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Stochastic Optimal Control in infinite dimension: why it is worth studying?

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Main goal

Present some applied problems (mainly in Economics and Finance) which can be naturally formulated as **stochastic control problems in infinite dimensional spaces** and briefly discuss how they can be approached and, sometimes, solved.

We will focus on the Dynamic Programming Approach.

Overview

- Two introductory models with solution:
  - Optimal consumption in growth models;
  - Pricing/hedging contingent claims.
- Related infinite dimensional problems:
  - Growth and optimal investments with age structure;
  - Pricing and hedging in delay/path dependent financial models and in forward mortality models.
  - Optimal portfolio and optimal advertising with history dependent dynamics.
- Discussion on the solution methods.
- An example of results: models with delay.

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Two introductory models: optimal consumption in growth models and pricing/hedging contingent claims

STARTING PROBLEM 1: OPTIMAL CONSUMPTION (RAMSEY GROWTH MODEL)

- State equation:

$$\begin{cases} x'(t) = f_P(x(t)) - c(t) =: i(t), \quad t > 0 \\ x(0) = x_0 > 0, \end{cases}$$

  where, at time $t \geq 0$, $c(t)$ is the **consumption rate** (the control), $x(t)$ is the **capital stock** (the state), $f_P : [0, +\infty) \longrightarrow [0, +\infty)$ is the **per capita production function** (usually increasing and concave). $x_{x_0,c(\cdot)}(t)$ denotes the unique solution at time t.

- Constraints $c(\cdot) \geq 0$, $x(\cdot) \geq 0$, (sometimes also $x'(\cdot) \geq 0$).

- Problem: *Maximize the* **intertemporal utility**

$$J(x_0; c(\cdot)) := \int_0^\infty e^{-\rho t} u(c(t)) dt$$

  where $\rho > 0$, and $u(\cdot) : [0, +\infty) \longrightarrow \mathbb{R}$ is increasing and concave.

  For example $u(c) = u_\sigma(c) := \frac{c^{1-\sigma}-1}{1-\sigma}$ for $\sigma > 0$.

Stochastic variants:

- (see e.g. [Malliaris-Brock '82, Morimoto '10]) add a white noise term $\sigma x(t)\dot{W}(t)$ in the state equation.

$$\begin{cases} dx(t) = [f_P(x(t)) - c(t)]dt + \sigma x(t) dW(t), \quad t > 0 \\ x(0) = x_0 > 0, \end{cases}$$

- (see e.g. [Constantinides '90, Obstfeld '94]) add a new control $\alpha(t)$ (investment in "risky" asset) and a white noise term $\sigma\alpha(t)\dot{W}(t)$ in the state equation. f_P is linear.

$$\begin{cases} dx(t) = [rx(t) + (\mu - r)\alpha(t) - c(t)]dt + \sigma\alpha(t) dW(t), \quad t > 0 \\ x(0) = x_0 > 0, \end{cases}$$

In both cases the goal is to maximize the expected value of the functional J above.

NOTE: in the two models above also Levy noise can be used [Steger 05, Posch-Walde '11].

DYNAMIC PROGRAMMING (DP1 - EASY)

- Consider the problem for generic initial datum x and define the ***value function***

$$V(x_0) = \sup_{c(\cdot) \text{ admissible at } x_0} J(x_0; c(\cdot)).$$

- Prove that the value function satisfies a functional equation which is the so called ***Dynamic Programming Principle*** (DPP).

$$V(x_0) = \sup_{c(\cdot) \text{ admissible at } x_0} \left\{ \int_0^t e^{-\rho s} u(c(s)) ds + e^{-\rho t} V(x_{x_0,c(\cdot)}(t)) \right\}.$$

- Write the PDE version of the DPP, the so called ***Hamilton-Jacobi-Bellman equation*** (HJB). If the value function V is smooth, then it solves the HJB equation.
  But we do not know ex ante that V is smooth!

DYNAMIC PROGRAMMING (DP2 - HARD)

- Prove that the associated HJB equation has a classical solution v (possibly, but not necessarily, $v = V$).
- Use the fact that v solves the HJB equation to prove, via the so called Verification Theorem that:
  - $v = V$ and
  - there exist a (possibly unique) optimal control strategy c^* in feedback form, i.e. $c^*(t) = G(x^*(t))$ where x^* is the associated state trajectory and G (the feedback map) depends on the derivative of v.

For the stochastic variants the procedure is the same. Methods of proof are more difficult and HJB is different as it contains a second order term (by Ito Formula). See the lectures of prof. Oksendal.

DP APPROACH FOR THE STARTING PROBLEM 1

- The associated HJB equation is

$$\rho v(x) = f_P(x)v'(x) + H_0(v'(x)), \quad x > 0 \tag{1}$$

where

$$H_0(p) := \sup_{c \geq 0} \{-cp + u(c)\}$$

with the boundary condition $v(0) = 0$. We call $G_0(p)$ the argmax of the Hamiltonian ($G_0(p) = \left\{ p^{-\frac{1}{\sigma}} \right\}$ when $u = u_\sigma$ and $p > 0$).

In the stochastic case, by Ito formula, the function H_0 also depends on v''. E.g. in the first variant we simply add $\frac{1}{2}\sigma^2 x^2 v''(x)$.

- The value function V is the unique classical (C^1) solution of the HJB equation above satisfying a suitable growth condition at infinity. When f_P is linear and $u = u_\sigma$, V can be computed explicitly: $V(x) = Ax^{1-\sigma}$, for given $A \in \mathbb{R}$.

- The feedback map G associated with V is given by

$$G(x) := G_0(V'(x)), \qquad \left(G(x) = V'(x)^{-\frac{1}{\sigma}} \quad if\ u = u_\sigma\right).$$

So, substituting $c(t) = G(x(t))$ in the state equation we get the Closed Loop Equation (CLE)

$$\begin{cases} x'(t) = f_P(x(t)) - G(x(t)), \\ x(0) = x_0 > 0, \end{cases}$$

which admits a unique solution $x^*(\cdot)$.

- The feedback strategy $c^*(t) := G(x^*(t))$ is admissible and is the unique optimal strategy.

Note: we need $V \in C^1$ to well define the optimal feedback map G. This is false in general (e.g. when f_P is not concave).

In the stochastic case, if the diffusion coefficient depends on the control (e.g. in the second variant), then also the second derivative V'' needed to define G.

STARTING PROBLEM 2: PRICING/HEDGING OF EUROPEAN CONTINGENT CLAIMS

Consider a market with two assets: a risk free asset U which has the deterministic dynamics

$$\mathrm{d}U_t = rU_t\,\mathrm{d}t \qquad U_0 = 1$$

and a stock price x, with the dynamics

$$\mathrm{d}x_t = \mu(t, x_t)x_t\,\mathrm{d}t + \nu(t, x_t)x_t\,\mathrm{d}B_t \qquad x_0 = y^0 \in \mathbb{R}^+ \tag{2}$$

with $\mu, \nu : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ deterministic functions and B standard Wiener process generating a filtration $\mathcal{F}_t$.

For any given $g : \mathbb{R} \longrightarrow \mathbb{R}$, consider the two problems:

1. find a "fair price" π_t at time t of the claim $g(x_T)$;
2. find a self-financing portfolio strategy: processes h_U and h_x such that (defining $V = h_U U + h_x$):
   - $\mathrm{d}V_t = h_{U,t}\,\mathrm{d}U_t + h_{x,t}\,\mathrm{d}x_t$ (self-financing condition);
   - $V_T = g(x_T)$ (replicability condition).

The Risk Neutral Valuation Formula

Standard arguments based on absence of arbitrage conditions imply (see prof. Eberlein lectures)

$$\pi_t = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[g(x_T^{0,y^0}) \middle| \mathcal{F}_t \right]$$

where $\mathbb{Q}$ is the (unique) equivalent martingale measure of the market, y^0 is the initial condition for x and $x_t^{s,y}$ is the solution of (2) with initial condition $x_s = y$.

The solution of a Kolmogorov PDE gives the price function

Under suitable differentiability conditions on the coefficients of the stock dynamics and on g it can be proved (as in the lectures of prof. Beck for the case of the heat equation) that the function

$$u(s, y^0) = \mathrm{e}^{-r(T-s)}\mathbb{E}^{\mathbb{Q}}\left[g(x_T^{s,y^0})\right]$$

solves the Kolmogorov-type equation

$$\begin{cases} \partial_t u + rx\partial_x u + \frac{1}{2}\nu^2(t,x)\partial_{xx}^2 u - ru = 0 \\ u(T,x) = g(x) \end{cases} \tag{3}$$

When ν is constant this is the well known Black-Scholes equation.

The solution of a Kolmogorov PDE gives the hedging strategy

The hedging strategy can be found via BSDE. Nevertheless, once having the solution of (3), an explicit form is given by

$$\begin{aligned} h_{U,t} &= \frac{u(t, x_t^{0,y^0}) - x_t^{0,y^0} \partial_{y^0} u(t, x_t^{0,y^0})}{U_t} \\ h_{x,t} &= \partial_{y^0} u(t, x_t^{0,y^0}). \end{aligned} \tag{4}$$

With this formulae it is possible to compute, explicitly or numerically, the price and the hedging strategy of a given contingent claim.

Remark: the process h_U can be constructed once we have h_x.

General Message 1:

To solve the optimal control problem

AND

to solve the pricing/hedging problem,

using “Dynamic Programming”,

we have to solve an “upper level" Partial Differential Equation (PDE) in “classical sense"

where the “space” variable lies in the state space of the problem.

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Related infinite dimensional problems 1: economic growth and optimal investments

GROWTH AND DELAY 1: TIME-TO-BUILD (Asea-Zak '99, Bambi '08, Bambi-Fabbri-Gozzi '12, Bambi-Gozzi-Licandro '14)

- State equation:

$$\begin{cases} x'(t) = f_P(x(t-d)) - c(t), & t > 0 \\ x(t) = x_0(t), & t \in [-d, 0] \end{cases}$$

  Again at time $t \geq 0$, $c(t)$ is the consumption rate (control), $x(t)$ is the capital stock (state). $f_P : [0, +\infty) \longrightarrow [0, +\infty)$ is increasing and concave (per capita production function).

  $x_{x_0(\cdot), c(\cdot)}(t)$ denotes the unique solution at time t.

  New feature: d is the time-to build: this gives a state equation with **delay in the state**.

- Constraints $c(\cdot) \geq 0$, $x(\cdot) \geq 0$, (sometimes also $x'(\cdot) \geq 0$).

- Problem: *Maximize*

$$\int_0^\infty e^{-\rho t} u(c(t))dt$$

where $\rho > 0$, and $u(\cdot) : [0, +\infty) \longrightarrow \mathbb{R}$ is increasing and concave.

For example $u(c) = u_\sigma(c) := \frac{c^{1-\sigma}-1}{1-\sigma}$ for $\sigma > 0$.

Goal: repeat the DP program as above. Note: here the problem is path dependent $\longrightarrow$ infinite dimensional initial condition!

- Extension/Variant 1.(Federico-Gozzi-Goldys '10). More general dependence on the past: $f_p(\cdot)$ depends on an integral of the whole path with respect to a given positive measure μ on $[-d, 0)$:

$$f_P\left(\int_{-d}^0 x(t+\xi)d\mu(\xi)\right).$$

In the above case $\mu = \delta_{-d}$.

- Extension/Variant 2. (Kydland-Prescott '82, Bambi-Di Girolami-Federico-Gozzi '14). Since the quantity $i(t) := f_p(x(t)) - c(t)$ represents the investments one can, in a more realistic way, put the delay on the investment writing it as an integral of the whole path with respect to a given positive measure μ on $[-d, 0)$.

$$x'(t) = \left(\int_{-d}^{0} i(t+\xi) d\mu(\xi) \right).$$

  Here we have **delay in the control**.

- Extension/Variant 3. (Augeraud-Bambi-Gozzi '14). Habit formation: delay appears in the functional J only. Here we have **delay in the control**.

- Extension/Variant 4. (Bambi-Gozzi work in preparation). Stochastic case of the all above. Here we may have **delay in the state and/or in the control**.

GROWTH AND DELAY 2: VINTAGE CAPITAL (Boucekkine et al '04, Fabbri-Gozzi '08, Boucekkine-Fabbri-Gozzi '10)

- State equation:

$$\begin{cases} x'(t) = i(t) - i(t-d)), \quad t > 0 \\ x(0) = x_0, \quad i(t) = i_0(t), \qquad t \in [-d, 0) \end{cases}$$

  At time $t \geq 0$, $i(t)$ is the investment rate (control), $x(t)$ is the capital stock (state).

  $x_{x_0(\cdot),c(\cdot)}(t)$ denotes the unique solution at time t.

  New feature: d is the time-to-scrap: this gives a state equation with **delay in the control**.

- Constraints: for every $t \geq 0$, $c(t) := f_P(x(t)) - i(t) \geq 0$ where $f_P : [0, +\infty) \longrightarrow [0, +\infty)$ is the per capita production function. Moreover $x(\cdot) \geq 0$.
- The functional to maximize is the same.

GROWTH AND DELAY 3: GROWTH AND WEALTH DISTRIBUTION (Boucekkine-Fabbri-Gozzi '13)

- State equation: the same as in case 2 but with different meaning: at time $t \geq 0$, $i(t)$ is the birth rate (control), $x(t)$ is the number of individuals (state).

  New feature: d is the time-to-live: this gives a state equation with **delay in the control**.

- Constraints: for every $t \geq 0$, $c(t) := a - b\frac{i(t)}{x(t)} \in [0, a]$ where $a, b > 0$. Moreover $x(\cdot) \geq 0$.

- The functional to maximize is

$$\int_0^\infty e^{-\rho t} u(c(t)) x(t)^\gamma dt$$

  where $\rho > 0$, $\gamma \in (0, 1)$, and $u(\cdot) : [0, +\infty) \longrightarrow \mathbb{R}$ is increasing and concave. γ measures the degree of altruism.

GROWTH 4: SPATIAL GROWTH
(Boucekkine-Camacho-Fabbri '13)

- State equation:

$$\begin{cases} \frac{\partial}{\partial t}x(t,\xi) = \frac{\partial^2}{\partial \xi^2}x(t,\xi) + f_P(x(t,\xi) - c(t,\xi), & t > 0,\ \xi \in [0,1] \\ x(0) = x_0(\xi), \qquad x(t,\xi) = 0 \quad if \quad \xi = 0,1. \end{cases}$$

  At time $t \geq 0$, $\xi \in [0,1]$, $c(t,\xi)$ is the consumption rate (control), $x(t,\xi)$ is the capital stock (state) and $f_P : [0,+\infty) \longrightarrow [0,+\infty)$ is the per capita production function.

  $x_{x_0(\cdot),c(\cdot,\cdot)}(t,\xi)$ denotes the unique solution at time t and space ξ.

  New feature: the variable ξ denotes the space: this gives a state equation in ***infinite dimension****.*

- Constraints: for every $t \geq 0$, $c(t) \geq 0$. Moreover $x(\cdot) \geq 0$.

- Same functional to maximize but with integration also in ξ.

INVESTMENTS WITH VINTAGE CAPITAL
(Barucci-Gozzi '99, Feichtinger-Hartl-Kort-Veliov '06, Faggian-Gozzi '09)

- State equation:

$$\begin{cases} \frac{\partial}{\partial t}x(t,s) = \frac{\partial}{\partial s}x(t,s) - \delta x(t,s) + i_1(t,s)), & t > 0, \ s \in [0,\bar{s}] \\ x(0,s) = x_0(s), \ s \in [0,\bar{s}], \quad x(t,0) = i_0(t), & t \geq 0 \end{cases}$$

  At time $t \geq 0$, and vintage s, $i(t,s)$ is the investment rate (control) in capital of vintage s, $x(t,s)$ is the capital stock (state).

  $x_{x_0(\cdot),i_0(\cdot),i_1(\cdot)}(t,s)$ denotes the unique solution.

  New feature: age structure in the capital and investment:
  $\longrightarrow$ **first order PDE**.

- Constraints: for every $t \geq 0$, $s \in [0,\bar{s}]$, $x(t,s) \geq 0$

- The functional to maximize is the profit.

AGE STRUCTURED AND SPATIAL DIFFUSION MODELS IN EMISSION CONTROL AND MANAGING OF RENEWABLE RESOURCES

(e.g. Anita-Iannelli '98, Georgiev-Margenov-Veliov '05, Behringer-Upmann '14)

Models of this kind, may contain both the age structure (e.g in finding the optimal harvesting of a fish population) and the space diffusion (e.g. in controlling the emissions of pollutant over the space).

They have been investigated only in few special cases.

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Related infinite dimensional problems 2:

Pricing and Hedging:

delay/path dependent models and

forward interest rate/mortality models

Path dependent stock dynamics (Hobson-Rogers 1998, Foschi-Pascucci 2007)

Let U as before. For x assume the dynamics ($d > 0$ given)

$$\begin{cases} \mathrm{d}x_t = \mu\left(t, x_t, (x_s)_{s\in[t-d,t)}\right) x_t\,\mathrm{d}t + \sigma\left(t, x_t, (x_s)_{s\in[t-d,t)}\right) x_t\,\mathrm{d}B_t \\ x_0 = y^0\; x_s = y^1_s \text{ for } s \in [-d, 0). \end{cases} \tag{5}$$

where (for fixed $p \geq 2$)

- $\mu : [0,T] \times \mathbb{R} \times L^2(-d,0;\mathbb{R}) \longrightarrow \mathbb{R}$ and $\nu : [0,T] \times \mathbb{R} \times L^2(-d,0;\mathbb{R}) \longrightarrow \mathbb{R}^+$ are measurable functions;
- $y^0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}_+)$;
- $y^1 \in L^p\left(\Omega, \mathcal{F}_0, \mathbb{P}; L^2(-d,0;\mathbb{R}_+)\right)$.Call $y = (y^0, y^1)$.

Consider a path-dependent claim

$$g : \mathbb{R} \times L^2(-d, 0; \mathbb{R}) \longrightarrow \mathbb{R}$$

and consider again the two problems:

1. find a "fair price" π_t at time t of the derivative $g\left(x_T^{0,y}, (x_s^{0,y})_{s\in[T-d,T)}\right)$;
2. find a self-financing portfolio strategy (h_U, h_x).

The price process is still given by the Risk Neutral Valuation Formula

$$\pi_t = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[g\left(x_T^{0,y}, (x_s^{0,y})_{s\in[T-d,T)}\right)\Big| \mathcal{F}_t\right] \tag{6}$$

where $\mathbb{Q}$ is the unique martingale measure.

Forward mortality rates
(Bauer-Benth-Kiesel 2012, Tappe-Weber '14)

Let $\mu_t(\tau, x)$ be the force of mortality for an x-year old individual at time t in τ years.

Following e.g. [Bauer-Benth-Kiesel, 2012] a good model for μ_t is the following

$$\begin{cases} d\mu_t = \left[\frac{\partial}{\partial \tau}\mu_t - \frac{\partial}{\partial x}\mu_t + \alpha_t\right] dt + \sigma dW_t, \quad t > 0,\ s \in [0, \bar{s}] \\ \mu_0 \text{ given for } (\tau, x) \in [0, +\infty)^2, \end{cases}$$

To take account of newborn cohort effects Bauer and Biffis (paper in preparation) propose to add the boundary condition

$$d\mu_t(\tau, 0) = B_t \mu_t(\tau, 0) dt + \Theta_t dZ_t.$$

where Z_t is another Brownian motion, possibly correlated with W_t.

This is a stochastic PDE with boundary noise term that can be treated adapting the tools introduced by [Da Prato-Zabczyk 2002] (see also Bonaccorsi-Mastrogiacomo 2010])

Also here, for insurance purposes, one is interested to solve an associated pricing/hedging problem.

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Related infinite dimensional problem 3: Optimal portfolio with execution delay and Optimal advertising with memory effects

OPTIMAL PORTFOLIO WITH EXECUTION DELAY

The state equation is

$$dx(t) = b_0 \pi(t-d)dt + \sigma \pi(t-d)dB(t)$$
$$x(0) = \eta_0, \;\; \pi(s) = \pi_0(s) \quad s \in [-d, 0),$$

We want to maximize the functional

$$E\left[U\left(x(T)\right)dt\right],$$

over the set of the admissible strategies. i.e. the ones that keep the wealth $x(\cdot)$ positive.

Problem under study with G. Fabbri, S. Federico and H. Pham.

Other Optimal Portfolio problems with delay/path dependent terms are studied in [Federico, 2010], [Biffis-Gozzi-Prosdocimi, in progress], [Gozzi-Prosdocimi-Sekine, in progress].

OPTIMIZING THE GOODWILL'S RETURNS

The state equation is

$$dx(t) = \left[a_0 x(t) + \int_{-d}^{0} a_1(\xi) x(t+\xi) d\xi + b_0 c(t) + \int_{-d}^{0} b_1(\xi) c(t+\xi) d\xi \right] dt + \sigma dB(t)$$

$$x(0) = \eta_0, \; x(s) = \eta_1(s), \; c(s) = c_0(s) \quad s \in [-d, 0),$$

We want to maximize the functional

$$E \left[\int_0^T l_0 \left(x(t), c(t) \right) dt \right],$$

over the set of the admissible strategies.

Problem studied with C. Marinelli and now (work in progress) with F. Masiero.

General Message 2:

Optimal control models dealing with key applied issues like:

- delay/path dependency,

- age structure,

- spatial heterogeneity

carry state variables (initial conditions) belonging naturally to infinite dimensional spaces.

Hence they can be rewritten and studied as optimal control of infinite dimensional systems.

How?

Discussion on the solution method

(using Dynamic Programming)

Step 1

Rewrite the state equations as ODE or SDE in a suitable infinite dimensional (Hilbert) space with a new state variable called $X(\cdot)$. The general form is of this type

$$\begin{cases} X'(s) = AX(s) + F(X(s), c(s)) \quad + \Sigma(X(s), c(s))dW(s), \\ X(t) = x \in H, \end{cases}$$

- H is a suitable Hilbert space (the new state space) depending on the specific problem.
- $c(\cdot) : [t, T] \longrightarrow C$ is the control (C is the control space, a Polish space in general).
- The operator A is a linear differential operator (e.g. the Laplacian in the case of spatial growth or the first derivative in the case of age structure equations).
- W is a cylindrical Wiener process, possibly defined in a different Hilbert space Ξ.
- The functions F and Σ depends on the nonlinear drift and diffusion coefficients.

Step 2

Similarly one rewrites the objective functional in the following general form ($\Phi = 0$ if $T = +\infty$)

$$J(t, x; c(\cdot)) := \mathbb{E}\left[\int_t^T L(t, X(t), c(t))dt + \Phi(X(T))\right]$$

The one defines the value function $V(t, x)$ as function of the initial data (t, x) and writes the corresponding HJB equation (Kolmogorov equation in the case of pricing/hedging)

The general form of the HJB is of this type

$$\begin{cases} \frac{\partial}{\partial t}V(t,x) + \left\langle Ax, \frac{\partial}{\partial x}V(t,x)\right\rangle + \mathcal{H}\left(t, x, \frac{\partial}{\partial x}V(t,x), \frac{\partial^2}{\partial x^2}V(t,x)\right), \\ V(T,x) = \Phi(x), \quad x \in H, \end{cases}$$

where the Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}(t, x, p, Q) := \sup_{c \in C}\left\{F(x, c), p + \frac{1}{2}\Sigma(x, c)Q\Sigma^*(x, c)\right\}$$

Step 3

“Solving” the HJB equation, finding, possibly, solutions with “enough regularity” to prove the Verification Theorem and to write the optimal feedback map and the closed loop equation, as in the first example.

The theory is more complicated but is doable, in some cases with satisfaction (e.g. explicit solutions).

All the problems presented above are open in their general setting. However one can treat many cases already interesting for the applications.

HJB EQUATION: WHICH KIND OF SOLUTIONS?

- If the value function is smooth (say C^1 in time and space), then it solves the HJB equation. However this argument is only formal: in general the value function is not smooth.

- Even if the value function is smooth, it is difficult to prove directly regularity results for the value function going beyond the continuity.

- A good concept of solution in the context of HJB equations seems to be the concept of ***viscosity solution*** (see e.g. [Crandall-Ishii-Lions, 1992], [Fabbri-Gozzi-Swiech 2015]). It does not require regularity (classical or generalized) for the definition of solution and can be used also in infinite dimension. But regularity results are very rare in this context.

To find results on existence of ''smooth'' solutions to the HJB equation one can use various approaches heavily depending on the underlying problem (see [Fabbri-Gozzi-Swiech 2015] for a survey).

- Explicit solutions (see e.g. [Fabbri-Gozzi, 2008] or [Biffis-Gozzi Prosdocimi, 2015])

- Convex regularization approach (see e.g. [Barbu-Da Prato, 1980]).

- Regularization of viscosity solutions (see e.g. [Federico-Goldys-Gozzi, 2010]). Only deterministic.

- Perturbation approach based on smoothing properties of suitable transition semigroups (see e.g. Cannarsa-Da Prato, 1992] or [Goldys-Gozzi, 2006]). Only stochastic.

- Using BSDEs (see e.g. [Fuhrman-Tessitore 2006]). Only stochastic.

QUESTION

What precise results can be proved and what is the use of them in the applied models?

ANSWER

We show some results in the case of delay/path dependent equations

Then we also give an overview of future targets.

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Short introduction to delay equations and to their infinite dimensional representation.

DELAY EQUATIONS

A Differential Delay Equation (DDE) is a Differential Equation in which the knowledge of the future depends also on the past of the state:

$$x'(t) = f\left(x(t), x(t+\xi)|_{\xi\in[-d,0)}\right).$$

In general for stating the evolution of the system such an equation requires as initial datum the knowledge of the whole past trajectory

$$x(\cdot)|_{[-d,0]}.$$

Thus the problem is basically infinite-dimensional.

DELAY EQUATIONS: FINITE-DIMENSIONAL REPRESENTATION 1: A SPECIAL CASE

There are some special case for which the evolution of the system can be reduced to a finite dimensional system, which is clearly more treatable. For example:

$$x'(t) = f\left(x(t), \int_{-\infty}^{0} e^{\lambda\xi} x(t+\xi) d\xi\right), \quad \lambda \geq 0.$$

In this case the variable

$$y(t) := \int_{-\infty}^{0} e^{\lambda\xi} x(t+\xi) d\xi$$

contains sufficient information for the evolution of the system, which could be rewritten as a 2-dimensional system

$$\begin{cases} x'(t) = f\left(x(t), y(t)\right), \\ y'(t) = -\lambda y(t) + x(t). \end{cases}$$

DELAY EQUATIONS: THE FINITE-DIMENSIONAL REPRESENTATION 2: RESULTS

Various papers treat this subject, for example

[Elsanosi-Oksendal-Sulem '00], [Larssen-Risebro '03],

[Bauer-Rieder 05], [Federico-Oksendal '11], [Federico-Tankov '14].

In particular in the last one (with first ideas in the fourth one) is is given a complete characterization of when the finite dimensional representation is possible. This is given in terms of invariant subspaces of the operator A defined later.

DELAY EQUATIONS: THE INFINITE DIMENSIONAL REPRESENTATION

A classical approach to treat Differential Delay Equations (DDE's), consists in rewriting them as Ordinary Differential Equations (ODE's) in a suitable infinite dimensional space. Here we choose an Hilbert space ($\mathbb{R} \times L^2$). Another common choice is C^0 (Banach nonreflexive space).

The idea behind this approach is to consider as state not only the present, but also the past, i.e. to define a new state variable representing the present and the past of the old state variable.

Consider

$$H = \mathbb{R} \times L^2([-d,0];\mathbb{R})$$

and denote by $\eta = (\eta_0, \eta_1(\cdot))$ the generic element of this space.

We want to write an ODE in H which is, at least formally, the infinite-dimensional counterpart of our one-dimensional DDE. More precisely we want that the solution $X(t)$ of the ODE in H is such that

$$X(t) := (X_0(t), X_1(t)) = \left(x(t), x(t+\cdot)|_{\cdot \in [-d,0]} \right),$$

where $x(t)$ is the solution of the original DDE:

$$x'(t) = f\left(x(t), x(t+\xi)|_{\xi \in [-d,0)} \right). \quad (7)$$

The equation for $X_0(\cdot)$ come from (7), while the equation for $X_1(\cdot)$ come from the fact that at each t $X_1(t)$ is the past of $X_0(t)$ i.e.

$$X_1(t)(\xi) = X_0(t+\xi).$$

The equation in H is

$$X'(t) = AX(t) + F(X(t)),$$

where

- A is a first order operator that generates a semigroup of shift operators moving the past:

$$D(A) = \Big\{\eta \in H \;\Big|\; \eta_1(\cdot) \in W^{1,2}([-d,0];\mathbb{R}),\;\; \eta_0 = \eta_1(0)\Big\},$$

$$\begin{aligned} A: \;\; D(A) \subset H \;\; &\longrightarrow \qquad H \\ (\eta_0, \eta_1(\cdot)) \;\; &\longmapsto \;\; (0, \eta_1'(\cdot)). \end{aligned}$$

  - On the first component A does not act;
  - On the second component A is the first derivative.

  The role of the boundary condition in $D(A)$: $\eta_0 = \eta_1(0)$ forces the past to follow the present.

- F is the counterpart of f in the DDE:

$$\begin{aligned} F: \ D(F) \subset H &\longrightarrow H, \\ (\eta_0, \eta_1(\cdot)) &\longmapsto \big(f(\eta_0, \eta_1(\cdot)), 0\big). \end{aligned}$$

  $D(F)$ must be a subspace of H where f is well-defined.

  For example if

$$f(\eta_0, \eta_1(\cdot)) = f_P(\eta(-d)),$$

  (like in the time to build model), then we may choose

$$D(F)) = W^{1,2}([-d, 0]; \mathbb{R})$$

  or

$$D(F)) = C^0([-d, 0]; \mathbb{R})$$

  .

DELAY IN THE CONTROL: THE INFINITE DIMENSIONAL REPRESENTATION

- Take a controlled DDE with delay in the control

$$x'(t) = f\left(x(t), x(t+\xi)|_{\xi\in[-d,0)}, c(t+\xi)|_{\xi\in[-d,0)}\right).$$

- Also in this case it is possible to rewrite the equation as a controlled ODE in a suitable infinite dimensional space (see e.g. Ichikawa 1982) with linearity in the control and with unbounded control operator.

- In the linear case (see e.g Vinter-Kwong) the rewriting can be simplified. allowing bounded control operators if the delay dependence in the control is non atomic.

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Some results for a special optimal control problem with delay terms.

THE CONTROL PROBLEM

- **State equation:**

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t),\ \int_{-d}^{0} x(t+\xi)a(\xi)d\xi\right) - c(t) \quad +\sigma dW(t), \\ x(0) = \eta_0,\ x(s) = \eta_1(s),\ s \in [-d,0). \end{cases}$$

- $a(\xi)d\xi$ is a positive Radon measure on [-d,0). Some results will hold when we assume also that $a \in W^{1,2}([-d,0];\mathbb{R})$, $a(\cdot) > 0$ on $(-d,0]$ and $a(-d) = 0$;
- $f_0 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is concave, Lipschitz, nondecreasing on the second variable and $f_0(0,0) \geq 0$. (f_0 can be extended to a Lipschitz continuous function on $\mathbb{R}^2$.)
- state constraint: $x(\cdot) \geq 0$;
- control constraint: $c(\cdot) \geq 0$.

- **Optimization problem:** Maximize, over the set

$$\mathcal{C}_{ad}(\eta) = \{c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R}^+) \mid x(\cdot;\eta,c(\cdot)) \geq 0\},$$

the functional

$$\int_0^{+\infty} e^{-\rho t}\left[U_1(c(t)) + U_2(x(t))\right] dt,$$

where

- $\rho > 0$;

- U_1, U_2 utility functions bounded from above (we may require weaker conditions relating the growth of U_1, U_2 to ρ).

- U_1 satisfying Inada's conditions: $U_1'(0^+) = +\infty$, $U_1'(+\infty) = 0$;

- for most of the results it can be $U_2 \equiv 0$.

INFINITE-DIMENSIONAL REPRESENTATION

We pass from the one-dimensional DDE to an infinite-dimensional DE (without delay): we define the Hilbert space

$$H = \mathbb{R} \times L^2([-d, 0]; \mathbb{R}).$$

The new state variable in this space is

$$X(t) = (X_0(t), X_1(t)) \in H.$$

Formally we want

$$X_0(t) = x(t); \quad X_1(t)(\xi) = x(t + \xi), \text{ for a.e. } \xi \in [-d, 0].$$

Define

- the closed unbounded operator

$$A : D(A) \subset H \to H,$$

where

$$D(A) = \left\{ (\eta_0, \eta_1(\cdot)) \in H \;\middle|\; \eta_1(\cdot) \in W^{1,2}([-d, 0]; \mathbb{R}),\ \ \eta_0 = \eta_1(0) \right\};$$

and A maps

$$D(A) \ni (\eta_0, \eta_1(\cdot)) \mapsto (r\eta_0, \eta_1'(\cdot)).$$

NOTE: A is the generator of a C_0-semigroup $S_A(\cdot)$ on H.

- the nonlinear map

$$F : H \to H,$$

$$\begin{pmatrix} \eta_0 \\ \eta_1(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} f(\eta_0, \eta_1(\cdot)) \\ 0 \end{pmatrix} := \begin{pmatrix} f_0\left(\eta_0, \int_{-d}^{0} a(\xi)\eta_1(\xi)d\xi\right) \\ 0 \end{pmatrix}.$$

NOTE: F is defined on the whole space H and is Lipschitz continuous as an application from H to H.

Define the infinite-dimensional ODE in the space H

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n} \quad + \Sigma dW(t), \\ X(0) = \eta = (\eta_0, \eta_1(\cdot)) \in H, \end{cases}$$

where $\hat{n} = (1, 0) \in H$.

(i) The role of A in the equation.

$$A(X_0(t), X_1(t)) = (rX_0(t), X_1(t)'(\cdot)).$$

-On the first component A carries the linear part of the evolution of the present.
-On the second component A moves the past as a shift.

(ii) The role of the boundary condition in $D(A)$. The boundary condition forces the past to follow the present, i.e. the last point of the past has to follow the same evolution of the present.

MILD SOLUTIONS AND EQUIVALENCE

Proposition 1 *For any* $\eta \in H$ *and* $c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R})$*, the equation*

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n} \quad + \Sigma dW(t), \\ X(0) = \eta = (\eta_0, \eta_1(\cdot)) \in H, \end{cases}$$

admits a unique mild solution $X(\cdot)$*, i.e.*

$$X(t) = S_A(t)\eta + \int_0^t S_A(t-\tau)F(X(\tau))d\tau + \int_0^t c(\tau)S_A(t-\tau)\hat{n}\,d\tau + W_A(t).$$

Moreover

$$X(t) = \left(X_0(t), X_1(t)(\xi)|_{\xi\in[-d,0]}\right) = \left(x(t), x(t+\xi)|_{\xi\in[-d,0]}\right),$$

where $x(\cdot)$ *is the unique solution of the one-dimensional delay equation.*

HJB EQUATION

Formally the HJB equation for the problem is

$$\rho v(\eta) = \langle A\eta, \nabla v(\eta)\rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)) \quad + \frac{1}{2}\mathrm{Tr}\Sigma\Sigma^* v_{\eta\eta}; \tag{8}$$

this requires in particular $\eta \in D(A)$.

In order to allow $\eta \in H$ we can rewrite it as

$$\rho v(\eta) = \langle \eta, A^*\nabla v(\eta)\rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)) \quad + \frac{1}{2}\mathrm{Tr}\Sigma\Sigma^* v_{\eta\eta};$$

requiring more regularity on the gradient of v.

Here $\mathcal{H}$ is the sup-Legendre transform of U_1, i.e.

$$\mathcal{H}(p) := \sup_{c \geq 0} \left(U_1(c) - cp\right), \quad p > 0.$$

Note that $\mathcal{H}$ depends only on the "present" component of the gradient of v.

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Regular solutions to the HJB equation.

HJB EQUATION: WHICH KIND OF SOLUTIONS?

- If the value function is smooth (say C^1 in time and space), then it solves the HJB equation. However this argument is only formal: in general the value function is not smooth.
- Even if the value function is smooth, it is difficult to prove directly regularity results for the value function going beyond the continuity.
- A possible strategy is to find results on existence of ''smooth'' solutions to the HJB equation. However classical PDEs theory does not adapt to PDEs of HJB type in general even in finite dimension. Some results are found in [Barbu-Da Prato, 1980] with a convex regularization approach that cannot be used here.
- A good concept of solution in the context of HJB equations seems to be the concept of ***viscosity solution*** (Crandall and Lions, '80). It does not require regularity (classical or generalized) for the definition of solution and can be used also in infinite dimension. But regularity results are very rare in this context.

Our case: f linear, $U_1 = u_\sigma$, $u_2 = 0$:

In this case, both in the deterministic and in the stochastic case, we can find an explicit solution v of the HJB equation (8) under some restrictions on the data. This is very good as the DP approach can be fully developed finding optimal feedback strategies.

For example in the papers [Bambi-Fabbri-Gozzi, ET, (2012)] (deterministic), [Biffis-Gozzi-Prosdocimi, in progress] (stochastic) the linear case is fully solved explaining applied issues (like the so-called consumption smoothing) that could not be treated with other techniques. Here is easy to see that

$$v(\eta_0, \eta_1) = \nu \left(\int_{-d}^{0} e^{s\xi} \eta_1(s) ds + \eta_0 \right)^{1-\sigma}$$

but the hard part is to prove that this is is indeed the value function and to find optimal feedbacks. **Hardness comes mainly from the constraints.**

Similar results were obtained in the other cases mentioned above (delay in the control, age structure, spatial growth).

f nonlinear:

In general we do not know whether there exists a classical solution of the HJB equation or not. Our approach in this case (developed in the papers [Federico-Gozzi-Goldys, SICON (2010)], [Federico-Gozzi-Goldys, SICON (2011)] and generalized in [Federico-Tacconi (2013)] in the case with delay in the control) is the following:

- Prove that the value function is a viscosity solution (possibly but not necessarily unique) of the HJB equation.
- Prove, by using this viscosity property, that the value function is indeed smooth and so it is a ''regular enough'' solution.
- Use the fact that the value function is a ''regular enough'' solution of the HJB equation to prove a verification theorem giving an optimal strategy for the problem.

Other approaches are found e.g. in [Gozzi-Masiero, in progress], [Gozzi-Prosdocimi-Sekine, in progress].

FUTURE TARGETS (DETERMINISTIC CASE)

- <u>Numerical:</u>
  - To analyze the behavior (convergence, rate of convergence) of numerical schemes for this problem. See e.g. [Falcone et al, 2010]
- <u>Theoretical:</u>
  - Find other examples with explicit solutions, e.g. when constraints are binding. (See e.g. [Boucekkine-Gozzi-Rosestolato, work in progress]).
  - Study the properties of the optimal strategies when explicit solutions are not available (dynamics of infinite dimensional systems: equilibriums, stability, attractors, etc.). (See e.g. [Bambi-Gozzi-Licandro, JET '14]).

FUTURE TARGETS: STOCHASTIC CASE

To treat the case when a finite dimensional (additive or multiplicative) white noise is added to the state equation: various projects/works in progress with Biffis, Federico, Masiero, Pham, Rosestolato, Russo, Swiech, Touzi on the applied models listed above and on theoretical results.
The main problems here are:

- the HJB is now degenerate second order (fully nonlinear if the control is in the diffusion coefficient, semilinear otherwise);

- If there is delay in the control the HJB equation does not satisfy the so-called "structure condition" needed e.g. to use the BSDE approach to the problem.

- the verification theorem is much harder even in finite dimension.

Work in progress on this:

- Biffis, Federico, G., Prosdocimi, on optimal portfolio choices with retirement.
- Fabbri, Federico, G., Pham, on optimal portfolio with execution delay.
- G., Masiero, on the case with delay in control,, regularity for HJB (with possible application to advertising models, see e.g. the work of Marinelli).
- Di Girolami, Federico, G., Russo, Rosestolato, Swiech on general regularity theorems and verification theorems.
- Cosso, Federico, G., Rosestolato, Touzi on the use of path dependent calculus (introduced by Dupire) to solve the associated Kolmogorov and HJB equations in the viscosity sense.

General Message 3:

We can solve satisfactorily

some infinite dimensional PDEs

AND

use the result to solve the applied problem above

Hence infinite dimensional problems are not impossible. They are possible (sometimes difficult).

Technique should studied and adapted carefully for each case

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THANKS A LOT FOR YOUR ATTENTION.