On trees with a given degree sequence

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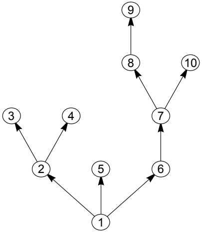
Our discrete tree variety



Trees as graphs:

A **tree** is a graph in which any two vertices can be joined by a unique path.
A **rooted tree** is a graph with a distinguished element ρ called the root.
The **genealogical partial order** is defined as: $u \preceq v$ if u belongs to the path joining ρ to v.

Plane trees:

A **plane tree** is a tree in which a total order $\leq$ has been given which is adapted to the genealogical partial order: $u \preceq v$ implies $u \leq v$.

Trees with a given degree sequence

Let $(V, E, \rho, \leq)$ be a plane tree.
Write $V = \{v_0, \dots, v_{n-1}\}$ where $\rho = v_0 < \cdots < v_{n-1}$ and

$$c_i = \#\{j \geq i : \{i,j\} \in E\}.$$

Degree sequence

The degree sequence $N_0, N_1, \dots$ is obtained by setting

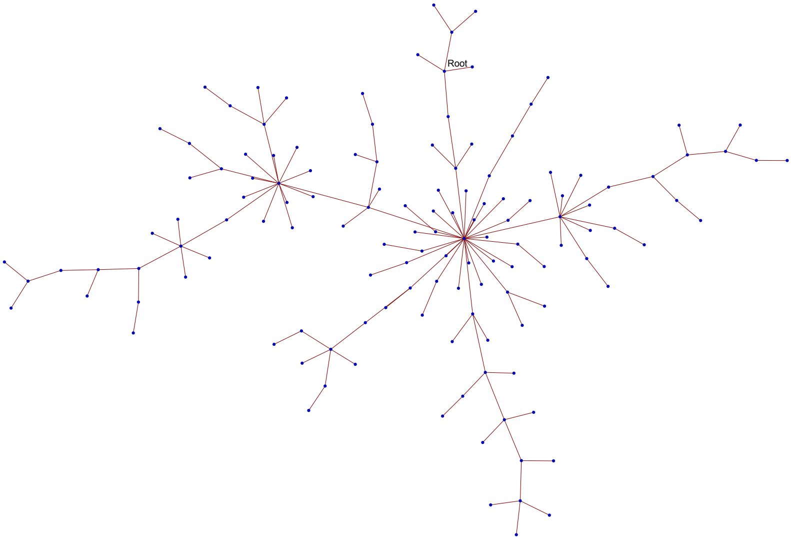
$$N_i = \#\{j : c_j = i\}.$$

It is characterized by: $\sum_i N_i = 1 + \sum_i iN_i$.
Every such sequence arises from a plane tree.

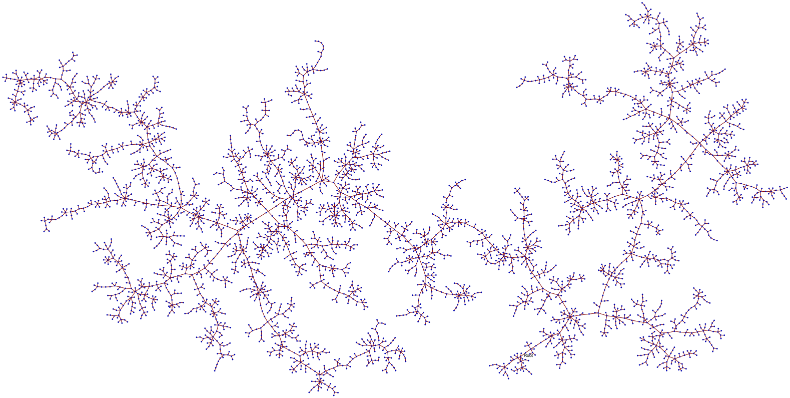
Tree with a given degree sequence

Let $s = (N_0, N_1, \dots)$ be a degree sequence. We will be interested in uniform trees from the set of plane trees having degree sequence s.

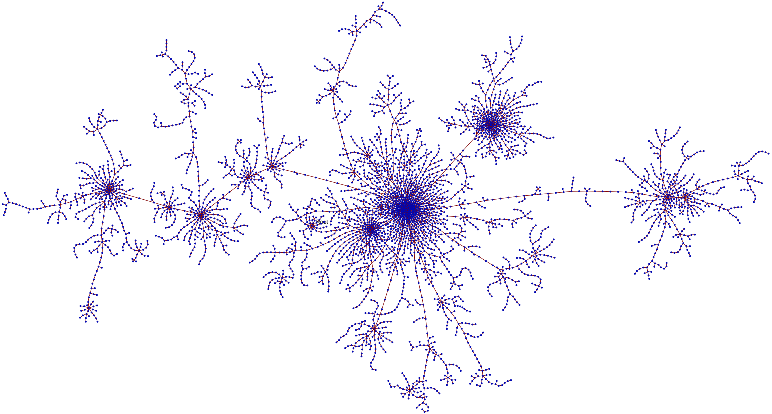
Examples



Examples



Examples



Conditioned Galton-Watson trees

Let $\mathscr{U} = \{\emptyset\} \cup \{u_1 \cdots u_n : n \geq 1, u_i \in \mathbb{Z}_+\}$ be the set of canonical labels of plane trees.
Let $\mu = (\mu_k, k \in \mathbb{N})$ be an offspring distribution and $(\xi_u, u \in \mathscr{U})$ be iid with law μ.

Galton-Watson trees

$$\Theta = \{\emptyset\} \cup \{uj \in \mathscr{U} : j \leq \xi_u\}.$$

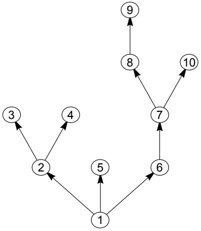
Conditioned Galton-Watson trees

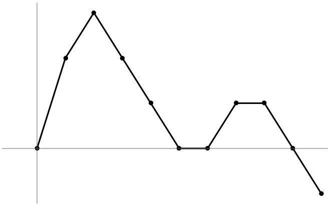
$$\Theta_n \stackrel{d}{=} \Theta \text{ conditioned on having } n \text{ vertices}$$

Proposition

Θ conditioned on having degree sequence s is uniform on trees with degree sequence s.

Coding walks for plane trees



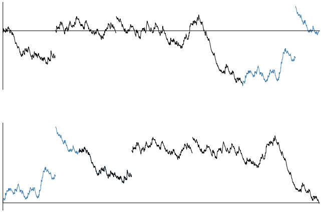


Let τ be a plane tree.
Order its individuals as $\emptyset = u_0 < \cdots < u_{n-1}$. Define
$x_i = k_{u_0}(\tau) + \cdots + k_{u_{i-1}}(\tau) - i$.

x is called the depth-first walk of the tree.

x satisfies: $x_i \geq 0$ for $i \leq n-1$ and $x_n = -1$.
Conversely, any such walk gives rise to a tree.

Coding walks for CGW trees



- Let μ be an offspring distribution and let $\tilde{\mu}_i = \mu_{i+1}$.
- Let S be a random walk conditioned to visit -1 at time n.
- Let V be the Vervaat transform of S:
  - If ρ is uniform on the set of times that S visits its minimum, consider

$$V_i = S_{\rho+i \mod n} - S_\rho$$

- Then random tree whose DFW is V is a CGW(μ, n).

Generating trees with a given degree sequence

Let $s = (N_0, N_1, \ldots)$ be a degree sequence of size $n = \sum_i N_i$.
Let $c = (c_1, \ldots, c_n)$ be an associated child sequence:

$$N_j = \#\{i : c_i = j\}.$$

Let C be obtained from c by a uniform random permutation and define:

$$X_i = C_1 + \cdots + C_i - i.$$

Let V be the Vervaat transform of X.
Then V is the depth-first walk (or breadth first walk) of a uniform tree with degree sequence s.

Exchangeable increment processes

Let $\beta = (\beta_1, \beta_2, \ldots)$ be such that $\sum \beta_i^2 < \infty$ and let $\sigma \in [0, \infty)$. Let b be a Brownian bridge and $U_1, U_2, \ldots$ uniform on $(0, 1)$; everything independent.

EI process

An EI process directed by (σ, β) admits the representation

$$X_t = \sigma b_t + \sum_i \beta_i \left[\mathbf{1}_{U_i \leq t} - t\right].$$

Simulation

Convergence of Depth-First Walks

Let $(\mathbf{s}^n)_n$ be a sequence of degree sequences, $\mathbf{s}^n = (N_0^n, N_1^n, \ldots)$. Let $\tilde{c}^n$ be the unique non-increasing associated child sequence. Let X^n be the DFW of a uniform tree with degree sequence $\mathbf{s}^n$. Assume the technical hypotheses:

1. $s_n = \sum_i N_i^n \to \infty$ as $n \to \infty$.
2. There exists a non-negative sequence $(b_n, n \geq 1)$, $b_n \to \infty$, $M \in \mathbb{N} \cup \{\infty\}$, $\sigma \in [0, \infty)$ and $\beta \in l_2^\downarrow$ such that

$$\frac{1}{b_n} \sum_i (i-1)^2 N_i^n \to \sigma^2 + \sum_{i=1}^{M} \beta_i^2 \qquad \frac{\tilde{c}_i}{b_n} \to \beta_i$$

3. Either $\sigma > 0$ or $\sum \beta_i = \infty$.

Then, $X^n_{s_n \cdot}/b_n$ converges on Skorohod space to the Vervaat transform of X, where X is a process with exchangeable increments on $[0,1]$ directed by σ and β.

The height function of a plane tree

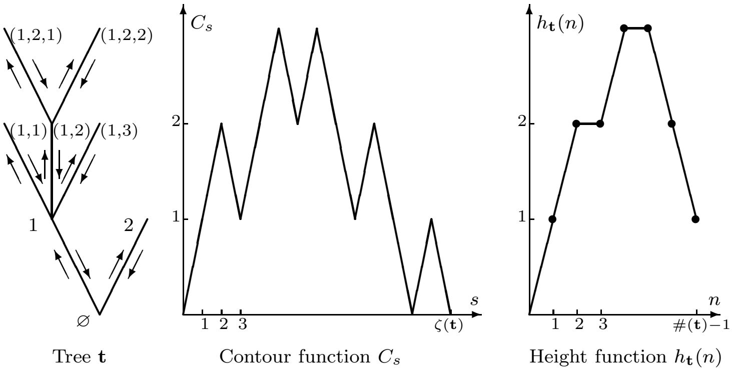


Figure: From *Random trees and applications* by Jean-François Le Gall

The main open problem

For a sequence of degree sequences (s^n), let τ_n be uniform on trees with degree sequence s^n.

Find conditions on (s^n) such that τ_n has a scaling limit.

The scaling limit should be a continuum tree: a metric space in which two elements can be joined by a unique path.

A first step

For a sequence of degree sequences $(\mathbf{s}^n)$ of sizes s_n, let τ_n be uniform on trees with degree sequence s^n.

Let U be uniform on $\{0, \ldots, s_n - 1\}$ and independent of τ_n.
Find conditions on (s^n) such that the height of individual U in τ_n has a scaling limit.

A related problem for CGW

The DFW of a μ_n-CGW, say V^n, is the Vervaat transform of a bridge from 0 to -1 of the associated random walk.
Find conditions on μ_n for the convergence of these bridges and of their Vervaat transformation.

The problem is relevant since the height process of the limit should *code* the limiting tree.

A subproblem

Let μ^n be a sequence of degree distributions. Let θ_n be a μ^n-GW tree of size s_n.
Find conditions on μ^n and s_n such that the sequence of trees θ_n has a limit.