as the jump distribution of ξ^{C}.

Lemma 4.13. *The random variable* $\exp(\Delta\xi^{\mathrm{C}})$ *is equal in distribution to*

$$\left(-\frac{X_\tau}{X_{\tau-}}\right)\left(-\hat{X}_{\hat{\tau}}\right),$$

where X *and* $\hat{X}$ *are taken to be independent with respective laws* P_1 *and* $\hat{\mathrm{P}}_1$.

Proof. Examining the proofs of Propositions 4.10 and 3.6—more precisely, substituting Y for R in (3.10)—we see that

$$\exp(\Delta\xi^{\mathrm{C}}_{S(\tau)}) = \frac{Y_\tau}{Y_{\tau-}} = \frac{X_\sigma}{X_{\tau-}}. \tag{4.4}$$

Applying the Markov property, and then using Lemma 4.12 with the α-self-similar process $\hat{X}$, we obtain

$$\begin{aligned} \mathrm{P}_1(X_\sigma \in \cdot \mid \mathscr{F}_\tau) &= \hat{\mathrm{P}}_{-y}(-\hat{X}_{\hat{\tau}} \in \cdot)\big|_{y=X_\tau} \\ &= \hat{\mathrm{P}}_1(y\hat{X}_{\hat{\tau}} \in \cdot)\big|_{y=X_\tau}. \end{aligned}$$

Then, by disintegration, for a Borel function f,

$$\begin{aligned} \mathrm{E}_1\left[f\left(\frac{X_\sigma}{X_{\tau-}}\right)\right] = \mathrm{E}_1\left[\mathrm{E}_1\left[f\left(\frac{X_\sigma}{X_{\tau-}}\right)\middle|\mathscr{F}_\tau\right]\right] &= \mathrm{E}_1\left[\int f\left(\frac{x}{X_{\tau-}}\right)\mathrm{P}_1[X_\sigma \in \mathrm{d}x \mid \mathscr{F}_\tau]\right] \\ &= \mathrm{E}_1\left[\int f\left(\frac{x}{X_{\tau-}}\right)\hat{\mathrm{P}}_1\left[y\hat{X}_{\hat{\tau}} \in \mathrm{d}x\right]\big|_{y=X_\tau}\right] \\ &= \mathrm{E}_1\left[\hat{\mathrm{E}}_1\left[f\left(\frac{y\hat{X}_{\hat{\tau}}}{z}\right)\right]\Bigg|_{y=X_\tau,\, z=X_{\tau-}}\right] \\ &= \mathrm{E}_1 \otimes \hat{\mathrm{E}}_1\left[f\left(\frac{X_\tau\hat{X}_{\hat{\tau}}}{X_{\tau-}}\right)\right]. \end{aligned}$$

Combining this with (4.4), we obtain that the law under P_1 of $\exp\big(\Delta\xi^{\mathrm{C}}_{S(\tau)}\big)$ is equal to that of $\dfrac{X_\tau\hat{X}_{\hat{\tau}}}{X_{\tau-}}$ under $\mathrm{P}_1 \otimes \hat{\mathrm{P}}_1$, which establishes the claim. □

The characteristic function of $\Delta\xi^{\mathrm{C}}$ can now be found by rewriting the expression in Lemma 4.13 in terms of overshoots and undershoots of stable Lévy processes, whose marginal and joint laws are given in Rogozin [73] and Doney and Kyprianou [32]. Following the notation of [32], let

$$\tau_a^+ = \inf\{t \geq 0 : X_t > a\},$$

and let $\hat{\tau}_a^+$ be defined similarly for $\hat{X}$.

Proposition 4.14. *The characteristic function of the jump distribution of ξ^{C} is given by*

$$\mathbb{E}_0\big[\exp\big(\mathrm{i}\theta\Delta\xi^{\mathrm{C}}\big)\big] = \frac{\sin(\pi\alpha\rho)}{\pi\Gamma(\alpha)}\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta). \tag{4.5}$$

Proof. In the course of the coming computations, we will make use several times of the beta integral,

$$\int_0^1 s^{x-1}(1-s)^{y-1}\,\mathrm{d}s = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \operatorname{Re} x, \operatorname{Re} y > 0.$$

See for example [38, formulas 8.830.1–3].

Now, for $\theta \in \mathbb{R}$,

$$\begin{aligned} \hat{\mathrm{E}}_1\Bigl(-\hat{X}_{\hat{\tau}}\Bigr)^{\mathrm{i}\theta} &= \mathrm{E}_0\Bigl(X_{\tau_1^+} - 1\Bigr)^{\mathrm{i}\theta} \\ &= \frac{\sin(\pi\alpha\rho)}{\pi}\int_0^\infty t^{\mathrm{i}\theta-\alpha\rho}(1+t)^{-1}\,\mathrm{d}t \\ &= \frac{\sin(\pi\alpha\rho)}{\pi}\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta). \end{aligned} \tag{4.6}$$

Furthermore,

$$\begin{aligned} \mathrm{E}_1\left(-\frac{X_\tau}{X_{\tau-}}\right)^{\mathrm{i}\theta} &= \hat{\mathrm{E}}_0\left(\frac{\hat{X}_{\hat{\tau}_1^+} - 1}{1-\hat{X}_{\hat{\tau}_1^+-}}\right)^{\mathrm{i}\theta} \\ &= K\int_0^1\int_y^\infty\int_0^\infty \frac{u^{\mathrm{i}\theta}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{v^{\mathrm{i}\theta}(v+u)^{1+\alpha}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y, \end{aligned} \tag{4.7}$$

where $K = \frac{\sin(\pi\alpha\hat{\rho})}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$. For the innermost integral above we have

$$\begin{aligned} \int_0^\infty \frac{u^{\mathrm{i}\theta}}{(u+v)^{1+\alpha}}\,\mathrm{d}u &\overset{w=u/v}{=} v^{\mathrm{i}\theta-\alpha}\int_0^\infty \frac{w^{\mathrm{i}\theta}}{(1+w)^{1+\alpha}}\,\mathrm{d}w \\ &= v^{\mathrm{i}\theta-\alpha}\frac{\Gamma(\mathrm{i}\theta+1)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha+1)}. \end{aligned}$$

The next iterated integral in (4.7) becomes, substituting $z = v/y - 1$,

$$\begin{aligned} \int_y^\infty v^{-\alpha}(v-y)^{\alpha\rho-1}\,\mathrm{d}v &= y^{-\alpha\hat{\rho}}\int_0^\infty \frac{z^{\alpha\rho-1}}{(1+z)^\alpha}\,\mathrm{d}z \\ &= y^{-\alpha\hat{\rho}}\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}{\Gamma(\alpha)}, \end{aligned}$$

and finally it remains to calculate

$$\int_0^1 y^{-\alpha\hat{\rho}}(1-y)^{\alpha\hat{\rho}-1}\,\mathrm{d}y = \Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho}).$$

Multiplying together these expressions and using the reflection identity

$$\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x),$$

we obtain

$$\mathrm{E}_1\left(-\frac{X_\tau}{X_{\tau-}}\right)^{\mathrm{i}\theta} = \frac{\Gamma(\mathrm{i}\theta+1)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha)}. \tag{4.8}$$

The result now follows from Lemma 4.13 by multiplying (4.6) and (4.8) together. □

Remark 4.15. The recent work of Chaumont et al. [25] on the so-called Lamperti–Kiu processes can be applied to give the same result. The quantity $\Delta\xi^{\mathrm{C}}$ in the present work corresponds to the independent sum $\xi^-_\zeta + U^+ + U^-$ in that paper, where U^+ and U^- are "log-Pareto" random variables and ξ^- is the killed Lamperti-stable process corresponding to $\hat{X}$ absorbed below zero; see [25, Corollary 11] for details. It is straightforward to show that the characteristic function of this sum is equal to the right-hand side of (4.5). We give more details on this work in section 6.1.3.

It is now possible to deduce the density of the jump distribution from its characteristic function. By substituting on the left and using the beta integral, it can be shown that

$$\int_{-\infty}^{\infty} e^{\mathrm{i}\theta x}\,\alpha e^x(1+e^x)^{-(\alpha+1)}\,\mathrm{d}x = \frac{\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha)},$$

$$\int_{-\infty}^{\infty} e^{\mathrm{i}\theta x}\,\frac{\sin(\pi\alpha\rho)}{\pi}e^{(1-\alpha\rho)x}(1+e^x)^{-1}\,\mathrm{d}x = \frac{\sin(\pi\alpha\rho)}{\pi}\Gamma(\alpha\rho-\mathrm{i}\theta)\Gamma(1-\alpha\rho+\mathrm{i}\theta),$$

and so the density of $\Delta\xi^{\mathrm{C}}$ can be seen as the convolution of these two functions. Moreover, it is even possible to calculate this convolution directly:

$$\begin{aligned}
&\mathbb{P}_0\big(\Delta\xi^{\mathrm{C}}\in\mathrm{d}x\big)/\mathrm{d}x \\
&= \frac{\alpha}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}\int_{-\infty}^{\infty} e^u(1+e^u)^{-(\alpha+1)}e^{(1-\alpha\rho)(x-u)}(1+e^{x-u})^{-1}\,\mathrm{d}u \\
&= \frac{\alpha}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)}e^{-\alpha\rho x}\int_0^{\infty} t^{\alpha\rho}(1+t)^{-(\alpha+1)}(te^{-x}+1)^{-1}\,\mathrm{d}t \\
&= \frac{\alpha\Gamma(\alpha\rho+1)\Gamma(\alpha\hat{\rho}+1)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha+2)}e^{-\alpha\rho x}{}_2F_1(1,\alpha\rho+1;\alpha+2;1-e^{-x}),
\end{aligned} \tag{4.9}$$

where the final line follows from [38, formula 3.197.5], and is to be understood in the sense of analytic continuation when $x<0$.

4.4 Wiener–Hopf factorisation of ξ

We deduce in explicit form the Wiener-Hopf factors of ξ from its characteristic exponent. Analytically, we will need to distinguish the cases $\alpha \in (0,1]$ and $\alpha \in (1,2)$; in probabilistic terms, these correspond to the regimes where X cannot and can hit zero, respectively. We will make use of the hypergeometric and extended hypergeometric classes of Lévy processes, which were introduced in section 2.7 and chapter 3 respectively.

4.4.1 Wiener–Hopf factorisation for $\alpha \in (0,1]$

Theorem 4.16 (Wiener–Hopf factorisation)**.**

(i) When $\alpha \in (0,1]$, the Wiener–Hopf factorisation of ξ has components

$$\kappa(\lambda) = \frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(\lambda)}, \qquad \hat{\kappa}(\lambda) = \frac{\Gamma(1 - \alpha\rho + \lambda)}{\Gamma(1 - \alpha + \lambda)}, \qquad \lambda \geq 0,$$

and ξ is a hypergeometric Lévy process with parameters

$$\big(\beta, \gamma, \hat{\beta}, \hat{\gamma}\big) = \big(1,\, \alpha\rho,\, 1 - \alpha,\, \alpha\hat{\rho}\big).$$

(ii) The ascending ladder height process is a Lamperti-stable subordinator with parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(0,\, \alpha\rho,\, 1,\, -\frac{1}{\Gamma(-\alpha\rho)},\, 0\right).$$

(iii) The descending ladder height process is a Lamperti-stable subordinator with parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(\frac{\Gamma(1-\alpha\rho)}{\Gamma(1-\alpha)}, \alpha\hat{\rho},\, \alpha,\, -\frac{1}{\Gamma(-\alpha\hat{\rho})},\, 0\right),$$

when $\alpha < 1$, and

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(0,\, \alpha\hat{\rho},\, \alpha,\, -\frac{1}{\Gamma(-\alpha\hat{\rho})},\, 0\right),$$

when $\alpha = 1$.

Proof. First we compute Ψ^{C} and Ψ^{L}, the characteristic exponents of ξ^{C} and ξ^{L}. As ξ^{C} is a compound Poisson process with jump rate c_-/α and jump distribution given by (4.5), we obtain, after using the reflection formula for the gamma function,

$$\Psi^{\mathrm{C}}(\theta) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}\left(1 - \frac{\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha)}\right),$$

for $\theta \in \mathbb{R}$. On the other hand, [46, Theorem 1] provides an expression for the characteristic exponent Ψ^* of the Lamperti-stable process ξ^* from section 2.5, and removing the killing from this gives us

$$\Psi^{\mathrm{L}}(\theta) = \frac{\Gamma(\alpha-\mathrm{i}\theta)}{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)}\frac{\Gamma(1+\mathrm{i}\theta)}{\Gamma(1-\alpha\hat{\rho}+\mathrm{i}\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}.$$

We can now compute, applying the reflection formula twice,

$$\begin{aligned}
\Psi(\theta) &= \Psi^{\mathrm{L}}(\theta) + \Psi^{\mathrm{C}}(\theta) \\
&= \Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta) \\
&\quad \times \left(\frac{1}{\Gamma(\alpha\hat{\rho}-\mathrm{i}\theta)\Gamma(1-\alpha\hat{\rho}+\mathrm{i}\theta)} - \frac{\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}\right) \\
&= \Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)\Gamma(1-\alpha\rho+\mathrm{i}\theta)\Gamma(\alpha\rho-\mathrm{i}\theta) \\
&\quad \times \left(\frac{\sin(\pi(\alpha\hat{\rho}-\mathrm{i}\theta))\sin(\pi(\alpha\rho-\mathrm{i}\theta))}{\pi^2} - \frac{\sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho)}{\pi^2}\right).
\end{aligned}$$

It may be proved, using product and sum identities for trigonometric functions, that

$$\sin(\pi(\alpha\hat{\rho}-\mathrm{i}\theta))\sin(\pi(\alpha\rho-\mathrm{i}\theta)) + \sin(\pi\mathrm{i}\theta)\sin(\pi(\alpha-\mathrm{i}\theta)) = \sin(\pi\alpha\hat{\rho})\sin(\pi\alpha\rho).$$

Again using the reflection formula twice, this leads to

$$\begin{aligned}
\Psi(\theta) &= \frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\Gamma(1+\mathrm{i}\theta)\Gamma(-\mathrm{i}\theta)}\frac{\Gamma(\alpha\rho-\mathrm{i}\theta)\Gamma(1-\alpha\rho+\mathrm{i}\theta)}{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1-\alpha+\mathrm{i}\theta)} \\
&= \frac{\Gamma(\alpha\rho-\mathrm{i}\theta)}{\Gamma(-\mathrm{i}\theta)} \times \frac{\Gamma(1-\alpha\rho+\mathrm{i}\theta)}{\Gamma(1-\alpha+\mathrm{i}\theta)}. \qquad (4.10)
\end{aligned}$$

Part (i) now follows by applying Proposition 2.12, and the rest of the theorem follows from Corollary 2.13. □

Proposition 4.17. *(i) The process ξ has Lévy density*

$$\pi(x) = \begin{cases} -\dfrac{1}{\Gamma(1-\alpha\hat{\rho})\Gamma(-\alpha\rho)} e^{-\alpha\rho x} {}_2F_1(1+\alpha\rho, 1; 1-\alpha\hat{\rho}; e^{-x}), & x > 0, \\ -\dfrac{1}{\Gamma(1-\alpha\rho)\Gamma(-\alpha\hat{\rho})} e^{(1-\alpha\rho)x} {}_2F_1(1+\alpha\hat{\rho}, 1; 1-\alpha\rho; e^{x}), & x < 0. \end{cases}$$

(ii) The ascending ladder height has Lévy density

$$\pi_H(x) = -\frac{1}{\Gamma(-\alpha\rho)} e^x (e^x - 1)^{-(\alpha\rho+1)}, \qquad x > 0.$$

The ascending renewal measure $V(\mathrm{d}x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{H_t \in \mathrm{d}x\}}\,\mathrm{d}t$ *is given by*

$$V(\mathrm{d}x)/\mathrm{d}x = \frac{1}{\Gamma(\alpha\rho)} (1 - e^{-x})^{\alpha\rho - 1}, \qquad x > 0.$$

(iii) The descending ladder height has Lévy density

$$\pi_{\hat{H}}(x) = -\frac{1}{\Gamma(-\alpha\hat{\rho})} e^{\alpha x} (e^x - 1)^{-(\alpha\hat{\rho}+1)}, \qquad x > 0.$$

The descending renewal measure is given by

$$\hat{V}(\mathrm{d}x)/\mathrm{d}x = \frac{1}{\Gamma(\alpha\hat{\rho})} (1 - e^{-x})^{\alpha\hat{\rho} - 1} e^{-(1-\alpha)x}, \qquad x > 0.$$

Proof. The Lévy density of ξ follows from Proposition 2.12(ii), and the expressions for π_H and $\pi_{\hat{H}}$ are obtained by substituting in (2.9). The renewal measures can be verified using the Laplace transform identity

$$\int_0^\infty e^{-\lambda x} V(\mathrm{d}x) = 1/\kappa(\lambda), \qquad \lambda \geq 0,$$

and the corresponding identity for the descending ladder height. □

4.4.2 Wiener–Hopf factorisation for $\alpha \in (1, 2)$

Theorem 4.18 (Wiener–Hopf factorisation)**.**

(i) When $\alpha \in (1, 2)$*, the Wiener–Hopf factorisation of* ξ *has components*

$$\kappa(\lambda) = (\alpha - 1 + \lambda)\frac{\Gamma(\alpha\rho + \lambda)}{\Gamma(1 + \lambda)}, \qquad \hat{\kappa}(\lambda) = \lambda \frac{\Gamma(1 - \alpha\rho + \lambda)}{\Gamma(2 - \alpha + \lambda)}, \qquad \lambda \geq 0,$$

and ξ *is an extended hypergeometric Lévy process with parameters*

$$(\beta, \gamma, \hat{\beta}, \hat{\gamma}) = (1,\, \alpha\rho,\, 1 - \alpha,\, \alpha\hat{\rho}).$$

(ii) The ascending ladder height process can be identified as the conjugate sub-

ordinator to $\mathcal{T}_{\alpha-1}\psi^*$ (see section 2.6.2), where

$$\psi^*(\lambda) = \frac{\Gamma(2-\alpha+\lambda)}{\Gamma(1-\alpha\hat{\rho}+\lambda)}, \qquad \lambda \geq 0$$

is the Laplace exponent of a Lamperti-stable process. This Lamperti-stable process has parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})}, 1-\alpha\rho,\ \alpha\hat{\rho},\ -\frac{1}{\Gamma(\alpha\rho-1)},\ 0\right).$$

(iii) The descending ladder process is the conjugate subordinator to a Lamperti-stable process with Laplace exponent

$$\phi^*(\lambda) = \frac{\Gamma(2-\alpha+\lambda)}{\Gamma(1-\alpha\rho+\lambda)}, \qquad \lambda \geq 0,$$

which has parameters

$$\big(q, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\big) = \left(\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\rho)},\ 1-\alpha\hat{\rho},\ \alpha\rho,\ -\frac{1}{\Gamma(\alpha\hat{\rho}-1)},\ 0\right).$$

Proof. If we return to the proof of Theorem 4.16(i), we observe that the derivation of (4.10) is valid for all $\alpha \in (0,2)$. However, the factorisation for $\alpha \in (0,1]$ does not apply when $\alpha \in (1,2)$. This may be seen in the fact that the expression for $\hat{\kappa}$ is equal to zero at $\alpha - 1 > 0$, which contradicts the requirement that it be the Laplace exponent of a subordinator.

Now, applying the identity $x\Gamma(x) = \Gamma(x+1)$ to each denominator in that expression, we obtain for $\theta \in \mathbb{R}$

$$\Psi(\theta) = (\alpha - 1 - \mathrm{i}\theta)\frac{\Gamma(\alpha\rho - \mathrm{i}\theta)}{\Gamma(1-\mathrm{i}\theta)} \times \mathrm{i}\theta\frac{\Gamma(1-\alpha\rho+\mathrm{i}\theta)}{\Gamma(2-\alpha+\mathrm{i}\theta)}.$$

The theorem is proved once we apply Proposition 3.1.

□

Remark 4.19. There is another way to view the ascending ladder height, which is often more convenient for calculation. Applying the second part of Proposition 2.7, we find that

$$\kappa(\lambda) = \mathcal{E}_{\alpha-1}\psi(\lambda) + \psi(\alpha-1),$$

where ψ is conjugate to ψ^*. Hence, H can be seen as the Esscher transform of the subordinator conjugate to ψ^*, with additional killing.

Proposition 4.20. *(i) The process ξ has Lévy density*

$$\pi(x) = \begin{cases} -\dfrac{1}{\Gamma(1-\alpha\hat{\rho})\Gamma(-\alpha\rho)} e^{-\alpha\rho x} {}_2F_1(1+\alpha\rho, 1; 1-\alpha\hat{\rho}; e^{-x}), & x > 0, \\ -\dfrac{1}{\Gamma(1-\alpha\rho)\Gamma(-\alpha\hat{\rho})} e^{(1-\alpha\rho)x} {}_2F_1(1+\alpha\hat{\rho}, 1; 1-\alpha\rho; e^{x}), & x < 0. \end{cases}$$

(ii) The ascending ladder height has Lévy density

$$\pi_H(x) = \frac{(e^x - 1)^{-(\alpha\rho+1)}}{\Gamma(1-\alpha\rho)} \big(\alpha - 1 + (1-\alpha\hat{\rho})e^x\big), \qquad x > 0.$$

The ascending renewal measure $V(\mathrm{d}x) = \mathbb{E}\int_0^\infty \mathbb{1}_{\{H_t \in \mathrm{d}x\}}\, \mathrm{d}t$ is given by

$$V(\mathrm{d}x)/\mathrm{d}x = e^{-(\alpha-1)x}\left[\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})} + \frac{1-\alpha\rho}{\Gamma(\alpha\rho)} \int_x^\infty e^{\alpha\hat{\rho}z}(e^z - 1)^{\alpha\rho - 2}\, \mathrm{d}z\right],$$

for $x > 0$.

(iii) The descending ladder height has Lévy density

$$\pi_{\hat{H}}(x) = \frac{e^{(\alpha-1)x}(e^x - 1)^{-(\alpha\hat{\rho}+1)}}{\Gamma(1-\alpha\hat{\rho})} \big(\alpha - 1 + (1-\alpha\rho)e^x\big), \qquad x > 0.$$

The descending renewal measure is given by

$$\hat{V}(\mathrm{d}x)/\mathrm{d}x = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\rho)} + \frac{1-\alpha\hat{\rho}}{\Gamma(\alpha\hat{\rho})} \int_x^\infty e^{\alpha\rho z}(e^z - 1)^{\alpha\hat{\rho}-2}\, \mathrm{d}z, \qquad x > 0.$$

Proof. We will prove (i), and then (iii) and (ii) in that order.

(i) This is a direct application of Proposition 3.1. However, there is an alternate proof which may be preferred as it requires only the decomposition in Proposition 4.10; we give this proof now.

Multiplying the jump density (4.9) of ξ^{C} by c_-/α, we obtain an expression for its Lévy density π^{C} in terms of a ${}_2F_1$ function. When we apply the relations [38, formulas 9.131.1–2], we obtain

$$\pi^{\mathrm{C}}(x) = \begin{cases} -\dfrac{1}{\Gamma(1-\alpha\hat{\rho})\Gamma(-\alpha\rho)} e^{-\alpha\rho x} {}_2F_1(1+\alpha\rho, 1; 1-\alpha\hat{\rho}; e^{-x}) \\ \quad + \dfrac{\Gamma(\alpha+1)}{\Gamma(1+\alpha\rho)\Gamma(-\alpha\rho)} e^{-\alpha x} {}_2F_1(1+\alpha\hat{\rho}, \alpha+1; 1+\alpha\hat{\rho}; e^{-x}), \\ \hfill x > 0, \\ -\dfrac{1}{\Gamma(1-\alpha\rho)\Gamma(-\alpha\hat{\rho})} e^{(1-\alpha\rho)x} {}_2F_1(1+\alpha\hat{\rho}, 1; 1-\alpha\rho; e^{x}) \\ \quad - \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} e^{x} {}_2F_1(1+\alpha\rho, \alpha+1; 1+\alpha\rho; e^{x}), \\ \hfill x < 0. \end{cases}$$

Recall that ${}_2F_1(a,b;a;z) = (1-z)^{-b}$. Then, comparing with (2.6), the equation reads

$$\pi^{\mathrm{C}}(x) = \pi(x) - \pi^{\mathrm{L}}(x), \qquad x \neq 0,$$

where $\pi^{\mathrm{L}} = \pi^*$ is the Lévy density of ξ^{L}. The claim then follows by the independence of ξ^{C} and ξ^{L}.

(iii) In [55, Example 2], the authors give the tail of the Lévy measure $\Pi_{\hat{H}}$, and show that it is absolutely continuous. The density $\pi_{\hat{H}}$ is obtained by differentiation.

In order to obtain the renewal measure, start with the following standard observation. For $\lambda \geq 0$,

$$\int_0^\infty e^{-\lambda x} \hat{V}(\mathrm{d}x) = \frac{1}{\hat{\kappa}(\lambda)} = \frac{\phi^*(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda x} \overline{\Pi}_{\phi^*}(x)\, \mathrm{d}x, \tag{4.11}$$

where $\overline{\Pi}_{\phi^*}(x) = q_{\phi^*} + \Pi_{\phi^*}(x,\infty)$, and q_{ϕ^*} and Π_{ϕ^*} are, respectively, the killing rate and Lévy measure of the subordinator corresponding to ϕ^*. Comparing with section 2.6.1, we have

$$q_{\phi^*} = \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\rho)}$$

and

$$\Pi_{\phi^*}(\mathrm{d}x)/\mathrm{d}x = -\frac{1}{\Gamma(\alpha\hat{\rho}-1)} e^{\alpha\rho x}(e^x-1)^{\alpha\hat{\rho}-2}, \qquad x > 0,$$

and substituting these back into (4.11) leads immediately to the desired expression for $\hat{V}$.

(ii) To obtain the Lévy density, it is perhaps easier to use the representation of H as a killed Esscher transform, noted in Remark 4.19. As in part (iii), applying [55, Example 2] gives

$$\pi_\psi(x) = \frac{e^{(\alpha-1)x}(e^x-1)^{-(\alpha\rho+1)}}{\Gamma(1-\alpha\rho)}\big(\alpha-1+(1-\alpha\hat{\rho})e^x\big), \qquad x>0,$$

where π_ψ is the Lévy density corresponding to $\psi(\lambda)=\lambda/\psi^*(\lambda)$. The effect of the Esscher transform on the Lévy measure gives

$$\pi_H(x) = e^{-(\alpha-1)x}\pi_\psi(x), \qquad x>0,$$

and putting everything together we obtain the required expression.

Emulating the proof of (iii), we calculate

$$\int_0^\infty e^{-\lambda x}U(\mathrm{d}x) = \frac{1}{\kappa(\lambda)} = \frac{\psi^*(\alpha-1+\lambda)}{\alpha-1+\lambda} = \int_0^\infty e^{-\lambda x}e^{-(\alpha-1)x}\overline{\Pi}_{\psi^*}(x)\,\mathrm{d}x,$$

using similar notation to previously, and the density of V follows. □

4.5 Proofs of main results

In this section, we use the Wiener-Hopf factorisation of ξ to prove Theorem 4.1, and then deduce Corollary 4.2. We then make use of a connection with the process conditioned to stay positive in order to prove Theorem 4.4.

Our method for proving each theorem will be to prove a corresponding result for the Lévy process ξ, and to relate this to the stable process X by means of the Lamperti transform and path-censoring. In this respect, the following observation is elementary but crucial. Let

$$\tau_0^b = \inf\{t\ge 0 : X_t \in (0,b)\}$$

be the first time at which X enters the interval $(0,b)$, where $b<1$, and

$$S_a^- = \inf\{s\ge 0 : \xi_s < a\}$$

the first passage of ξ below the negative level a. Notice that, if $e^a=b$, then

$$S_a^- < \infty, \text{ and } \xi_{S_a^-} \le x \iff \tau_0^b<\infty, \text{ and } X_{\tau_0^b} \le e^x.$$

We will use this relationship several times.

Our first task is to prove Theorem 4.1. We split the proof into two parts, based on the value of α. In principle, the method which we use for $\alpha\in(0,1]$ extends to the $\alpha\in(1,2)$ regime; however, it requires the evaluation of an integral including the descending renewal measure. For $\alpha\in(1,2)$ we have been unable to calculate this in closed form, and have instead used a method based on the Laplace transform. Conversely, the second method could be applied in the case $\alpha \in (0,1]$; however, it is less transparent.

Proof of Theorem 4.1, $\alpha \in (0,1]$**.** We begin by finding a related law for ξ. By [6, Proposition III.2], for $a < 0$,

$$\begin{aligned}\mathbb{P}_0(\xi_{S_a^-} \in \mathrm{d}w) &= \mathbb{P}_0(-\hat{H}_{S_a^+} \in \mathrm{d}w) \\ &= \int_{[0,-a]} \hat{V}(\mathrm{d}z)\pi_{\hat{H}}(-w-z)\,\mathrm{d}w.\end{aligned}$$

Using the expressions obtained in section 4.4 and changing variables,

$$\begin{aligned}&\mathbb{P}_0\big(\xi_{S_a^-} \in \mathrm{d}w\big) \\ &= \frac{\alpha\hat{\rho}e^{-\alpha w}\,\mathrm{d}w}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}\int_0^{1-e^a} t^{\alpha\hat{\rho}-1}(e^{-w}-1-e^{-w}t)^{-\alpha\hat{\rho}-1}\,\mathrm{d}t \\ &= \frac{\alpha\hat{\rho}\,\mathrm{d}w}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}e^{-\alpha\rho w}(e^{-w}-1)^{-1}\int_0^{\frac{1-e^a}{1-e^w}} s^{\alpha\hat{\rho}-1}(1-s)^{-\alpha\hat{\rho}-1}\,\mathrm{d}s \\ &= \frac{1}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}(1-e^a)^{\alpha\hat{\rho}}e^{(1-\alpha\rho)w}(1-e^w)^{-1}(e^a-e^w)^{-\alpha\hat{\rho}}\,\mathrm{d}w, \quad (4.12)\end{aligned}$$

where the last equality can be reached by [38, formula 8.391] and the formula ${}_2F_1(a,b;a;z) = (1-z)^{-b}$.

Denoting by $f(a,w)$ the density on the right-hand side of (4.12), the relationship between $\xi_{S_a^-}$ and $X_{\tau_0^b}$ yields that

$$g(b,z) := \mathrm{P}_1(X_{\tau_0^b} \in \mathrm{d}z)/\mathrm{d}z = z^{-1}f(\log b, \log z), \qquad b<1,\ z \in (0,b).$$

Finally, using the scaling property we obtain

$$\begin{aligned}\frac{\mathrm{P}_x\big(X_{\tau_{-1}^1} \in \mathrm{d}y\big)}{\mathrm{d}y} &= \frac{1}{x+1}g\left(\frac{2}{x+1}, \frac{y+1}{x+1}\right) \\ &= \frac{1}{y+1}f\left(\log\left(\frac{2}{x+1}\right), \log\left(\frac{y+1}{x+1}\right)\right) \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}}(x-y)^{-1},\end{aligned}$$

for $y \in (-1,1)$. □

Proof of Theorem 4.1, $\alpha \in (1,2)$**.** We begin with the so-called "second factorisation identity" [51, Exercise 6.7] for the process ξ, adapted to passage below a level:

$$\int_0^\infty \int \exp(qa - \beta y)\, \mathbb{P}_0(a - \xi_{S_a^-} \in \mathrm{d}y)\, \mathrm{d}a = \frac{\hat{\kappa}(q) - \hat{\kappa}(\beta)}{(q-\beta)\hat{\kappa}(q)},$$

where $a < 0$ and $q, \beta > 0$. A lengthy calculation, which we omit, inverts the two Laplace transforms to give the overshoot distribution for ξ,

$$\begin{aligned} f(a,w) &:= \frac{\mathbb{P}_0(a - \xi_{S_a^-} \in \mathrm{d}w)}{\mathrm{d}w} \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi} e^{-(1-\alpha\rho)w}(1-e^{-w})^{-\alpha\hat{\rho}} \\ &\quad \times \Bigg[e^{(1-\alpha)a}(1-e^a)^{\alpha\hat{\rho}} e^{-w}(e^{-a} - e^{-w})^{-1} \\ &\qquad - (\alpha\rho - 1)\int_0^{1-e^a} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\, \mathrm{d}t \Bigg], \end{aligned}$$

for $a < 0, w > 0$. Essentially the same argument as in the $\alpha \in (0,1]$ case gives the required hitting distribution for X,

$$\begin{aligned} \frac{\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y)}{\mathrm{d}y} &= \frac{1}{y+1} f\left(\log\left(\frac{2}{x+1}\right), \log\left(\frac{2}{y+1}\right)\right) \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat{\rho}} \\ &\quad \times \Bigg[(y+1)(x-1)^{\alpha\hat{\rho}}(x+1)^{\alpha\rho-1}(x-y)^{-1} \\ &\qquad - (\alpha\rho-1)2^{\alpha-1}\int_0^{\frac{x-1}{x+1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\, \mathrm{d}t \Bigg], \end{aligned} \tag{4.13}$$

for $x > 1$, $y \in (-1,1)$.

By the substitution $t = \frac{s-1}{s+1}$,

$$\begin{aligned} &2^{\alpha-1}\int_0^{\frac{x-1}{x+1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\, \mathrm{d}t \\ &= 2\int_1^x (s-1)^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-2}\, \mathrm{d}s \\ &= \int_1^x (s-1)^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1}\, \mathrm{d}s - \int_1^x (s-1)^{\alpha\hat{\rho}}(s+1)^{\alpha\rho-2}\, \mathrm{d}s. \end{aligned}$$

Now evaluating the second term on the right hand side above via integration by parts and substituting back into (4.13) yields the required law. □

Remark 4.21. It is worth noting that for the meromorphic class of Lévy processes of Kuznetsov et al. [48], which we discussed in section 2.6.4, the law of the position of first entry into an interval was computed as a convergent series of exponential densities by solving a pair of simultaneous non-linear equations; see Rogozin [72] for the original use of this method, in the context of first passage of stable processes when exiting a finite interval. In principle the method of solving two simultaneous non-linear equations (that is, writing the law of first entry in $(-1,1)$ from $x > 1$ in terms of the law of first entry in $(-1,1)$ from $x < -1$ and vice versa) may provide a way of proving Theorem 4.1. However it is unlikely that this would present a more convenient approach because of the complexity of the two non-linear equations involved and because of the issue of proving uniqueness of their solution.

Proof of Corollary 4.2. This will follow by integrating out Theorem 4.1. First making the substitutions $z = (y+1)/2$ and $w = \frac{1-z}{1-2z/(x+1)}$, we obtain

$$\begin{aligned}
&\mathrm{P}_x(\tau_{-1}^1 < \infty) \\
&\quad = \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}} \int_{-1}^{1} (1+u)^{-\alpha\rho}(1-u)^{-\alpha\hat{\rho}}(x-u)^{-1}\,\mathrm{d}u \\
&\quad = \frac{\sin(\pi\alpha\hat{\rho})}{\pi}(x+1)^{\alpha\rho}(x-1)^{\alpha\hat{\rho}}2^{1-\alpha} \int_{0}^{1} z^{-\alpha\rho}(1-z)^{-\alpha\hat{\rho}}\left(1-\frac{2}{x+1}z\right)^{-1}\mathrm{d}z \\
&\quad = \frac{\sin(\pi\alpha\hat{\rho})}{\pi}\left(\frac{2}{x+1}\right)^{1-\alpha} \int_{0}^{1} w^{-\alpha\hat{\rho}}(1-w)^{-\alpha\rho}\left(1-\frac{2}{x+1}w\right)^{\alpha-1}\mathrm{d}w \\
&\quad = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)} \int_{0}^{\frac{2}{x+1}} s^{-\alpha}(1-s)^{\alpha\hat{\rho}-1}\,\mathrm{d}s,
\end{aligned}$$

where the last line follows by [38, formulas 3.197.3, 8.391]. Finally, substituting $t = 1-s$, it follows that

$$\mathrm{P}_x(\tau_{-1}^1 = \infty) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)} \int_{0}^{\frac{x-1}{x+1}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,\mathrm{d}t,$$

and this was our aim. □

Proof of Proposition 4.3. In Port [67, §3, Remark 3], the author establishes, for $s > 0$, the hitting distribution of $[0,s]$ for a spectrally positive α-stable process started at $x < 0$. In our situation, we have a spectrally negative stable process X, and so the dual process $\hat{X}$ is spectrally positive, and we obtain:

$$\begin{aligned}
\mathrm{P}_x(X_{\tau_{-1}^1} \in \mathrm{d}y) &= \hat{\mathrm{P}}_{1-x}(\hat{X}_{\tau_0^2} \in 1-\mathrm{d}y) \\
&= f_{1-x}(1-y)\,\mathrm{d}y + \gamma(1-x)\,\delta_{-1}(\mathrm{d}y),
\end{aligned}$$

using the notation from [67] in the final line. Port gives expressions for f_{1-x} and γ which differ somewhat from the density and atom seen in our Proposition 4.3;

we have

$$f_{1-x}(1-y) = \frac{\sin(\pi(\alpha-1))}{\pi}(x-1)^{\alpha-1}(1-y)^{1-\alpha}(x-y)^{-1}\mathbb{1}_{(-1,1)}(y),$$

which is obtained from Port's by evaluating an integral, and our expression

$$\gamma(1-x) = \frac{\sin\pi(\alpha-1)}{\pi}\int_0^{\frac{x-1}{x+1}} t^{\alpha-2}(1-t)^{1-\alpha}\,\mathrm{d}t$$

is obtained by computing $\gamma(1-x) = 1 - \int_{-1}^1 f_{1-x}(1-y)\,\mathrm{d}y$.

We now prove weak convergence. For this purpose, the identity (4.13) is more convenient than the final expression in Theorem 4.1. Let us denote the right-hand side of (4.13), treated as the density of a measure on $[-1,1]$, by the function $g_\rho\colon [-1,1]\to\mathbb{R}$, so that

$$\begin{aligned} g_\rho(y) = {} & \frac{\sin(\pi\alpha\hat\rho)}{\pi}(x-1)^{\alpha\hat\rho}(x+1)^{\alpha\rho-1}(1+y)^{1-\alpha\rho}(1-y)^{-\alpha\hat\rho} \\ & + (1-\alpha\rho)\frac{\sin(\pi\alpha\hat\rho)}{\pi}2^{\alpha-1}(1+y)^{-\alpha\rho}(1-y)^{-\alpha\hat\rho}\int_0^{\frac{x-1}{x+1}} t^{\alpha\hat\rho-1}(1-t)^{1-\alpha}\,\mathrm{d}t, \end{aligned}$$

for $y\in(-1,1)$, and we set $g_\rho(-1)=g_\rho(1)=0$ for definiteness.

As we take the limit $\rho\to 1/\alpha$, $g_\rho(y)$ converges pointwise to $f_{1-x}(1-y)$. Furthermore, the functions g_ρ are dominated by a function $h\colon[-1,1]\to\mathbb{R}$ of the form

$$h(y) = C(1-y)^{1-\alpha}(x-y)^{-1} + D(1+y)^{-1}(1-y)^{1-\alpha}, \qquad y\in(-1,1)$$

for some $C,D\ge 0$ depending only on x and α; again we set $h(-1)=h(1)=0$.

Let $z>-1$. The function h is integrable on $[z,1]$, and therefore dominated convergence yields

$$\int_{[z,1]} g_\rho(y)\,\mathrm{d}y \to \int_{[z,1]} f_{1-x}(1-y)\,\mathrm{d}y = \mathrm{P}_x(X_{\tau_{-1}^1}\ge z),$$

while

$$\int_{[-1,1]} g_\rho(y)\,\mathrm{d}y = 1 = \mathrm{P}_x(X_{\tau_{-1}^1}\ge -1),$$

and this is sufficient for weak convergence. □

We now turn to the problem of first passage upward before hitting a point. To tackle this problem, we will use the *stable process conditioned to stay positive*, which was defined in section 2.5. If X is the stable process, we denote the stable process conditioned to stay positive by $X^\uparrow$; it is the Doob h-transform of the process killed upon exiting $[0,\infty)$ with respect to the invariant function $h^\uparrow(x) = x^{\alpha\hat\rho}$.

In fact we will make use of this construction for the dual process $\hat X$, with invariant function $\hat h^\uparrow(x) = x^{\alpha\rho}$, and accordingly we will denote the conditioned process by $\hat X^\uparrow$ and use $(\hat{\mathrm{P}}^\uparrow_x)_{x>0}$ for its probability laws. It is known that the process $\hat X^\uparrow$ is an α-pssMp which drifts to $+\infty$.

We have already discussed the Lamperti–Lévy process $\xi^\uparrow$ in section 2.5; analogously, denote the Lévy process associated to $\hat X^\uparrow$ by $\hat\xi^\uparrow$ with probability laws $(\hat{\mathbb{P}}^\uparrow_y)_{y>0}$. The crucial observation here is that $\hat X^\uparrow$ hits the point 1 if and only if its Lamperti transform, $\hat\xi^\uparrow$, hits the point 0.

We now have all the apparatus in place to begin the proof.

Proof of Theorem 4.4. For each $y \in \mathbb{R}$, let T_y be the first hitting time of the point y, and let τ_y^+ and τ_y^- be the first hitting times of the sets (y,∞) and $(-\infty,y)$, respectively. When $\alpha \in (1,2)$, these are all a.s. finite stopping times for the stable process X and its dual $\hat X$. Then, when $x \in (-\infty,1)$,

$$\begin{aligned}\mathrm{P}_x(T_0 < \tau_1^+) = \mathrm{P}_{x-1}(T_{-1} < \tau_0^+) = \hat{\mathrm{P}}_{1-x}(T_1 < \tau_0^-) \\ = \hat h^\uparrow(1-x)\hat{\mathrm{E}}_{1-x}\left[\mathbb{1}_{\{T_1<\infty\}}\frac{\hat h^\uparrow(\hat X_{T_1})}{\hat h^\uparrow(1-x)}, \, T_1 < \tau_0^-\right] \\ = (1-x)^{\alpha\rho}\hat{\mathrm{P}}^\uparrow_{1-x}(T_1 < \infty),\end{aligned} \tag{4.14}$$

where we have used the definition of $\hat{\mathrm{P}}^\uparrow_\cdot$ at T_1. (Note that, to unify notation, the various stopping times refer to the canonical process for each measure.)

We now use several facts from potential theory which may be found in Bertoin [6, Proposition II.18 and Theorem II.19]. Provided that the potential measure $U = \hat{\mathbb{E}}^\uparrow_0 \int_0^\infty \mathbb{1}_{\{\hat\xi^\uparrow \in \cdot\}}\,\mathrm{d}t$ is absolutely continuous and there is a bounded continuous version of its density, say u, then the following holds:

$$\hat{\mathrm{P}}^\uparrow_{1-x}(T_1 < \infty) = \hat{\mathbb{P}}^\uparrow_{\log(1-x)}(T_0 < \infty) = Cu\big(-\log(1-x)\big), \tag{4.15}$$

where C is the capacity of the set $\{0\}$ for the process $\hat\xi^\uparrow$.

Therefore, we have reduced our problem to that of finding a bounded, continuous version of the potential density of $\hat\xi^\uparrow$ under $\hat{\mathbb{P}}^\uparrow_0$. Provided the renewal measures of $\hat\xi^\uparrow$ are absolutely continuous, it is readily deduced from an identity of Silverstein (see Bertoin [6, Theorem VI.20] or Silverstein [77, Theorem 6]) that a potential density u exists and is given by

$$u(y) = \begin{cases} k\int_0^\infty v(y+z)\hat v(z)\,\mathrm{d}z, & y>0, \\ k\int_{-y}^\infty v(y+z)\hat v(z)\,\mathrm{d}z, & y<0, \end{cases}$$

where v and $\hat v$ are the ascending and descending renewal densities of the process

$\hat{\xi}^\uparrow$, and k is the constant in the Wiener-Hopf factorisation (2.8) of $\hat{\xi}^\uparrow$.

The work of Kyprianou et al. [57] gives the Wiener–Hopf factorisation of $\hat{\xi}^\uparrow$, shows that the renewal measures are absolutely continuous and computes their densities, albeit for a different normalisation of the stable process X. In our normalisation, the renewal densities are given by

$$v(z) = \frac{1}{\Gamma(\alpha\hat{\rho})}(1-e^{-z})^{\alpha\hat{\rho}-1}, \qquad \hat{v}(z) = \frac{1}{\Gamma(\alpha\rho)}e^{-z}(1-e^{-z})^{\alpha\rho-1},$$

and $k = 1$; see Corollary 2.13. It then follows that

$$u(y) = \begin{cases} \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(1-e^{-y})^{\alpha-1}e^{\alpha\rho y}\int_0^{e^{-y}} t^{\alpha\rho-1}(1-t)^{-\alpha}\,\mathrm{d}t, & y > 0, \\ \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(1-e^{y})^{\alpha-1}e^{(1-\alpha\hat{\rho})y}\int_0^{e^{y}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,\mathrm{d}t, & y < 0. \end{cases}$$

This u is the bounded continuous density which we seek, so by substituting into (4.15) and (4.14), we arrive at the hitting probability

$$\mathrm{P}_x(T_0 < \tau_1^+) = \begin{cases} C' x^{\alpha-1}\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{-\alpha}\,\mathrm{d}t, & 0 < x < 1, \\ C'(-x)^{\alpha-1}\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\,\mathrm{d}t, & x < 0, \end{cases} \tag{4.16}$$

where $C' = \frac{C}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}$. It only remains to determine the unknown constant here, which we will do by taking the limit $x \uparrow 0$ in (4.16). First we manipulate the second expression above, by recognising that $1 = t + (1-t)$ and integrating by parts. For $x < 0$,

$$\begin{aligned} &\mathrm{P}_x(T_0 < \tau_1^+) \\ &\quad = C'(-x)^{\alpha-1}\left[\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}}(1-t)^{-\alpha}\,\mathrm{d}t + \int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\,\mathrm{d}t\right] \\ &\quad = \frac{C'}{\alpha-1}\left[(1-x)^{\alpha\rho-1} - (1-\alpha\rho)(-x)^{\alpha-1}\int_0^{(1-x)^{-1}} t^{\alpha\hat{\rho}-1}(1-t)^{1-\alpha}\,\mathrm{d}t\right]. \end{aligned}$$

Now taking $x \uparrow 0$, we find that $C' = \alpha - 1$.

Finally, we obtain the expression required by performing the integral substitution $s = 1/(1-t)$ in (4.16). □

Remark 4.22. We have only proved Theorem 4.4 for a stable process with parameters in $\mathcal{A}_{\mathrm{st}}$. If, instead, X is a totally asymmetric stable process, it either creeps upward, which is to say that each positive level is passed continuously, or it creeps downward. In either case it follows that the event $\{T_0 < \tau_1^+\}$ is equal to the event $\{\tau_0^- < \tau_1^+\}$. Hence, finding the equivalent to Theorem 4.4 is the same as solving the two-sided exit problem, and this was done by Rogozin [73].

Proof of Proposition 4.6. As we saw in section 3.3, the Lamperti representation gives the following relation between $\mathcal{I}$ and the process ξ:

$$I(\alpha\xi) := \int_0^\infty e^{\alpha\xi_s}\,\mathrm{d}s = \inf\{u \geq 0 : Y_u = 0\} = \int_0^{T_0} \mathbb{1}_{\{X_t > 0\}}\,\mathrm{d}t = \mathcal{I}.$$

The first expression for $\mathcal{M}$ follows immediately from Proposition 3.4, since $-\xi$ is an extended hypergeometric Lévy process. The expressions in terms of k and l arise from repeated application of the identities (2.17) and (2.18) for the double gamma function. □

Remark 4.23. We again remark on the situation where X is totally asymmetric. In this case, the random variable $\mathcal{I}$ is equal in law to the first passage time τ_0^-. The distribution of this random variable may be characterised using the theory of scale functions; see [50, Theorems 2.6 and 3.3] and [51, Exercise 8.2].

Chapter 5

Potentials of killed processes

For a Lévy process X, the measure

$$U^A(x, \mathrm{d}y) = \mathrm{E}_x \int_0^\infty \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \mathbb{1}_{\{\forall s \le t : X_s \in A\}} \, \mathrm{d}t,$$

called the *potential* (or *resolvent*) *measure of* X *killed outside* A, is a quantity of interest related to exit problems.

The main cases where the potential measure can be computed explicitly are as follows. If X is a Lévy process with known Wiener–Hopf factors, it can be obtained when A is half-line or $\mathbb{R}$; see [6, Theorem VI.20]. When X is a totally asymmetric Lévy process with known scale functions, it can be obtained for A a bounded interval, a half-line or $\mathbb{R}$; see [51, Section 8.4]. Finally, [4] details a technique to obtain a potential measure for a reflected Lévy process killed outside a bounded interval from the same quantity for the unreflected process.

In this chapter, we take X to be a stable process, and consider two cases. In section 5.1, we derive U^A when $A = (-\infty, -1) \cup (1, \infty)$, from which one may, via spatial homogeneity and scaling, compute the potential for similar sets; while in section 5.2, we derive U^A and several related quantities when A is a bounded interval.

The proofs in the remainder of this chapter rely upon a simple observation, which we now explain. Suppose that Y is a pssMp with associated laws $(\mathrm{P}_x)_{x>0}$, and Lamperti representation ξ with laws $(\mathbb{P}_y)_{y \in \mathbb{R}}$. Write S and T for the time-changes appearing in the Lamperti transform; that is, S is as in (2.4) and T is its inverse. Then it is simple to see that $e^{\alpha \xi_{S(t)}} \, \mathrm{d}S(t) = \mathrm{d}t$. Now if τ is a random time (it will typically be the hitting time of a set), we may calculate

$$\begin{aligned} \mathrm{E}_x \int_0^\tau \mathbb{1}_{\{Y_t \in \mathrm{d}y\}} \, \mathrm{d}t &= \mathbb{E}_{\log x} \int_0^\tau \mathbb{1}_{\{\exp(\xi_{S(t)}) \in \mathrm{d}y\}} e^{\alpha \xi_{S(t)}} \, \mathrm{d}S(t) \\ &= y^\alpha \mathbb{E}_{\log x} \int_0^{S(\tau)} \mathbb{1}_{\{\exp(\xi_s) \in \mathrm{d}y\}} \, \mathrm{d}s. \end{aligned}$$

This method allows us to compute certain killed potentials of self-similar processes in terms of killed potentials for Lévy processes; and, as we have seen in the proof of Theorem 4.4, there are already several tools available for the latter task.

5.1 The stable process killed on entering a bounded interval

Making use of the path-censored stable process, we derive the following killed potential.

Theorem 5.1. *Let $\alpha \in (0,1]$, $x > 1$ and $y > 1$. Then,*

$$\begin{aligned} &\mathrm{E}_x \int_0^{\tau_{-1}^{1}} \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t / \mathrm{d}y \\ &= \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\dfrac{x-y}{2}\right)^{\alpha-1} \displaystyle\int_1^{\frac{1-xy}{y-x}} (t-1)^{\alpha\rho-1}(t+1)^{\alpha\hat{\rho}-1} \, \mathrm{d}t, & 1 < y < x, \\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left(\dfrac{y-x}{2}\right)^{\alpha-1} \displaystyle\int_1^{\frac{1-xy}{x-y}} (t-1)^{\alpha\hat{\rho}-1}(t+1)^{\alpha\rho-1} \, \mathrm{d}t, & y > x. \end{cases} \end{aligned}$$

To obtain this potential for $x < -1$ and $y < -1$, one may easily appeal to duality. In the case that $x < -1$ and $y > 1$, one notes that

$$\mathrm{E}_x \int_0^{\tau_{-1}^{1}} \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t = \mathrm{E}_x \mathrm{E}_\Delta \int_0^{\tau_{-1}^{1}} \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t, \tag{5.1}$$

where the quantity Δ is randomised according to the distribution of the random variable $X_{\tau_{-1}^+} \mathbb{1}_{\{X_{\tau_{-1}^+} > 1\}}$, with

$$\tau_{-1}^+ = \inf\{t \geq 0 : X_t > -1\}.$$

Although the distribution of $X_{\tau_{-1}^+}$ is available from [73], and hence the right hand side of (5.1) can be written down explicitly, it does not seem to be easy to find a convenient closed form expression for the corresponding potential density.

Regarding this potential, let us finally remark that the same method by which we prove Theorem 5.1 may also be applied even when $\alpha \in (1,2)$ in order to give an explicit result in terms of multiple integrals; however, in this case it does not seem possible to obtain any compact expression for the density.

Remark 5.2. Consider the case where X only jumps in one direction. Since $\alpha \leq 1$, X is either a subordinator or the negative of a subordinator, and the potential measure in question is equal to a suitable restriction of the unkilled potential measure of X. In the case where $\alpha > 1$, we could not give a concise result for the potential measure in the generality of Theorem 5.1. If, however, X is totally asymmetric, the theory of scale functions may be applied; the measure of Theorem 5.1 is equal to $\mathrm{E}_x \int_0^{\tau_1^-} \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t$, and a formula for this may be found in [50, Theorem 2.7].

Proof (of Theorem 5.1). We begin by determining a killed potential for ξ. Let

$$u(p,w)\,\mathrm{d}w = \mathbb{E}_p \int_0^{S_0^-} \mathbb{1}_{\{\xi_s \in \mathrm{d}w\}} \, \mathrm{d}s, \qquad p,\, w > 0,$$

if this density exists. Applying Proposition 2.14 to the hypergeometric Lévy process ξ, we immediately obtain

$$u(p,w) = \begin{cases} \dfrac{(e^{p-w}-1)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{\frac{1-e^{-w}}{1-e^{-p}}} t^{\alpha\rho-1}(1-t)^{-\alpha}\mathrm{d}t, & 0 < w < p, \\ \dfrac{(1-e^{p-w})^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{\frac{1-e^{-p}}{1-e^{-w}}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\mathrm{d}t, & w > p. \end{cases}$$

We can now start to calculate the killed potential for X. Let

$$\bar{u}(b,z)\,\mathrm{d}z = \mathrm{E}_1 \int_0^{\tau_0^b} \mathbb{1}_{\{X_t \in \mathrm{d}z\}} \, \mathrm{d}t, \qquad 0 < b < 1,\, z > b.$$

Recall now that $\mathrm{d}A_t = \mathbb{1}_{\{X_t > 0\}} \, \mathrm{d}t$, and γ denotes the right-inverse of A; the path-censored stable process Y satisfies $Y_t = X_{\gamma(t)} \mathbb{1}_{\{t < \tau_0\}}$ for $t \geq 0$. Bearing in mind the discussion at the beginning of this chapter, we make the following calculation.

$$\begin{aligned} \bar{u}(b,z)\,\mathrm{d}z &= \mathrm{E}_1 \int_0^{\tau_0^b(X)} \mathbb{1}_{\{X_t \in \mathrm{d}z\}} \, \mathrm{d}A_t \\ &= \mathrm{E}_1 \int_0^{\tau_0^b(Y)} \mathbb{1}_{\{Y_t \in \mathrm{d}z\}} \, \mathrm{d}t \\ &= \mathbb{E}_0 \int_0^{T(S_a^-)} \mathbb{1}_{\{\exp(\xi_{S(t)}) \in \mathrm{d}z\}} \exp(\alpha \xi_{S(t)}) \, \mathrm{d}S(t) \\ &= z^\alpha \mathbb{E}_0 \int_0^{S_a^-} \mathbb{1}_{\{\exp(\xi_s) \in \mathrm{d}z\}} \, \mathrm{d}s \\ &= z^\alpha \mathbb{E}_{-a} \int_0^{S_0^-} \mathbb{1}_{\{\exp(\xi_s + a) \in \mathrm{d}z\}} \, \mathrm{d}s, \end{aligned}$$

where $a = \log b$, and, for clarity, we have written $\tau_0^b(Z)$ for the hitting time of

$(0, b)$ calculated for a process Z. Hence,

$$\bar{u}(b, z) = z^{\alpha-1} u(\log b^{-1}, \log b^{-1} z), \qquad 0 < b < 1,\ z > b.$$

Finally, a scaling argument yields the following. For $x \in (0, 1)$ and $y > 1$,

$$\begin{aligned}
\mathrm{E}_x \int_0^{\tau_{-1}^1} \mathbb{1}_{\{X_t \in \mathrm{d}y\}} \, \mathrm{d}t/\mathrm{d}y
&= (x+1)^{\alpha-1} \bar{u}\left(\frac{2}{x+1}, \frac{y+1}{x+1} \right) \\
&= (y+1)^{\alpha-1} u\left(\log \frac{x+1}{2}, \log \frac{y+1}{2} \right) \\
&= \begin{cases} \dfrac{(x-y)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{\frac{y-1}{y+1}\frac{x+1}{x-1}} t^{\alpha\rho-1}(1-t)^{-\alpha} \, \mathrm{d}t, & 1 < y < x, \\ \dfrac{(y-x)^{\alpha-1}}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \displaystyle\int_0^{\frac{y+1}{y-1}\frac{x-1}{x+1}} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha} \, \mathrm{d}t, & y > x. \end{cases}
\end{aligned}$$

The integral substitution $t = \frac{s-1}{s+1}$ gives the form in the theorem. □

5.2 The stable process and the reflected stable process killed on exiting a bounded interval

In this section, we consider the case where X is a stable process and A is a bounded interval. We compute the measure $U^{[0,1]}$, from which U^A may be obtained for any bounded interval A; and from this we compute the joint law at first exit of $[0, 1]$ of the overshoot, undershoot and undershoot from the maximum. Furthermore, we give the potential measure and triple law also for the process reflected in its infimum.

The potential measure has been already been computed when X is symmetric; see Blumenthal et al. [11, Corollary 4] and references therein, as well as Baurdoux [4]. We extend these results to asymmetric stable processes by rewriting the potential measure of interest via the Lamperti representation ξ^* of the killed stable process X^*. It is then enough to know the killing rate of ξ^* and the solution of certain exit problems for X. We then use [4] to compute potentials for the reflected process.

We end this section by remarking on the relationship between the potential measure for the stable process and that of the stable process conditioned to stay positive or hit zero continuously.

We will now give our results. Let

$$\sigma^{[0,1]} = \inf\{t \geq 0 : X_t \notin [0,1]\},$$

and define the killed potential measure and its density

$$U_1(x, \mathrm{d}y) := U^{[0,1]}(x, \mathrm{d}y) = \mathrm{E}_x \int_0^{\sigma^{[0,1]}} \mathbb{1}\big[X_t \in \mathrm{d}y\big] \, \mathrm{d}t = u_1(x,y)\, \mathrm{d}y,$$

provided the density u_1 exists.

Theorem 5.3. *For $0 < x, y < 1$,*

$$u_1(x,y) = \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(x-y)^{\alpha-1} \displaystyle\int_0^{\frac{y(1-x)}{x-y}} s^{\alpha\rho-1}(s+1)^{\alpha\hat{\rho}-1}\, \mathrm{d}s, & y < x, \\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}(y-x)^{\alpha-1} \displaystyle\int_0^{\frac{x(1-y)}{y-x}} s^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1}\, \mathrm{d}s, & x < y. \end{cases}$$

When X is symmetric, this reduces, by spatial homogeneity and scaling of X, and substituting in the integral, to [11, Corollary 4].

Proof (of Theorem 5.3). To avoid the proliferation of symbols, here and elsewhere in the section we shall only distinguish processes typographically by the measures associated with them; the exception is that self-similar processes will be distinguished from processes obtained by Lamperti transform. Thus, the time

$$\tau_1^+ = \inf\{t \geq 0 : X_t > 0\}$$

always refers to the canonical process of the measure it appears under, and will be used for self-similar processes; and

$$S_0^+ = \inf\{s \geq 0 : \xi_s > 0\}, \quad \text{and} \quad S_0^- = \inf\{s \geq 0 : \xi_s < 0\}$$

will likewise be used for processes obtained by Lamperti transform.

Our proof makes use of the pssMp (X, P^*) and its Lamperti transform $(\xi, \mathbb{P}^*)$, which were discussed in section 2.5; in particular, recall that $(\xi, \mathbb{P}^*)$ is killed at rate

$$q = c_-/\alpha = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}. \tag{5.2}$$

Let $0 < x,\ y < 1$. Then

$$\begin{aligned} U_1(x, \mathrm{d}y) &= \mathrm{E}_x \int_0^{\sigma^{[0,1]}} \mathbb{1}[X_t \in \mathrm{d}y]\, \mathrm{d}t \\ &= \mathrm{E}^*_x \int_0^{\tau_1^+} \mathbb{1}[X_t \in \mathrm{d}y]\, \mathrm{d}t, \end{aligned}$$

using nothing more than the definition of (X, P^*). We now use the Lamperti representation to relate this to $(\xi, \mathbb{P}^*)$. This process is killed at the rate q given in (5.2), and so it may be represented as an unkilled Lévy process $(\xi, \mathbb{P})$ which is sent to some cemetery state at the independent exponental time $\mathbf{e}_q$. We now make the following calculation, as we remarked at the start of the chapter.

$$\begin{aligned} U_1(x, \mathrm{d}y) &= \mathbb{E}^*_{\log(x)} \int_0^{T(S_0^+)} \mathbb{1}\big[e^{\xi_{S(t)}} \in \mathrm{d}y\big] e^{\alpha \xi_{S(t)}}\, \mathrm{d}S(t) \\ &= y^\alpha \mathbb{E}_{\log(x)} \int_0^{S_0^+} \mathbb{1}\big[e^{\xi_s} \in \mathrm{d}y\big] \mathbb{1}[\mathbf{e}_q > s]\, \mathrm{d}s \\ &= y^\alpha \hat{\mathbb{E}}_{\log(1/x)} \int_0^{S_0^-} \mathbb{1}\big[\xi_s \in \log(1/\mathrm{d}y)\big] e^{-qs}\, \mathrm{d}s, \end{aligned}$$

where $\hat{\mathbb{E}}$ refers to the Lévy process dual to ξ. Examining the proof of Theorem VI.20 in Bertoin [6] reveals that, for any $a > 0$,

$$\begin{aligned} &\hat{\mathbb{E}}_a \int_0^{S_0^-} \mathbb{1}[\xi_s \in \cdot\,]\, e^{-qs}\, \mathrm{d}s \\ &\qquad = \frac{1}{q} \int_{[0,\infty)} \hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \mathrm{d}w) \int_{[0,a]} \hat{\mathbb{P}}_0(-\underline{\xi}_{\mathbf{e}_q} \in \mathrm{d}z) \mathbb{1}[a + w - z \in \cdot\,], \end{aligned}$$

where for each $t \geq 0$, $\overline{\xi}_t = \sup\{\xi_s : s \leq t\}$ and $\underline{\xi}_t = \inf\{\xi_s : s \leq t\}$. Then, provided that the measures $\hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot)$ and $\hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot)$ possess respective densities g_S and g_I (as we will shortly see they do), it follows that for $a > 0$,

$$\hat{\mathbb{E}}_a \int_0^{S_0^-} \mathbb{1}[\xi_s \in \mathrm{d}v]\, e^{-qs}\, \mathrm{d}s = \frac{\mathrm{d}v}{q} \int_{(a-v)\vee 0}^{a} \mathrm{d}z\, g_I(-z) g_S(v - a + z).$$

We may apply this result to our potential measure U_1 in order to find its density, giving

$$u_1(x, y) = \frac{1}{q} y^{\alpha - 1} \int_{\frac{y}{x}\vee 1}^{\frac{1}{x}} t^{-1} g_I(\log t^{-1}) g_S(\log(tx/y))\, \mathrm{d}t. \tag{5.3}$$

It remains to determine the densities g_S and g_I of the measures $\hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot)$ and $\hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot)$. These can be related to functionals of X by the Lamperti transform:

$$\begin{aligned} \hat{\mathbb{P}}_0(\overline{\xi}_{\mathbf{e}_q} \in \cdot) &= \mathbb{P}_0(-\underline{\xi}_{\mathbf{e}_q} \in \cdot) = \mathrm{P}_1(-\log \underline{X}_{\tau_0^- -} \in \cdot) \\ \hat{\mathbb{P}}_0(\underline{\xi}_{\mathbf{e}_q} \in \cdot) &= \mathbb{P}_0(-\overline{\xi}_{\mathbf{e}_q} \in \cdot) = \mathrm{P}_1(-\log \bar{X}_{\tau_0^-} \in \cdot), \end{aligned} \tag{5.4}$$

where $\underline{X}$ and $\bar{X}$ are defined in the obvious manner.

The laws of the rightmost random variables in (5.4) are available explicitly, as we now show. For the law of $\underline{X}_{\tau_0^- -}$, we transform it into an overshoot problem and make use of Example 7 in Doney and Kyprianou [32], as follows. We omit the calculation of the integral, which uses [38, 8.380.1].

$$\begin{aligned} \mathrm{P}_1(\underline{X}_{\tau_0^- -} \in \mathrm{d}y) &= \hat{\mathrm{P}}_0(1 - \bar{X}_{\tau_1^+ -} \in \mathrm{d}y) \\ &= K \int_y^\infty \mathrm{d}v \int_0^\infty \mathrm{d}u\, (v-y)^{\alpha\rho - 1}(v+u)^{-(\alpha+1)}(1-y)^{\alpha\hat{\rho}-1}\,\mathrm{d}y \\ &= \frac{\sin(\pi\alpha\hat{\rho})}{\pi} y^{-\alpha\hat{\rho}}(1-y)^{\alpha\hat{\rho}-1}\,\mathrm{d}y. \end{aligned} \tag{5.5}$$

For the law of $\bar{X}_{\tau_0^-}$, consider the following calculation.

$$\mathrm{P}_1(\bar{X}_{\tau_0^-} \geq y) = \mathrm{P}_1(\tau_y^+ < \tau_0^-) = \mathrm{P}_{1/y}(\tau_1^+ < \tau_0^-).$$

This final quantity depends on the solution of the two-sided exit problem for the stable process; it is computed in Rogozin [73], where it is denoted $f_1(1/y, \infty)$. Note that [73] contains a typographical error: in Lemma 3 of that work and the discussion after it, the roles of q (which is ρ in our notation) and $1-q$ should be swapped. In the corrected form, we have

$$\begin{aligned} \mathrm{P}_1(\bar{X}_{\tau_0^-} \geq y) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^{1/y} u^{\alpha\hat{\rho}-1}(1-u)^{\alpha\rho-1}\,\mathrm{d}u \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_y^\infty t^{-\alpha}(t-1)^{\alpha\rho-1}\,\mathrm{d}t, \end{aligned} \tag{5.6}$$

which gives us the density.

Now we substitute (5.5) and (5.6) into (5.3):

$$u_1(x,y) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} x^{\alpha\hat{\rho}-1} y^{\alpha\rho} \int_{\frac{y}{x}\vee 1}^{\frac{1}{x}} t^{-\alpha}(t-1)^{\alpha\rho-1}\left(t - \frac{y}{x}\right)^{\alpha\hat{\rho}-1}\,\mathrm{d}t.$$

The expression in the statement follows by a short manipulation of this integral. □

Remark 5.4. As we have already noted, it is not difficult to find the potential in Theorem 5.3 when X has only one-sided jumps. If X is a subordinator or the negative of a subordinator, then the potential is simply a restriction of the unkilled potential measure of X; while if X is a totally symmetric stable process, one may apply the theory of scale functions, making use of, for example, [50, Theorem 2.7].

With very little extra work, Theorem 5.3 yields an apparently stronger result. Let

$$\tau_0^- = \inf\{t \geq 0 : X_t < 0\}; \qquad \bar{X}_t = \sup_{s \leq t} X_s, \qquad t \geq 0,$$

and write

$$\mathrm{E}_x \int_0^{\tau_0^-} \mathbb{1}[X_t \in \mathrm{d}y,\, \bar{X}_t \in \mathrm{d}z]\,\mathrm{d}t = u(x,y,z)\,\mathrm{d}y\,\mathrm{d}z,$$

if the right-hand side exists. Then we have the following.

Corollary 5.5. *For $x > 0$, $y \in [0, z)$, $z > x$,*

$$u(x,y,z) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} x^{\alpha\hat{\rho}} y^{\alpha\rho} \frac{(z-x)^{\alpha\rho-1}(z-y)^{\alpha\hat{\rho}-1}}{z^\alpha}\,\mathrm{d}y\,\mathrm{d}z. \tag{5.7}$$

Proof. We obtain

$$\mathrm{E}_x \int_0^{\tau_0^-} \mathbb{1}[X_t \in \mathrm{d}y,\, \bar{X}_t \leq z]\,\mathrm{d}t = z^{\alpha-1} u_1(x/z, y/z),$$

via rescaling the left-hand side, and the density is found by differentiating the right-hand side in z. □

From this density, one may recover the following hitting distribution, which originally appeared in Kyprianou et al. [57, Corollary 15]. Let

$$\tau_1^+ = \inf\{t \geq 0 : X_t > 1\}.$$

Corollary 5.6. *For $u \in [0, 1-x)$, $v \in (u, 1]$, $y \geq 0$,*

$$\begin{aligned} &\mathrm{P}_x(1 - \bar{X}_{\tau_1^+ -} \in \mathrm{d}u,\, 1 - X_{\tau_1^+ -} \in \mathrm{d}v,\, X_{\tau_1^+} - 1 \in \mathrm{d}y,\, \tau_1^+ < \tau_0^-) \\ &\quad = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \frac{x^{\alpha\hat{\rho}}(1-v)^{\alpha\rho}(1-u-x)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(1-u)^\alpha (v+y)^{\alpha+1}}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}y. \end{aligned} \tag{5.8}$$

Proof. Following the proof of [6, Proposition III.2], one may show that the left-hand side of (5.8) is equal to $u(x, 1-v, 1-u)\pi(v+y)$, where π is the Lévy density of X. □

Remark 5.7. The proof of Corollary 5.6 suggests an alternative derivation of Theorem 5.3. Since the identity (5.8) is already known, one may deduce $u(x, y, z)$ from it by following the proof backwards. The potential $u_1(x, y)$ without $\bar{X}$ may then be obtained via integration. However, we prefer to offer a self-contained proof based on well-known hitting distributions for the stable process.

Now let Z denote the stable process X reflected in its infimum, that is,

$$Z_t = X_t - \underline{X}_t, \qquad t \geq 0,$$

where $\underline{X}_t = \inf\{X_s, 0 \leq s \leq t\} \wedge 0$ for $t \geq 0$. Z is a self-similar Markov process.

Let $T_1^+ = \inf\{t > 0 : Z_t > 1\}$ denote the first passage time of Z above the level 1, and define

$$R_1(x, \mathrm{d}y) = \mathrm{E}_x \int_0^{T_1^+} \mathbb{1}[Z_t \in \mathrm{d}y] \, \mathrm{d}t = r_1(x, y) \, \mathrm{d}y.$$

We may then use the results of Baurdoux [4] to find r_1. Note that, as Z is self-similar, R_1 suffices to deduce the potential of Z killed at first passage above any level.

Theorem 5.8. *For* $0 < y < 1$,

$$r_1(0, y) = \frac{1}{\Gamma(\alpha)} y^{\alpha\rho - 1} (1 - y)^{\alpha\hat{\rho}}.$$

Hence, for $0 < x,\ y < 1$,

$$r_1(x,y) = \begin{cases} \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \Bigg[(x-y)^{\alpha-1} \displaystyle\int_0^{\frac{y(1-x)}{x-y}} s^{\alpha\rho-1}(s+1)^{\alpha\hat{\rho}-1} \, \mathrm{d}s \\ \qquad + y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}} \displaystyle\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1} \, \mathrm{d}t \Bigg], & y < x, \\ \dfrac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \Bigg[(y-x)^{\alpha-1} \displaystyle\int_0^{\frac{x(1-y)}{y-x}} s^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1} \, \mathrm{d}s \\ \qquad + y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}} \displaystyle\int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1} \, \mathrm{d}t \Bigg], & x < y. \end{cases}$$

Proof. According to Baurdoux [4, Theorem 4.1], since X is regular upwards, we have the following formula for $r_1(0, y)$:

$$r_1(0, y) = \lim_{z \downarrow 0} \frac{u_1(z, y)}{\mathrm{P}_z(\tau_1^+ < \tau_0^-)}.$$

We have found u_1 above, and as we already mentioned, we have from Rogozin

[73] that

$$\mathrm{P}_x(\tau_1^+ < \tau_0^-) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^x t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}\,\mathrm{d}t.$$

We may then make the following calculation, using l'Hôpital's rule on the second line since the integrals converge,

$$\begin{aligned} r_1(0,y) &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \lim_{z\downarrow 0} \frac{\displaystyle\int_0^{\frac{z(1-y)}{y-z}} s^{\alpha\hat{\rho}-1}(s+1)^{\alpha\rho-1}\,\mathrm{d}s}{\displaystyle\int_0^z t^{\alpha\hat{\rho}-1}(1-t)^{\alpha\rho-1}\,\mathrm{d}t} \\ &= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} \lim_{z\downarrow 0} \frac{z^{\alpha\hat{\rho}-1}(1-y)^{\alpha\hat{\rho}-1}(y-z)^{1-\alpha\hat{\rho}} \frac{\partial}{\partial z}\left[\frac{z(1-y)}{y-z}\right]}{z^{\alpha\hat{\rho}-1}\frac{\partial}{\partial z}\left[z\right]} \\ &= \frac{1}{\Gamma(\alpha)} y^{\alpha\rho-1}(1-y)^{\alpha\hat{\rho}}. \end{aligned}$$

Finally, the full potential density $r_1(x,y)$ follows simply by substituting in the following formula, from the same theorem in [4]:

$$r_1(x,y) = u_1(x,y) + \mathrm{P}_x(\tau_0^- < \tau_1^+) r_1(0,y). \qquad \square$$

Writing

$$\mathrm{E}_x \int_0^\infty \mathbb{1}[Y_t \in \mathrm{d}y,\ \bar{Z}_t \in \mathrm{d}z]\,\mathrm{d}t = r(x,y,z)\,\mathrm{d}y\,\mathrm{d}z,$$

where $\bar{Z}_t$ is the supremum of Z up to time t, we obtain the following corollary, much as we had for X.

Corollary 5.9. *For $y \in (0,z)$, $z \geq 0$,*

$$r(0,y,z) = \frac{\alpha\hat{\rho}}{\Gamma(\alpha)} y^{\alpha\rho-1}(z-y)^{\alpha\hat{\rho}-1},$$

and for $x > 0$, $y \in (0,z)$, $z \geq x$,

$$\begin{aligned} r(x,y,z) = \frac{1}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} y^{\alpha\rho-1}(z-y)^{\alpha\hat{\rho}-1} \Bigg[& x^{\alpha\hat{\rho}}(z-x)^{\alpha\rho-1} z^{1-\alpha} \\ & + \alpha\hat{\rho} \int_0^{1-\frac{x}{z}} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\,\mathrm{d}t \Bigg] \end{aligned}$$

We also have the following corollary, which is the analogue of Corollary 5.6.

Corollary 5.10. *For $u \in (0,1]$, $v \in (u,1)$, $y \geq 0$,*

$$\mathrm{P}_0(1 - \bar{Z}_{T_1^+} \in \mathrm{d}u,\, 1 - Z_{T_1^+} \in \mathrm{d}v,\, Z_{T_1^+} - 1 \in \mathrm{d}y) = \frac{\alpha \cdot \alpha\hat{\rho}}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \frac{(1-v)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(v+y)^{\alpha+1}},$$

and for $x \geq 0$, $u \in [0, 1-x)$, $v \in (u,1)$, $y \geq 0$,

$$\begin{aligned}\mathrm{P}_x(1 - \bar{Z}_{T_1^+} \in \mathrm{d}u,\, 1 - Z_{T_1^+} \in \mathrm{d}v,\, Z_{T_1^+} - 1 \in \mathrm{d}y) \\ = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\rho)} \frac{(1-v)^{\alpha\rho-1}(v-u)^{\alpha\hat{\rho}-1}}{(v+y)^{\alpha+1}} \Bigg[x^{\alpha\hat{\rho}}(1-u-x)^{\alpha\rho-1}(1-u)^{1-\alpha} \\ + \alpha\hat{\rho} \int_0^{1-\frac{x}{1-u}} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\, \mathrm{d}t \Bigg].\end{aligned}$$

The marginal in $\mathrm{d}v\,\mathrm{d}y$ appears in Baurdoux [4, Corollary 3.5] for the case where X is symmetric and $x = 0$. The marginal in $\mathrm{d}y$ is given in Kyprianou [52] for the process reflected in the supremum; this corresponds to swapping ρ and $\hat{\rho}$. However, unless $x = 0$, it appears to be difficult to integrate in Corollary 5.10 and obtain the expression found in [52].

Finally, one may integrate in Theorem 5.8 and obtain the expected first passage time for the reflected process.

Corollary 5.11. *For $x \geq 0$,*

$$\mathrm{E}_x[T_1^+] = \frac{1}{\Gamma(\alpha+1)} \left[x^{\alpha\hat{\rho}}(1-x)^{\alpha\rho} + \alpha\hat{\rho} \int_0^{1-x} t^{\alpha\rho-1}(1-t)^{\alpha\hat{\rho}-1}\, \mathrm{d}t \right].$$

In particular,

$$\mathrm{E}_0[T_1^+] = \frac{1}{\Gamma(\alpha)} \frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho}+1)}{\Gamma(\alpha+1)}.$$

We now make some remarks on potentials for conditioned stable processes. There are two cases to consider. Recall that the *stable process conditioned to stay positive* is the process given (see section 2.5.2) by the Doob h-transform of the measures $\mathrm{P}_x(\cdot,\, t < \tau_0^-)$ via the harmonic function

$$h^{\uparrow}(x) = x^{\alpha\hat{\rho}}, \qquad x > 0.$$

Write $(\mathrm{P}_x^{\uparrow})_{x>0}$ for the associated laws. We define the potential

$$U_+^{\uparrow}(x, \mathrm{d}y) = \mathrm{E}_x^{\uparrow} \int_0^{\tau_1^+} \mathbb{1}[X_t \in \mathrm{d}y]\, \mathrm{d}t = u_+^{\uparrow}(x,y)\, \mathrm{d}y, \qquad 0 < x,\, y < 1.$$

Since τ_1^+ is a stopping time, it follows from the h-transform that

$$\mathrm{P}_x^{\uparrow}(X_t \in \mathrm{d}y,\, t < \tau_1^+) = \frac{h^{\uparrow}(y)}{h^{\uparrow}(x)} \mathrm{P}_x(X_t \in \mathrm{d}y,\, t < \sigma^{[0,1]})$$

and in terms of the potential, this gives the following relation between the process conditioned to stay positive and the process killed on exiting $(0, \infty)$:

$$u_+^{\uparrow}(x, y) = \frac{h^{\uparrow}(y)}{h^{\uparrow}(x)} u_1(x, y), \qquad 0 < x,\, y < 1.$$

The next case is conditioning to hit zero continuously. Again, recall the *stable process conditioned to hit zero continuously*, given (see section 2.5.3) by the Doob h-transform of the measures $\mathrm{P}_x(\cdot,\, t < \tau_0^-)$ via the harmonic function

$$h^{\searrow}(x) = x^{\alpha\hat{\rho}-1}, \qquad x > 0.$$

Write $(\mathrm{P}_x^{\searrow})_{x>0}$ for the associated laws. As before, we define the potential

$$U_+^{\searrow}(x, \mathrm{d}y) = \mathrm{E}_x^{\searrow} \int_0^{\tau_1^+} \mathbb{1}[X_t \in \mathrm{d}y]\, \mathrm{d}t = u_+^{\searrow}(x, y)\, \mathrm{d}y, \qquad 0 < x,\, y < 1,$$

and it is simple to see that

$$u_+^{\searrow}(x, y) = \frac{h^{\searrow}(y)}{h^{\searrow}(x)} u_1(x, y) = (y/x)^{-1} u_+^{\uparrow}(x, y) \qquad 0 < x,\, y < 1.$$

Remark 5.12. Let us note another way to derive Theorem 5.3 and its corollaries. In [14], Caballero and Chaumont consider the stable process conditioned to stay positive and conditioned to hit zero continuously, and compute their Lamperti representations, say $\xi^{\uparrow}$ and $\xi^{\searrow}$. The Wiener-Hopf factorisations of these processes are also known; see for example [46]. It is hence a simple matter to compute the renewal measures of $\xi^{\uparrow}$, say, and use the Silverstein identity [6, Theorem VI.20] to compute $U_+^{\uparrow}$. One can then obtain U_1 and $U_+^{\searrow}$ by using in reverse the harmonic transforms we have already mentioned.

Chapter 6

The hitting time of zero

Consider a stable process X with $\alpha \in (1,2)$. We are interested in computing the distribution of

$$T_0 = \inf\{t \geq 0 : X_t = 0\},$$

the first hitting time of zero for X. When $\alpha > 1$, this random variable is a.s. finite, while when $\alpha \leq 1$, points are polar for the stable process, so $T_0 = \infty$ a.s.; this explains our exclusion of such processes.

When X is symmetric, the distribution of T_0 is not difficult to find using the Lamperti representation of the radial part of X, which we considered in section 3.3, and we will discuss this shortly. In the general case where X may be asymmetric, we present in this chapter a method making use of the so-called Lamperti–Kiu transform and Markov additive processes.

Let us also note here that, in the symmetric case, the distribution of T_0 has been characterised in Yano et al. [81, Theorem 5.3], and the Mellin transform appears in Cordero [29, equation (1.36)]; however, these authors proceed via a different method.

For the spectrally one-sided case, which our range of parameters omits, the law of T_0 has been computed by Peskir [66]. Nonetheless, as we explain in Remark 6.9, our methodology can also be used in this case.

First we reiterate the characterisation we gave in section 3.3 of the law of T_0 when X is symmetric. As in section 3.3, define a process R by

$$R_t = |X_t| \mathbb{1}_{\{t < T_0\}}, \qquad t \geq 0,$$

the *radial part of* X. The process R is a pssMp whose Lamperti representation, say ξ^R, is a Lévy process. It is then known that T_0 has the same distribution as the random variable

$$I(\alpha\xi^R) := \int_0^\infty \exp(\alpha\xi^R_t)\,\mathrm{d}t,$$

the so-called exponential functional of $\alpha\xi^R$.

In order to find the distribution of T_0, we must compute the Mellin transform $\mathbb{E}[I(\alpha\xi^R)^{s-1}]$ for a suitable range of s. In Proposition 3.11, we computed the Mellin transform of $I(\alpha\xi^{R'})$, and a simple rescaling and reformulation gives the result for the symmetric case: for $\operatorname{Re} s \in (-1/\alpha, 2-1/\alpha)$,

$$\begin{aligned} \mathrm{E}_1[T_0^{s-1}] &= 2^{-\alpha(s-1)} \frac{\sqrt{\pi}}{\Gamma(\frac{1}{\alpha})\Gamma(1-\frac{1}{\alpha})} \frac{\Gamma(1+\frac{\alpha}{2}-\frac{\alpha s}{2})}{\Gamma(\frac{1-\alpha}{2}+\frac{\alpha s}{2})} \Gamma(\tfrac{1}{\alpha}-1+s) \frac{\Gamma(2-\frac{1}{\alpha}-s)}{\Gamma(2-s)} \\ &= \sin(\pi/\alpha) \frac{\sin\big(\frac{\pi}{2}(1-\alpha+\alpha s)\big)}{\sin\Big(\frac{\pi}{\alpha}\big(1-\alpha+\alpha s\big)\Big)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}. \end{aligned}$$

This characterises the distribution.

The rest of this chapter is organised as follows.

In section 6.1 we derive the distribution of T_0 for a general stable process, which may not be symmetric. The reasoning is similar to that presented above. The process R still satisfies the scaling property, but it is no longer a Markov process; however, due to the recent work of Chaumont et al. [25], there is still a type of Lamperti representation for X, not in terms of a Lévy process, but in terms of a so-called Markov additive process, say ξ^X. Again, the distribution of T_0 is equal to that of $I(\alpha\xi^X)$, and we develop techniques to compute a vector-valued Mellin transform for the exponential function of this Markov additive process. Further, we invert the Mellin transform of $I(\alpha\xi^X)$ in order to deduce explicit series representations for the law of T_0.

In certain scenarios the distribution T_0 is a very convenient quantity to have, and we consider some applications in section 6.2: for example, we give an alternative description of the stable process conditioned to avoid zero, and we give some identities in law similar to the result of Bertoin and Yor [8] for the entrance law of a pssMp started at zero.

Finally, we discuss the relationship between conditionings at zero of self-similar Markov processes and the existence of a Cramér number for their Lamperti or Lamperti–Kiu transforms, and give suggestions for future work.

6.1 The asymmetric stable process

This section is laid out as follows. We devote the first two subsections to a discussion of Markov additive processes and their exponential functionals, and then discuss real self-similar Markov processes and the Lamperti–Kiu representation. Finally, in the last subsection, we apply the theory which we have developed to the problem of determining the law of T_0 for a general two-sided jumping stable process with $\alpha \in (1, 2)$.

6.1.1 Markov additive processes

Let E be a finite state space and $(\mathscr{H}_t)_{t\geq 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R}\times E$ with law $\mathbb{P}$ is called a *Markov additive process (MAP)* with respect to $(\mathscr{H}_t)_{t\geq 0}$ if $(J(t))_{t\geq 0}$ is a continuous-time Markov chain in E, and the following property is satisfied, for any $i \in E$ and $s, t \geq 0$:

$$\text{given } \{J(t) = i\}, \text{ the pair } (\xi(t+s) - \xi(t), J(t+s)) \text{ is independent of } \mathscr{H}_t, \\ \text{and has the same distribution as } (\xi(s) - \xi(0), J(s)) \text{ given } \{J(0) = i\}. \quad (6.1)$$

Aspects of the theory of Markov additive processes are covered in a number of texts, among them [1] and [2]. We will mainly use the notation of [40], which principally works under the assumption that ξ is spectrally negative; the results which we quote are valid without this hypothesis, however.

Let us introduce some notation. We write $\mathbb{P}_i = \mathbb{P}(\cdot \mid \xi(0) = 0, J(0) = i)$; and if μ is a probability distribution on E, we write

$$\mathbb{P}_\mu = \mathbb{P}(\cdot \mid \xi(0) = 0, J(0) \sim \mu) = \sum_{i \in E} \mu(i)\mathbb{P}_i.$$

We adopt a similar convention for expectations.

It is well-known that a Markov additive process (ξ, J) also satisfies (6.1) with t replaced by a stopping time. Furthermore, it has the structure given in the following proposition; see [2, §XI.2a] and [40, Proposition 2.5].

Proposition 6.1. *The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$ and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J, such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J, the process ξ has the representation*

$$\xi(t) = \mathbb{1}_{\{n>0\}}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n), \qquad t \in [T_n, T_{n+1}),\ n \geq 0.$$

For each $i \in E$, it will be convenient to define, on the same probability space, ξ_i as a Lévy process whose distribution is the common law of the ξ_i^n processes in the above representation; and similarly, for each $i, j \in E$, define U_{ij} to be a random variable having the common law of the U_{ij}^n variables.

Let us now fix the following setup. Firstly, we confine ourselves to irreducible Markov chains J. Let the state space E be the finite set $\{1, \dots, N\}$, for some $N \in \mathbb{N}$. Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j\in E}$. For each $i \in E$, the Laplace$^+$ exponent of the Lévy process ξ_i will be written ϕ_i, which is to say that $e^{\phi_i(z)} = \mathbb{E}(e^{z\xi_i(1)})$, for all $z \in \mathbb{C}$ for which the right-hand side exists. For each pair of $i, j \in E$, define the Laplace$^+$ transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij}, where this exists; write $G(z)$ for the $N \times N$ matrix whose (i, j)th element is $G_{ij}(z)$. We will adopt the convention that $U_{ij} = 0$ if

$q_{ij} = 0$, $i \neq j$, and also set $U_{ii} = 0$ for each $i \in E$.

A multidimensional analogue of the Laplace$^+$ exponent of a Lévy process is provided by the matrix-valued function

$$F(z) = \operatorname{diag}(\phi_1(z), \ldots, \phi_N(z)) + Q \circ G(z), \tag{6.2}$$

for all $z \in \mathbb{C}$ where the elements on the right are defined, where $\circ$ indicates elementwise multiplication, also called Hadamard multiplication. It is then known that

$$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{F(z)t}\right)_{ij}, \qquad i, j \in E,$$

for all $z \in \mathbb{C}$ where one side of the equality is defined. For this reason, F is called the *matrix exponent* of the MAP ξ.

We now describe the existence of the *leading eigenvalue* of the matrix F, which will play a key role in our analysis of MAPs. This is sometimes also called the *Perron-Frobenius eigenvalue*; see [2, §XI.2c] and [40, Proposition 2.12].

Proposition 6.2. *Suppose that $z \in \mathbb{C}$ is such that $F(z)$ is defined. Then, the matrix $F(z)$ has a real simple eigenvalue $k(z)$, which is larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector $v(z)$ may be chosen so that $v_i(z) > 0$ for every $i \in E$, and normalised such that*

$$\pi v(z) = 1 \tag{6.3}$$

where π is the equilibrium distribution of the chain J.

This leading eigenvalue features in the following probabilistic result, which identifies a martingale (sometimes known as the Wald martingale) and change of measure analogous to the exponential martingale and Esscher transformation of a Lévy process; cf. [2, Proposition XI.2.4, Theorem XIII.8.1].

Proposition 6.3. *Let*

$$M(t, \gamma) = e^{\gamma[\xi(t) - \xi(0)] - k(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \qquad t \geq 0,$$

for some γ such that the right-hand side is defined. Then,

(i) $M(\cdot, \gamma)$ is a unit-mean martingale with respect to $(\mathscr{H}_t)_{t \geq 0}$ under any initial distribution of $(\xi(0), J(0))$.

(ii) Define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^\gamma}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{H}_t} = M(t, \gamma).$$

Under $\mathbb{P}^\gamma$, the process ξ is still a Markov additive process. We shall write a superscript (γ) to indicate a quantity calculated under $\mathbb{P}^\gamma$; thus, one may calculate for each $i, j \in E$:

- $\mathbb{P}^\gamma(U_{ij} \in \mathrm{d}x) = \dfrac{e^{\gamma x}}{G_{ij}(\gamma)} \mathbb{P}(U_{ij} \in \mathrm{d}x)$, *and hence* $G_{ij}^{(\gamma)}(z) = \dfrac{G_{ij}(z+\gamma)}{G_{ij}(\gamma)}$,
- $q_{ij}^{(\gamma)} = \dfrac{v_j(\gamma)}{v_i(\gamma)} q_{ij} G_{ij}(\gamma)$ *and*
- $\phi_i^{(\gamma)}(z) = \phi_i(z+\gamma) - \phi_i(\gamma)$.

Furthermore,

$$F^{(\gamma)}(z) = \Delta_{v(\gamma)}^{-1}[F(z+\gamma) - k(\gamma)\mathrm{Id}]\Delta_{v(\gamma)},$$

where $\Delta_{v(\gamma)} = \mathrm{diag}(v_i(\gamma), i \in E)$*; and hence,*

$$k^{(\gamma)}(z) = k(z+\gamma) - k(\gamma).$$

Making use of this, the following proposition can be obtained; the properties of k given here are commonly used in the literature, but for convenience, we also provide a short proof.

Proposition 6.4. *Suppose that F is defined in some open interval D of $\mathbb{R}$. Then, the leading eigenvalue k of F is smooth and convex on D.*

Proof. Smoothness follows from results on the perturbation of eigenvalues; see [40, Proposition 2.13] for a full proof.

To prove that k is convex, we begin by proving that for $\lambda \in (0,1)$ such that z, $\lambda z \in D$,

$$k(\lambda z) \leq \lambda k(z). \tag{6.4}$$

Taking expectations in the martingale from Proposition 6.3, and using the normalisation condition (6.3), we see that, starting J in equilibrium,

$$\mathbb{E}_\pi[e^{z\xi(1)} v_{J(1)}(z)] = e^{k(z)}.$$

Now let $p = 1/\lambda > 1$, and let q be its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{q} = 1$. We apply Hölder's inequality as follows.

$$\begin{aligned} e^{k(z/p)} = \mathbb{E}_\pi[e^{(z/p)\xi(1)} v_{J(1)}(z)] &\leq \big(\mathbb{E}_\pi[e^{(zp/p)\xi(1)} v_{J(1)}^{p/p}(z)]\big)^{1/p} \big(\mathbb{E}_\pi[v_{J(1)}^{q/q}(z)]\big)^{1/q} \\ &= \big(\mathbb{E}_\pi[e^{z\xi(1)} v_{J(1)}(z)]\big)^{1/p} (\pi v(z))^{1/q} \\ &= \big(\mathbb{E}_\pi[e^{z\xi(1)} v_{J(1)}(z)]\big)^{1/p} \\ &= e^{k(z)/p} \end{aligned}$$

Replacing $1/p$ by λ, we obtain (6.4).

Now let $u, v \in D$ and $\lambda \in (0,1)$; recall from Proposition 6.3 that for the leading eigenvalue under the change of measure, $k^{(u)}(z) = k(z+u) - k(u)$. We then calculate:

$$\begin{aligned} k((1-\lambda)u + kv) - k(u) &= k^{(u)}(\lambda(v-u)) \\ &\leq \lambda k^{(u)}(v-u) \\ &= \lambda(k(v) - k(u)) \end{aligned}$$

and so

$$k((1-\lambda)u + \lambda v) \leq (1-\lambda)k(u) + \lambda k(v),$$

which completes the proof. □

6.1.2 The Mellin transform of the exponential functional

In section 3.3, we studied the exponential functional of a certain Lévy process associated to the radial part of the stable process; now we are interested in obtaining some results which will assist us in computing the law of an exponential functional associated to Markov additive processes.

For a MAP ξ, let

$$I(-\xi) = \int_0^\infty \exp(-\xi(t))\, \mathrm{d}t.$$

One way to characterise the law of $I(-\xi)$ is via its Mellin transform, which we write as $\mathcal{M}(s)$. This is the vector in $\mathbb{R}^N$ whose ith element is given by

$$\mathcal{M}_i(s) = \mathbb{E}_i[I(-\xi)^{s-1}], \qquad i \in E.$$

We will shortly obtain a functional equation for $\mathcal{M}$, analogous to the functional equation (2.19) which we saw in section 3.2. For Lévy processes, proofs of the result can be found in Carmona et al. [18, Proposition 3.1], Maulik and Zwart [60, Lemma 2.1] and Rivero [69, Lemma 2]; our proof follows the latter, making changes to account for the Markov additive property.

We make the following assumption, which is analogous to the Cramér condition for a Lévy process; recall that k is the leading eigenvalue of the matrix F, as discussed in section 6.1.1.

Assumption 6.5 (Cramér condition for a Markov additive process)**.** There exist $z_0 < 0$ such that $F(s)$ exists on $(z_0, 0)$, and some $\theta \in (0, -z_0)$, such that $k(-\theta) = 0$. We say that the Markov additive process (ξ, J) satisfies the Cramér condition with Cramér number $-\theta$.

Since the leading eigenvalue k is smooth and convex where it is defined, it follows also that $k(-s) < 0$ for $s \in (0, \theta)$. In particular, this renders the matrix $F(-s)$ negative definite, and hence invertible. Furthermore, it follows that $k'(0-) > 0$, and hence (see [2, Corollary XI.2.7] and [40, Lemma 2.14]) that ξ drifts to $+\infty$ independently of its initial state. This implies that $I(-\xi)$ is an a.s. finite random variable.

Proposition 6.6. *Suppose that ξ satisfies the Cramér condition (Assumption 6.5) with Cramér number $-\theta$, such that $\theta \in (0,1)$. Then, $\mathcal{M}(s)$ is finite and analytic when* $\operatorname{Re} s \in (0, 1+\theta)$*, and the following vector-valued functional equation holds:*

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \qquad s \in (0,\theta).$$

Proof. At the end of the proof, we shall require the existence of certain moments of the random variable

$$Q_t = \int_0^t e^{-\xi(u)}\,\mathrm{d}u,$$

and so we shall begin by establishing this.

Suppose that $s \in (0,\theta]$, and let $p > 1$. Then, by the Cramér condition, it follows that $k(-s/p) < 0$, and hence for any $u \geq 0$, $e^{-uk(-s/p)} \geq 1$.

Recall that for fixed z, the process

$$M(u,z) = e^{z\xi(u)-k(z)u}\frac{v_{J(u)}(z)}{v_{J(0)}(z)}, \qquad u \geq 0$$

is a martingale (the Wald martingale) under any initial distribution $(\xi(0), J(0))$, and set

$$V(z) = \min_{j\in E} v_j(z) > 0,$$

so that for each $j \in E$, $v_j(z)/V(z) \geq 1$.

We now have everything in place to make the following calculation, which uses the Cramér condition in the second line, and the Doob maximal inequality in connection with the Wald martingale in the third line; note that $M(\cdot, -s/p)^p$ is a submartingale.

$$\begin{aligned}
\mathbb{E}_i[Q_t^s] &\leq t^s\ \mathbb{E}_i\left[\sup_{u\leq t}\left[e^{-s\xi(u)/p}\right]^p\right] \\
&\leq t^s\mathbb{E}_i\left[\sup_{u\leq t}\left[M(u,-s/p)v_i(-s/p)(V(-s/p))^{-1}\right]^p\right] \\
&\leq t^s v_i(-s/p)^p V(-s/p)^{-p}\left(\frac{p}{p-1}\right)^p \mathbb{E}_i\left[M(t,-s/p)^p\right] \\
&\leq t^s V(-s/p)^{-p}\left(\frac{p}{p-1}\right)^p e^{-tpk(-s/p)}\max_{j\in E} v_j(-s/p)^p\mathbb{E}_i\left[e^{-s\xi(t)}\right] < \infty.
\end{aligned}$$

We now commence with the main argument in the proof. To begin with, for all $s > 0$, $t \geq 0$,

$$\Big(\int_0^\infty e^{-\xi(u)}\,\mathrm{d}u\Big)^s - \Big(\int_t^\infty e^{-\xi(u)}\,\mathrm{d}u\Big)^s \\ = s\int_0^t e^{-s\xi(u)}\Big(\int_0^\infty e^{-(\xi(u+v)-\xi(u))}\,\mathrm{d}v\Big)^{s-1}\,\mathrm{d}u.$$

For each $i \in E$, we take expectations and apply the Markov additive property.

$$\begin{aligned}\mathbb{E}_i&\Big[\Big(\int_0^\infty e^{-\xi(u)}\,\mathrm{d}u\Big)^s - \Big(\int_t^\infty e^{-\xi(u)}\,\mathrm{d}u\Big)^s\Big] \\ &= s\sum_{j\in E}\int_0^t \mathbb{E}_i\Big[e^{-s\xi(u)}; J(u) = j\Big]\mathbb{E}_j\Big[\int_0^\infty e^{-\xi(v)}\,\mathrm{d}v\Big]^{s-1}\,\mathrm{d}u \\ &= s\int_0^t \sum_{j\in E}\Big(e^{F(-s)u}\Big)_{ij}\mathbb{E}_j\big[I(-\xi)^{s-1}\big]\,\mathrm{d}u.\end{aligned}$$

Assuming now that $0 < s < \theta < 1$, it follows that $\big||x|^s - |y|^s\big| \leq |x-y|^s$ for any $x, y \in \mathbb{R}$, and so we see that for each $i \in E$, the left-hand side of the above equation is bounded by $\mathbb{E}_i(Q_t^s) < \infty$. Since $e^{F(-s)u}$ has no zero columns, it follows that $\mathbb{E}_i[I(-\xi)^{s-1}] < \infty$ for each $i \in E$, also.

If we now take $t \to \infty$, the left-hand side is monotone increasing, while on the right, the Cramér condition ensures that $F(-s)$ is negative definite, which is a sufficient condition for convergence, giving the limit:

$$\mathcal{M}(s+1) = -s(F(-s))^{-1}\mathcal{M}(s), \qquad s \in (0,\theta).$$

Furthermore, as we know the right-hand side is finite, this functional equation allows us to conclude that $\mathcal{M}(s) < \infty$ for all $s \in (0, 1+\theta)$. It then follows from the general properties of Mellin transforms of probability measures that $\mathcal{M}(s)$ is finite and analytic for all $s \in \mathbb{C}$ such that $\operatorname{Re} s \in (0, 1+\theta)$. □

6.1.3 Real self-similar Markov processes

In chapters 3–5, we studied a number Lévy processes associated through the Lamperti representation to positive, self-similar Markov processes. Here we see that Markov additive processes also admit an interpretation as Lamperti-type representations of *real* self-similar Markov processes.

The structure of real self-similar Markov processes has been investigated by Chybiryakov [28] in the symmetric case, and Chaumont et al. [25] in general. Here, we give an interpretation of these authors' results in terms of a two-state Markov additive process. We begin with some relevant definitions, and state some of the results of these authors.

A *real self-similar Markov process* with *self-similarity index* $\alpha > 0$ is a standard (in the sense of [10]) Markov process $X = (X_t)_{t\geq 0}$ with probability laws $(\mathrm{P}_x)_{x\in\mathbb{R}\setminus\{0\}}$ which satisfies the *scaling property*, that for all $x \in \mathbb{R}\setminus\{0\}$ and $c > 0$,

$$\text{the law of } (cX_{tc^{-\alpha}})_{t\geq 0} \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

In [25] the authors define four classes of processes:

C.1 $\mathrm{P}_x(\exists t > 0 : X_t X_{t-} < 0)$ is equal to 1 when $x > 0$ and 0 when $x < 0$.

C.2 $\mathrm{P}_x(\exists t > 0 : X_t X_{t-} < 0)$ is equal to 1 when $x < 0$ and 0 when $x > 0$.

C.3 $\mathrm{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 0$ for all $x \neq 0$.

C.4 $\mathrm{P}_x(\exists t > 0 : X_t X_{t-} < 0) = 1$ for all $x \neq 0$.

The authors confine their attention to class **C.4**, where, with probability one, the rssMp X changes sign infinitely often; processes in the other classes have a somewhat simpler structure. As with the stable process, define

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

For such processes, X may be identified up to the time T_0 as the time-changed exponential of a certain complex-valued process $\mathcal{E}$, which we call the *Lamperti–Kiu representation* of X. The following result is taken from [25].

Proposition 6.7 ([25, Theorem 6])**.** *Let X be a rssMp in class **C.4**, and let $x \neq 0$. It is possible to define independent sequences $(\xi^{\pm,k})_{k\geq 0}$, $(\zeta^{\pm,k})_{k\geq 0}$ and $(U^{\pm,k})_{k\geq 0}$ of iid random objects with the following proprties:*

1. *The elements of these sequences are distributed such that: the $\xi^{\pm}$ are real-valued Lévy processes; $\zeta^{\pm}$ are exponential random variables with parameters $q^{\pm}$; and $U^{\pm}$ are real-valued random variables.*

2. *For each $x \neq 0$, define the following objects:*

$$(\xi^{(k)}, \zeta^{(k)}, U^{(k)}) = \begin{cases} (\xi^{+,k}, \zeta^{+,k}, U^{+,k}), & \textit{if } \operatorname{sgn}(x)(-1)^k = 1 \\ (\xi^{-,k}, \zeta^{-,k}, U^{-,k}), & \textit{if } \operatorname{sgn}(x)(-1)^k = -1, \end{cases}$$

$$\mathcal{T}_0 = 0, \quad \mathcal{T}_n = \sum_{k=0}^{n-1} \zeta^{(k)},$$

$$N_t = \max\{n \geq 0 : \mathcal{T}_n \leq t\},$$

$$\sigma_t = t - \mathcal{T}_{N_t},$$

$$\mathcal{E}_t = \xi^{(N_t)}_{\sigma_t} + \sum_{k=0}^{N_t-1} (\xi^{(k)}_{\zeta^{(k)}} + U^{(k)}) + \mathrm{i}\pi N_t, \qquad t \geq 0,$$

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s |\exp(\alpha \mathcal{E}_u)| \, \mathrm{d}u > t|x|^{-\alpha} \right\}, \qquad t < T_0.$$

Then, the process X under the measure P_x has the representation

$$X_t = x \exp(\mathcal{E}_{\tau(t)}), \qquad 0 \leq t < T_0.$$

The abundance of notation necessary to be precise in this context may obscure the fundamental idea, which is as follows. At any given time, the process $\mathcal{E}$ evolves as a Lévy process $\xi^{\pm}$, moving along a line $\operatorname{Im} z = \pi N$, up until an exponential 'clock' $\zeta^{\pm}$ (corresponding to the process X changing sign) rings. At this point the imaginary part of $\mathcal{E}$ is incremented by π, the real part jumps by $U^{\pm}$, and the process begins to evolve as the other Lévy process, $\xi^{\mp}$.

Particularly in light of the above discussion, our interpretation is simple to state.

Proposition 6.8. *Let X be an rssMp, with Lamperti–Kiu representation $\mathcal{E}$. Define furthermore*

$$[n] = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Then, for each $x \neq 0$, the process

$$(\xi(t), J(t)) = \big(\operatorname{Re} \mathcal{E}_t, \big[\operatorname{Im} \mathcal{E}_t/\pi + \mathbb{1}_{\{x>0\}}\big]\big),$$

with the right-hand side as in Proposition 6.7, is a Markov additive process with state space $E = \{1, 2\}$, and X under P_x has the representation

$$X_t = x \exp\big(\xi(\tau(t)) + \mathrm{i}\pi (J(\tau(t)) + 1)\big), \qquad 0 \leq t < T_0,$$

where we note that $(\xi(0), J(0))$ is equal to $(0, 1)$ if $x > 0$, or $(0, 2)$ if $x < 0$. Furthermore, the time-change τ has the representation

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s \exp(\alpha \xi(u)) \, \mathrm{d}u > t|x|^{-\alpha} \right\}, \qquad t < T_0, \tag{6.5}$$

in terms of the real-valued process ξ.

Furthermore, we observe from the expression (6.5) for the time-change τ that under P_x, for any $x \neq 0$, the following identity holds for T_0, the hitting time of zero:

$$|x|^\alpha T_0 = \int_0^\infty e^{\alpha \xi(u)} \, \mathrm{d}u.$$

Implicit in this statement is that the MAP on the right-hand side has law $\mathbb{P}_1$ if $x > 0$, and law $\mathbb{P}_2$ if $x < 0$. This observation will be exploited in the coming section, in which we put together the theory we have outlined so far.

6.1.4 The hitting time of zero

We now return to the central problem of this chapter: computing the distribution of T_0 for a stable process. We already have in hand the representation of T_0 for an rssMp as the exponential functional of a MAP, as well as a functional equation for this quantity which will assist us in the computation.

Let X be the stable process with parameters $(\alpha, \rho) \in \mathcal{A}_{\mathrm{st}}$ and $\alpha > 1$. We will restrict our attention to X under the measures $\mathrm{P}_{\pm 1}$; the results for other initial values can be derived via scaling.

Since X is an rssMp, it has a representation in terms of a MAP (ξ, J); futher-more, under $\mathrm{P}_{\pm 1}$,

$$T_0 = \int_0^\infty e^{\alpha\xi(s)}\,\mathrm{d}s = I(\alpha\xi);$$

to be precise, under P_1 the process ξ is under $\mathbb{P}_1$, while under P_{-1} it is under $\mathbb{P}_2$.

In [25, §4.1], the authors calculate the characteristics of the Lamperti–Kiu representation for X, that is, the processes $\xi^\pm$, and the jump distributions $U^\pm$ and rates $q^\pm$. Using this information, and the representation (6.2), one sees that the MAP $(-\alpha\xi, J)$ has matrix exponent

$$F(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat\rho + \alpha z)\Gamma(1-\alpha\hat\rho - \alpha z)} & \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)} \\ \dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha(1+z))\Gamma(1-\alpha z)}{\Gamma(\alpha\rho + \alpha z)\Gamma(1-\alpha\rho - \alpha z)} \end{pmatrix},$$

for $\operatorname{Re} z \in (-1, 1/\alpha)$.

Remark 6.9. It is well-known that, when X does not have one-sided jumps, it changes sign infinitely often; that is, the rssMp X is in [25]'s class **C.4**. When the stable process has only one-sided jumps, which corresponds to the parameter values $\rho = 1 - 1/\alpha$, $1/\alpha$, then it jumps over 0 at most once before hitting it; the rssMp is therefore in class **C.1** or **C.2** according to the classification of [25]. The Markov chain component of the corresponding MAP then has one absorbing state, and hence is no longer irreducible. Although it seems plain that our calculations can be carried out in this case, we omit it for the sake of simplicity. As we remarked in the introduction, it is considered in [66].

We now analyse F in order to deduce the Mellin transform of T_0. By considering the equation $\det F(z) = 0$ with $z = 1/\alpha - 1$, it is not difficult to deduce that $k(1/\alpha - 1) = 0$; that is, $-\alpha\xi$ satisfies the Cramér condition (Assumption 6.5) with Cramér number $-\theta$, where $\theta = 1 - 1/\alpha$.

Let

$$f_1(s) := \mathrm{E}_1[T_0^{s-1}] = \mathbb{E}_1[I(\alpha\xi)^{s-1}], \quad f_2(s) := \mathrm{E}_{-1}[T_0^{s-1}] = \mathbb{E}_2[I(\alpha\xi)^{s-1}],$$

which by Proposition 6.6 are defined when $\mathrm{Re}\, s \in (0, 2-1/\alpha)$.

Setting $\mathbf{A}(s) = -s(F(-s))^{-1}$, we have

$$\mathbf{A}(s) = -\frac{1}{\pi\alpha}\Gamma(1-\alpha+\alpha s)\Gamma(1-\alpha s)\begin{bmatrix}\sin(\pi\alpha(\rho-s)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & \sin(\pi\alpha(\hat{\rho}-s))\end{bmatrix}$$

for $\mathrm{Re}\, s \in (1-2/\alpha, 1-1/\alpha)$, and the proposition states that

$$\begin{bmatrix} f_1(s+1) \\ f_2(s+1)\end{bmatrix} = \mathbf{A}(s)\begin{bmatrix} f_1(s) \\ f_2(s)\end{bmatrix}, \qquad s \in (0, 1-1/\alpha). \tag{6.6}$$

The following theorem is the main result of this chapter.

Theorem 6.10. *For $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$ we have*

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})}\frac{\sin\big(\pi\hat{\rho}(1-\alpha+\alpha s)\big)}{\sin\big(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\big)}\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}. \tag{6.7}$$

The corresponding expression for $\mathrm{E}_{-1}[T_0^{s-1}]$ can be obtained from (6.7) by exchanging $\hat{\rho}$ and ρ.

The rest of this section is devoted to proving the uniqueness of the solution (6.7) of the equation (6.6). The results are as in [49].

Let us denote the function on the right-hand side of (6.7) by $h_1(s)$, and by $h_2(s)$ the function obtained from $h_1(s)$ by replacing $\hat{\rho} \mapsto \rho$. Before we are able to prove Theorem 6.10, we need to establish several properties of these functions.

Lemma 6.11.

(i) There exists $\varepsilon \in (0, 1-1/\alpha)$ such that the functions $h_1(s)$, $h_2(s)$ are analytic and zero-free in the vertical strip $0 \le \mathrm{Re}(s) \le 1+\varepsilon$.

(ii) For any $-\infty < a < b < +\infty$ there exists $C > 0$ such that

$$e^{-\pi|\mathrm{Im}(s)|} < |h_i(s)| < e^{-\frac{\pi}{2}(\alpha-1)|\mathrm{Im}(s)|}, \qquad i = 1, 2,$$

for all s in the vertical strip $a \le \mathrm{Re}(s) \le b$ satisfying $|\mathrm{Im}(s)| > C$.

Proof. It is clear from the definition of $h_1(s)$ that it is a meromorphic function. Its zeroes are contained in the set

$$\{2, 3, 4, \dots\} \cup \{1 - 1/\alpha + n/(\alpha\hat{\rho}) : n \in \mathbb{Z}, n \neq 0\}$$

and its poles are contained in the set

$$\{1 + n/\alpha : n \geq 1\} \cup \{n - 1/\alpha : n \in \mathbb{Z}, n \neq 1\}.$$

In particular, there are no zeroes or poles of $h_1(s)$ in the strip $0 \leq \mathrm{Re}(s) \leq 1$. The same is clearly true for $h_2(s)$, which proves part (i).

We now make use of the asymptotic formula (3.17), as we did in section 3.3. Applying this to $h_1(s)$ we find that, as $s \to \infty$, we have

$$\log|h_1(s)| = -\frac{\pi}{2}(1 + \alpha - 2\alpha\hat{\rho})|\mathrm{Im}\, s| + O(\log|\mathrm{Im}\, s|),$$

uniformly in the strip $a \leq \mathrm{Re}(s) \leq b$. Since for $\alpha > 1$, any admissible parameters α, ρ must satisfy $\alpha - 1 < \alpha\hat{\rho} < 1$, this shows that

$$\alpha - 1 < 1 + \alpha - 2\alpha\hat{\rho} < 3 - \alpha < 2,$$

and completes the proof of part (ii). □

Lemma 6.12. *The functions $h_1(s)$, $h_2(s)$ satisfy the system of equations (6.6).*

Proof. Denote the elements of the matrix $\mathbf{A}(s)$ by $A_{ij}(s)$. Multiplying the first row of $\mathbf{A}(s)$ by the column vector $[h_1(s), h_2(s)]^T$, and we obtain, using the fact that $\sin(\pi\rho) = \sin(\pi\hat{\rho})$,

$$\begin{aligned}
&A_{11}(s)h_1(s) + A_{12}(s)h_2(s) \\
&\quad = -\frac{1}{\pi\alpha}\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})}\frac{\Gamma(1-\alpha s)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)}\left[\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}\Gamma(1-\alpha+\alpha s)\right] \\
&\qquad \times \left\{ \sin(\pi\alpha(\rho - s))\sin\big(\pi\hat{\rho}(1-\alpha+\alpha s)\big) + \sin(\pi\alpha\hat{\rho})\sin\left(\pi\rho(1-\alpha+\alpha s)\right) \right\}.
\end{aligned}$$

Applying identity $\Gamma(z+1) = z\Gamma(z)$ and reflection formula for the gamma function, we rewrite the expression in the square brackets as follows:

$$\begin{aligned}
\left[\frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}\Gamma(1-\alpha+\alpha s)\right] &= \frac{\alpha\Gamma(\alpha-\alpha s)}{\Gamma(1-s)}\Gamma(1-\alpha+\alpha s) \\
&= \frac{\pi\alpha}{\sin(\pi\alpha(1-s))\Gamma(1-s)}.
\end{aligned}$$

Applying product-sum identities for trigonometric functions, we obtain

$$\begin{aligned}
\sin(\pi\alpha(\rho - s))\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right) + \sin(\pi\alpha\hat{\rho})\sin\left(\pi\rho(1-\alpha+\alpha s)\right)& \\
= \sin(\pi\alpha(1-s))\sin(\pi\hat{\rho}(1+\alpha s)).&
\end{aligned}$$

Combining the above three formulas we conclude

$$A_{11}(s)h_1(s)+A_{12}(s)h_2(s) = -\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat\rho)}\frac{\sin\left(\pi\hat\rho(1+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)}\frac{\Gamma(1-\alpha s)}{\Gamma(1-s)} = h_1(s+1).$$

The derivation of the identity

$$A_{21}(s)h_1(s) + A_{22}(s)h_2(s) = h_2(s+1)$$

is identical. □

Proof of Theorem 6.10. Our goal now is to establish the uniqueness of solutions to the system (6.6) in a certain class of meromorphic functions, which contains both $h_i(s)$ and $f_i(s)$. This will imply $h_i(s) \equiv f_i(s)$. Our argument is similar in spirit to the proof of Proposition 2 in [46].

First of all, we check that there exists $\varepsilon \in (0, 1/2 - 1/(2\alpha))$ such that the functions $f_1(s)$, $f_2(s)$ are analytic and bounded in the open strip

$$\mathcal{S}_\varepsilon = \{s \in \mathbb{C} : \varepsilon < \mathrm{Re}(s) < 1 + 2\varepsilon\}.$$

This follows from Proposition 6.6 and the estimate

$$|f_1(s)| = |\mathrm{E}_1[T_0^{s-1}]| \le \mathrm{E}_1[|T_0^{s-1}|] = \mathrm{E}_1[T_0^{\mathrm{Re}(s)-1}] = f_1(\mathrm{Re}(s)).$$

The same applies to f_2. Given the results of Lemma 6.11, we can also assume that ε is small enough that the functions $h_i(s)$ are analytic, zero-free and bounded in the strip $\mathcal{S}_\varepsilon$.

Let us define $D(s) := f_1(s)h_2(s) - f_2(s)h_1(s)$ for $s \in \mathcal{S}_\varepsilon$. From the above properties of $f_i(s)$ and $h_i(s)$ we conclude that $D(s)$ is analytic and bounded in $\mathcal{S}_\varepsilon$. Our first goal is to show that $D(s) \equiv 0$.

If both s and $s+1$ belong to $\mathcal{S}_\varepsilon$, then the function $D(s)$ satisfies the equation

$$D(s+1) = -\frac{1}{\alpha^2}\frac{\Gamma(1-\alpha+\alpha s)\Gamma(1-\alpha s)}{\Gamma(\alpha-\alpha s)\Gamma(\alpha s)}D(s), \tag{6.8}$$

as may be established by taking determinants in the matrix equation

$$\begin{bmatrix} f_1(s+1) & h_1(s+1) \\ f_2(s+1) & h_2(s+1)\end{bmatrix} = \mathbf{A}(s)\begin{bmatrix} f_1(s) & h_1(s) \\ f_2(s) & h_2(s)\end{bmatrix}.$$

Define also

$$G(s) := \frac{\Gamma(s-1)\Gamma(\alpha-\alpha s)}{\Gamma(1-s)\Gamma(-\alpha+\alpha s)}\sin\left(\pi\left(s+\tfrac{1}{\alpha}\right)\right).$$

It is simple to check that:

(i) G satisfies the functional equation (6.8),

(ii) G is analytic and zero-free in the strip $\mathcal{S}_\varepsilon$, and

(iii) $|G(s)| \to \infty$ as $|\mathrm{Im}\, s| \to \infty$, uniformly in the strip $\mathcal{S}_\varepsilon$.

For the last point, one may use (3.17), derived from Stirling's asymptotic formula.

We now define $H(s) := D(s)/G(s)$ for $s \in \mathcal{S}_\varepsilon$. The property (ii) guarantees that H is analytic in the strip $\mathcal{S}_\varepsilon$, while property (i) and (6.8) can be applied to show that $H(s+1) = H(s)$ if both s and $s+1$ belong to $\mathcal{S}_\varepsilon$. Therefore, we can extend $H(s)$ to an entire function satisfying $H(s+1) = H(s)$ for all $s \in \mathbb{C}$. Using the periodicity of $H(s)$, property (iii) of the function $G(s)$ and the fact that the function $D(s)$ is bounded in the strip $\mathcal{S}_\varepsilon$, we conclude that $H(s)$ is bounded on $\mathbb{C}$ and $H(s) \to 0$ as $|\mathrm{Im}\, s| \to \infty$. Since H is entire, it follows that $H \equiv 0$.

So far, we have proved that for all $s \in \mathcal{S}_\varepsilon$ we have $f_1(s)h_2(s) = f_2(s)h_1(s)$. Let us define $w(s) := f_1(s)/h_1(s) = f_2(s)/h_2(s)$. Since both $f_i(s)$ and $h_i(s)$ satisfy the same functional equation (6.6), if s and $s+1$ belong to $\mathcal{S}_\varepsilon$ we have

$$\begin{aligned} w(s+1)h_1(s+1) &= f_1(s+1) \\ &= A_{11}(s)f_1(s) + A_{12}(s)f_2(s) \\ &= w(s)[A_{11}(s)h_1(s) + A_{12}(s)h_2(s)], \end{aligned}$$

and therefore $w(s+1) = w(s)$. Using again the fact that f_i and h_i are analytic in this strip and h_i is also zero free there, we conclude that $w(s)$ is analytic in $\mathcal{S}_\varepsilon$, and the periodicity of w implies that it may be extended to an entire periodic function. Lemma 6.11(ii) together with the uniform boundedness of $f_i(s)$ in $\mathcal{S}_\varepsilon$ imply that there exists a constant $C > 0$ such that for all $s \in \mathcal{S}_\varepsilon$,

$$|w(s)| < Ce^{\pi|\mathrm{Im}(s)|}.$$

By the periodicity of w, we conclude that the above bound holds for all $s \in \mathbb{C}$. Since w is periodic with period one, this bound implies that w is a constant function (this argument may be found in the proof of Proposition 2 in [46]). Finally, we know that $f_i(1) = h_i(1) = 1$, and so we conclude that $w(s) \equiv 1$. Hence, $f_i(s) \equiv h_i(s)$ for all $s \in \mathcal{S}_\varepsilon$. Since $h_i(s)$ are analytic in the wider strip $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$, we use the result [59, Theorem 2] and argue, as in the proof of Proposition 3.11, that (6.7) holds for all s such that $-1/\alpha < \mathrm{Re}(s) < 2 - 1/\alpha$. □

Remark 6.13. Since the proof of Theorem 6.10 is based on a verification technique, it does not reveal how we derived the formula on the right-hand side of (6.7). For this a trial and error approach was necessary. As the expression in (6.7) is already known in the two spectrally one-sided cases ($\rho = 1 - 1/\alpha$ and $\rho = 1/\alpha$) due to the theory of scale functions and the paper of Peskir [66], respectively, and in the symmetric case ($\rho = 1/2$) due to [81, 29], we sought a function which interpolated these three cases and satisfied the functional equation (6.6).

We turn our attention to computing the density of T_0. Let us define

$$\|x\| = \min_{n\in\mathbb{Z}}|x-n|,$$

which is the fractional part of either x or $-x$, whichever is smaller. We let

$$\mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} \tfrac{1}{n} \log\|nx\| = 0\}).$$

This set was introduced in Hubalek and Kuznetsov [39], where it was shown that $\mathcal{L}$ is a subset of the Liouville numbers, which implies $x \in \mathcal{L}$ if and only if the coefficients of the continued fraction representation of x grow extremely fast. It is known that $\mathcal{L}$ is dense, yet it is a rather small set: it has Hausdorff dimension zero, and therefore its Lebesgue measure is also zero.

For $\alpha \in \mathbb{R}$ we also define

$$\mathcal{K}(\alpha) = \{N \in \mathbb{N} : \|(N-\tfrac{1}{2})\alpha\| > \exp(-\tfrac{\alpha-1}{2}(N-2)\log(N-2))\}.$$

Proposition 6.14. *Assume that $\alpha \notin \mathbb{Q}$.*

(i) The set $\mathcal{K}(\alpha)$ is unbounded and has density equal to one, in the sense that

$$\lim_{n\to\infty} \frac{\#(\mathcal{K}(\alpha)\cap[1,n])}{n} = 1.$$

(ii) If $\alpha \notin \mathcal{L}$, the set $\mathbb{N}\setminus\mathcal{K}(\alpha)$ is finite.

Proof. Fix $\varepsilon \in (0, 1/2)$. It follows from the ergodic theorem that for any irrational α the sequence $\|(N-\frac{1}{2})\alpha\|$ is uniformly distributed on the interval $(0, 1/2)$. Therefore, the density of the set of all N such that

$$\|(N-\tfrac{1}{2})\alpha\| > \varepsilon$$

is equal to $1-2\varepsilon$. At the same time, for all N large enough, we will certainly have $\exp(-\frac{\alpha-1}{2}(N-2)\log(N-2)) < \varepsilon$. Thus the density of the set of all N such that

$$\|(N-\tfrac{1}{2})\alpha\| > \exp(-\tfrac{\alpha-1}{2}(N-2)\log(N-2))$$

is greater than $1-2\varepsilon$, which proves part (i).

To prove part (ii), assume that $\alpha \notin \mathcal{L}$. Since $\lim_{n\to\infty}\frac{1}{n}\log\|n\alpha\| = 0$, there exists $C > 0$ such that for all n we have $\|n\alpha\| > C2^{-n}$. Then for all N we have

$$\|(N-\tfrac{1}{2})\alpha\| \geq \frac{1}{2}\|(2N-1)\alpha\| > C2^{-2N}.$$

Since for all N large enough it is true that

$$C2^{-2N} > \exp(-\tfrac{\alpha-1}{2}(N-2)\log(N-2)),$$

we conclude that all N large enough will be in the set $\mathcal{K}(\alpha)$, therefore the set $\mathbb{N} \setminus \mathcal{K}(\alpha)$ is finite. □

Theorem 6.15. *The random variable T_0 is absolutely continuous under P_1. Denote its density by $p\colon [0,\infty) \to [0,\infty)$.*

(i) If $\alpha \notin \mathbb{Q}$ then for all $t > 0$ we have

$$p(t) = \lim_{\substack{N\in\mathcal{K}(\alpha)\\ N\to\infty}} \Bigg[\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat\rho)} \sum_{1\le k<\alpha(N-\frac12)-1} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \times \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} - \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat\rho)} \sum_{1\le k<N} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}} \Bigg]. \quad (6.9)$$

The above limit is uniform for $t \in [\varepsilon,\infty)$ and any $\varepsilon > 0$.

(ii) If $\alpha = m/n$, with $m,n \in \mathbb{N}$ coprime, then for all $t > 0$ we have

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat\rho)} \sum_{\substack{k\ge1\\ k\not\equiv -1 \pmod m}} \sin(\pi\hat\rho(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \times \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} - \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat\rho)} \sum_{\substack{k\ge1\\ k\not\equiv 0 \pmod n}} \frac{\sin(\pi\alpha\hat\rho k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}} - \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi^2\alpha\sin(\pi\hat\rho)} \sum_{k\ge1} (-1)^{km} \frac{\Gamma\left(kn-\frac{1}{\alpha}\right)}{(km-2)!} R_k(t) t^{-kn-1+\frac{1}{\alpha}}, \quad (6.10)$$

where

$$R_k(t) := \pi\alpha\hat\rho\cos(\pi\hat\rho km) - \sin(\pi\hat\rho km)\left[\pi\cot\left(\tfrac{\pi}{\alpha}\right) - \boldsymbol{\psi}\left(kn-\tfrac{1}{\alpha}\right) + \alpha\boldsymbol{\psi}(km-1) + \log(t)\right]$$

and $\boldsymbol{\psi}$ is the digamma function. The three series in (6.10) converge uniformly for $t \in [\varepsilon,\infty)$ and any $\varepsilon > 0$.

(iii) *For all values of α and any $c > 0$, the following asymptotic expansion holds as $t \downarrow 0$:*

$$p(t) = \frac{\alpha \sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{1 \le n < 1+c} \sin(\pi\alpha\hat{\rho}n) \frac{\Gamma(\alpha n + 1)}{\Gamma\left(n + \frac{1}{\alpha}\right)} (-1)^{n-1} t^{n-1+\frac{1}{\alpha}} + O(t^{c+\frac{1}{\alpha}}).$$

Proof. Recall that $h_1(s) = \mathrm{E}_1[T_0^{s-1}]$ denotes the function in (6.7). According to Lemma 6.11(ii), for any $x \in \mathbb{R}$, $h_1(x + \mathrm{i}y)$ decreases to zero exponentially fast as $|y| \to \infty$. This implies that the density of T_0 exists and is a smooth function. (See, for example, Kawata [43, §11.6]. It is in any case known that T_0 is absolutely continuous, since X has absolutely continuous transition measures; see Monrad [61].) It also implies that $p(t)$ can be written as the inverse Mellin transform,

$$p(t) = \frac{1}{2\pi\mathrm{i}} \int\limits_{1+\mathrm{i}\mathbb{R}} h_1(s) t^{-s} \mathrm{d}s. \tag{6.11}$$

The function $h_1(s)$ is meromorphic, and it has poles at points

$$\{s_n^{(1)} := 1 + n/\alpha : n \ge 1\} \cup \{s_n^{(2)} := n - 1/\alpha : n \ge 2\} \cup \{s_n^{(3)} := -n - 1/\alpha : n \ge 0\}$$

If $\alpha \notin \mathbb{Q}$, all these points are distinct and $h_1(s)$ has only simple poles. When $\alpha \in \mathbb{Q}$, some of $s_n^{(1)}$ and $s_m^{(2)}$ will coincide, and $h_1(s)$ will have double poles at these points.

Let us first consider the case $\alpha \notin \mathbb{Q}$, so that all poles are simple. Let $N \in \mathcal{K}(\alpha)$ and define $c = c(N) = N + \frac{1}{2} - \frac{1}{\alpha}$. Lemma 6.11(ii) tells us that $h_1(s)$ decreases exponentially to zero as $|\mathrm{Im}\, s| \to \infty$, uniformly in any finite vertical strip. Therefore, we can shift the contour of integration in (6.11) and obtain

$$\begin{aligned} p(t) = &- \sum\nolimits_n \mathrm{Res}_{s=s_n^{(1)}} (h_1(s) t^{-s}) - \sum\nolimits_m \mathrm{Res}_{s=s_m^{(2)}} (h_1(s) t^{-s}) \\ &+ \frac{1}{2\pi\mathrm{i}} \int_{c(N)+\mathrm{i}\mathbb{R}} h_1(s) t^{-s} \mathrm{d}s, \end{aligned} \tag{6.12}$$

where $\sum_n$ and $\sum_m$ indicate summation over $n \ge 1$ such that $s_n^{(1)} < c(N)$ and over $m \ge 2$ such that $s_m^{(2)} < c(N)$, respectively. Computing the residues we obtain the two sums in the right-hand side of (6.9).

Now our goal is to show that the integral term

$$I_N(t) := \frac{1}{2\pi\mathrm{i}} \int_{c(N)+\mathrm{i}\mathbb{R}} h_1(s) t^{-s} \mathrm{d}s$$

converges to zero as $N \to \infty$, $N \in \mathcal{K}(\alpha)$. We require the following inequalities:

$$|\sin(\pi x)| \geq \|x\|, \quad |\sin(x)|\cosh(y) \leq |\sin(x+\mathrm{i}y)| \leq \cosh(y), \qquad x, y \in \mathbb{R}.$$

The first of these is simple, and the second follows from the identities

$$\sin(x+\mathrm{i}y) = \sin x \cosh y + \mathrm{i} \cos x \sinh y, \quad |\sin(x+\mathrm{i}y)|^2 = \cosh^2 y - \cos^2 x.$$

We use these inequalites together with the reflection formula for the gamma function to estimate $h_1(s^*)$, where $s^* = c(N) + \mathrm{i}u$ and $u \in \mathbb{R}$, as follows:

$$\begin{aligned} |h_1(s^*)| &= \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \left| \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s^*)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s^*)\right)} \frac{\sin(\pi s^*)}{\sin(\pi\alpha(s^*-1))} \frac{\Gamma(s^*-1)}{\Gamma(\alpha(s^*-1))} \right| \\ &\leq \frac{C_1}{\|\alpha(N-\frac{1}{2})\|} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} \left| \frac{\Gamma(s^*-1)}{\Gamma(\alpha(s^*-1))} \right|, \end{aligned} \tag{6.13}$$

for some constant $C_1 > 0$. Now, using Stirling's formula (3.7), we find that when $\operatorname{Re} s > 0$,

$$\frac{\Gamma(s)}{\Gamma(\alpha s)} = \sqrt{\alpha} e^{-s((\alpha-1)\log(s)+A)+O(s^{-1})}, \qquad s \to \infty,$$

where $A := 1 - \alpha + \alpha\log(\alpha) > 0$. Therefore, there exists a constant $C_2 > 0$ such that for $\operatorname{Re}(s) > 0$ we can estimate

$$\begin{aligned} \left|\frac{\Gamma(s)}{\Gamma(\alpha s)}\right| &< C_2 \left|\exp\left\{-s\left((\alpha-1)\log(s)+A\right)\right\}\right| \\ &= C_2 \exp\left\{-(\alpha-1)\operatorname{Re}(s\log(s)) - A\operatorname{Re}(s)\right\} \\ &= C_2 \exp\left\{-(\alpha-1)\operatorname{Re}(s)\log|s| - A\operatorname{Re}(s) + (\alpha-1)\arg(s)\operatorname{Im}(s)\right\} \\ &< C_2 \exp\left\{-(\alpha-1)\operatorname{Re}(s)\log(\operatorname{Re}(s)) + (\alpha-1)\frac{\pi}{2}|\operatorname{Im}(s)|\right\}. \end{aligned}$$

Taking this together with (6.13) and using the fact that $N \in \mathcal{K}(\alpha)$, we find that, for $u \in \mathbb{R}$,

$$\begin{aligned} |h_1(c(N)+\mathrm{i}u)| &< \frac{C_1 C_2}{\|\alpha(N-\frac{1}{2})\|} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{-(\alpha-1)(c(N)-1)\log(c(N)-1)+(\alpha-1)\frac{\pi}{2}|u|} \\ &< C_1 C_2 e^{-\frac{\alpha-1}{2}(N-2)\log(N-2)} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{(\alpha-1)|u|\frac{\pi}{2}}. \end{aligned}$$

Note that the function in the right-hand side of the above inequality decreases to zero exponentially fast as $|u| \to \infty$ (since $\alpha\hat{\rho} + \frac{1}{2}(\alpha-1) - \alpha < 0$), and hence in particular is integrable on $\mathbb{R}$. Thus we can estimate

$$\begin{aligned}|I_N(t)| &= \frac{t^{-c(N)}}{2\pi}\left|\int_{\mathbb{R}} h_1(c(N)+\mathrm{i}u))t^{-\mathrm{i}u}\,\mathrm{d}u\right| \\ &< \frac{t^{-c(N)}}{2\pi}\int_{\mathbb{R}} |h_1(c(N)+\mathrm{i}u))|\,\mathrm{d}u \\ &< \frac{t^{-c(N)}}{2\pi} C_1 C_2 e^{-\frac{\alpha-1}{2}(N-2)\log(N-2)} \int_{\mathbb{R}} \frac{\cosh(\pi\alpha\hat{\rho}u)}{\cosh(\pi\alpha u)} e^{(\alpha-1)|u|\frac{\pi}{2}}\,\mathrm{d}u. \quad (6.14)\end{aligned}$$

When $N \to \infty$, the right-hand side of (6.14) converges to zero for any fixed $t > 0$, and indeed uniformly for t in any set bounded away from zero. This ends the proof of part (i).

The proof of part (ii) is very similar, and we offer only a sketch. It also begins with (6.12) and uses the above estimate for $h_1(s)$. The only difference is that when $\alpha \in \mathbb{Q}$ some of $s_n^{(1)}$ and $s_m^{(2)}$ will coincide, and $h_1(s)$ will have double poles at these points. The terms with double poles give rise to the third series in (6.10). In this case all three series are convergent, and we can express the limit of partial sums as a series in the usual sense.

The proof of part (iii) is much simpler: we shift the contour of integration in (6.12) in the opposite direction ($c < 0$). The proof is identical to the proof of Theorem 9 in [45]. □

The next corollary shows that, for almost all irrational α, the expression (6.9) can be written in a simpler form.

Corollary 6.16. *If $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then*

$$\begin{aligned}p(t) = &\frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat{\rho})} \sum_{k\ge 1} \sin(\pi\hat{\rho}(k+1)) \frac{\sin\left(\frac{\pi}{\alpha}k\right)}{\sin\left(\frac{\pi}{\alpha}(k+1)\right)} \frac{\Gamma\left(\frac{k}{\alpha}+1\right)}{k!}(-1)^{k-1} t^{-1-\frac{k}{\alpha}} \\ &- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat{\rho})} \sum_{k\ge 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}}. \quad (6.15)\end{aligned}$$

The two series in the right-hand side of the above formula converge uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$.

Proof. As we have shown in Proposition 6.14, if $\alpha \notin \mathcal{L} \cup \mathbb{Q}$ then the set $\mathbb{N} \setminus \mathcal{K}(\alpha)$ is finite. Therefore we can remove the restriction $N \in \mathcal{K}(\alpha)$ in (6.9).

The remaining part of the proof is based on the following simple fact: If the two series $\sum a_n$ and $\sum b_n$ converge, then for any increasing, unbounded sequences c_n and d_n,

$$\lim_{N\to\infty}\left[\sum_{n=1}^{c_N} a_n + \sum_{n=1}^{d_N} b_n\right] = \sum_{n\ge 1} a_n + \sum_{n\ge 1} b_n.$$

Therefore, to finish the proof it is enough to show that both series in (6.15) converge.

In [39, Proposition 1] it was shown that $x \in \mathcal{L}$ iff $x^{-1} \in \mathcal{L}$. Therefore, according to our assumption, neither α nor $1/\alpha$ are in the set $\mathcal{L}$. From the definition of $\mathcal{L}$ we see that there exists $C > 0$ such that $\|\alpha n\| > C2^{-n}$ and $\|\alpha^{-1} n\| > C2^{-n}$ for all integers n. Using the estimate $|\sin(\pi x)| \geq \|x\|$ and Stirling's formula (3.7), one sees that both series in (6.15) converge (uniformly for $t \in [\varepsilon, \infty)$ and any $\varepsilon > 0$), which ends the proof of the corollary. □

Remark 6.17. When $\alpha \in \mathcal{L}$, the series on the right-hand side of (6.15) may fail to converge. An example where this occurs is given after Theorem 5 in [45].

6.2 Applications

6.2.1 Conditioning to avoid zero

In section 3.4, we introduced Pantí's conditioning to avoid zero for a Lévy process. For a stable process this is given by a Doob h-transform of the process killed on hitting zero, with respect to the invariant function $h^{\updownarrow}$ expressed as the formula

$$h^{\updownarrow}(x) = -\Gamma(1-\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}x^{\alpha-1}$$

when $x > 0$, and the same formula with ρ replacing $\hat{\rho}$ when $x < 0$.

Our aim in this section is to prove the following variation of Proposition 3.12(ii), making use of our expression for the density of T_0, in the stable case only. Our presentation here owes much to Yano et al. [82, §4.3].

Proposition 6.18. *Let X be a stable process adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$.*

(i) Define the function

$$Y(s,x) = \frac{\mathrm{P}_x(T_0 > s)}{h^{\updownarrow}(x) n(\zeta > s)} \qquad s > 0,\, x \neq 0.$$

Then, for any $x \neq 0$,

$$\lim_{s\to\infty} Y(s,x) = 1, \tag{6.16}$$

and furthermore, Y is bounded on its whole domain.

(ii) For any $x \neq 0$, any P_x-a.s. finite stopping time T, and $\Lambda \in \mathscr{F}_T$,

$$\mathrm{P}^{\updownarrow}_x(\Lambda) = \lim_{s\to\infty} \mathrm{P}_x(\Lambda \mid T_0 > T + s).$$

Proof. We begin by proving

$$h^{\updownarrow}(x) = \lim_{s\to\infty} \frac{\mathrm{P}_x(T_0 > s)}{n(\zeta > s)}, \tag{6.17}$$

for $x > 0$, noting that when $x < 0$, we may deduce the same limit by duality.

Let us denote the density of the measure $\mathrm{P}_x(T_0 \in \cdot)$ by $p(x, \cdot)$. A straightforward application of scaling shows that

$$\mathrm{P}_x(T_0 > t) = \mathrm{P}_1(T_0 > x^{-\alpha}t), \qquad x > 0,\, t \geq 0,$$

and so we may focus our attention on $p(1,t)$, which is the quantity given as $p(t)$ in Theorem 6.15. In particular, we have

$$p(1,t) = -\frac{\sin^2(\pi/\alpha)}{\pi \sin(\pi\hat{\rho})} \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha)} \frac{\Gamma(1-1/\alpha)}{\Gamma(\alpha-1)} t^{1/\alpha-2} + O(t^{-1/\alpha-1}).$$

Denote the coefficient of $t^{1/\alpha-2}$ in the first term of this expression by P. Note that all limits in Theorem 3.15 exist uniformly in $t \in [\varepsilon, \infty)$, so we can integrate in t on any interval $[t_0, \infty)$ for $t_0 > 0$. This allows us to compute asymptotic expressions for $\mathrm{P}_x(T_0 > t)$ from the expansions in Theorem 6.15 by integrating term-by-term.

To obtain an expression for $n(\zeta > t)$, we turn to Fitzsimmons and Getoor [34], in which the authors compute explicitly the density of $n(\zeta \in \cdot)$ for a stable process; see p. 84 in that work, where n is denoted P^* and ζ is denoted R. The authors work with a different normalisation of the stable process; they have $c = 1$. In our context, their result says

$$n(\zeta \in \mathrm{d}t) = \frac{\alpha - 1}{\Gamma(1/\alpha)} \frac{\sin(\pi/\alpha)}{\cos(\pi(\rho - 1/2))} t^{1/\alpha-2}\, \mathrm{d}t, \qquad t \geq 0. \tag{6.18}$$

Denote the coefficient in the above power law by W.

We can now compute $h^{\updownarrow}$. We will use elementary properties of trigonometric functions and the reflection identity for the gamma function. For $x > 0$,

$$\begin{aligned}
\frac{\mathrm{P}_x(T_0 > t)}{n(\zeta > t)} &= \frac{P}{W} x^{\alpha-1} + O(t^{1-2/\alpha}) \\
&= -\frac{\cos(\pi(\rho-1/2))\sin(\pi\alpha\hat{\rho})}{\Gamma(\alpha)\sin(\pi\hat{\rho})\sin(\pi\alpha)} x^{\alpha-1} + o(1) \\
&= -\frac{1}{\Gamma(\alpha)} \frac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha)} x^{\alpha-1} + o(1) \\
&= -\Gamma(1-\alpha) \frac{\sin(\pi\alpha\hat{\rho})}{\pi} x^{\alpha-1} + o(1).
\end{aligned}$$

This proves (6.17) for $x > 0$, and it is simple to deduce via duality that the limit holds for $x \neq 0$.

We now turn our attention to the slightly more delicate result about Y. It is clear that the limit in (6.16) holds, so we only need to prove that Y is bounded. We begin by noting that, for fixed $x \neq 0$, $Y(t,x)$ is bounded in t since the function is continuous and converges to 1 as $t \to \infty$. Now, due to the expression (6.18) and the scaling property of X, we have the relation $Y(t,x) = Y(|x|^{\alpha-1}t, \operatorname{sgn} x)$. This then shows that Y is bounded as a function of two variables.

With this in hand, we move on to the calculation of the limiting measure. This proceeds along familiar lines, using the strong Markov property:

$$\begin{aligned}
&\mathrm{P}_x(\Lambda \mid T_0 > T+s) \\
&\quad = \mathrm{E}_x\left[\frac{\mathrm{P}_x(\mathbb{1}_\Lambda, T_0 > T+s \mid \mathscr{F}_T)}{\mathrm{P}_x(T_0 > T+s)}\right] \\
&\quad = \mathrm{E}_x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{T_0>T\}} \frac{\mathrm{P}_{X_T}(T_0 > s)}{\mathrm{P}_x(T_0 > T+s)}\right] \\
&\quad = \mathrm{E}_x\left[\mathbb{1}_\Lambda \mathbb{1}_{\{T_0>T\}} h^\updownarrow(X_T) Y(s, X_T) \frac{n(\zeta > s)}{n(\zeta > T+s)} \frac{1}{h(x)Y(s+T,x)}\right].
\end{aligned}$$

Now, as $h^\updownarrow$ is invariant for the stable process killed at zero, (3.19) also holds at the stopping time T, and in particular the random variable $h^\updownarrow(X_T)$ is integrable; meanwhile, Y is bounded, as are the final two ratios, at least as $s \to \infty$. If we now apply the dominated convergence theorem, we obtain the result we seek. □

We offer a brief comparison to conditioning a Lévy process to stay positive. In this case, Chaumont [20, Remark 1] observes that the analogue of Proposition 6.18(ii) holds under Spitzer's condition, and in particular for a stable process. However, it appears that in general, a key role is played by the exponential random variable analogous to that appearing in Proposition 3.12(ii).

6.2.2 The entrance law of the excursion measure

It is known, from a more general result [27, (2.8)] on Markov processes in weak duality, that for any Borel function f,

$$\int_0^\infty e^{-qt} n(f(X_t))\,\mathrm{d}t = \int_{\mathbb{R}} f(x) \hat{\mathrm{E}}_x\left[e^{-qT_0}\right] \mathrm{d}x.$$

(This formulation is from [82, (3.9)].) In terms of densities, this may be written

$$\begin{aligned}
n(X_t \in \mathrm{d}x)\,\mathrm{d}t &= \hat{\mathrm{P}}_x(T_0 \in \mathrm{d}t)\,\mathrm{d}x \\
&= |x|^{-\alpha} \mathrm{P}_{\operatorname{sgn}(-x)}(T_0 \in |x|^{-\alpha}\mathrm{d}t)\,\mathrm{d}x
\end{aligned}$$

Therefore, our expressions in Theorem 6.15 for the density of T_0 yield expressions for the density of the entrance law of the excursions of the stable process from zero.

6.2.3 Identities in law using the exponential functional

In a series of papers (Bertoin and Yor [8], Caballero and Chaumont [15] and Chaumont et al. [26]) it is proved that under certain conditions, the laws $(\mathrm{P}_x)_{x>0}$ of an α-pssMp X admit a weak limit P_0 as $x \downarrow 0$, in the Skorokhod space of càdlàg paths. If ξ is the Lamperti transform of X under P_1, then provided that $\mathbb{E}|\xi_1| < \infty$ and $m := \mathbb{E}\xi_1 > 0$, it is known that the entrance law of P_0 satisfies

$$\mathrm{E}_0(f(X_t^\alpha)) = \frac{1}{\alpha m}\mathbb{E}(I(-\alpha\xi)^{-1} f(t/I(-\alpha\xi))),$$

for any $t > 0$ and Borel function f. Similar expressions exist under less restrictive conditions on ξ.

It is tempting to speculate that any rssMp may admit a weak limit P_0 along similar lines, but we do not propose any results in this direction; instead, we demonstrate similar formulae for the entrance law $n(X_t \in \cdot)$ of the stable process, and the corresponding measure $\mathrm{P}_0^\updownarrow$ for the stable process conditioned to avoid zero.

Let X be a stable process, possibly asymmetric. From the previous subsection, we have that

$$n(f(X_t)) = \int_{-\infty}^{\infty} |x|^{-\alpha} p(\operatorname{sgn}(-x), |x|^{-\alpha} t) f(x)\,\mathrm{d}x.$$

Substituting in the integral, and recalling that the law of T_0 for the stable process is equal to the law of the exponential functional $I(\alpha\xi)$ of the Markov additive process associated with it, we obtain

$$\begin{aligned} n(f(X_t)) &= \frac{1}{\alpha}\int_0^\infty p(1,u) f(-(u/t)^{-1/\alpha}) u^{-1/\alpha} t^{1/\alpha - 1}\,\mathrm{d}u \\ &\quad + \frac{1}{\alpha}\int_0^\infty p(-1,u) f((u/t)^{-1/\alpha}) u^{-1/\alpha} t^{1/\alpha-1}\,\mathrm{d}u \\ &= \frac{1}{\alpha}\mathbb{E}_1\Big[f\big(-(t/I(\alpha\xi))^{1/\alpha}\big) I(\alpha\xi)^{-1/\alpha} t^{1/\alpha-1}\Big] \\ &\quad + \frac{1}{\alpha}\mathbb{E}_2\Big[f\big((t/I(\alpha\xi))^{1/\alpha}\big) I(\alpha\xi)^{-1/\alpha} t^{1/\alpha-1}\Big]. \end{aligned}$$

Recall from [64] that the law $\mathrm{P}_0^\updownarrow$ of the stable process conditioned to avoid zero is given by the following harmonic transform of the stable excursion measure n:

$$\mathrm{P}_0^\updownarrow(\Lambda) = n(\mathbb{1}_\Lambda h^\updownarrow(X_t), t < \zeta), \qquad t \geq 0,\ \Lambda \in \mathscr{F}_t,$$

with $h^{\updownarrow}$ as in (3.18). Therefore, applying the above result to the Borel function $h^{\updownarrow} f$, we obtain

$$\begin{aligned}\mathrm{E}_0^{\updownarrow}(f(X_t)) &= n(h^{\updownarrow}(X_t)f(X_t)) \\ &= \Gamma(-\alpha)\frac{\sin(\pi\alpha\rho)}{\pi}\mathbb{E}_1\Big[I(\alpha\xi)^{-1}f\big(-(t/I(\alpha\xi))^{1/\alpha}\big)\Big] \\ &\quad + \Gamma(-\alpha)\frac{\sin(\pi\alpha\hat{\rho})}{\pi}\mathbb{E}_2\Big[I(\alpha\xi)^{-1}f\big((t/I(\alpha\xi))^{1/\alpha}\big)\Big],\end{aligned}$$

where we emphasise that $I(\alpha\xi)$ (under $\mathbb{E}_i$) is still the exponential functional of the Markov additive process associated to X.

6.3 The Cramér number and conditionings at zero

In this section we make some brief remarks on the relationship between the Cramér number of a Lévy process or a Markov additive process, and conditioning of the pssMp or rssMp associated to it, taking as our example the stable process which has been our object of study. At the end, we suggest further work with a view to making the discussion rigorous.

We begin by reviewing known facts about Lévy processes.

Let ξ be a Lévy process. For the sake of simplicity, we will work under the measure $\mathbb{P} = \mathbb{P}_0$, under which the process starts from zero. Denote by ϕ the Laplace$^+$ exponent of ξ, that is, $e^{\phi(z)} = \mathbb{E}e^{z\xi_1}$. Suppose that ξ satisfies the Cramér condition with Cramér number $\theta > 0$, in the following sense: there exists some $\varepsilon > 0$ such that ϕ is defined on $(0, \theta+\varepsilon)$, and $\phi(\theta) = 0$. Since ϕ is smooth and convex where it is defined, ξ drifts to $-\infty$, and furthermore, under $\mathbb{P}$, the process

$$M(t,\theta) = e^{\theta\xi_t}, \qquad t \geq 0,$$

is a martingale in the natural filtration $(\mathscr{H}_t)_{t\geq 0}$ of ξ; it is sometimes called the Wald martingale. We denote by $\mathbb{P}^{\natural}$ the measure obtained by 'tilting' the process ξ using $M(\cdot, \theta)$:

$$\mathbb{P}^{\natural}(\Lambda) = \mathbb{E}[\mathbb{1}_\Lambda M(t,\theta)], \qquad t \geq 0,\, \Lambda \in \mathscr{H}_t.$$

This is essentially the Esscher transformation which we discussed in section 2.6.2, but with a different sign convention. It can be shown that, under $\mathbb{P}^{\natural}$, ξ is a Lévy process which drifts to $+\infty$, and is commonly thought of as ξ *conditioned to drift to* $+\infty$. Indeed, when ξ is spectrally negative, the following is true.

Lemma 6.19 ([6, Lemma VII.7]). *Let ξ be a spectrally negative Lévy process with*

Cramér number θ. Then for every $t > 0$ and $\Lambda \in \mathscr{H}_t$,

$$\lim_{x\to\infty} \mathbb{P}(\Lambda | \overline{\xi}_\infty > x) = \mathbb{P}^\natural(\Lambda),$$

which is to say, $\mathbb{P}^\natural$ is the law of ξ conditioned to pass every positive level.

On the other hand, when ξ is not spectrally negative, we are not aware of any existing results showing that $\mathbb{P}^\natural$ is a limit of conditionings, but one still has the following:

Lemma 6.20 ([7])**.** *Let ξ be a Lévy process with Cramér number θ. Then*

$$\lim_{x\to\infty} e^{\theta x}\mathbb{P}(\overline{\xi}_\infty > x) = D$$

for some constant $D \geq 0$, and $D > 0$ if and only if $\mathbb{E}^\natural|\xi_1| < \infty$.

To see the relationship with pssMps, let us consider a symmetric stable process, X, with $\alpha > 1$. We write $R = |X|$, just as we had in section 3.3. Furthermore, let ξ^R be the Lamperti representation of the pssMp R. We saw in Theorem 3.8 that the characteristic exponent of ξ^R is given by

$$\Psi^R(\theta) = 2^\alpha \frac{\Gamma(\alpha/2 - \mathrm{i}\theta/2)}{\Gamma(-\mathrm{i}\theta/2)} \frac{\Gamma(1/2 + \mathrm{i}\theta/2)}{\Gamma((1-\alpha)/2 + \mathrm{i}\theta/2)}, \qquad \theta \in \mathbb{R},$$

and it is clear from this that R satisfies the Cramér condition with Cramér number $\theta = \alpha - 1$.

Then, for each $y \in \mathbb{R}$, the process

$$M(s, \alpha - 1) = e^{(\alpha-1)(\xi^R_s - y)}, \qquad s \geq 0,$$

is a martingale under $\mathbb{P}_y$ in the natural filtration $(\mathscr{H}_t)_{t\geq 0}$ of ξ^R. Furthermore, since for every $t \geq 0$, the random variable $S(t)$ appearing in the Lamperti representation is an $(\mathscr{H}_t)_{t\geq 0}$-stopping time, it follows that for each $x = e^y$, the process

$$N(t, \alpha - 1) = \left(\frac{R_t}{x}\right)^{\alpha-1} = e^{(\alpha-1)(\xi^R_{S(t)} - x)}, \qquad t \geq 0,$$

is also a martingale, under P_x in the natural filtration of R.

The martingale $N(\cdot, \alpha - 1)$ is then precisely $h^\updownarrow(X_t)/h^\updownarrow(x)$, the conditioning of Pantí in the symmetric stable case. This is far from being a reproof of the fact that the invariant function $h^\updownarrow$ conditions a symmetric stable process to avoid zero, but it is suggestive, and we continue this line of thought.

We turn to the situation for Markov additive processes and rssMps. Suppose now that (ξ, J) is a Markov additive process in the framework of section 6.1.1, satisfying the Cramér condition with Cramér number $\theta > 0$. In this situation this means that the matrix exponent F is defined on $(0, \theta + \varepsilon)$ for some ε, and that the leading eigenvalue k satisfies $k(\theta) = 0$. As in the case of a Lévy process, ξ then drifts to $-\infty$ almost surely.

We have already discussed, in Proposition 6.3, the Wald martingale for a Markov additive process; evaluated at the Cramér number, it is the process

$$M(t, \theta) = e^{\theta(\xi(t) - \xi(0))} \frac{v_{J(t)}(\theta)}{v_{J(0)}(\theta)}, \qquad t \geq 0, \tag{6.19}$$

where $v(\theta)$ is a right-eigenvector of $F(\theta)$ associated to the simple eigenvalue 0. Let us denote by $\mathbb{P}^\natural$ the law of the Markov additive process which arises after changing measure with respect to $M(\cdot, \theta)$. Certainly, ξ under $\mathbb{P}^\natural$ drifts to $+\infty$, and it is natural to think of it, by analogy with Lévy processes, as the process conditioned to drift to $+\infty$.

To explore the connection with rssMps, let us again return to our example of the stable process X with $\alpha > 1$. The process X is an rssMp, and the associated MAP (ξ, J) was was computed in section 6.1.4. It has matrix exponent F given by

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix},$$

for $\operatorname{Re} z \in (-1, \alpha)$. In particular, the process satisfies the Cramér condition with Cramér number $\alpha - 1 > 0$. It is not difficult to compute the eigenvector $v(\alpha - 1)$, and substituting in (6.19) yields the following expression for the Wald martingale:

$$M(t, \alpha - 1) = \begin{cases} e^{(\alpha-1)\xi(t)}, & J(0) = 1, J(t) = 1, \\ \dfrac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})} e^{(\alpha-1)\xi(t)}, & J(0) = 1, J(t) = 2, \\ \dfrac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\rho)} e^{(\alpha-1)\xi(t)}, & J(0) = 2, J(t) = 1, \\ e^{(\alpha-1)\xi(t)}, & J(0) = 2, J(t) = 2, \end{cases} \qquad t \geq 0,$$

where it is assumed $\xi(0) = 0$. Since, again, the random variables $\tau(t)$ appearing in the Lamperti–Kiu representation are stopping times for all $t \geq 0$, we have the equivalent martingale for the process X started at any point $x \neq 0$:

$$N(t, \alpha - 1) = C_t \left| \frac{X_t}{x} \right|^{\alpha - 1}, \tag{6.20}$$

where

$$C_t = \begin{cases} 1, & xX_t > 0, \\ \dfrac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat{\rho})}, & x > 0, X_t < 0, \\ \dfrac{\sin(\pi\alpha\hat{\rho})}{\sin(\pi\alpha\rho)}, & x < 0, X_t > 0. \end{cases}$$

This is precisely $h^{\updownarrow}(X_t)/h^{\updownarrow}(x)$, the martingale which is used by Pantí [63] to condition a stable process to avoid zero.

Let us consider what occurs when $\alpha < 1$. Now the Lévy process or Markov additive process, ξ, will still satisfy a Cramér condition with Cramér number $\alpha - 1$, but now $\alpha - 1 < 0$. This essentially reverses the situation. The process ξ drifts to $+\infty$, which corresponds to the fact that the process X is transient. The process $M(\cdot, \alpha - 1)$ remains a martingale, but the measure $\mathbb{P}^\natural$ now represents the process ξ conditioned to drift to $-\infty$.

Looking now at the associated self-similar process, we consider first the simpler case, in which ξ is a Lévy process and its associated pssMp R is the radial part of a symmetric stable process. Then the new measure induced by the martingale $N(\cdot, \alpha - 1)$ is precisely the law of the pssMp R conditioned to hit zero (continuously), as in Chaumont and Rivero [23, Theorem 14].

If we are instead working with the general case, in which ξ is a Markov additive process whose associated rssMp is the stable process X, the process $N(\cdot, \alpha - 1)$ of (6.20) remains a martingale, and it is natural to imagine that the new measure induced by this martingale is the law of a transient stable process conditioned to hit zero. However, we are not aware of any existing research on this question.

Finally, we summarise our suggestions for further work. The discussion we have just made has been very heuristic. It would be interesting to show rigorously that, if an rssMp X is associated with a Markov additive process ξ possessing a Cramér number $\theta > 0$, then the h-transform $x \mapsto w(\operatorname{sgn} x)x^\theta$, where the function $w\colon \{-1, 1\} \to [0, \infty)$ is obtained from the Wald martingale of ξ, induces a conditioning of X to avoid zero. One consequence of this would be a reproof of the result of Pantí in the special case of a stable process. Indeed, even in the case where X is a pssMp and ξ its associated Lévy process, this would, to our knowledge, be a new result.

On the other hand, if X is an rssMp associated to a Markov additive process with a *negative* Cramér number, one expects the corresponding h-function to condition X to hit zero continuously. This would complement the results of Chaumont and Rivero [23] already mentioned.

Bibliography

[1] S. Asmussen. *Ruin probabilities*, volume 2 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co. Inc., River Edge, NJ, 2000. ISBN 981-02-2293-9. doi:10.1142/9789812779311.

[2] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. ISBN 0-387-00211-1.

[3] E. W. Barnes. The genesis of the double gamma functions. *Proc. London Math. Soc.*, S1-31(1):358, 1899. ISSN 0024-6115. doi:10.1112/plms/s1-31.1.358.

[4] E. J. Baurdoux. Some excursion calculations for reflected Lévy processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, 6:149–162, 2009. ISSN 1980-0436.

[5] J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.*, 47(1):17–35, 1993. ISSN 0304-4149. doi:10.1016/0304-4149(93)90092-I.

[6] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. ISBN 0-521-56243-0.

[7] J. Bertoin and R. A. Doney. Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, 21(5):363–365, 1994. ISSN 0167-7152. doi:10.1016/0167-7152(94)00032-8.

[8] J. Bertoin and M. Yor. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, 17(4):389–400, 2002. ISSN 0926-2601. doi:10.1023/A:1016377720516.

[9] J. Bertoin and M. Yor. Exponential functionals of Lévy processes. *Probab. Surv.*, 2:191–212, 2005. ISSN 1549-5787. doi:10.1214/154957805100000122.

[10] R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.

[11] R. M. Blumenthal, R. K. Getoor, and D. B. Ray. On the distribution of first hits for the symmetric stable processes. *Trans. Amer. Math. Soc.*, 99: 540–554, 1961. ISSN 0002-9947.

[12] K. Bogdan, K. Burdzy, and Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields*, 127(1):89–152, 2003. ISSN 0178-8051. doi:10.1007/s00440-003-0275-1.

[13] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 1996. ISBN 3-7643-5463-1.

[14] M. E. Caballero and L. Chaumont. Conditioned stable Lévy processes and the Lamperti representation. *J. Appl. Probab.*, 43(4):967–983, 2006. ISSN 0021-9002. doi:10.1239/jap/1165505201.

[15] M. E. Caballero and L. Chaumont. Weak convergence of positive self-similar Markov processes and overshoots of Lévy processes. *Ann. Probab.*, 34(3): 1012–1034, 2006. ISSN 0091-1798. doi:10.1214/009117905000000611.

[16] M. E. Caballero, J. C. Pardo, and J. L. Pérez. On Lamperti stable processes. *Probab. Math. Statist.*, 30(1):1–28, 2010. ISSN 0208-4147.

[17] M. E. Caballero, J. C. Pardo, and J. L. Pérez. Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli*, 17(1):34–59, 2011. ISSN 1350-7265. doi:10.3150/10-BEJ275.

[18] P. Carmona, F. Petit, and M. Yor. On the distribution and asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Bibl. Rev. Mat. Iberoamericana, pages 73–130. Rev. Mat. Iberoamericana, Madrid, 1997.

[19] J. M. Chambers, C. L. Mallows, and B. W. Stuck. A method for simulating stable random variables. *J. Amer. Statist. Assoc.*, 71(354):340–344, 1976. ISSN 0162-1459.

[20] L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.*, 64(1):39–54, 1996. ISSN 0304-4149. doi:10.1016/S0304-4149(96)00081-6.

[21] L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.*, 10:no. 28, 948–961 (electronic), 2005. ISSN 1083-6489.

[22] L. Chaumont and J. C. Pardo. The lower envelope of positive self-similar Markov processes. *Electron. J. Probab.*, 11:no. 49, 1321–1341, 2006. ISSN 1083-6489. doi:10.1214/EJP.v11-382.

[23] L. Chaumont and V. Rivero. On some transformations between positive self-similar Markov processes. *Stochastic Process. Appl.*, 117(12):1889–1909, 2007. ISSN 0304-4149. doi:10.1016/j.spa.2007.03.007.

[24] L. Chaumont, A. E. Kyprianou, and J. C. Pardo. Some explicit identities associated with positive self-similar Markov processes. *Stochastic Process. Appl.*, 119(3):980–1000, 2009. ISSN 0304-4149. doi:10.1016/j.spa.2008.05.001.

[25] L. Chaumont, H. Pantí, and V. Rivero. The Lamperti representation of real-valued self-similar Markov processes. arXiv:arXiv:1111.1272v2 [math.PR], 2011.

[26] L. Chaumont, A. Kyprianou, J. C. Pardo, and V. Rivero. Fluctuation theory and exit systems for positive self-similar Markov processes. *Ann. Probab.*, 40(1):245–279, 2012. ISSN 0091-1798. doi:10.1214/10-AOP612.

[27] Z.-Q. Chen, M. Fukushima, and J. Ying. *Extending Markov processes in weak duality by Poisson point processes of excursions*, volume 2 of *Abel Symp.*, pages 153–196. Springer, Berlin, 2007. doi:10.1007/978-3-540-70847-6_7.

[28] O. Chybiryakov. The Lamperti correspondence extended to Lévy processes and semi-stable Markov processes in locally compact groups. *Stochastic Process. Appl.*, 116(5):857–872, 2006. ISSN 0304-4149. doi:10.1016/j.spa.2005.11.009.

[29] F. Cordero. *On the excursion theory for the symmetric stable Lévy processes with index* $\alpha \in]1,2]$ *and some applications.* PhD thesis, Université Pierre et Marie Curie – Paris VI, 2010.

[30] R. A. Doney. On Wiener-Hopf factorisation and the distribution of extrema for certain stable processes. *Ann. Probab.*, 15(4):1352–1362, 1987. ISSN 0091-1798.

[31] R. A. Doney. *Fluctuation theory for Lévy processes*, volume 1897 of *Lecture Notes in Mathematics.* Springer, Berlin, 2007. ISBN 978-3-540-48510-0.

[32] R. A. Doney and A. E. Kyprianou. Overshoots and undershoots of Lévy processes. *The Annals of Applied Probability*, 16(1):91–106, 2006. doi:10.1214/105051605000000647.

[33] P. J. Fitzsimmons. On the existence of recurrent extensions of self-similar Markov processes. *Electron. Comm. Probab.*, 11:230–241, 2006. doi:10.1214/ECP.v11-1222.

[34] P. J. Fitzsimmons and R. K. Getoor. Occupation time distributions for Lévy bridges and excursions. *Stochastic Process. Appl.*, 58(1):73–89, 1995. ISSN 0304-4149. doi:10.1016/0304-4149(95)00013-W.

[35] R. K. Getoor. First passage times for symmetric stable processes in space. *Trans. Amer. Math. Soc.*, 101:75–90, 1961. doi:10.2307/1993412.

[36] R. K. Getoor. Continuous additive functionals of a Markov process with applications to processes with independent increments. *J. Math. Anal. Appl.*, 13:132–153, 1966. doi:10.1016/0022-247X(66)90079-5.

[37] A. V. Gnedin. Regeneration in random combinatorial structures. *Probab. Surv.*, 7:105–156, 2010. ISSN 1549-5787. doi:10.1214/10-PS163.

[38] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007. ISBN 978-0-12-373637-6. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger.

[39] F. Hubalek and A. Kuznetsov. A convergent series representation for the density of the supremum of a stable process. *Electron. Commun. Probab.*, 16:84–95, 2011. ISSN 1083-589X. doi:10.1214/ECP.v16-1601.

[40] J. Ivanovs. *One-sided Markov additive processes and related exit problems.* PhD thesis, Universiteit van Amsterdam, 2011.

[41] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I. Fourier analysis and semigroups.* Imperial College Press, London, 2001. ISBN 1-86094-293-8. doi:10.1142/9781860949746.

[42] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications. Springer-Verlag, New York, second edition, 2002. ISBN 0-387-95313-2.

[43] T. Kawata. *Fourier analysis in probability theory.* Academic Press, New York, 1972. Probability and Mathematical Statistics, No. 15.

[44] A. Kuznetsov. Wiener-Hopf factorization for a family of Lévy processes related to theta functions. *J. Appl. Probab.*, 47(4):1023–1033, 2010. ISSN 0021-9002.

[45] A. Kuznetsov. On extrema of stable processes. *Ann. Probab.*, 39(3):1027–1060, 2011. ISSN 0091-1798. doi:10.1214/10-AOP577.

[46] A. Kuznetsov and J. C. Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. arXiv:1012.0817v1 [math.PR], 2010.

[47] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, and K. van Schaik. A Wiener-Hopf Monte Carlo simulation technique for Lévy processes. *Ann. Appl. Probab.*, 21(6):2171–2190, 2011. ISSN 1050-5164. doi:10.1214/10-AAP746.

[48] A. Kuznetsov, A. E. Kyprianou, and J. C. Pardo. Meromorphic Lévy processes and their fluctuation identities. *Ann. Appl. Probab.*, 22(3):1101–1135, 2012. doi:10.1214/11-AAP787.

[49] A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, and A. R. Watson. The hitting time of zero for a stable process. arXiv:1212.5153v1 [math.PR], 2012.

[50] A. Kuznetsov, A. E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II*, Lecture Notes in Mathematics, pages 97–186. Springer, Berlin and Heidelberg, 2013. ISBN 978-3-642-31406-3. doi:10.1007/978-3-642-31407-0_2.

[51] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications.* Universitext. Springer-Verlag, Berlin, 2006. ISBN 978-3-540-31342-7.

[52] A. E. Kyprianou. First passage of reflected strictly stable processes. *ALEA Lat. Am. J. Probab. Math. Stat.*, 2:119–123, 2006. ISSN 1980-0436.

[53] A. E. Kyprianou. Fluctuations of Lévy processes with applications. Second edition, in preparation, 2013.

[54] A. E. Kyprianou and P. Patie. A Ciesielski-Taylor type identity for positive self-similar Markov processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47 (3):917–928, 2011. ISSN 0246-0203. doi:10.1214/10-AIHP398.

[55] A. E. Kyprianou and V. Rivero. Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.*, 13:1672–1701, 2008. ISSN 1083-6489.

[56] A. E. Kyprianou, J. C. Pardo, and A. R. Watson. Hitting distributions of α-stable processes via path censoring and self-similarity. arXiv:1112.3690v2 [math.PR], 2012. To appear, *Ann. Probab.*

[57] A. E. Kyprianou, J. C. Pardo, and V. Rivero. Exact and asymptotic n-tuple laws at first and last passage. *Ann. Appl. Probab.*, 20(2):522–564, 2010. ISSN 1050-5164. doi:10.1214/09-AAP626.

[58] J. Lamperti. Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 22:205–225, 1972.

[59] E. Lukacs and O. Szász. On analytic characteristic functions. *Pacific J. Math.*, 2:615–625, 1952. ISSN 0030-8730.

[60] K. Maulik and B. Zwart. Tail asymptotics for exponential functionals of Lévy processes. *Stochastic Process. Appl.*, 116(2):156–177, 2006. ISSN 0304-4149. doi:10.1016/j.spa.2005.09.002.

[61] D. Monrad. Lévy processes: absolute continuity of hitting times for points. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 37(1):43–49, 1976/77.

[62] F. W. J. Olver. *Asymptotics and special functions.* Academic Press, New York–London, 1974. Computer Science and Applied Mathematics.

[63] H. Pantí. On Lévy processes conditioned to avoid zero. arXiv:1304.3191v1 [math.PR], 2012.

[64] H. Pantí. *A Lamperti type representation of real-valued self-similar Markov processes and Lévy processes conditioned to avoid zero.* PhD thesis, CIMAT A.C., Guanajuato, Mexico, August 2012.

[65] J. C. Pardo. The upper envelope of positive self-similar Markov processes. *J. Theoret. Probab.*, 22(2):514–542, 2009. ISSN 0894-9840. doi:10.1007/s10959-008-0152-z.

[66] G. Peskir. The law of the hitting times to points by a stable Lévy process with no negative jumps. *Electron. Commun. Probab.*, 13:653–659, 2008. ISSN 1083-589X. doi:10.1214/ECP.v13-1431.

[67] S. C. Port. Hitting times and potentials for recurrent stable processes. *J. Analyse Math.*, 20:371–395, 1967. ISSN 0021-7670.

[68] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005. ISSN 1350-7265. doi:10.3150/bj/1120591185.

[69] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. II. *Bernoulli*, 13(4):1053–1070, 2007. ISSN 1350-7265. doi:10.3150/07-BEJ6082.

[70] L. C. G. Rogers. Wiener-Hopf factorization of diffusions and Lévy processes. *Proc. London Math. Soc. (3)*, 47(1):177–191, 1983. ISSN 0024-6115. doi:10.1112/plms/s3-47.1.177.

[71] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. ISBN 0-521-77594-9. Reprint of the second (1994) edition.

[72] B. A. Rogozin. The distribution of the first ladder moment and height and fluctuation of a random walk. *Theory of Probability and its Applications*, 16 (4):575–595, 1971. ISSN 0040585X. doi:10.1137/1116067.

[73] B. A. Rogozin. The distribution of the first hit for stable and asymptotically stable walks on an interval. *Theory of Probability and its Applications*, 17 (2):332–338, 1972. ISSN 0040585X. doi:10.1137/1117035.

[74] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. ISBN 0-521-55302-4.

[75] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions: theory and applications*, volume 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2010. ISBN 978-3-11-021530-4.

[76] M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988. ISBN 0-12-639060-6.

[77] M. L. Silverstein. Classification of coharmonic and coinvariant functions for a Lévy process. *Ann. Probab.*, 8(3):539–575, 1980. ISSN 0091-1798.

[78] R. Song and Z. Vondraček. Potential theory of special subordinators and subordinate killed stable processes. *J. Theoret. Probab.*, 19(4):817–847, 2006. ISSN 0894-9840. doi:10.1007/s10959-006-0045-y.

[79] V. Vigon. *Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf*. PhD thesis, INSA de Rouen, 2002.

[80] J. Vuolle-Apiala. Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, 22(2):546–565, 1994. ISSN 0091-1798.

[81] K. Yano, Y. Yano, and M. Yor. On the laws of first hitting times of points for one-dimensional symmetric stable Lévy processes. In *Séminaire de Probabilités XLII*, volume 1979 of *Lecture Notes in Math.*, pages 187–227. Springer, Berlin, 2009. doi:10.1007/978-3-642-01763-6_8.

[82] K. Yano, Y. Yano, and M. Yor. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan*, 61(3):757–798, 2009. ISSN 0025-5645.

[83] V. M. Zolotarev. *One-dimensional stable distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986. ISBN 0-8218-4519-5. Translated from the Russian by H. H. McFaden. Translation edited by Ben Silver.