Probabilistic Aspects of Voting

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Outline

- 1. Introduction to voting theory
- 2. Probability and voting
  - 2.1. Aggregating noisy signals: The Condorcet Jury Theorem
  - 2.2. Let's calculate probability: Bertrand's Ballot Theorem
  - 2.3. Probability is a richer language: Gibbard's random dictator theorem
- 3. Final remarks

1.What is voting theory?

- In the end democracy pins down to voting!
- Then, how should we organize voting properly? i.e. What is a good voting procedure? and Which voting system is the best? etc.
- As a byproduct of developments in the age of enlightenment a formal approach to this question is emerged with contributions of
  - Marquis de Condorcet (1743 – 1794)
  - Jean Charles de Borda (1733 – 1799)
  - Joseph Bertrand (1822 – 1900)
  - Charles Dodgson (1832 – 1898), etc.
- The formal approach is based on the following analysis:
  - Which voting scheme has which property?

1.Voting Schemes

- A general rule: When we have two alternatives the simple majority rule does the job!
- What if we have more than two alternatives?
- Firstly, the simple majority does not work!
  - Condorcet paradox: Suppose there are 3 voters and 3 alternatives, A, B, C and the rankings are $(ABC), (BCA), (CAB)$, respectively. Then majority prefers A to B, B to C and C to A.
- Yet Condorcet proposed the following method: Collect the ballots (i.e. the rankings and ties are allowed), and apply majority rule on all pairwise comparisons of alternatives. If there is a winner, it must be chosen. (If not, then use the Kemeny-Young extension!)

1.The Condorcet vs. Borda

- Suppose after collecting ballots outcome is as follows:

# of voters	2	2	3	2
Ballots	$(B \succ C \succ A)$	$(B \succ A \succ C)$	$(A \succ B \succ C)$	$(C \succ A \succ B)$

- The Condorcet winner is A.
- But one can argue that B is not inferior to A. Indeed that is what Borda rule says: In case of $m \in \mathbb{N}$ alternatives assign the score of $m - i$ to the i'th ranked alternative in every ballot and rank alternatives according to their total scores.
- The Borda winner is B.
- Both methods are known to have some drawbacks! …

2.1. The Condorcet's jury theorem

- <u>Motivation</u>: When is a group decision better than individual decision? What is the optimal size of a committee?
- <u>Problem</u>: Consider a jury with three members each of which has the probability p of making the right decision, and $1 - p$ of getting the wrong. Assume also that the probabilities are independent. If the committee outcome is based on the majority rule, what is the probability of jury getting the right decision?
  - <u>Answer</u>: $P_3 = p^3 + 3p^2(1-p)$ and $P_3 > p$ iff $p > \frac{1}{2}$

2.1. The Condorcet's jury theorem

- Theorem (Condorcet, 1785): Suppose there are $n \in \mathbb{N}$ (assume n is odd) jurors and their votes are $i.i.d$ draws from the Bernoulli distribution with success probability p. Let P_n be the probability that the majority of the jury members vote for success. Then,
  - If $0.5 < p < 1$ and $n \geq 3$, then $P_n > p$, P_n increases with n and $P_n \to 1$ as $n \to \infty$;
  - If $0.5 > p > 0$ and $n \geq 3$, then $P_n < p$, P_n dicreases with n and $P_n \to 0$ as $n \to \infty$; and
  - If $0.5 = p$, or $p = 1$, then $P_n = \mathrm{p}$ for all $n \in \mathbb{N}$.
- Proof: Notice that $P_n = \sum_{x=\frac{n+1}{2}}^{n} f(x)$ where $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$ and recall the LLN.

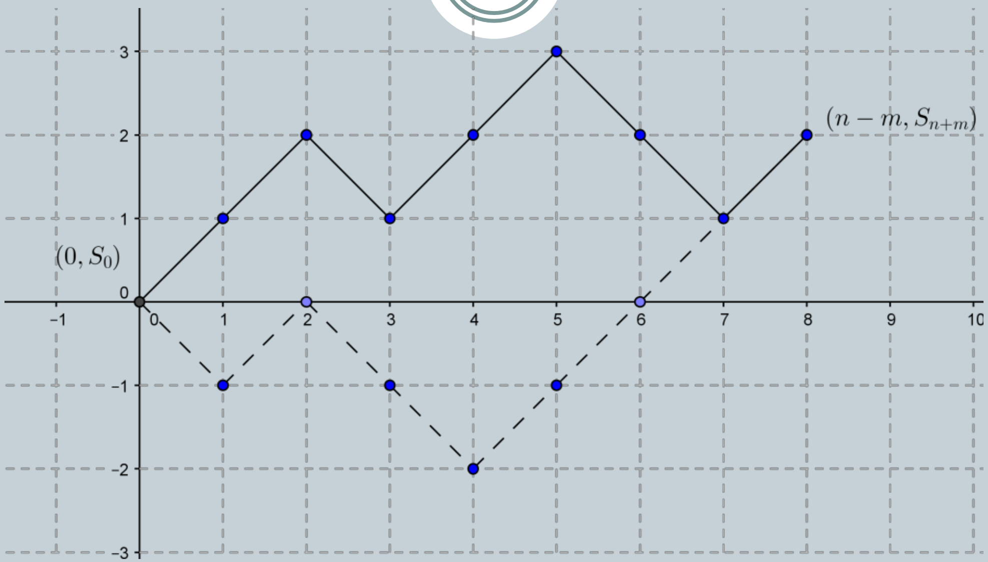
2.1. The Condorcet's jury theorem

- CJT is nice in the sense that it gives a formal basis for group action (i.e. democracy).
- From voting theory perspective, it is a theorem about majority rule. Indeed one can further show that **majority rule is the best estimator** in this context (Proof by the Neyman-Pearson lemma!).
- It allows for many extensions. For example, Owen et al., (1989) shows that when jurors have different levels of competence each greater than 0.5 , or any case, its average is greater than 0.5, group deciding via majority rule is better than average member, and its competence increases with group size and approaches to 1.
  - REF: Owen G., Grofman B. and S.L.Feld (1989) Proving distribution free generalization of the CJT, *Math. Soc. Sciences*, 17: 1 – 16

2.2. Bertand's Ballot Theorem

- Theorem (Bertrand, 1887): Suppose there are $n + m$ voters and two candidates A, B receiving n, m votes respectively with $n > m$ (so A is the winner). If voters cast their ballots in a random order the probability that A has more votes than B at all times during the election is $\frac{n-m}{n+m}$.
  - Proof: Let X_i be the random variable that takes value 1 if i'th voter votes for A and -1, if otherwise. Consider the sum $S_k = X_1 + \cdots + X_k$ and clearly $S_{n+m} = n - m$. On a two dimensional grid consider points $(0, S_0), (1\ S_1), \dots, (n - m, S_{n+m})$ and we call the line connecting these points as a **path**.

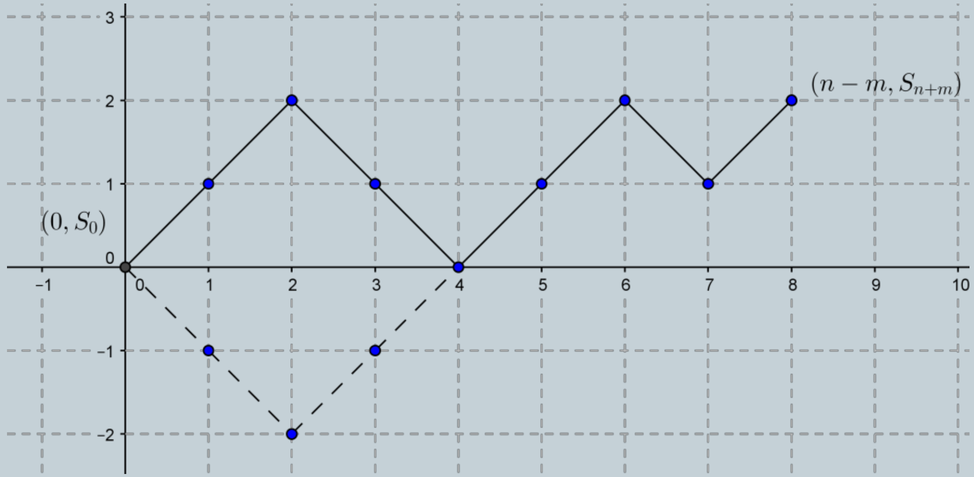
2.2. Bertand's Ballot Theorem



- Then our problem reduces to counting the number of paths that lie strictly above X-axis (except the origin), and that of all paths, and finding their ratio.
- Counting the latter is easy: $\binom{n+m}{n}$.
- Count the former as follows: First count the number of paths that intersect with X-axis and then subtract it from $\binom{n+m}{n}$.

2.2. Bertand's Ballot Theorem

- Reflection principle: The number paths that intersect with X-axis is twice the number of paths starting at $(-1,-1)$ and ends $(n-m, n+m)$.



- Thus, $p = \frac{\binom{n+m}{n} - 2\binom{n+m-1}{n}}{\binom{n+m}{n}} = \frac{n-m}{n+m}$.

2.2. Bertand's Ballot Theorem

- This problem quite delicate and relevant for both combinatorics and probability theory. Thus, elections can lead to interesting problems!
- It also allows for various generalizations including continuous versions (see REF below).
- From the point of voting the reverse problem sounds also interesting: Given the past history, what is it chance of a candidate (a party) winning in the next?
  - REF: Addario-Berry L. and B.A.Reed (2008) Ballot theorems, old and new. In *Horizons of Combinatorics*, Bolyai Soc. Math. Stud. Vol. 17: 9 – 35.

2.3. Gibbard's Random Dictatorship Theorem

- Setting: N is the set of voters, and **A** is the set of alternatives with n and $m > 2$ elements, respectively. Voter $i = 1, \dots, n$ has a strict preferences ordering over **A**. Let $\boldsymbol{L}(\mathbf{A})$ is the set of all possible strict orderings on $\boldsymbol{A}$ and $\boldsymbol{P}(\boldsymbol{A})$ be the set all probability distributions over $\boldsymbol{A}$.
- A decision scheme is a mapping $f\colon \boldsymbol{L}(\boldsymbol{A})^n \to \boldsymbol{P}(\boldsymbol{A})$.
- Payoff (or utility): Given f, at any profile $l \in \boldsymbol{L}(\boldsymbol{A})^n$ voter $i \in N$ receives
  - $U_i(f(l), l) = \sum_{j=1}^{m} u_i(x_j, l) \cdot p(x_j, l)$ where $u_i(.,.)\colon \boldsymbol{A} \times \boldsymbol{L}(\boldsymbol{A})^n \to \mathbb{R}$ is a non-random utility representation.
- Axioms:
  - Strategy Proof: Take any pair $l, l' \in \boldsymbol{L}(\boldsymbol{A})^n$ which are identical except voter i's ranking. If for some $u_i(.,.)$ representing i's ranking we have $U_i(f(l'), l) > U_i(f(l), l)$ then f is manipulable for her at $l \in \boldsymbol{L}(\boldsymbol{A})^n$. f is STP if it is never manipulable.
  - (Ex post) Pareto: For any $x, y \in \boldsymbol{A}$ and any $l \in \boldsymbol{L}(\boldsymbol{A})^n$ if every voter prefers x to y at l, then $p(y, l) = 0$.

2.3. Gibbard's Random Dictatorship Theorem

- Randomly dictatorial decision scheme: A dictatorial decision scheme is the one that picks one voter and always chooses her best alternative as an outcome. $f: \boldsymbol{L}(\boldsymbol{A})^n \to \boldsymbol{P}(\boldsymbol{A})$ is r.d. if it is a convex combination of some dictatorial decision schemes.
- Theorem (Gibbard, 1977):

  Let $m > 2$. Then $f: \boldsymbol{L}(\boldsymbol{A})^n \to \boldsymbol{P}(\boldsymbol{A})$ satisfies STP and Pareto iff it is randomly dictatorial.

  - Proof: See
    - Gibbard A. (1977) Manipulation of schemes that mix voting with chance, *Econometrica* 45: 665 – 681
    - Tanaka Y. (2003) An alternative proof of Gibbard's random dictatorship theorem, *Rev. Econ. Design* 8: 319 – 328.

2.3. Gibbard's Random Dictatorship Theorem

- It is an extension of the so called Gibbard-Satterhwaite impossibility theorem (see the REF below).
- Thus, it is a theorem about the notion of STP.
- A continuous analog of this theorem is yet to be formulated!
  - REF:
    - Ninjbat U. (2012) Another direct proof for the Gibbard-Satterthwaite theorem, *Econ. Letters* 116(3): 418 – 421.
    - Ninjbat U. (2015) Impossibility theorems are modified and unified, to appear in *Soc. Choice Welf.*

3. Final comments

- Diversity and unity are equally important in doing research!
- Accordingly, we presented three results in voting theory with elements probability in it which suggest that voting and probability are mutually relevant:
  - Probability is relevant for voting (see CJT)
  - Voting is relevant for probability (see Ballot theorem)
  - Its likely that the most of classical results admit a probabilistic version (see Gibbard's RDT)
- There is not much stochastic analysis (explicit) in here! But there certainly is a room for it!
  - It makes sense to think ballots as realizations of some random variables
  - Idea of conditioning also makes lots of sense in this context, etc.

Some more references

General introduction:

- Wallis W.D. (2014) *The Mathematics of Elections and Voting*, Springer.
- Nitzan S. (2010) *Collective Preferences and Choice*, CUP.

Statistical approach:

- Balinksi M., R.Laraki (2011) *Measuring, Ranking and Electing*, MIT Press.
- Pivato M. (2013) Voting rules as statistical estimators, *Soc. Choice Welf*. 40(2): 581 – 630.
- Häggström O., Kalai G., Mossel E. (2006) A law of large numbers for weighted majority, *Advances in applied mathematics* 37(1): 112 – 123.