Recent progress in high-dimensional percolation

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Joint work with:
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- ▷ Takashi Hara (Kyushu)
- ▷ Gordon Slade (UBC Vancouver)
- ▷ Antal Járai (Bath)

Percolation on $\mathbb{Z}^d$

Bonds join x to y for $x, y \in \mathbb{Z}^d$. Make bonds (x, y) independently

occupied	with probability p,
vacant	with probability $1 - p$,

where $p \in [0, 1]$ is percolation parameter.

Key examples:

▷ nearest-neighbor percolation;

▷ spread-out percolation, where range of bonds grows proportionally with parameter L, and L is often taken to be large: Bonds between x and y when $0 < \|x\|_\infty \leq L$.

[1] Heydenreich and van der Hofstad. Progress in high-dimensional percolation and random graphs.
Lecture notes CRM-PIMS Summer School in Probability 2015.
http://www.win.tue.nl/∼rhofstad/survey-high-d-percolation.pdf

Phase transition on $\mathbb{Z}^d$

Critical value $p_d(\mathbb{Z}^d)$ for percolation on $\mathbb{Z}^d$ is

$$p_d(\mathbb{Z}^d) = \inf\{p\colon \mathbb{P}_p(0 \longleftrightarrow \infty) > 0\}$$

Moreover, $p_c(\mathbb{Z}^d) \in (0,1)$ for $d \geq 2$.

$\mathcal{C}(x) = \{x\colon 0 \longleftrightarrow x\}$ is cluster of x, and $|\mathcal{C}(x)|$ its size.

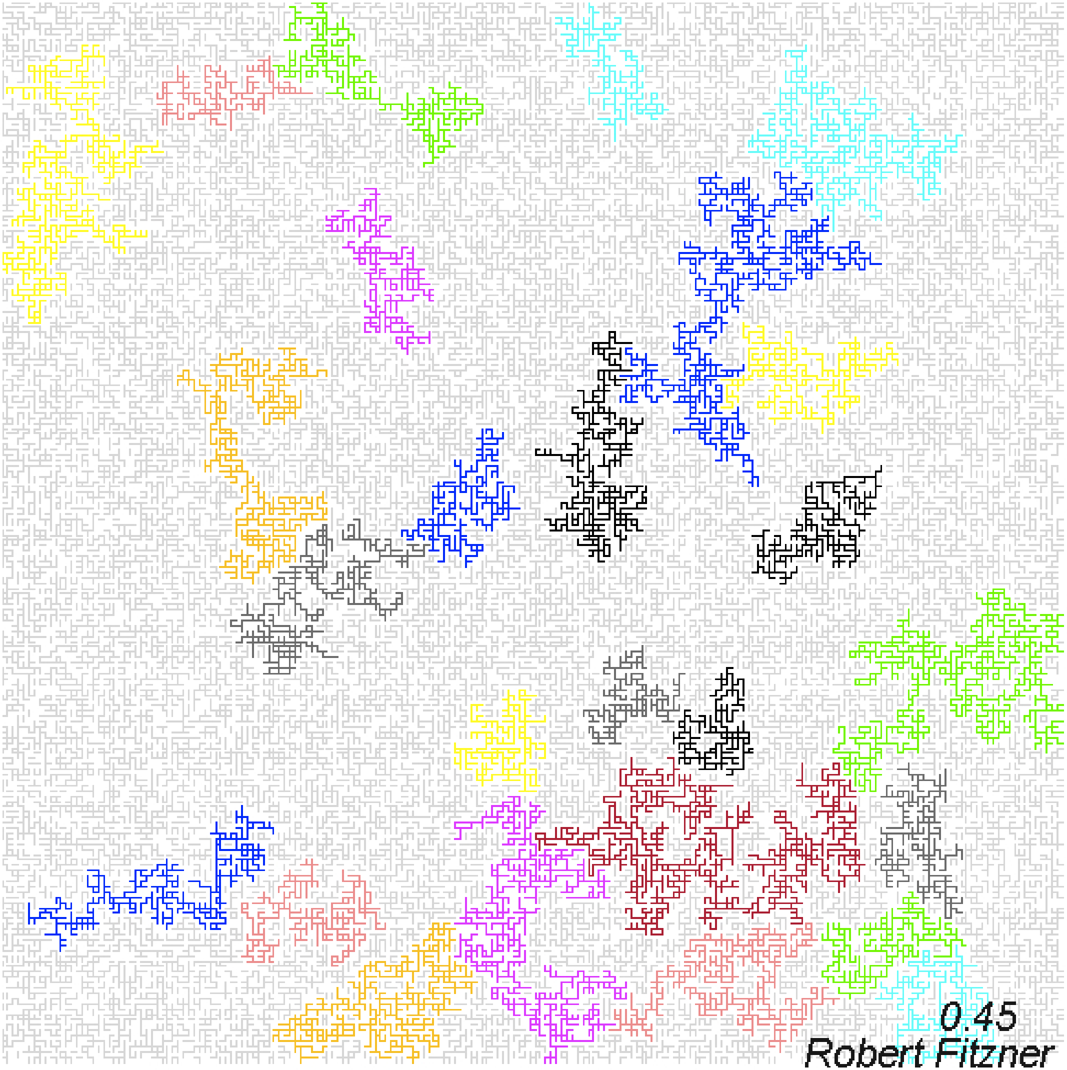
Aizenman and Barsky (87) or Menshikov (86):

$$p_c(\mathbb{Z}^d) = \sup\{p\colon \mathbb{E}_p|\mathcal{C}(0)| < \infty\}.$$

Beautiful new proof Duminil-Copin & Tassion (2015)!
We generally do not know that $\mathbb{P}_{p_c}(|\mathcal{C}(0)| = \infty) = 0$.

$\triangleright$ Central question:

What is structure of large critical clusters?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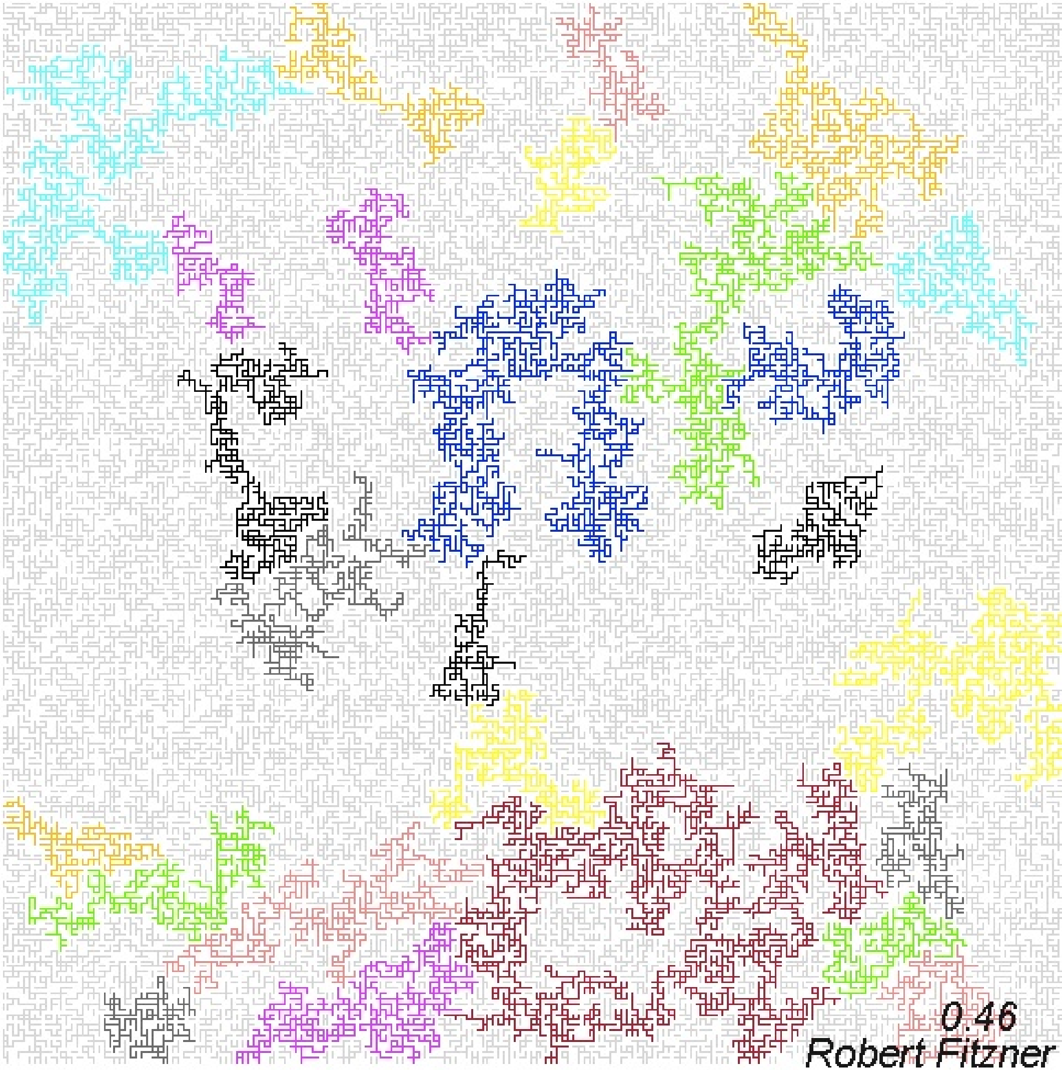
Robert Fitzner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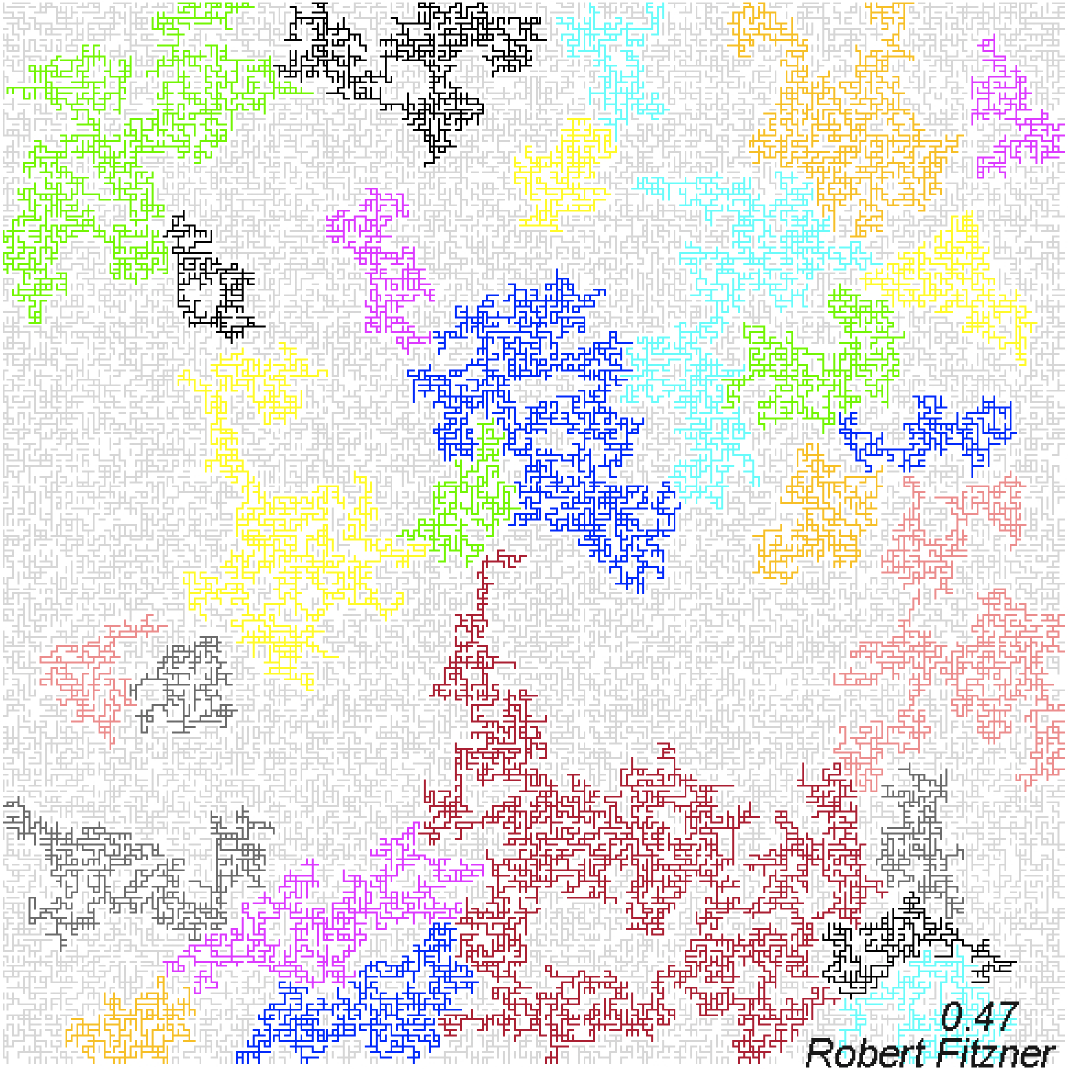
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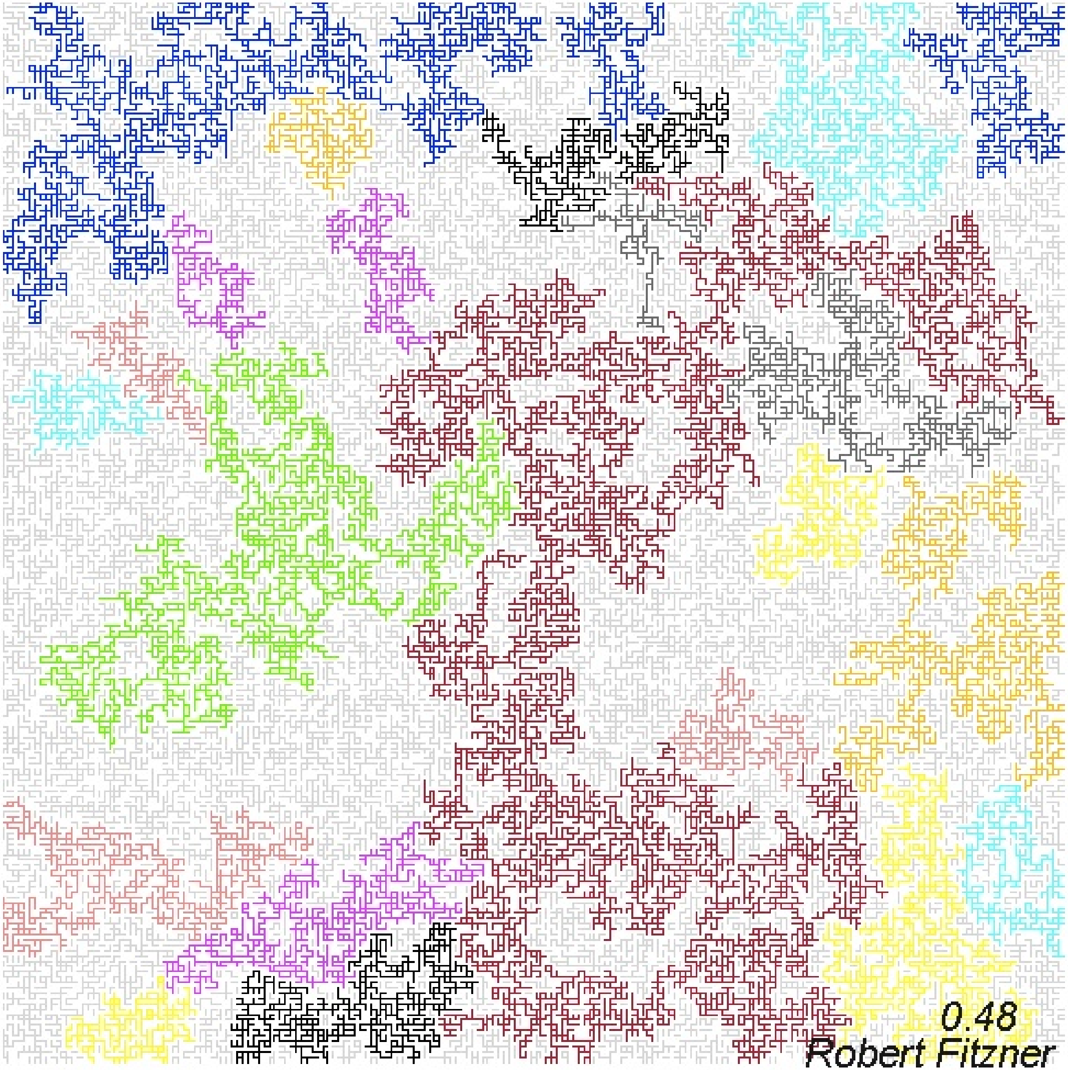
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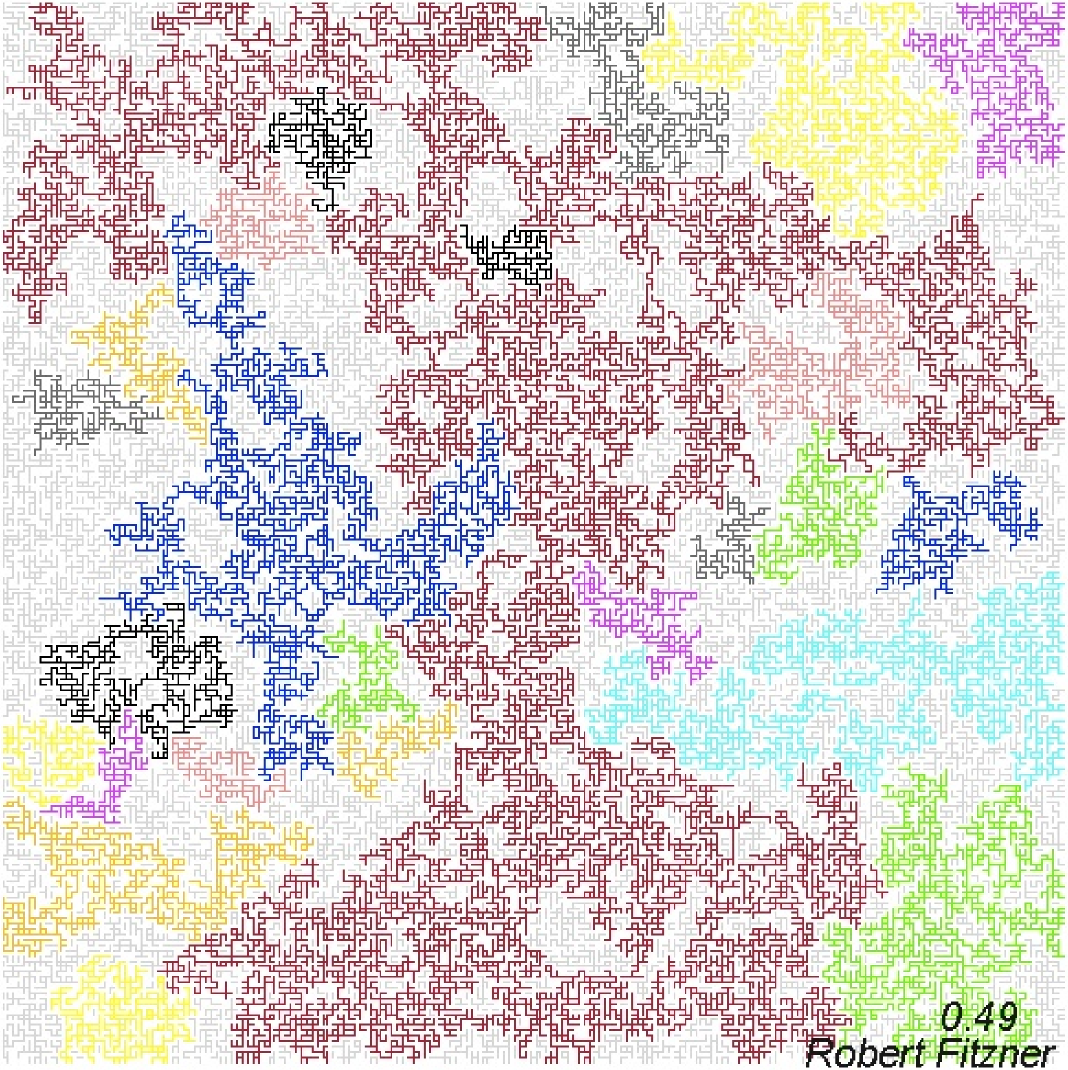
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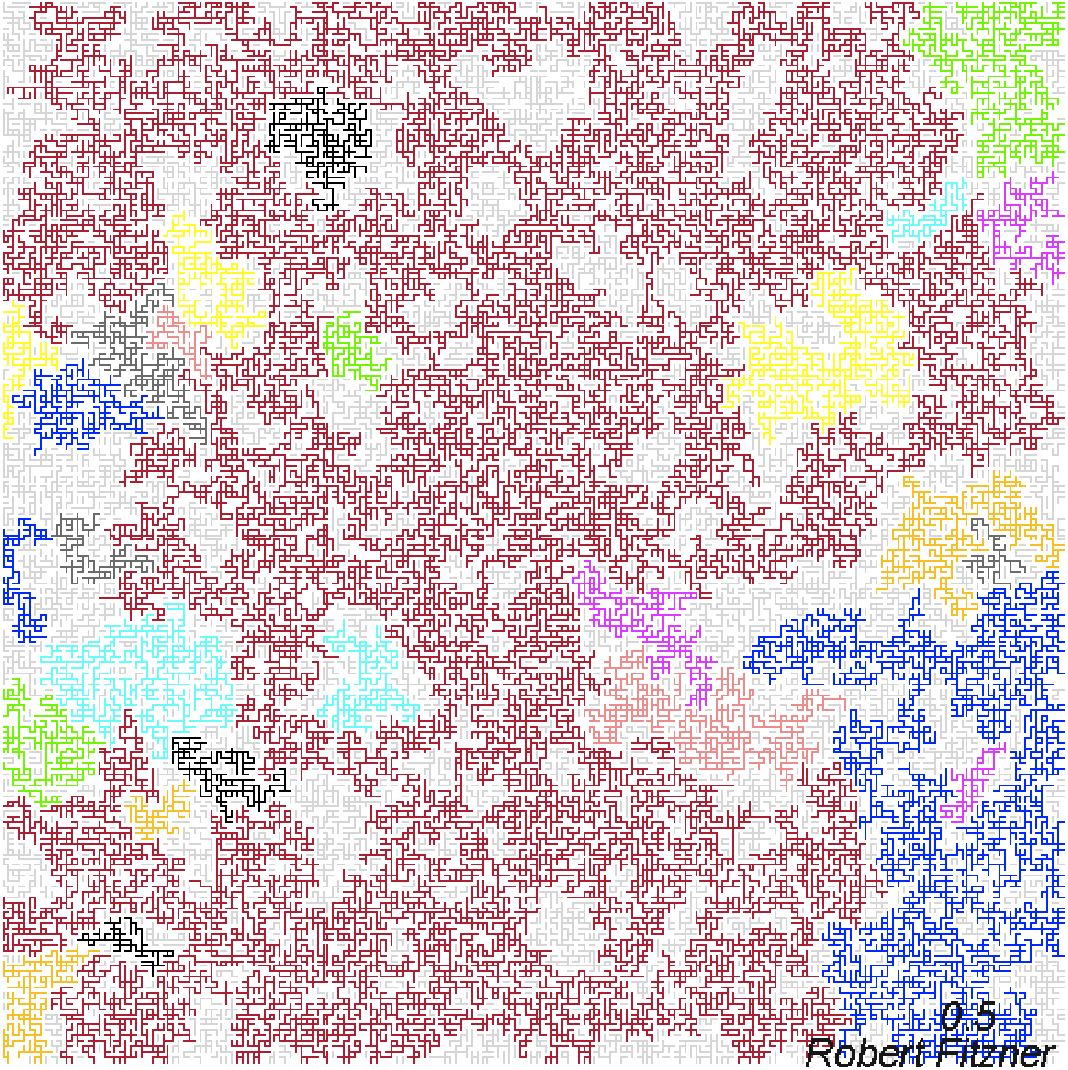
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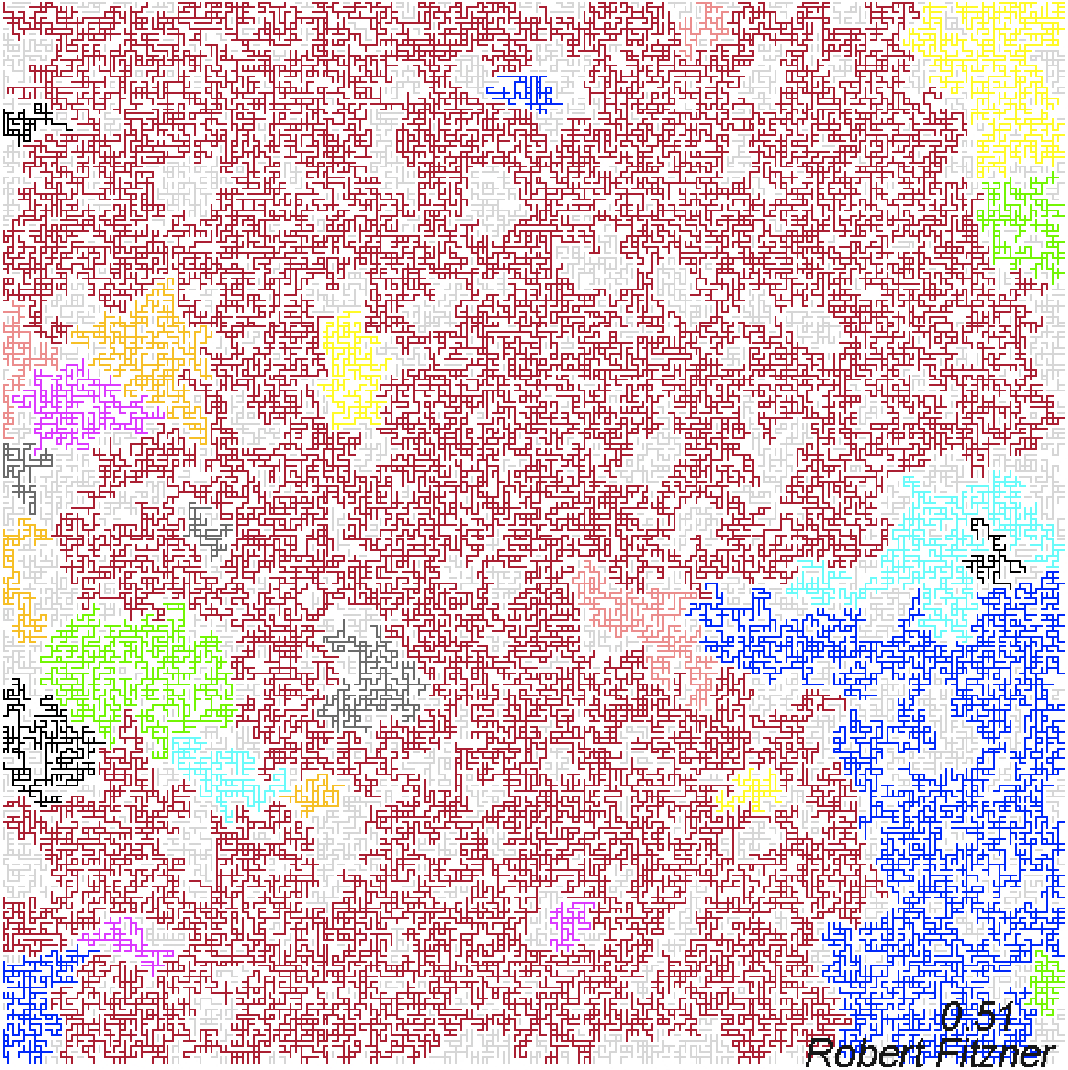
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Critical exponents

Let $\mathcal{C}(0)$ be cluster of 0, i.e.,

$$\mathcal{C}(0) = \{x \in \mathbb{Z}^d \colon 0 \longleftrightarrow x\},$$

where $0 \longleftrightarrow x$ when there is path of occupied bonds between $0, x$.

Let

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty), \qquad \chi(p) = \mathbb{E}_p|\mathcal{C}(0)|.$$

Close to criticality, these are believed to show power-laws:

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) \sim (p - p_c)^{\beta} \quad p \searrow p_c, \qquad \mathbb{E}_p|\mathcal{C}(0)| \sim (p_c - p)^{-\gamma} \quad p \nearrow p_c,$$

where β, γ are believed to be universal, i.e., they only depend on dimension and not on any other details of model.

Other critical exponents

Many more critical exponents are believed to exist:

$$\begin{aligned} &\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \sim n^{-1/\delta}, && n \to \infty, \\ &\mathbb{P}_{p_c}(0 \longleftrightarrow x) \sim |x|^{-(d-2-\eta)} && |x| \to \infty, \\ &\mathbb{P}_{p_c}(\exists x \in \mathcal{C}(0) \colon |x| \geq m) \sim m^{-\rho} && m \to \infty. \end{aligned}$$

Unfortunately, our understanding of these objects is limited to

two-dimensions and high-dimensions.

In particular, there we do know that $\mathbb{P}_{p_c}(|\mathcal{C}(0)| = \infty) = 0$, while, for $d = 3$, we do not even know that $\mathbb{P}_{p_c}(|\mathcal{C}(0)| = \infty) \downarrow 0$, as $p \searrow p_c$.

Existence critical exponents is much stronger:

Holy grail percolation theory!

Two-dimensions

Theorem 1. (Schramm00, Smirnov01, SW01, LSW02) For site percolation on two-dimensional triangular lattice, critical exponents $\beta, \gamma, \eta, \delta, \rho$ exist in logarithmic sense, and take on values

$$\beta = \frac{5}{36}, \qquad \gamma = \frac{43}{18}, \qquad \eta = \frac{5}{24}, \quad \delta = \frac{91}{5}, \qquad \rho = \frac{48}{5}.$$

In particular,

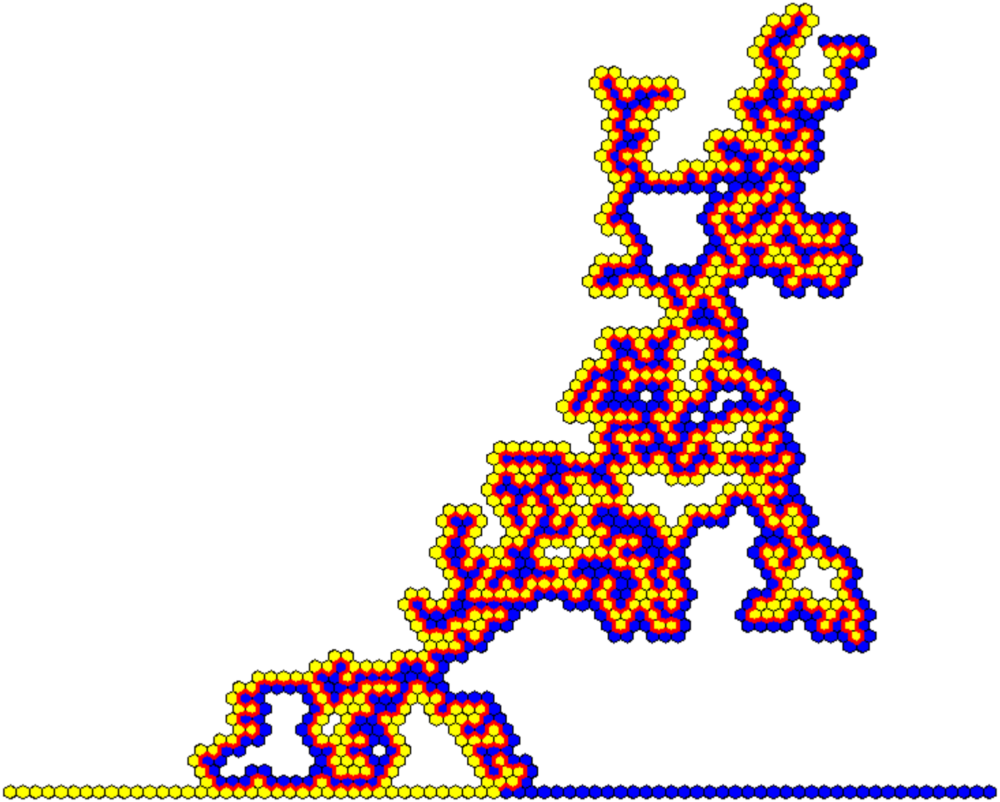
$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) = (p - p_c)^{\frac{5}{36} + o(1)}, \qquad \mathbb{E}_p[|\mathcal{C}(0)|] = (p_c - p)^{-\frac{43}{18} + o(1)},$$

$$\mathbb{P}_{p_c}(0 \longleftrightarrow x) = |x|^{-\frac{5}{24} + o(1)}, \qquad \mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) = n^{-\frac{5}{91} + o(1)}.$$

Proof: conformal invariance and Schramm/Stochastic Loewner Evolution (SLE).

Not known for square lattice...

Interfaces of clusters



(Picture Matthijs Joosten)

High-dimensions

Physics prediction:

Percolation in high-dimensions $(d > d_c = 6)$ behaves as it does on regular infinite tree:

▷ No infinite critical cluster;

▷ Critical exponents agree with tree embedded into space.

Informal reason:

When dimension is high, space is so vast that faraway pieces of percolation cluster no longer interact.

Thus, geometry "trivializes", and answer to most questions is same as for percolation on an infinite regular tree, embedded into space.

Percolation on tree

Let T_r be r-ary infinite tree, where, for simplicity, we assume that root has degree $r-1$.

Perform bond percolation on this tree with percolation probability $p \in [0,1]$. Denote root of T_r by 0.

Percolation on tree has phase transition at $p_c = 1/(r-1)$, i.e.,

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) > 0 \qquad \text{iff} \qquad p > p_c = 1/(r-1),$$

and

$$\mathbb{E}_p|\mathcal{C}(0)| < \infty \qquad \text{iff} \qquad p < p_c = 1/(r-1).$$

Phase transition tree

By realizing that percolation on tree is branching process,

$$\theta(p) = \mathbb{P}_p(|\mathcal{C}(0)| = \infty) \sim (p - p_c), \qquad \chi(p) = \mathbb{E}_p|\mathcal{C}(0)| \sim (p_c - p)^{-1}.$$

Indeed,

$$\chi(p) = 1 + (r-1)p\chi(p),$$

so that, for $p < 1/(r-1)$,

$$\chi(p) = 1/(1-(r-1)p),$$

while $\chi(p) = \infty$ for $p > 1/(r-1)$.

Phase transition tree

By conditioning on number of occupied bonds incident to root,

$$\begin{aligned} 1-\theta(p) &= \sum_{m=0}^{r-1} \binom{r-1}{m} p^m (1-p)^{r-1-m} (1-\theta(p))^m \\ &= \big[p(1-\theta(p)) + (1-p)\big]^{r-1} \equiv G(1-\theta(p)), \end{aligned}$$

which has unique solution $1-\theta(p)=1$ when $G'(1)=(r-1)p \le 1$, and two solutions when $G'(1)=(r-1)p > 1$.

In former case, $\theta(p)=0$, while in latter case, $\theta(p)>0$. By investigating properties of $x \mapsto G(x)$ for $x \nearrow 1$, we conclude that, as $p \searrow p_c$,

$$\theta(p) \sim 2(r-1)(p-p_c).$$

Critical exponents tree

Conclusion: By branching process methodology,

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) \sim (p - p_c), \qquad \mathbb{E}_p|\mathcal{C}(0)| \sim (p_c - p)^{-1}.$$

Moreover,

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \sim n^{-1/2}, \qquad \mathbb{P}_{p_c}\big(\text{diameter}(|\mathcal{C}(0)|) \geq n\big) \sim n^{-1}.$$

Critical exponents:

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2, \qquad \rho_{\text{int}} = 1.$$

Percolation on $\mathbb{Z}^d$

Bonds join x to y for $x, y \in \mathbb{Z}^d$. Make bonds (x, y) independently

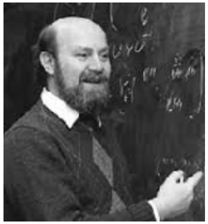
occupied with probability p,
vacant with probability $1 - p$,

where $p \in [0, 1]$ is percolation parameter.

Key examples:
$\triangleright$ nearest-neighbor percolation;
$\triangleright$ spread-out percolation, where range of bonds grows proportionally with parameter L, and L is often taken to be large:
Bonds between x and y when $0 < \|x\|_\infty \leq L$.

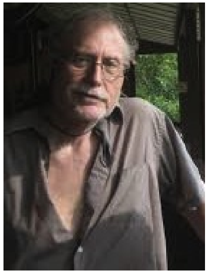
Critical exponents

Theorem 2. (Aizenman-Newman84, Barsky-Aiz91)
When triangle condition holds, i.e.,



$$\triangle(p_c) = \sum_{x,y\in\mathbb{Z}^d} \mathbb{P}_{p_c}(0 \longleftrightarrow x)\mathbb{P}_{p_c}(x \longleftrightarrow y)\mathbb{P}_{p_c}(y \longleftrightarrow 0) < \infty,$$

the following asymptotics hold:



$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) \asymp (p - p_c)^1, \qquad \mathbb{E}_p|\mathcal{C}(0)| \asymp (p_c - p)^{-1},$$

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \asymp n^{-1/2} :$$

critical exponents β, γ, δ exist and take on tree values

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2.$$

Proofs use differential inequalities, comparing critical exponents to those on infinite tree, assuming triangle condition.

Critical exponents

Theorem 3. (Hara-Slade90, HHS03, H08)
For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbor percolation for $d \geq 19$, the triangle condition holds and

$$\mathbb{P}_p(|\mathcal{C}(0)| = \infty) \asymp (p - p_c)^1, \qquad \mathbb{E}_p|\mathcal{C}(0)| \asymp (p_c - p)^{-1},$$

$$\mathbb{P}_{p_c}(|\mathcal{C}(0)| \geq n) \asymp n^{-1/2}, \qquad \mathbb{P}_{p_c}(0 \longleftrightarrow x) \asymp |x|^{-(d-2)} :$$

critical exponents $\beta, \gamma, \delta, \eta$ exist and take on tree values

$$\beta = 1, \qquad \gamma = 1, \qquad \delta = 2, \qquad \eta = 0.$$

Proofs use lace expansion, a perturbative method to prove that critical two-point function is alike that of critical branching random walk, thus verifying triangle condition.

Progress in high dim's

(a) Continuity nearest-neighbor percolation function above 10 dimensions.

(b) Arm exponents in high-dimensions.

(c) Existence incipient infinite cluster, behavior random walk on it.

(d) Percolation on high-dimensional tori.

Part (a): Nearest-neighbor

Theorem 4. (Fitzner-vdH 15) Theorems 2–3 extend to nearest-neighbour percolation for $d > 10$.

Proof by Hara-Slade was never published, so project improves transparancy in high-dimensional percolation.

▷ Download Mathematica code from site Robert Fitzner!

Proof uses
(a) Non-backtracking lace expansion (NoBLE) taking more of interaction explicitly into account;
(b) sharp bounds on NoBLE coefficients;
(c) analysis of NoBLE assuming explicit numerical conditions;
(d) a computer-assisted proof that verifies these conditions.

Part (b): Arm exponents

Theorem 5. (Kozma-Nachmias 09) Fix $d > 6$ for sufficiently spread-out model, or $d \geq 11$ for nearest-neighbor model. Then, as $R \to \infty$,

$$\mathbb{P}_{p_c}(\exists x \colon d_{\mathcal{C}(0)}(0, x) \geq R) \asymp R^{-1},$$

where $d_{\mathcal{C}(0)}(0, x)$ is intrinsic or graph distance of cluster $\mathcal{C}(0)$.

Theorem 6. (Kozma-Nachmias 09) Fix $d > 6$ for sufficiently spread-out model, or $d \geq 11$ for nearest-neighbor model. Then, as $R \to \infty$,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial Q_R) \asymp R^{-2},$$

where Q_R is Euclidean ball of radius R.

Implication $d_c \geq 6$

Note that, with $R = |x|/2$,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow x) \leq \mathbb{P}_{p_c}(0 \longleftrightarrow \partial Q_R)^2.$$

By Theorem 3, in high dimensions,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow x) \asymp |x|^{-(d-2)},$$

while, by Theorem 5, again in high dimensions,

$$\mathbb{P}_{p_c}(0 \longleftrightarrow \partial Q_R)^2 \sim |x|^{-4}.$$

Only possible when $d \geq 6$: critical exponents in Theorems 3 and 5 cannot hold for $d < 6$. Condition believed to be sharp.

Part (c): Existence IIC

For cylinder events E, define

$$(\mathrm{IIC}) \qquad \mathbb{P}_\infty(E) = \lim_{|x|\to\infty} \mathbb{P}_{p_c}(E \mid 0 \longleftrightarrow x).$$

Theorem 7. (vdH-Járai (03)) For spread-out percolation with L sufficiently large and $d > 6$, or nearest-neighbour percolation for $d \geq 15$, above limit exists for every cylinder event E. Moreover, $\mathbb{P}_\infty$ extends to probability measure on full sigma-algebra of events, and $\mathbb{P}_\infty(|\mathcal{C}(0)| = \infty) = 1$.

Result relies on asymptotics critical two-point function Theorem 3

$$\mathbb{P}_{p_c}(0 \longleftrightarrow x) \sim |x|^{-(d-2)}.$$

Random walk on IIC

Theorem 8. (Kozma-Nachmias 09) Fix $d > 6$ and sufficiently spread-out model, or $d \geq 15$ and nearest-neighbor model. Then, in probability,

$$\lim_{r\to\infty} \frac{\log E^0[\text{exit time of intrinsic ball radius } r]}{\log r} = 3,$$

$$\lim_{n\to\infty} \frac{\log \#\,\text{sites visited by random walk to time } n}{\log n} = 2/3,$$

where E^0 denotes conditional law of RW on IIC.

Part (d): Percolation torus

Random subgraph of finite tori

$$\mathbb{T}_{r,d} = (\mathbb{V}, \mathbb{B}) \qquad \text{where} \qquad \mathbb{V} = \{0, \dots, r-1\}^d.$$

Make bonds (x, y) for $x, y \in \mathbb{T}_{r,d}$ independently

occupied with probability p,
vacant with probability $1 - p$,

where $p \in [0, 1]$ is percolation parameter.

Key examples:
▷ nearest-neighbor percolation;
▷ spread-out percolation: Bonds between x and y when $0 < \|x\|_\infty \le L$ and L is taken to be large.

Percolation high-d tori

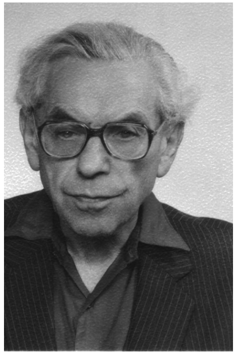
Branching random walk is mean-field model for high-dimensional critical percolation.

Torus is finite set: different scaling behavior for percolation on torus.

In high dimensions, geometry trivializes, so one might expect critical percolation on high-dimensional torus to be related to

percolation on complete graph of same size.

Erdős-Rényi graph



Erdős-Rényi random graph is random subgraph of complete graph on V vertices where each of $\binom{V}{2}$ edges is occupied with probab. p.

Phase transition: (Erdős and Rényi (60))
For $p = (1+\varepsilon)/V$, largest component is
(a) $\Theta_{\mathbb{P}}(\log V)$ for $\varepsilon < 0$;
(b) $\Theta_{\mathbb{P}}(V)$ for $\varepsilon > 0$.

Scaling window: (Bollobás (84) and Łuczak (90))
For $p = (1/V)(1+\lambda/V^{1/3})$, largest component is $\Theta_{\mathbb{P}}(V^{2/3})$, with expected cluster size $\Theta(V^{1/3})$.

High-d tori

Define largest connected component for percolation on torus by

$$|\mathcal{C}_{\max}| = \max_{v} |\mathcal{C}_{\mathbb{T}}(v)|,$$

where $|\mathcal{C}_{\mathbb{T}}(v)|$ is size connected component of v on torus.

Theorem 9. (Heydenreich+vdH 07+09) For $d \gg 6$ and nearest-neighbor bonds, or for $d > 6$ and sufficiently spread-out bonds, there exists $b > 0$ such that for all $\omega \geq 1$ and as $r \to \infty$

$$\mathbb{P}_{p_c(\mathbb{Z}^d)}\Big(\frac{1}{\omega}V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}\Big) \geq 1 - \frac{b}{\omega}.$$

Moreover, $|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated: hallmark of critical behavior.

Proof: Coupling relating $|\mathcal{C}(v)|$ on torus and on $\mathbb{Z}^d$.

Networks

Percolation on networks is also used as model for resilience of networks. This line of research fits well within recently awarded

Gravitation project Networks.

This project focusses on

fundamental mathematical and computer science challenges posed by networks, and studies these by combining stochastic modelling and algorithmics.

For more information, including vacancies, see

www.thenetworkcenter.nl