RW in dynamic random environment generated by the reversal of discrete time contact process

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joint with Matthias Birkner, Jiří Černý and Nina Gantert

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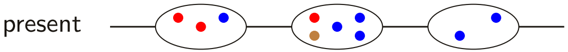
Motivation

Theoretical population genetics tries to explain variability observed in nature by elementary evolutionary mechanisms such as

- genetic drift (= change in type frequency due to random sampling),
- mutation, selection, recombination,
- migration, . . .



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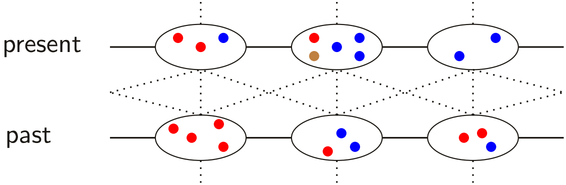

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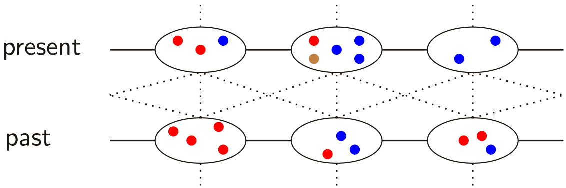

- Good understanding of variability requires good understanding of genealogical relationship of individuals (cf. Amaury Lambert's talk).
- In the spatial setting one needs to study coalescing random walks.

Introduction

- **Aim:** Study ancestral lineages in locally regulated populations[1] with fluctuating population size. We consider the discrete-time contact process without types.
- **Problem:** Pick an individual from the upper invariant distribution and denote by X_n the position of the ancestor of that individual n generations ago. Describe the behaviour of X_n. Do LLN and CLT for X_n hold?
- **Note:** X_n is a random walk in a Markovian random environment given by the time reversal of the original population process.

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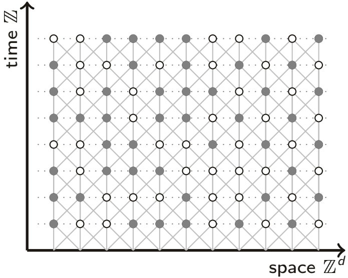
[1] locally regulated populations are supercritical in sparsely populated and subcritical in crowded regions

Oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$

- $\omega(x, n)$, $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ i.i.d. Ber(p)
- (x, n) is open if $\omega(x, n) = 1$ and closed if $\omega(x, n) = 0$
- for $n \leq m$ write $(x, n) \to (y, m)$ if there is $x_n = x, x_{n+1}, \ldots, x_m = y$ with $\omega(x_k, k) = 1$ and $\|x_{k+1} - x_k\| \leq 1$



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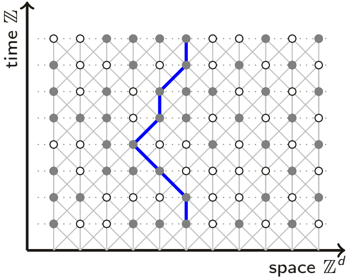

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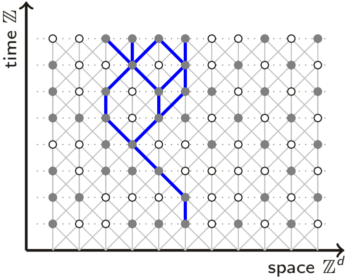


Theorem

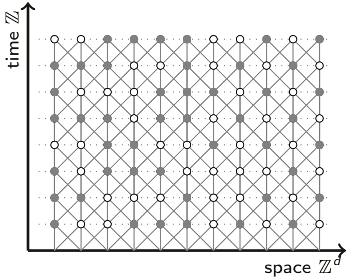
There is $p_c \in (0,1)$ s.th. for $p > p_c$

$$\mathbb{P}_p\big((0,0) \to \mathbb{Z}^d \times \{+\infty\}\big) > 0$$

(and $= 0$ for $p \le p_c$).

In what follows we fix $p > p_c$.

Discrete time contact process



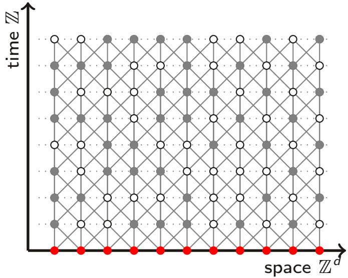
- Starting with some initial set $A \subset \mathbb{Z}^d$ at time $m \in \mathbb{Z}$, for $n \geq m$ set

$$A_n := \{y \in \mathbb{Z}^d : \exists x \in A, \text{ s.th. } (x, m) \to (y, n)\}$$

(here we artificially set $\omega(\cdot, m) = \mathbb{1}_A(\cdot)$)

- Let $p > p_c$, $A = \mathbb{Z}^d$ and $m \to -\infty$. Then $(\eta_n)_{n \in \mathbb{Z}}$ with $\eta_n = \mathbb{1}_{A_n}$ is the stationary contact process with $\mathcal{L}(\eta_n) =$ upper invariant law.
- **Interpretation:** η_n is the set of wet sites if water flows up from $\mathbb{Z}^d \times \{-\infty\}$ through open sites.

Discrete time contact process



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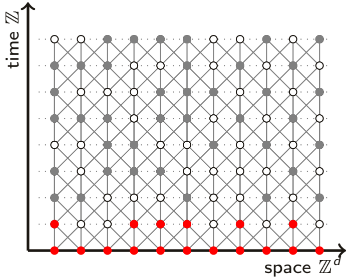

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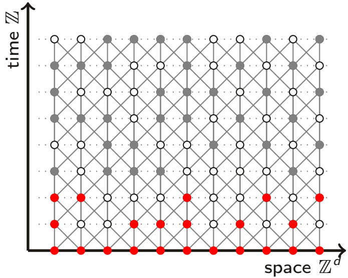

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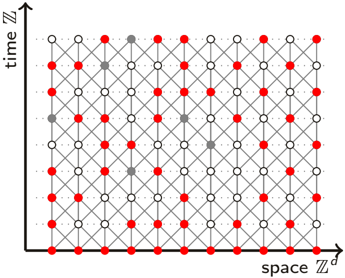

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Alternative local construction of (η_n)

- Conditioned on η_n the distribution of η_{n+1} is

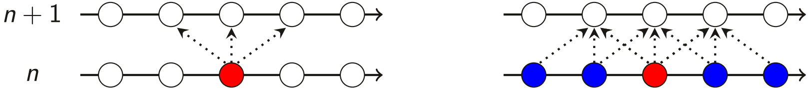
$$\mathbb{P}(\eta_{n+1}(x) = 1|\eta_n) = p \cdot \mathbb{1}_{\{\exists y, \|x-y\| \leq 1, \eta_n(y)=1\}}.$$

Interpretation:

- with prob. p at $(x, n+1)$ there are enough resources for one individual
  - if occupied sites in the neighbourhood in the previous generation exist, then choose a parent at random
  - otherwise $(x, n+1)$ remains vacant
- with prob. $1-p$ at $(x, n+1)$ no resources are available
  - $(x, n+1)$ remains vacant irrespective of the neighbourhood

Contact process as locallly regulated population

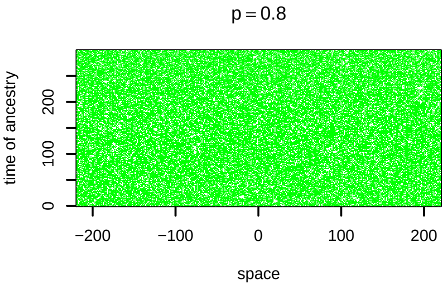
- Contact process is locally regulated: Expected offspring number is
  - $3p > 1$ in empty neighbourhoods
  - $3\frac{1}{3}p = p < 1$ in crowded neighbourhoods



Backbone: cluster of percolating sites

Define the cluster of percolating sites on $\mathbb{Z}^d \times \mathbb{Z}$ by

$$\mathcal{C} := \{(x, n) \in \mathbb{Z}^d \times \mathbb{Z} : (x, n) \to \mathbb{Z} \times \{+\infty\}\}.$$



- **Interpretation:** $\mathcal{C}$ is the set of all wet sites if water flows down from $\mathbb{Z}^d \times \{+\infty\}$ through open sites. When time is running upwards, the process of configurations of occupied (green) sites is time-reversal of the discrete time contact process.
- **Note:** $\mathcal{C}$ is not the "usual" percolation cluster.

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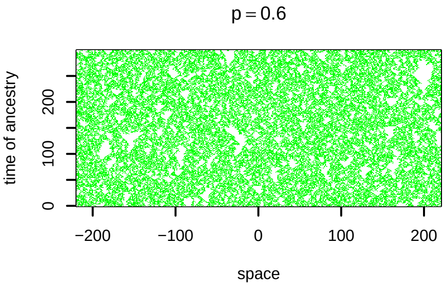

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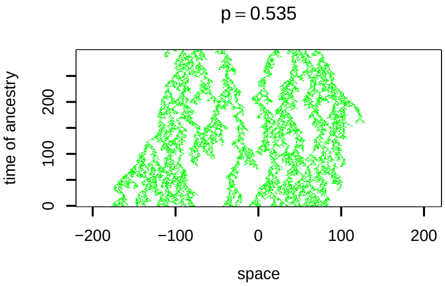

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Random walk on $\mathcal{C}$

- Define the neighbourhood of (x, n) by

$$U(x, n) := \{(y, n+1) : \|x - y\| \leq 1\}.$$

- On $B := \{(0,0) \in \mathcal{C}\}$ define the directed random walk $(X_n)_{n=0,1,\ldots}$ on $\mathcal{C}$ by

$$X_0 = 0,$$

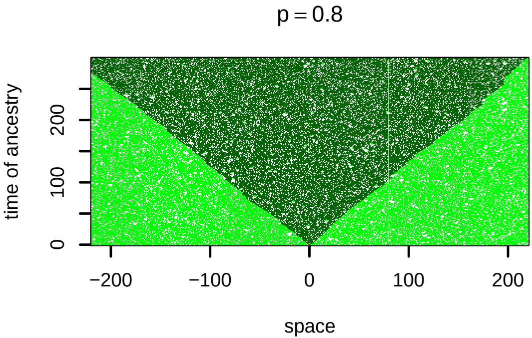
and for $(y, n+1) \in \mathcal{C} \cap U(x, n)$

$$\mathbb{P}\left(X_{n+1} = y | X_n = x, \mathcal{C}\right) = \frac{1}{|\mathcal{C} \cap U(x, n)|}.$$

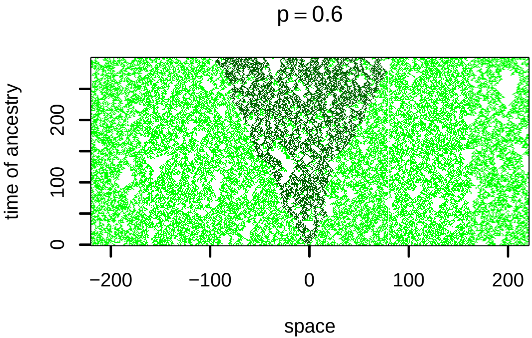
- **Interpretation:** (X_n) is the ancestral lineage of an individual located at space-time origin.
- **Remark:** More general finite and symmetric U's, leading to different p_c and geometry of clusters, can be considered, e.g.

$$U(x, n) := \{(y, n+1) : \|x - y\| = 1\}.$$

Possible paths for the RW



Possible paths for the RW



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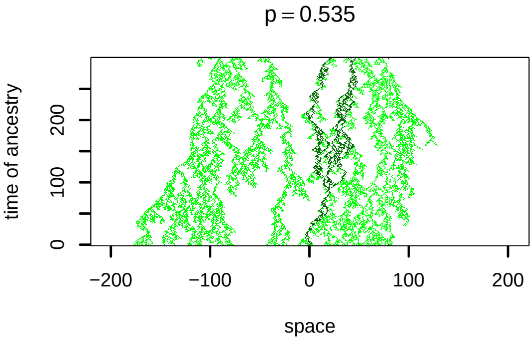


Annealed and quenched LLN and CLT

- P_ω law of RW with transition probab. $\mathbb{P}(X_{n+1} = y | X_n = x, \omega)$
- E_ω corresponding expectation

Theorem (LLN)

(i) $\mathbb{P}\big(X_n/n \to 0 \,\big|\, B\big) = 1$,

(ii) $P_\omega\big(X_n/n \xrightarrow{n\to\infty} 0\big) = 1$ *for* $\mathbb{P}(\,\cdot\,|B)$*-a.a.* ω.

Theorem (annealed and quenched CLT)

For $f \in C_b(\mathbb{R}^d)$

(i) $\mathbb{E}\big[f\left(X_n/\sqrt{n}\right) \,\big|\, B\big] \xrightarrow{n\to\infty} \Phi(f)$,

(ii) $E_\omega\big[f(X_n/\sqrt{n})\big] \xrightarrow{n\to\infty} \Phi(f)$ *for* $\mathbb{P}(\,\cdot\,|B)$*-a.a.* ω.

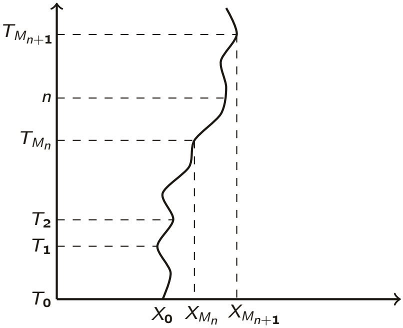
Here $\Phi(f) = \int f(x)\Phi(dx)$ *and* Φ *is a non-trivial centred* d*-dimensional normal law.*

Note: (i) $\Leftrightarrow$ (ii) in LLN; (ii) $\Rightarrow$ (i) but (i) $\not\Rightarrow$ (ii) in CLT.

Key steps of the proofs

- For LLN and annealed (averaged) CLT we need to identify a suitable regeneration structure and obtain moment estimates on regeneration times and corresponding spatial increments.
- For quenched CLT using two inedependent RW's on the same cluster show that $\mathbb{E}\left[\left(E_\omega\left[f\left(X_n/\sqrt{n}\right)\right] - \Phi(f)\right)^2 \middle| B\right]$ summable.
- **Note:** The steps are common in the RWRE literature but the details highly depend on the particular processes.

Use of regeneration structure for LLN



- Let $T_0 = 0 < T_1 < T_2 < \dots$ be a sequence of regeneration times, at which the process starts *anew*. Define M_n so that $T_{M_n} \le n < T_{M_{n+1}}$.
- Then

$$X_n = \quad (X_n - X_{M_n}) + \qquad \underbrace{\sum_{k=1}^{M_n} (X_{T_k} - X_{T_{k-1}})}_{\text{sum of i.i.d. r.v.'s}}$$

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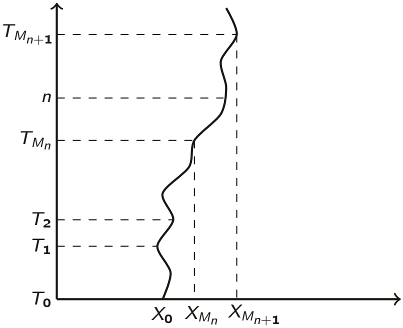

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- Then

$$\frac{1}{n} X_n = \frac{1}{n}(X_n - X_{M_n}) + \frac{M_n}{n}\frac{1}{M_n}\underbrace{\sum_{k=1}^{M_n}(X_{T_k} - X_{T_{k-1}})}_{\text{sum of i.i.d. r.v.'s}}$$

Remarks regarding regeneration times

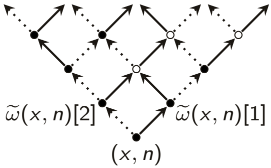
- The random walk cannot be constructed by local rules.
- At regeneration times not only the walk, but also the environment have start *anew*.
- We construct preliminary paths of the RW by local rules using ω's and some auxiliary randomness. These will become segments of the *real path* at regeneration times.

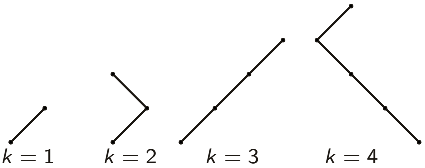
A local construction of the walk

- For $(x, n) \in \mathbb{Z}^d \times \mathbb{Z}$ let $\widetilde{\omega}(x, n)$ be an independent uniform ordering of elements of $U(x, n)$.
- For (x, n) define a (directed) path $\gamma_k^{(x,n)}$ of k steps that begin on open sites, choosing directions according to $\widetilde{\omega}$:
  - $\gamma_k^{(x,n)}(0) = (x, n)$,
  - $\gamma_k^{(x,n)}(j) = (y, n + j)$ then $\gamma_k^{(x,n)}(j + 1) = (z, n + j + 1)$, where $(z, n + j + 1)$ is the element of

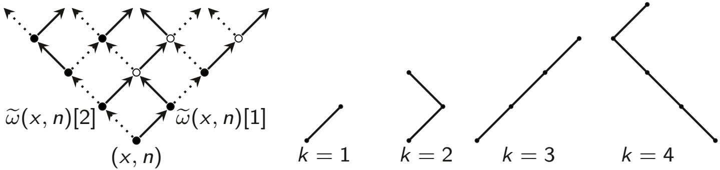
    $$\left\{(z', n + j + 1) \in U(y, n + j) : (z', n + j + 1) \to \mathbb{Z}^d \times \{n + k - 1\}\right\}$$

    with the smallest index in $\widetilde{\omega}(y, n + j)$
- Construction is local because only ω's and $\widetilde{\omega}$'s in time slices $\{n, \ldots, n + k - 1\}$ are used.





From local to global construction



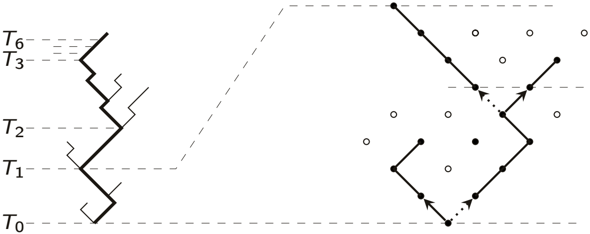
- $\gamma_k^{(x,n)}(k)$ = endpoint of the local k-step construction
- If $(x,n) \in \mathcal{C}$ then $\gamma_\infty^{(x,n)}(j) := \lim_{k\to\infty} \gamma_k^{(x,n)}(j)$ exists for all j.
- If $\gamma_k^{(x,n)}(k) \in \mathcal{C}$ then $\gamma_k^{(x,n)}(j) = \gamma_\infty^{(x,n)}(j)$ for all $j \le k$.
- On $B = \{(0,0) \in \mathcal{C}\}$

$$(X_k, k) := \gamma_\infty^{(0,0)}(k), \; k = 0, 1, 2, \ldots$$

  is a space-time version of the path of the directed RW on $\mathcal{C}$.
- If $\gamma_k^{(0,0)}(k) \in \mathcal{C}$ then $(X_j, j) = \gamma_k^{(0,0)}(j)$ for all $j \le k$.

Regeneration times

$$
\begin{aligned}
T_0 &:= 0, & Y_0 &:= 0,\\
T_1 &:= \min\left\{k > 0 : \gamma_k^{(0,0)}(k) \in \mathcal{C}\right\}, & Y_1 &:= X_{T_1} := \gamma_{T_1}^{(0,0)}(T_1),\\
T_2 &:= T_1 + \min\left\{k > 0 : \gamma_k^{(Y_1,T_1)}(k) \in \mathcal{C}\right\}, & Y_2 &:= X_{T_2} := \dots\\
&\vdots & &\vdots
\end{aligned}
$$



- Randomised version of Kuczek's (1989) construction
- Cont. time versions appeared in Neuhauser (1992) and Valesin (2010)

Main result towards LLN and annealed CLT

Proposition

Under $\mathbb{P}(\cdot \mid B)$

- the sequence $\big((Y_i - Y_{i-1}, T_i - T_{i-1})\big)_{i \geq 1}$ is i.i.d.,
- Y_1 is symmetrically distributed.

Furthermore there are $c, C \in (0, \infty)$ s.th.

$$\mathbb{P}(\|Y_1\| > n \mid B),\ \mathbb{P}(T_1 > n \mid B) \leq Ce^{-cn} \quad \text{for } n \in \mathbb{N}.$$

Proof sketch.

- For tail bounds use the fact that finite clusters are small.
- Symmetry follows from the symmetric construction of the paths.
- i.i.d. property holds because the local path construction uses disjoint time-slices.

□

Proof ideas for the quenched case

- Need to show that $\mathbb{E}\left[\left(E_\omega\left[f\left(X_n/\sqrt{n}\right)\right]-\Phi(f)\right)^2\middle|B\right]$ is summable.

  We prove this first along a subsequence and then use concentration arguments.
- For estimates we take independent RW's (X_n) and (X'_n) on the same cluster $\mathcal{C}$, i.e. they use the same ω, but independent $\widetilde{\omega}$ resp. $\widetilde{\omega}'$.
- One can define joint regeneration times and prove a proposition analogous to the one walk case.
- With high probability (X_n) and (X'_n) can be coupled with two independent walks on two independent clusters.
  - In case $d \geq 2$ walks spend enough time away from each other.
  - In case $d = 1$ we use a martingale decomposition of the difference.

Outlook & references

Problem:

- From population genetics point of view the joint behaviour of N ancestral lineages, i.e. N coalescing RW's is interesting.

References and generalisations:

- Birkner, Černý, D. and Gantert (2013), Directed random walk on the backbone of an oriented percolation cluster. *Electron. J. Probab.*
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- Miller (2015), Random walks on weighted, oriented percolation clusters, `http://arxiv.org/abs/1506.01879`