On Recent Developments on Markov Processes and Applications: hand in (post lectures) task.

My question

The Lamperti-Kiu representation for one dimensional self-similar Markov processes takes the form

$$X_t = \mathrm{e}^{\xi_{\varphi(t)}} J_{\varphi(t)}, \qquad t < \zeta$$

where $\zeta = \inf\{t > 0 : X_t = 0\}$, $\varphi(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u}\mathrm{d}u > t\}$ and (ξ, J) is a Markov additive Lévy process with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}$, $i \in \{-1, 1\}$. When it exists, its matrix characteristic exponent satisfies

$$\boldsymbol{\Psi}(z) = \begin{pmatrix} \psi_1(z) & 0 \\ 0 & \psi_{-1}(z) \end{pmatrix} + \begin{pmatrix} -q_{1,-1} & q_{1,-1} \\ q_{-1,1} & -q_{-1,1} \end{pmatrix} \circ \begin{pmatrix} 1 & E[\mathrm{e}^{zU_{1,-1}}] \\ E[\mathrm{e}^{zU_{-1,1}}] & 1 \end{pmatrix},$$

where

$$\mathbf{E}_{0,i}[\mathrm{e}^{z\xi_t}; J_t = j] = \left(\mathrm{e}^{\boldsymbol{\Psi}(z)t}\right)_{i,j} \qquad i, j \in \{1, -1\}$$

and $\circ$ means element wise multiplication.

For stable processes with positive and negative jumps, it known that

$$\boldsymbol{\Psi}(z) = \begin{bmatrix} \dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\[2ex] -\dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & \dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

Task: Prove this identity.

Your answer

Please present your solution fully as an essay (almost like writing a mini-paper), with a commentary on how you have developed your computations. Below are some facts that you may wish to take into consideration, in which we use the notation $\overline{X}_t = \sup_{s \le t} X_s$ and $\underline{X}_t = \inf_{s \le t} X_s$ for the stable process X.

- In the slides, we have considered the positive self-similar Markov process $X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \ge 0$ and discussed its Lampert representation as a positive self-similar Markov process. More precisely, we showed that
$$X_t^* := X_t \mathbf{1}_{(\underline{X}_t > 0)} = \mathrm{e}^{\xi^*_{\varphi^*(t)}}, \qquad t < \zeta^*,$$
where $\zeta^* = \inf\{t > 0 : X_t^* = 0\}$, $\varphi^*(t) = \inf\{s > 0 : \int_0^s \mathrm{e}^{\alpha\xi_u^*}\mathrm{d}u > t\}$ and ξ^* is a Lévy process with killing at an independent and exponentially distributed random time. Moreover, we computed it characteristic exponent such that
$$E[\mathrm{e}^{\mathrm{i}\theta\xi_t^*}] = \mathrm{e}^{-\Psi^*(\theta)t}, \qquad t \ge 0$$
and
$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)} = q^* + \text{LK formula of unkilled process},$$
for $z \in \mathbb{R}$. The killing rate $q^* = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})}$.

- The distributions $U_{-1,1}$ and $U_{1,-1}$ can be described in terms of the path of the stable process X. Specifically, one should show as an intermediary step that

$$\mathrm{e}^{U_{1,-1}} =^d \frac{|X_{\tau_0^-}|}{X_{\tau_0^- -}},$$

where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$.

- In the above claim, why does the point of issue of X not matter? This is because of the scaling property that $(cX_{c^{-\alpha}t}, t \geq 0)$ under $\mathbb{P}_x$ is equal in law to $(X_t, t \geq 0)$ under $\mathbb{P}_{cx}$. Indeed, suppose we include the point of issue of X in its notation, writing instead $(X_t^{(x)}, t \geq 0)$. Then note that

$$\begin{aligned}\tau_0^-(x) &:= \inf\{s > 0 : X_s^{(x)} < 0\} \\ &= \inf\{c^{-\alpha}(c^\alpha s) > 0 : cX^{(x)}_{c^{-\alpha}(c^\alpha s)} < 0\} \\ &= c^{-\alpha} \inf\{t > 0 : cX^{(x)}_{c^{-\alpha}t} < 0\} \\ &=^d c^{-\alpha} \inf\{t > 0 : X_t^{(cx)} < 0\} \\ &= c^{-\alpha}\tau_0^-(cx).\end{aligned}$$

As a consequence

$$X^{(x)}_{\tau_0^-(x)} = \frac{1}{c} cX^{(x)}_{c^{-\alpha}c^\alpha\tau_0^-(x)} =^d \frac{1}{c} X^{(cx)}_{\tau_0^-(cx)}$$

and hence

$$\frac{|X^{(x)}_{\tau_0^-(x)}|}{X_{\tau_0^-(x)-}} = \frac{|X^{(cx)}_{\tau_0^-(cx)}|}{X^{(cx)}_{\tau_0^-(cx)-}}.$$

- A useful distribution to have to hand is the following: For $y \in [0, x]$, $v \geq y$ and $u > 0$,

$$\begin{aligned}&\mathbb{P}(X_{\tau_x^+} - x \in \mathrm{d}u, x - X_{\tau_x^+ -} \in \mathrm{d}v, x - \overline{X}_{\tau_x^+ -} \in \mathrm{d}y) \\ &= \frac{\sin \alpha\rho\pi}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha\rho)\Gamma(\alpha(1-\rho))} \frac{(x-y)^{\alpha\rho-1}(v-y)^{\alpha(1-\rho)-1}}{(v+u)^{1+\alpha}} \,\mathrm{d}y \,\mathrm{d}v \,\mathrm{d}u.\end{aligned}$$

(See the quintuple law of Chapter 7 of [1], and Exercise 7.4 therein.)

References

[1] Kyprianou A. E. (2014) *Fluctuations of Lévy processes with applications: Introductory lectures*, second edition. Universitext, Springer.