Modeling Financial Time-Series with COGARCH(p, q) processes

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Outline

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- Introduction
  - Stylized features of financial data
  - GARCH models
  - Lévy processes

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- Simulation and Application
- Conclusions and Future work

Introduction

Stylized Features

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- Tail heaviness

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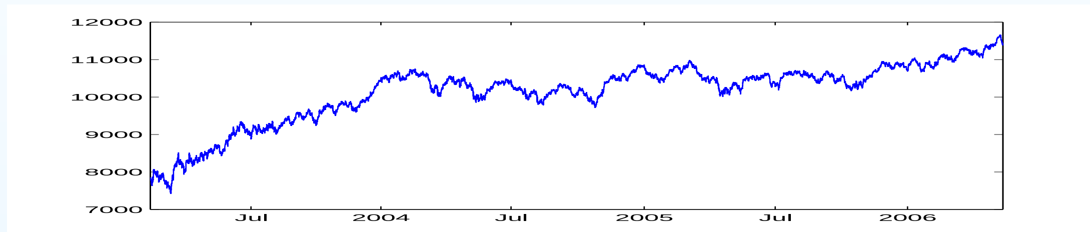
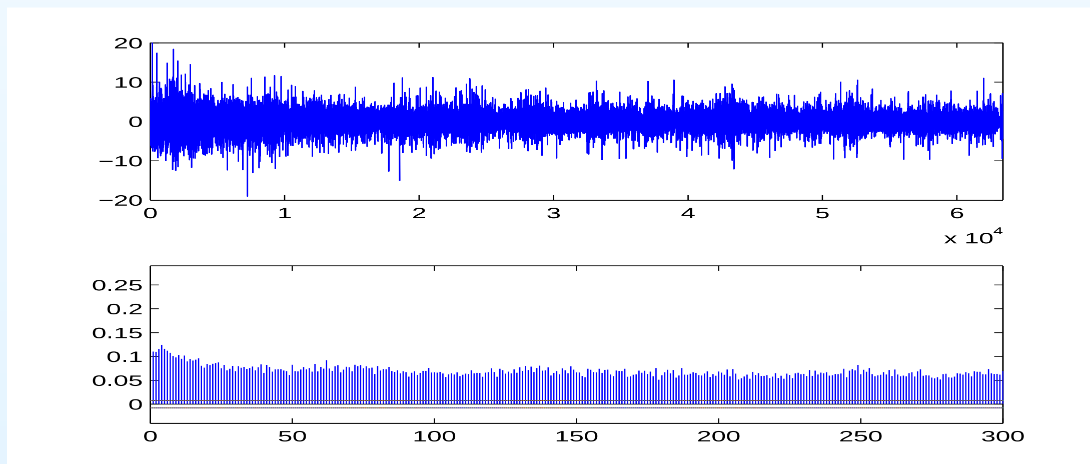
- Volatility clustering
- Dependence without correlation

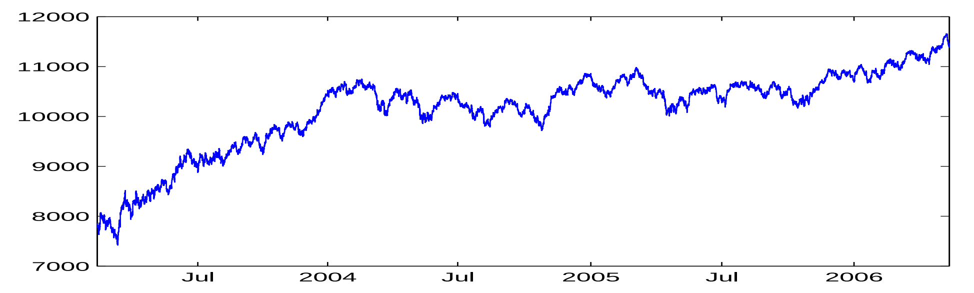


DJIA recorded from 02/13/2003 to 05/12/2006.



DJIA log-returns and the ACF of the squared log returns.

Dow Jones Industrial Average



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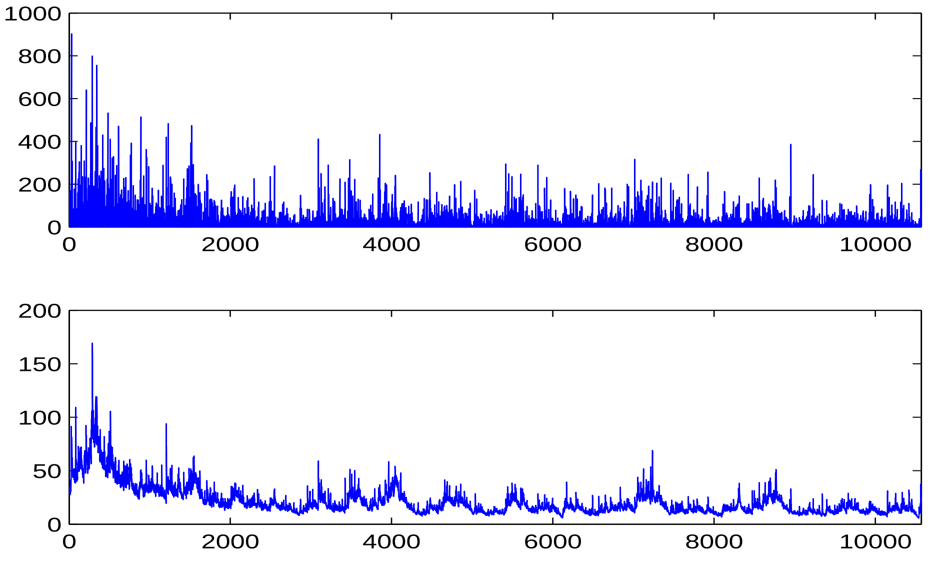

DJIA squared log-returns, and the estimated volatilities.

GARCH model

$X_n = \ln P_n - \ln P_{n-1}$ where P_n are asset price, $n = 1, 2, \ldots$

GARCH(p, q) model:

$$\begin{aligned} X_n &= \sqrt{v_n}\, \varepsilon_n, \\ v_n &= \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{n-i}^2 + \sum_{j=1}^{q} \beta_j v_{n-j}, \end{aligned}$$

where $\{\varepsilon_n\} \overset{i.i.d.}{\sim} N(0, 1)$, $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, p$

Example

ARCH(1) process:

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- X_n is not i.i.d.,
- X_n is “heavy-tailed”,
- X_n^2 has the ACF of an AR(1).

Lévy processes

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- L has stationary increments,
- L is continuous in probability.

Lévy processes

Characteristic function of L_t:

$$\varphi_t(\theta) = E(\exp(i\theta L_t)) = \exp(t\xi(\theta)), \quad \theta \in \mathbb{R},$$

where $\xi(\theta)$ satisfies the Lévy - Khinchin formula,

$$\xi(\theta) = i\gamma_L\theta - \tau_L^2\frac{\theta^2}{2} + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x\mathbf{1}_{|x|\leq 1}\right) \, d\nu_L(x).$$

$\xi(\theta)$ is called a characteristic exponent or Lévy symbol.

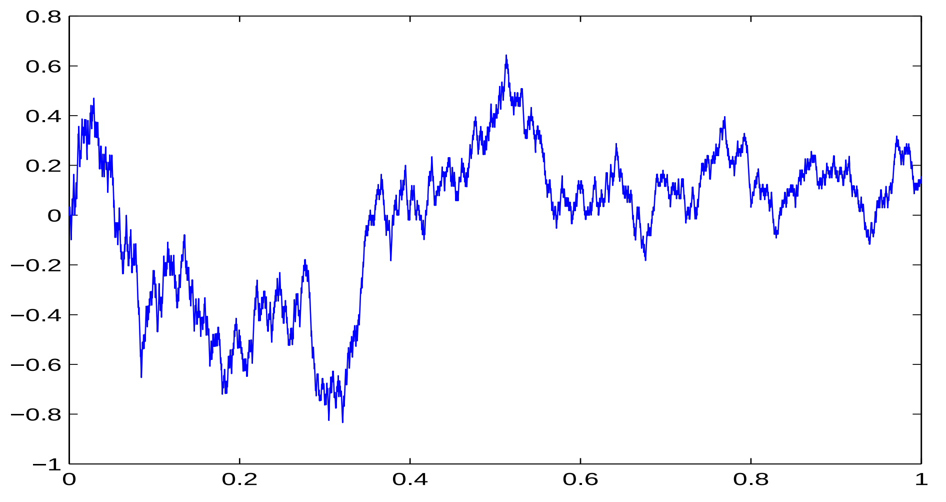
$(\gamma_L, \tau_L^2, \nu_L)$ is called the characteristic triplet of L.

ν_L on $\mathbb{R}$ is called the Lévy measure.

Example: Brownian Motion

The Lévy symbol: $\xi(\theta) = i\gamma_B\theta - \tau_B^2\frac{\theta^2}{2}$.

The characteristic triplet: $(\gamma_B, \tau_B^2, 0)$.



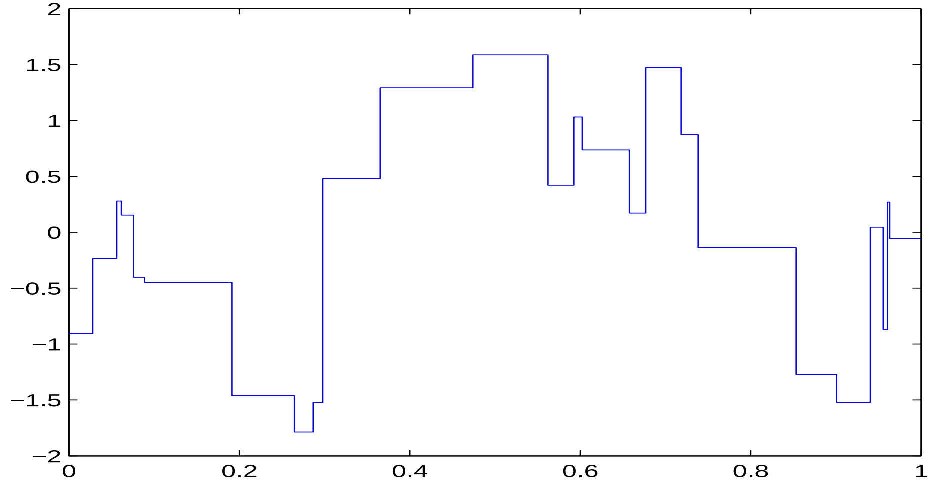
A sample path of a standard Brownian motion $(\gamma_B = 0, \tau_B^2 = 1)$.

Example: Compound Poisson

$$L_t := \sum_{i=1}^{N_t} Y_i, \text{ where } \{Y_n\} \overset{i.i.d.}{\sim} F_Y, \text{ indep. of } N = (N_t)_{t\geq 0}.$$

The Lévy symbol: $\xi(\theta) = \int_{\mathbb{R}} (e^{i\theta x} - 1)\lambda F_Y(dx)$.

The characteristic triplet: $(\gamma_L, 0, \lambda F_Y)$.



A sample path of a compound Poisson process with jump rate $\lambda = 25$ and standard normal jumps.

Continuous-time approaches

Nelson's diffusion limit

$G_t = \ln P_t$ where P_t, $t \geq 0$ are asset price.

GARCH(1, 1) diffusion limit satisfies

$$\begin{aligned} dG_t &= \sigma_t \, dW_t^{(1)}, \\ d\sigma_t^2 &= \theta(\gamma - \sigma_t^2)dt + \rho\sigma_t^2 \, dW_t^{(2)}, \quad t \geq 0, \end{aligned}$$

where $(W_t^{(1)})_{t\geq 0}$ and $(W_t^{(2)})_{t\geq 0}$ are independent Brownian motions.

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- GARCH process is driven by a single noise sequence.
- The diffusion limit is driven by two independent Brownian motions.

COGARCH(1,1) model

Due to Klüppelberg et. al. (2005). Start with GARCH(1,1):

$$\xi_n = \sigma_n \varepsilon_n, \quad \sigma_n^2 = \alpha_0 + \alpha_1 \xi_{n-1}^2 + \beta_1 \sigma_{n-1}^2, \quad n \in \mathbb{N}_0,$$

From the recursion,

$$\sigma_n^2 = \alpha_0 \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\beta_1 + \alpha_1 \varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\beta_1 + \alpha_1 \varepsilon_j^2)$$

$$= \left(\sigma_0^2 + \alpha_0 \int_0^n \exp\left[-\sum_{j=0}^{\lfloor s \rfloor} \log(\beta_1 + \alpha_1 \varepsilon_j^2)\right] ds\right) \exp\left[\sum_{j=0}^{n-1} \log(\beta_1 + \alpha_1 \varepsilon_j^2)\right].$$

Denote $X_n := -\sum_{j=0}^{n-1} \log(\beta_1 + \alpha_1 \varepsilon_j^2)$.

COGARCH(1,1) model

$$\begin{aligned} dG_t &= \sigma_t \, dL_t, \quad t > 0, \quad G_0 = 0, \\ \sigma_t^2 &= \left(\sigma_0^2 + \alpha_0 \int_0^t e^{X_s} \, ds \right) e^{-X_{t-}}, \quad t \geq 0, \end{aligned}$$

where

$$X_t := - \sum_{0<s\leq t} \log \left(\beta_1 + \alpha_1 (\Delta L_s)^2 \right).$$

The volatility process σ_t^2 satisfies

$$\sigma_t^2 = \alpha_0 t - \log \beta_1 \int_0^t \sigma_s^2 ds + (\alpha_1/\beta_1) \sum_{0<s<t} \sigma_s^2 (\Delta L_s)^2 + \sigma_0^2, \quad t \geq 0.$$

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- Heavy tails and volatility clusters at high levels
- Volatility process has an ACF of an CAR(1)
- Squared increment process has an ACF of an ARMA(1,1)

COGARCH(p, q) processes

Construction

Start with a GARCH(p, q) process $(\xi_n)_{n \in \mathbb{N}_0}$ defined by

$$\xi_n = \sigma_n \varepsilon_n,$$
$$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i \xi_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2, \quad n \geq \max\{p, q\}.$$

The volatility process can be viewed as a “self-exciting” ARMA($q, p-1$) process driven by the noise sequence $(\sigma_{n-1}^2 \varepsilon_{n-1}^2)_{n \in \mathbb{N}}$.

Construction cont.d

CARMA($q, p-1$) process $(\psi_t)_{t\geq 0}$ with driving Lévy process L:

$$\begin{aligned} \psi_t &= c + \mathbf{a}'\zeta_t, \\ d\zeta_t &= B\zeta_t dt + \mathbf{e}\, dL_t, \end{aligned}$$

where $\mathbf{a}' = [\alpha_1, \ldots, \alpha_q]$, $\alpha_j := 0$ for $j > p$, $\mathbf{e} = [0, \cdots, 0, 1]'$ and

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\beta_q & -\beta_{q-1} & -\beta_{q-2} & \cdots & -\beta_1 \end{bmatrix}.$$

Need to replace (L_t) by a continuous-time analog of the driving process $(\sigma^2_{n-1}\varepsilon^2_{n-1})_{n\in\mathbb{N}}$.

Construction cont.d

In discrete-time,

$$R_n^{(d)} := \sum_{i=0}^{n-1} \xi_i^2 = \sum_{i=0}^{n-1} \sigma_i^2 \varepsilon_i^2.$$

Its continuous-time analog:

$$R_t = \sum_{0<s\leq t} \sigma_{s-}^2 (\Delta L_s)^2 = \int_0^t \sigma_{s-}^2 d[L,L]_s^{(d)},$$

i.e.

$$dR_t = \sigma_{t-}^2 \, d[L,L]_t^{(d)},$$

where $[L,L]_t^{(d)}$ is the discrete part of the quadratic covariation of L.

COGARCH(p, q) equations

COGARCH(p, q) ($q \geq p \geq 1$) process with parameters B, $\mathbf{a}$, α_0 and driving Lévy process L is defined as follows:

$$dG_t = \sqrt{V_t}\, dL_t, \quad t > 0, \quad G_0 = 0,$$
$$V_t = \alpha_0 + \mathbf{a}'\mathbf{Y}_{t-}, \quad t > 0, \quad V_0 = \alpha_0 + \mathbf{a}'\mathbf{Y}_0,$$

where the state process $\mathbf{Y} = (\mathbf{Y}_t)_{t \geq 0}$ is the unique càdlàg solution of the stochastic differential equation

$$d\mathbf{Y}_t = B\mathbf{Y}_{t-}\, dt + \mathbf{e}(\alpha_0 + \mathbf{a}'\mathbf{Y}_{t-})\, d[L, L]_t^{(d)}, \quad t > 0,$$

with initial value $\mathbf{Y}_0$, independent of the driving Lévy process $(L_t)_{t \geq 0}$.

- When $p = q = 1$ it is reduced to the COGARCH$(1, 1)$ model.

Non-negativity of the volatility

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$$V_t > 0 \quad \text{a.s.} \quad \forall\, t \geq 0$$

for any driving Lévy process if

$$\mathbf{a}' e^{Bt} \mathbf{e} \geq 0, \quad \forall\, t \geq 0.$$

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- In the COGARCH$(2,2)$ case, the necessary and sufficient condition is

$$\alpha_2 \geq 0 \quad \text{and} \quad \alpha_1 \geq -\alpha_2 \lambda(B).$$

Stationarity property

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$$\mathbf{Y}_t \xrightarrow{d} \mathbf{Y}_\infty, \quad \text{as} \quad t \to \infty,$$

where $\mathbf{Y}_\infty$ is a finite r.v., if

$$\int_{\mathbb{R}} \log(1 + \|S^{-1}\mathbf{e}\mathbf{a}' S\|_r \, y^2) \, d\nu_L(y) < -\lambda = -\lambda(B),$$

for some $r \in [1, \infty]$ and some matrix S such that $S^{-1}BS$ is diagonal.

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for some $r \in [1, \infty]$ and some matrix S such that $S^{-1}BS$ is diagonal.

- In the COGARCH(1,1) case, this is a necessary and sufficient condition for existence of a stationary solution.
- If $\mathbf{Y}_0 \stackrel{d}{=} \mathbf{Y}_\infty$ then $(\mathbf{Y}_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are strictly stationary.

Second-order property

cov$(\mathbf{Y}_\infty)$ exists if $EL_1^4 < \infty$ and

$$\|S^{-1}\mathbf{e}\mathbf{a}'S\|_r^2\,\rho < 2(-\lambda - \|S^{-1}\mathbf{e}\mathbf{a}'S\|_r\mu),$$

where $\mu := EL_1^2$, $\rho := EL_1^4$.

If $(V_t)_{t\geq 0}$ is the stationary volatility process, then

$$\text{cov}(V_{t+h}, V_t) = \frac{\alpha_0^2\beta_q^2}{(\beta_q - \mu\alpha_1)^2(1-m)}\,\text{cov}(\psi_{t+h}, \psi_t), \quad t, h \geq 0,$$

where $m :=$ var (ψ_t).

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where $m := \text{var}\,(\psi_t)$.

- Volatility process has an ACF of a CARMA($q, p-1$). (Volatility process of a GARCH(p, q) has an ACF of an ARMA($q, p-1$)).

Squared increment process

$$G_t^{(r)} := G_{t+r} - G_t = \int_{(t,t+r]} \sqrt{V_s}\, dL_s, \quad t \geq 0.$$

For any $t \geq 0$ and $h \geq r > 0$,

$$\begin{aligned} E(G_t^{(r)}) &= 0, \\ E((G_t^{(r)})^2) &= \frac{\alpha_0 \beta_q r}{\beta_q - \mu \alpha_1} E L_1^2, \\ \text{cov}(G_t^{(r)}, G_{t+h}^{(r)}) &= 0. \end{aligned}$$

Squared increment process

$$\text{cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = EL_1^2\, \mathbf{a}' e^{\widetilde{B}h} \widetilde{B}^{-1}(I - e^{-\widetilde{B}r})\, \text{cov}(\mathbf{Y}_r, G_r^2), \quad h \geq$$

$$\text{var}((G_t^{(r)})^2) = 6EL_1^2\, \mathbf{a}'\mathbf{K}_r + 2(rEL_1^2 EV_\infty)^2 + rEL_1^4 EV_\infty^2,$$

where

$$\widetilde{B} := B + \mu \mathbf{e}\mathbf{a}',$$

$$\text{cov}(\mathbf{Y}_r, G_r^2) = [(I - e^{\widetilde{B}r})\text{cov}(\mathbf{Y}_\infty) - \widetilde{B}^{-1}(e^{\widetilde{B}r} - I)\text{cov}(\mathbf{Y}_\infty)B']\mathbf{e}$$

and

$$\mathbf{K}_r := \left[(rI - \widetilde{B}^{-1}(e^{\widetilde{B}r} - I))\text{cov}(\mathbf{Y}_\infty) - \widetilde{B}^{-1}(\widetilde{B}^{-1}(e^{\widetilde{B}r} - I) - rI)\text{cov}(\mathbf{Y}_\infty) B\right.$$

ACF of the squared increments

The ACF of the squared increment process:

$$\rho(h) = c_1 e^{\tilde{\lambda}_1 h} + \cdots + c_q e^{\tilde{\lambda}_q h}, \quad h \geq r,$$

where $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_q$ are the eigenvalues of $\widetilde{B}$. The ACF can also be written as

$$\rho(h) = \frac{\mathbf{a}' P_h \, \mathbf{a} + \mathbf{a}' \mathbf{Q}_h}{\mathbf{a}' P_0 \, \mathbf{a} + \mathbf{a}' \mathbf{Q}_0 + R_0}, \quad h \geq r,$$

where $P_h, \mathbf{Q}_h, P_0, \mathbf{Q}_0$ and R_0 depend only on $\tilde{\beta}_1, \ldots, \tilde{\beta}_q$.

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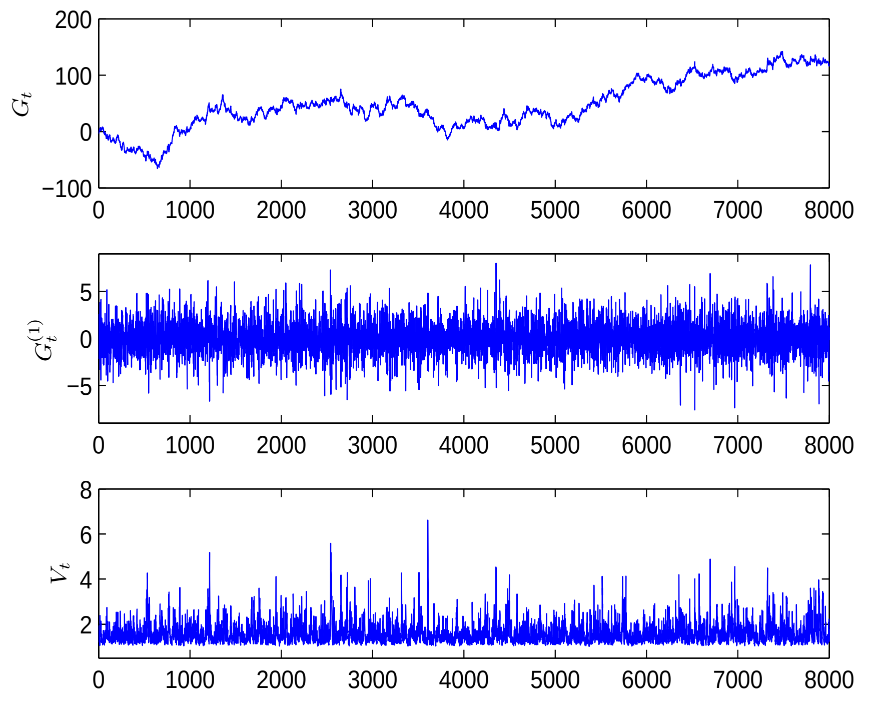
where $P_h, \mathbf{Q}_h, P_0, \mathbf{Q}_0$ and R_0 depend only on $\tilde{\beta}_1, \ldots, \tilde{\beta}_q$.

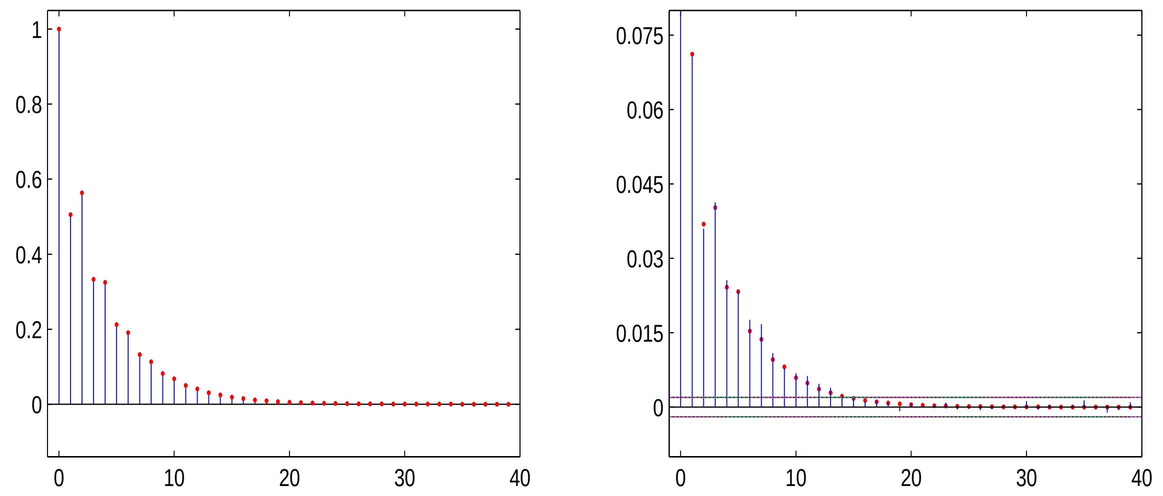
- Squared increment process has an ACF of an ARMA(q, q). (Square of a GARCH(p, q) process has an ACF of an ARMA(q, q).

Example: COGARCH(1,3)



The simulated compound-Poisson driven COGARCH(1,3) process with jump-rate 2, normally distributed jumps with mean zero and variance 0.74 and coefficients $\alpha_0 = \alpha_1 = 1$, $\beta_1 = 1.2$, $\beta_2 = .48 + \pi^2$ and $\beta_3 = .064 + .4\pi^2$.

Example: COGARCH(1,3)



The sample ACFs of the volatilities (V_t) (left) and of the squared COGARCH increments $((G_{t+1} - G_t)^2)$ (right) of a realisation of length 1,000,000 of the COGARCH(1,3) process.

Mixing: COGARCH(2,2)

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- For a compound-Poisson driven COGARCH(2,2) process, $(\mathbf{Y}_t)_{t\geq 0}$ is strongly mixing with geometric rate.
- The volatility process $(V_t)_{t\geq 0}$ and the squared increment process $(G_{rn}^{(r)})_{n\in\mathbb{N}}$ also inherits the strong mixing property as well as the rate.

Parameter Estimation

Least squares method

- The noise sequence in the squared increment process $((G_t^{(r)})^2)_{t\geq 0}$ is not an i.i.d. or a martingale difference.

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- Under strong mixing and moment conditions, the least squares estimators (LSE) of ARMA representations in which the noise is the linear innovation process are strongly consistent and asymptotically normal. (Francq and Zakoïan (1998)).

Strong consistency and asymptotic normality

Francq and Zakoïan (1998):

- For a strictly stationary ergodic process $(X_t)_{t\in\mathbb{Z}}$ satisfying the weak ARMA representation the LSE are strongly consistent.

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- For a strictly stationary ergodic process $(X_t)_{t\in\mathbb{Z}}$ satisfying the weak ARMA representation the LSE are strongly consistent.
- If in addition, $(X_t)_{t\in\mathbb{Z}}$ satisfies $E|X_t|^{4+2\nu} < \infty$ and strongly mixing with the mixing rate such that $\sum_{k=0}^{\infty} \alpha_k^{\nu/(2+\nu)} < \infty$ for some $\nu > 0$ then the LSE are asymptotically normal.

Estimating the parameters

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- Estimate $\tilde{\beta}$ by fitting an ARMA(2,2) to the observed squared increments and minimizing the least-squares sum.

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- Estimate $\tilde{\beta}$ by fitting an ARMA(2,2) to the observed squared increments and minimizing the least-squares sum.

- Estimate $\mathbf{a}$ by matching the ACF of the fitted ARMA(2,2) model and

$$\rho(h) = \frac{\mathbf{a}' P_h \mathbf{a} + \mathbf{a}' \mathbf{Q}_h}{\mathbf{a}' P_0 \mathbf{a} + \mathbf{a}' \mathbf{Q}_0 + R_0}, \quad h \geq r.$$

Estimating the parameters

- Estimate $\tilde{\beta}$ by fitting an ARMA(2,2) to the observed squared increments and minimizing the least-squares sum.

- Estimate $\mathbf{a}$ by matching the ACF of the fitted ARMA(2,2) model and

$$\rho(h) = \frac{\mathbf{a}' P_h \, \mathbf{a} + \mathbf{a}' \mathbf{Q}_h}{\mathbf{a}' P_0 \, \mathbf{a} + \mathbf{a}' \mathbf{Q}_0 + R_0}, \quad h \geq r.$$

- Finally, estimate α_0 using

$$E((G_t^{(r)})^2) = \frac{\alpha_0 \beta_q r}{\beta_q - \mu \alpha_1} \, E L_1^2.$$

(Assume $\mu = EL_1^2 = 1$).

Example: COGARCH(2,2)

$\alpha_0 = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0.2$ and compound Poisson driving process with std normal jumps and $EL_1^2 = 1$.

ACF of the squared increment:

$$\rho(h) = 0.1040e^{-0.1127h} - 0.0811e^{-0.8873h}, \quad h = 1, 2, \ldots$$

ARMA(2,2) parameters:

$$\phi_1 = 1.3052, \quad \phi_2 = -0.3679, \quad \theta_1 = -1.2642, \quad \theta_2 = 0.3669.$$

Hence,

$$\tilde{\beta}_1 = \log \xi_1 \xi_2 = 1 \quad \text{and} \quad \tilde{\beta}_2 = \log \xi_1 \log \xi_2 = 0.1,$$

where $\xi_1^{-1} = 0.8934$ and $\xi_2^{-1} = 0.4118$ are the autoregressive roots of the ARMA(2,2).

Example: COGARCH(2,2)

ACF as a function of $\mathbf{a}$:

$$\rho(h) = \frac{17.2007\alpha_1^2 - 0.2185\alpha_2^2 + 3.8771\alpha_1 - 0.4370\alpha_2}{14.6900\alpha_1^2 + 0.2853\alpha_2^2 + 2.3673\alpha_1 + 6.5707\alpha_2 + 5} e^{-0.1127h} + \frac{-2.3295\alpha_1^2 + 1.8340\alpha_2^2 - 4.1338\alpha_1 + 3.6680\alpha_2}{14.6900\alpha_1^2 + 0.2853\alpha_2^2 + 2.3673\alpha_1 + 6.5707\alpha_2 + 5} e^{-0.8873h}$$

Matching the two ACFs yields

$$-150.7545\alpha_1^2 + 2.3868\alpha_2^2 - 34.9245\alpha_1 + 10.7736\alpha_2 + 5 = 0,$$

$$-14.0288\alpha_1^2 + 22.8957\alpha_2^2 - 48.5971\alpha_1 + 51.7914\alpha_2 + 5 = 0,$$

giving $\mathbf{a} = (0.1, 0)$. Further, find

$$\beta_1 = \tilde{\beta}_1 + \alpha_2 = 1, \quad \beta_2 = \tilde{\beta}_2 + \alpha_1 = 0.2 \quad \text{and} \quad \alpha_0 = 1.$$

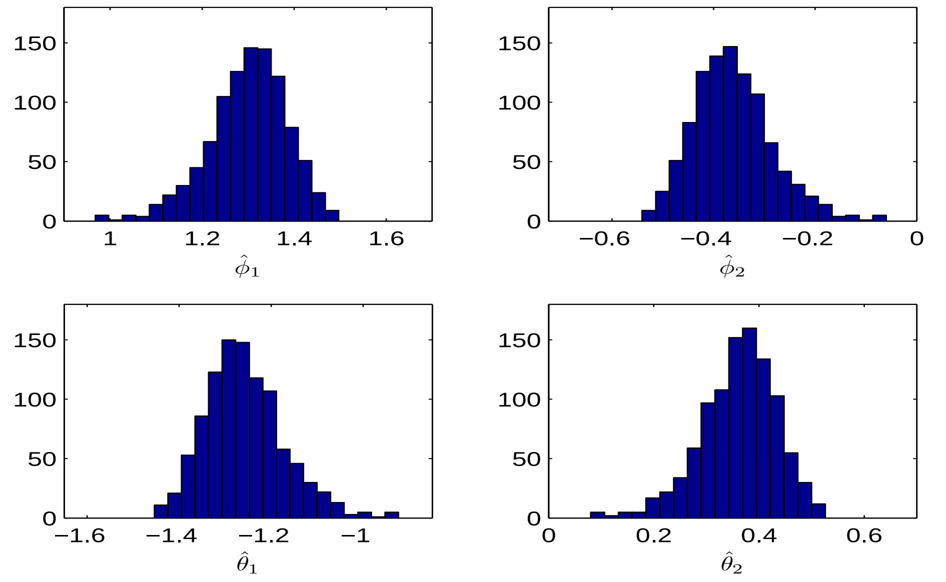
Simulation result

$\alpha_0 = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 0$, $\beta_1 = 1$, $\beta_2 = 0.2$ and compound Poisson driving process with std normal jumps and $EL_1^2 = 1$.

$1,000,000$ realizations of $G_i^{(1)}$, $i = 0, \ldots, 999,999$ were simulated. The estimation was repeated 2000 times.

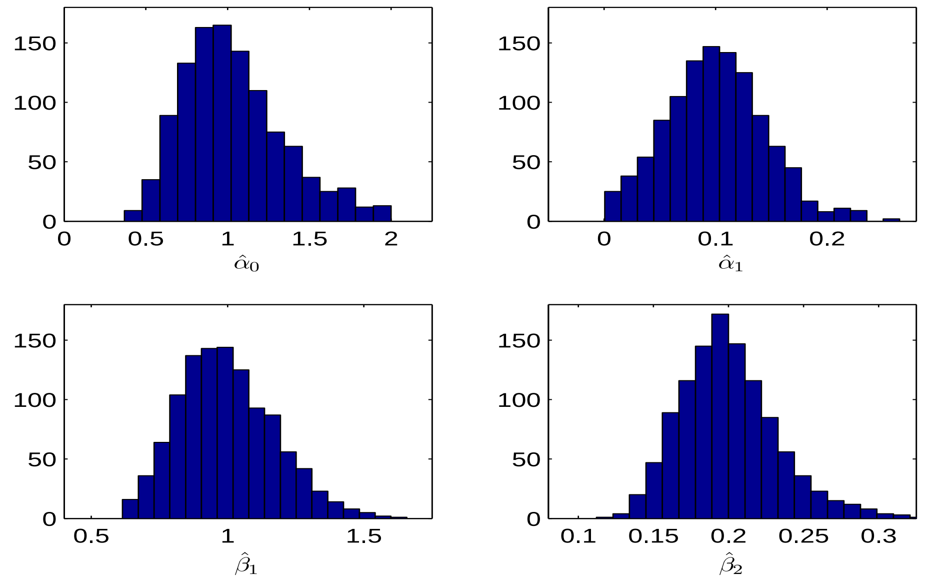
	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\beta}_2$
Mean	1.0683	0.0961	1.0123	0.1986
	(0.0100)	(0.0010)	(0.0054)	(0.0009)
Bias	0.0683	-0.0039	0.0123	-0.0014
	(0.0100)	(0.0013)	(0.0054)	(0.0009)
MSE	0.1099	0.0018	0.0300	0.0009
	(0.0059)	(0.00001)	(0.0014)	(0.0001)
MAE	0.2509	0.0337	0.1383	0.0242
	(0.0069)	(0.0008)	(0.0033)	(0.0006)

Simulation result



Least-squares estimators of the ARMA(2,2) parameters. The true values are $\phi_1 = 1.3052$, $\phi_2 = -0.3679$, $\theta_1 = -1.2642$ and $\theta_2 = 0.3669$.

Simulation result



Least-squares estimators of the COGARCH parameters. The true values are $\alpha_0 = 1$, $\alpha_1 = 0.1$, $\beta_1 = 1$ and $\beta_2 = 0.2$.

Volatility Estimation

Let h be a positive integer and $G_0, G_{\frac{1}{h}}, \ldots, G_1, G_{1+\frac{1}{h}}, \ldots$ are the observed log returns. Then

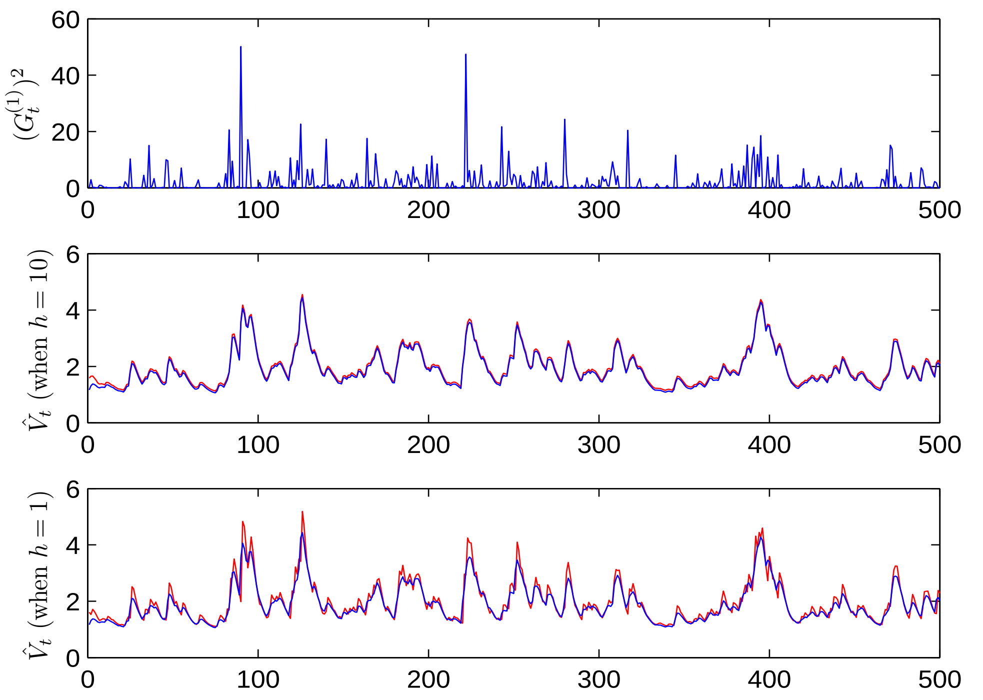
$$\hat{\mathbf{Y}}_{t+1} = e^{\hat{B}}\hat{\mathbf{Y}}_t + \sum_{i=1}^{h}(I + \frac{1}{h}\hat{B})^{h-i}\mathbf{e}\big(G^{(\frac{1}{h})}_{t+\frac{i-1}{h}}\big)^2$$

and

$$\hat{V}_{t+1} = \hat{\alpha}_0 + \hat{\mathbf{a}}'\hat{\mathbf{Y}}_{t+1}, \quad t = 0, 1, \ldots$$

where $\hat{\mathbf{Y}}_0$ is an initial starting value.

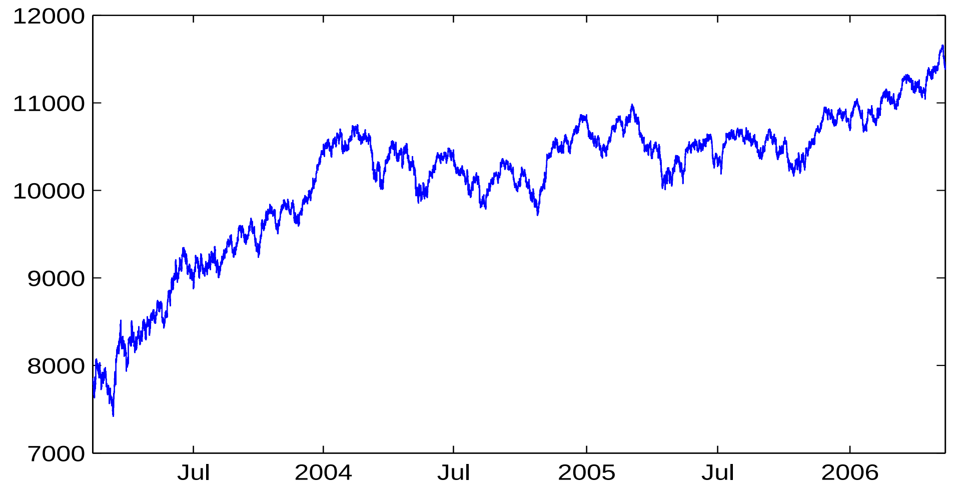
Volatility Estimation



Sample path of the squared increment process (top graph), theoretical (blue line) and the estimated volatility (red line) based on the observations $G_{\frac{1}{h}}, G_{\frac{2}{h}}, \ldots, G_{500}$ and the estimated coefficients. The middle graph is for $h = 10$ and the bottom graph is for $h = 1$.

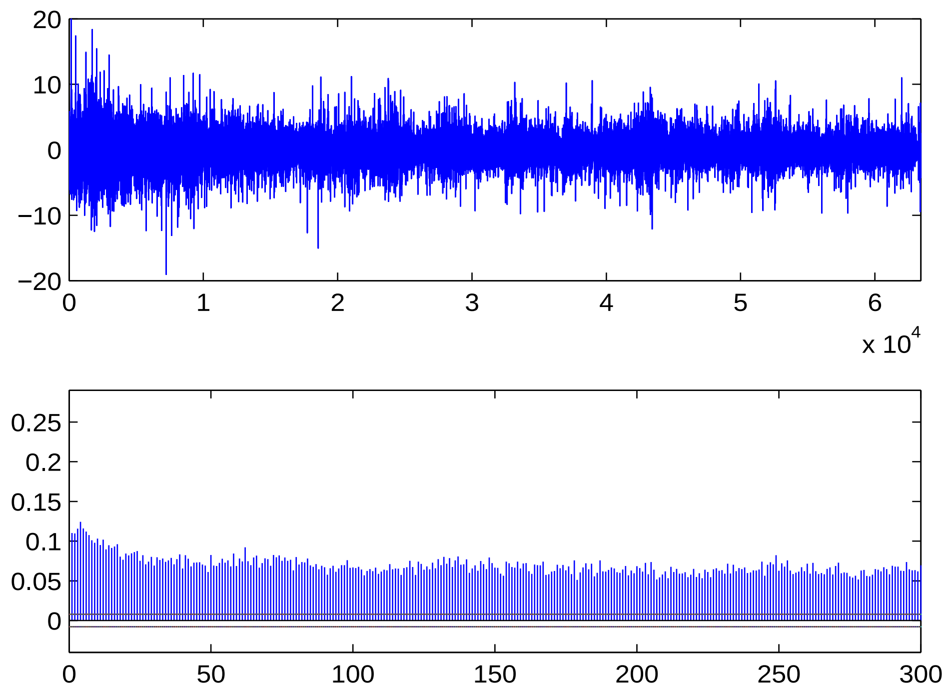
Real Data Analysis

Dow Jones 5-minute data



Dow Jones Industrial Average recorded from February 12th, 2003 to May 12th, 2006. (813 trading days with 78 5-minute observations per day, resulting in total of 63414 5-minute observations).

Dow Jones 5-minute data



Dow Jones log returns (the top graph) and the ACF of the squared log returns.

Estimated Coefficients

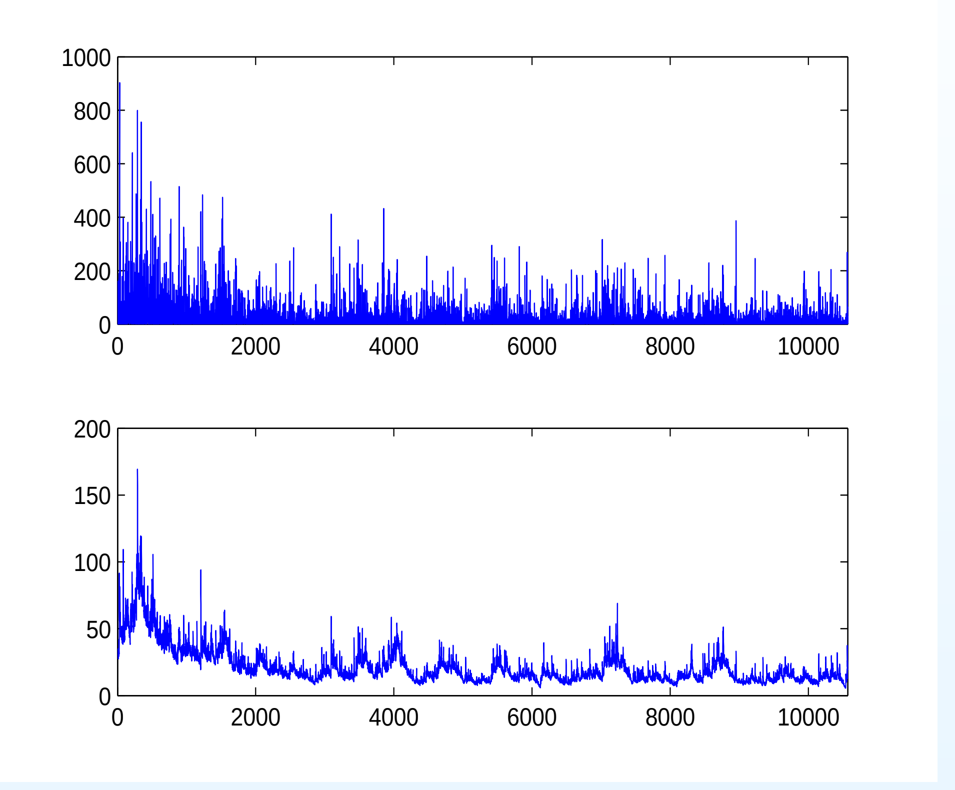
COGARCH(2, 2) model was fitted to the 30-minute log returns $G_0^{(1)}, G_1^{(1)}, \ldots, G_{10568}^{(1)}$.

The driving compound Poisson process has jump-rate $c = 2$ and normally distributed jumps with mean zero and variance 0.7071.

$$\hat{\alpha}_0 = 0.9760, \hat{\alpha}_1 = 0.0117, \hat{\alpha}_2 = 0.1860,$$

$$\hat{\beta}_1 = 0.6088, \hat{\beta}_2 = 0.0122.$$

Volatility Estimation



Dow Jones squared 30-minute log returns (top graph), the estimated volatilities based on the estimated coefficients and 5-minute log returns (middle graph). The unit of time is 30 minutes and there are $h = 6$ observations per unit interval.

Goodness of fit

Estimated residuals:

$$G_{t-1}^{(1)}/\sqrt{\hat{V}_t}, \quad t = 1, 2, \ldots, 10569.$$

Sample mean: -0.0232,

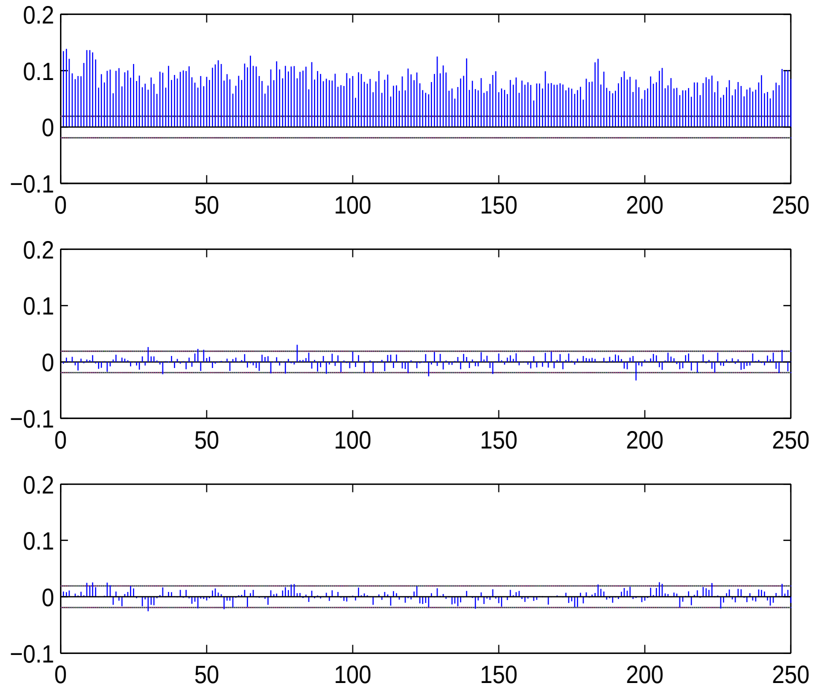
Sample standard deviation: 0.9325,

Ljung-Box test statistic: $Q_{LB} = 222.0025$ with lag 189,

McLeod-Li test statistic: $Q_{ML} = 214.0934$ with lag 189.

The critical value at 0.05 level was 222.0757.

Goodness of fit



ACF of the squared 30-minute log returns (top graph), the residuals (middle graph) and the squared residuals (bottom graph).

Conclusions and Future Works

Conclusions

- A family of continuous time GARCH processes, generalizing the COGARCH$(1,1)$ process was introduced and studied.
- A least-squares method to estimate the parameters of a COGARCH(2,2) process was proposed making use of the ACF structure of the squared increment process.
- When the driving Lévy process is compound Poisson, then the state process and the squared increments are strongly mixing with exponential rate, ensuring strong consistency and asymptotic normality of the LSE.
- The COGARCH(2, 2) model with compound poisson driving process was applied to a real data.

Future Works

- Investigate the degree to which the stationarity condition can be relaxed when $q > 1$.
- Investigate the strong mixing property of COGARCH(p, q) processes with $q > 2$ and with general driving Lévy process.
- Investigate the connections between higher order COGARCH and GARCH processes.
- Comparisons of COGARCH models fitted to observations of the same process made at different frequencies.

Thank you!