Functional central limit theorems for supercritical superprocesses

Yan-Xia Ren

Peking University

National University of Mongolia, Aug. 3, 2015

Outline

References

This talk is based on the following two papers:

[1] Y.-X. Ren, R. Song and R. Zhang: Central limit theorems for supercritical superprocesses. *Stoch. Proc. Appl.*, 125(2015) 428-457.

[2] Y.-X. Ren, R. Song and R. Zhang: Functional central limit theorems for supercritical superprocesses. Preprint, 2014.

Outline

Previous works on CLT for BMP and superprocesses

Kesten and Stigum (1966) gave a central limit theorem for **multi-type discrete time** processes by using the Jordan canonical form of the expectation matrix M.

Then Athreya (1969, 1969, 1971) proved central limit theorems for **multi-type continuous time** branching processes, also using the Jordan canonical form and the eigenvectors of the matrix M_t, the mean matrix at time t.

Asmussen and Hering (1983) established spatial central limit theorems for general supercritical branching Markov processes under a certain condition.

Janson (2004) extended the results mentioned above and established **functional** central limit theorems for multitype branching processes.

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Ren, Song and Zhang (2012-2014, 4 papers) established central limit theorems for supercritical branching Markov processes and superprocesses. We also extend the functional central limit theorems of Janson (2004) to supercritical superprocesses with space-dependent branching mechanisms.

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In this talk I will introduce our reuslts for superprocesses.

First we would like to find **conditions on the spatial processes** such that the spatial central limit theorem(CLT) works for supercritical **superprocesses** with space-dependent branching mechanisms.. The conditions should be easy to check and satisfied by a lot of Markov processes.

We also would like to give the **functional CLT** for supercritical superprocesses (to give a superprocess version of Jason's work).

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Superprocesses

E: a locally compact separable metric space.
m: a σ-finite Borel measure on E with full support.
∂: a separate point not contained in E. ∂ will be interpreted as the cemetery point.

$\xi = \{\xi_t, \Pi_x\}$: a Hunt process on E.
$\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ.
$\{P_t : t \geq 0\}$: the semigroup of ξ.

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The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by three objects:

(i) a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E, which is symmetric with respect to m.

(ii) a branching rate function $\beta(x)$ on E which is a non-negative bounded measurable function.

(iii) a branching mechanism φ of the form

$$\varphi(x,z) = -a(x)z + b(x)z^2 + \int_{(0,+\infty)} (e^{-zy} - 1 + zy)n(x,dy), x \in E, z > 0, \quad (1)$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0, \infty)$ satisfying

$$\sup_{x \in E} \int_{(0,+\infty)} y^2 n(x,dy) < \infty. \quad (2)$$

$\mathcal{M}_F(E)$ denote the space of finite measures on E.
$\langle f,\mu\rangle := \int_E f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle = \mu(E)$.

The superprocess X is a Markov process taking values in $\mathcal{M}_F(E)$. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. Then for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu\left(e^{-\langle f,X_t\rangle}\right) = \langle u_f(t,\cdot),\mu\rangle, \tag{3}$$

where $u_f(t,x)$ is the unique positive solution to the equation

$$u_f(t,x) + \Pi_x \int_0^{t\wedge\zeta} \varphi(\xi_s, u_f(t-s,\xi_s))\beta(\xi_s)ds = \Pi_x f(\xi_t), \tag{4}$$

Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and} \quad A(x) := \beta(x)\left(2b(x) + \int_0^\infty y^2 n(x,dy)\right).$$

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For any $f \in \mathcal{B}_b(E)$ and $(t,x) \in (0,\infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t) \right].$$

First moment: For any $f \in \mathcal{B}_b(E)$,

$$\mathbb{P}_\mu \langle f, X_t \rangle = \langle T_t f, \mu \rangle.$$

Second moment: For any $f \in \mathcal{B}_b(E)$,

$$\mathbb{V}\mathrm{ar}_\mu \langle f, X_t \rangle = \langle \mathbb{V}\mathrm{ar}_{\delta_\cdot} \langle f, X_t \rangle, \mu \rangle = \int_E \int_0^t T_s[A(T_{t-s}f)^2](x)\, ds \mu(dx), \quad (5)$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$.

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Assumptions on the spatial process

We assume that that there exists a family of continuous strictly positive symmetric functions $\{p_t(x,y) : t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p_t(x,y) f(y) m(dy).$$

Define

$$\widetilde{a}_t(x) := p_t(x,x).$$

Assumption 1

(i) For any $t > 0$, we have $\int_E \widetilde{a}_t(x)\, m(dx) < \infty$.

(ii) There exists $t_0 > 0$ such that $\widetilde{a}_{t_0}(x) \in L^2(E, m)$.

Remark (ii) above is equivalent to

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One can check that there exists a family $\{q_t(x,y) : t > 0\}$ of continuous strictly positive symmetric functions on $E \times E$ such that

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It follows from the assumptions (i) and (ii) in the previous subsection that a_t enjoys the following properties:

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The first eigenvalue $-\lambda_1$ is simple and the eigenfunction ϕ_1 associated with $-\lambda_1$ can be chosen to be strictly positive everywhere and continuous. We will assume that $\|\phi_1\|_2 = 1$. ϕ_1 is sometimes denoted as $\phi_1^{(1)}$.

For $k > 1$, let $\{\phi_j^{(k)}, j = 1, 2, \cdots n_k\}$ be an orthonormal basis of the eigenspace associated with $-\lambda_k$.

$\{\phi_j^{(k)}, j = 1, 2, \cdots n_k; k = 1, 2, \dots\}$ **forms a complete orthonormal basis of** $L^2(E, \mu)$ and all the eigenfunctions are continuous.

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More Assumptions

Assumption 2 The superprocess is supercritical, that is, $\lambda_1 < 0$. Then

$$\mathbb{P}_{\delta_x}\langle f, X_t\rangle = T_t f(x) \sim e^{-\lambda_1 t} \quad \text{as } t \to \infty.$$

Assumption 3 for any $t > 0$ and $x \in E$,

$$\mathbb{P}_{\delta_x}\{\|X_t\| = 0\} \in (0, 1). \tag{6}$$

Remark Here is a sufficient condition for (6). Suppose that $\Phi(z) = \inf_{x\in E} \psi(x, z)\beta(x)$ can be written in the form:

$$\Phi(z) = \widetilde{a}z + \widetilde{b}z^2 + \int_0^\infty (e^{-zy} - 1 + zy)\widetilde{n}(dy)$$

with $\widetilde{a} \in \mathbb{R}$, $\widetilde{b} \geq 0$ and $\widetilde{n}$ being a measure on $(0, \infty)$ satisfying $\int_0^\infty (y \wedge y^2)\widetilde{n}(dy) < \infty$. If $\widetilde{b} + \widetilde{n}(0, \infty) > 0$ and $\Phi(z)$ satisfies $\int^\infty \frac{1}{\Phi(z)} dz < \infty$ then (6) holds.

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Main Results

Define

$$H_t^{k,j} := e^{\lambda_k t}\langle \phi_j^{(k)}, X_t\rangle, \quad t \geq 0.$$

For any nonzero $\mu \in \mathcal{M}_F(E)$, $H_t^{k,j}$ is a martingale under $\mathbb{P}_\mu$.

Strong Law of Large Numbers: If $\lambda_1 > 2\lambda_k$, then $\sup_{t>3t_0} \mathbb{P}_\mu(H_t^{k,j})^2 < \infty$. Thus the limit

$$H_\infty^{k,j} := \lim_{t\to\infty} H_t^{k,j} \quad \text{exists } \mathbb{P}_\mu\text{-a.s. and in } L^2(\mathbb{P}_\mu).$$

In particular, we write $W_t := H_t^{1,1} = e^{\lambda_1 t}\langle \phi_1, X_t\rangle$ and $W_\infty := H_\infty^{1,1}$. $\{W_t : t \geq 0\}$ is a positive martingale and

$$W_t \to W_\infty, \quad \mathbb{P}_\mu\text{-a.s. and in } L^2(\mathbb{P}_\mu).$$

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Question 1: Is it possible to get

$$\frac{W_t - W_\infty}{e^{\lambda_1 t/2}} \stackrel{d}{\to} \text{a normal distribution}, \quad t \to \infty?$$

We will use $(\cdot,\cdot)_m$ to denote inner product in $L^2(E,m)$.

Question 2: For any bounded measurable $f \geq 0$, it will be proved that

$$e^{\lambda_1 t}\langle f, X_t\rangle \to (\phi_1, f)_m W_\infty \quad \text{as } t \to \infty.$$

When $(\phi_1, f)_m = 0$, what is the proper scaling rate for $\langle f, X_t\rangle$?

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Question 1: Is it possible to get

$$\frac{W_t - W_\infty}{e^{\lambda_1 t/2}} \xrightarrow{d} \text{a normal distribution}, \quad t \to \infty?$$

We will use $(\cdot, \cdot)_m$ to denote inner product in $L^2(E, m)$.

Question 2: For any bounded measurable $f \geq 0$, it will be proved that

$$e^{\lambda_1 t} \langle f, X_t \rangle \to (\phi_1, f)_m W_\infty \quad \text{as } t \to \infty.$$

When $(\phi_1, f)_m = 0$, what is the proper scaling rate for $\langle f, X_t \rangle$?

Main Results

Denote

$$\mathcal{C}_l := \left\{ g(x) = \sum_{k:\lambda_1 > 2\lambda_k} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : b_j^k \in \mathbb{R} \right\},$$

$$\mathcal{C}_c := \left\{ g(x) = \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : 2\lambda_k = \lambda_1, b_j^k \in \mathbb{R} \right\}$$

and

$$\mathcal{C}_s := \left\{ g(x) = \sum_{k:\lambda_1 < 2\lambda_k} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \text{ with } \sum_{k:\lambda_1 < 2\lambda_k} \sum_{j=1}^{n_k} (b_j^k)^2 < \infty \right\}.$$

Main Results

For $f \in \mathcal{C}_s$, define

$$\sigma_f^2 := \int_0^\infty e^{\lambda_1 s}(A(T_s f)^2, \phi_1)_m \, ds$$

For $h \in \mathcal{C}_c$, define

$$\rho_h^2 := \left(Ah^2, \phi_1\right)_m .$$

For $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, we define

$$\beta_g^2 := \int_0^\infty e^{-\lambda_1 s} \left(A(I_s g)^2, \phi_1\right)_m ds, \quad \text{where}$$

$$I_s g(x) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} e^{\lambda_k s} b_j^k \phi_j^{(k)}(x)$$

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Main Results

Theorem 1

If $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$ and $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, then $\sigma_f^2 < \infty$, $\rho_h^2 < \infty$ and $\beta_g^2 < \infty$. Let

$$F_t(g) := \sum_{k:2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} b_j^k H_\infty^{k,j}.$$

Then, it holds that, under $\mathbb{P}_\nu(\cdot \mid \mathcal{E}^c)$, as $t \to \infty$,

$$\left(e^{\lambda_1 t}\langle \phi_1, , X_t\rangle, \frac{\langle g, X_t\rangle - F_t(g)}{\sqrt{\langle \phi_1, X_t\rangle}}, \frac{\langle h, X_t\rangle}{\sqrt{t\langle \phi_1, X_t\rangle}}, \frac{\langle f, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}} \right) \xrightarrow{d} (W^*, G_3(g), G_2(h),\ G_1(f)), \tag{7}$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$, $G_3(g) \sim \mathcal{N}(0, \beta_g^2)$, $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W^*, $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent.

Main Results

For $f_1, f_2 \in \mathcal{C}_s$, define

$$\sigma(f_1, f_2) = \int_0^\infty e^{\lambda_1 s} (A(T_s f_1)(T_s f_2), \phi_1)_m \, ds$$

Then, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,

$$\left(\frac{\langle f_1, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle f_2, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (G_1(f_1), G_1(f_2)), \quad t \to \infty,$$

where $(G_1(f_1), G_1(f_2))$ is a bivariate normal random variable with covariance

$$\mathrm{Cov}(G_1(f_1), G_1(f_2)) = \sigma(f_1, f_2).$$

For $h_1, h_2 \in \mathcal{C}_c$, define

$$\rho(h_1, h_2) = (Ah_1 h_2, \phi_1)_m.$$

Then we have, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,

$$\left(\frac{\langle h_1, X_t \rangle}{\sqrt{t\langle \phi_1, X_t \rangle}}, \frac{\langle h_2, X_t \rangle}{\sqrt{t\langle \phi_1, X_t \rangle}} \right) \stackrel{d}{\to} (G_2(h_1), G_2(h_2)), \quad t \to \infty,$$

where $(G_2(h_1), G_2(h_2))$ is a bivariate normal random variable with covariance

$$\mathrm{Cov}(G_2(h_1), G_2(h_2)) = \rho(h_1, h_2).$$

For $g_1, g_2 \in \mathcal{C}_l$, define

$$\beta(g_1, g_2) = \int_0^\infty e^{-\lambda_1 s}(A(I_s g_1)(I_s g_2), \phi_1)_m \, ds.$$

Then we have, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,

$$\left(\frac{\langle g_1, X_t\rangle - F_t(g_1)}{\sqrt{\langle \phi_1, X_t\rangle}}, \frac{\langle g_2, X_t\rangle - F_t(g_2)}{\sqrt{\langle \phi_1, X_t\rangle}} \right) \xrightarrow{d} (G_3(g_1), G_3(g_2)),$$

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$$\mathrm{Cov}(G_3(g_1), G_3(g_2)) = \beta(g_1, g_2).$$

Main Results

We denote by $\mathbb{D}(\mathbb{R}^d)$ the space of all cadlag functions from $[0, \infty)$ into $\mathbb{R}^d$, equipped with the Skorokhod topology.

Our next aim is to establish functional central limit theorem: for "good" test function f such that $\langle f, X_t \rangle$ is right continuous and has left limit, we hope that prove

$$C_{t+\cdot}(\langle f, X_{t+\cdot} \rangle - A_{t+\cdot}) \to \sqrt{W_\infty} G(\cdot) \quad \text{(in distribution)}$$

with G being some Gaussian process.

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Main Results

Fix a $q > \max\{K, -2\lambda_1\}$. For any $p \geq 1$ and $f \in L^p(E, m)$, define

$$U_q f(x) := \begin{cases} \int_0^\infty e^{-qs} T_s f(x)\, ds, & \text{if } \int_0^\infty e^{-qs} T_s(|f|)(x)\, ds < \infty, \\ 0, & \text{otherwise}. \end{cases}$$

Then $U_q f \in L^p(E, m)$.

Lemma

If $f \in L^2(E, m)$, then for any $\mu \in \mathcal{M}_F(E)$, the function $\langle U_q f, X_\cdot \rangle$ is in $\mathbb{D}(\mathbb{R})$, $\mathbb{P}_\mu$-a.s.

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Main Results

For $f \in \mathcal{C}_s$, define

$$\sigma_{f,\tau} := e^{\lambda_1 \tau/2} \int_0^\infty e^{\lambda_1 s} (A(T_s f)(T_{s+\tau} f), \phi_1)_m \, ds. \tag{8}$$

We write $\sigma_{f,0}$ as σ_f^2.

For $h \in \mathcal{C}_c$, define

$$\rho_h^2 := (Ah^2, \phi_1)_m. \tag{9}$$

Recall that for $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, we put

$$F_t(g) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} e^{-\lambda_k t} b_j^k H_\infty^{k,j}, \quad t \geq 0.$$

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Main Results

For $g \in \mathcal{C}_l$, define

$$\beta_{g,\tau} := e^{-\lambda_1 \tau/2} \int_0^{\infty} e^{-\lambda_1 s} (A(I_s g)(I_{s+\tau} g), \phi_1)_m \, ds. \tag{10}$$

where $I_u g(x) := \sum_{k:2\lambda_k < \lambda_1} \sum_{j=1}^{n_k} e^{\lambda_k u} b_j^k \phi_j^{(k)}(x), x \in E, u \geq 0.$

We write $\beta_g^2 := \beta_{g,0}$.

For $f \in \mathcal{C}_s$, $g \in \mathcal{C}_l$ and $0 \leq \tau_1 \leq \tau_2$, define

$$\eta_{\tau_1,\tau_2}(f,g) := -e^{\lambda_1(\tau_1+\tau_2)/2} \int_{\tau_1}^{\tau_2} e^{-\lambda_1 u} \left(A(T_{\tau_2-u} f)(I_{u-\tau_1} g), \phi_1\right)_m du. \tag{11}$$

Main Results

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Main Results

Theorem 2

Assume that $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$, $g \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_F(E)$. For any $t > 0$, define

$$Y_t^{1,f}(\tau) := e^{\lambda_1(t+\tau)/2}\langle f, X_{t+\tau}\rangle, \quad \tau \geq 0,$$

$$Y_t^{2,h}(\tau) := t^{-1/2}e^{\lambda_1(t+\tau)/2}\langle h, X_{t+\tau}\rangle, \quad \tau \geq 0,$$

and

$$Y_t^{3,g}(\tau) := e^{\lambda_1(t+\tau)/2}\left(\langle g, X_{t+\tau} - F_{t+\tau}(g)\rangle\right), \quad \tau \geq 0.$$

Then, for each fixed $t \in [0, \infty)$, $\left(W_t, Y_t^{1,U_qf}(\cdot), Y_t^{2,h}(\cdot), Y_t^{3,g}(\cdot)\right)$ is a $\mathbb{D}(\mathbb{R}^4)$-valued random variable under $\mathbb{P}_\mu$, where W_t is regarded as a constant process. Furthermore, under $\mathbb{P}_\mu$,

$$\left(W_t, Y_t^{1,U_qf}(\cdot), Y_t^{2,h}(\cdot), Y_t^{3,g}(\cdot)\right)$$

$$\xrightarrow{d,\mathbb{D}(\mathbb{R}^4)} \left(W_\infty, \sqrt{W_\infty}G^{1,U_qf}(\cdot), \sqrt{W_\infty}G^{2,h}, \sqrt{W_\infty}G^{3,g}(\cdot)\right).$$

Main Results

Theorem 2 (continue)

Here $G^{2,h} \sim \mathcal{N}(0, \rho_h^2)$ is a constant process, and $\{(G^{1,U_q f}(\tau), G^{3,g}(\tau)) : \tau \geq 0\}$ is a continuous $\mathbb{R}^2$-valued Gaussian process with mean 0 and covariance functions given by

$$E(G^{1,U_q f}(\tau_1)G^{1,U_q f}(\tau_2)) = \sigma_{U_q f, \tau_2 - \tau_1}, \quad \text{for } 0 \leq \tau_1 \leq \tau_2, \tag{12}$$

$$E(G^{3,g}(\tau_1)G^{3,g}(\tau_2)) = \beta_{g, \tau_2 - \tau_1}, \quad \text{for } 0 \leq \tau_1 \leq \tau_2, \tag{13}$$

and

$$E(G^{3,g}(\tau_1)G^{1,U_q f}(\tau_2)) = \begin{cases} \eta_{\tau_1, \tau_2}(U_q f, g), & \text{if } 0 \leq \tau_1 < \tau_2, \\ 0, & \text{if } \tau_1 \geq \tau_2 \geq 0. \end{cases} \tag{14}$$

Moreover, W_∞, $G^{2,h}$ and $(G^{1,U_q f}, G^{3,g})$ are independent.

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Thank you!