Introduction to Stochastic integration.

Mongolia 2015

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First course

- Stochastic processes.
- Main properties and examples.
- Martingales
- discrete-time martingales
- Inequalities (Doob).

Second course

- Brownian motion: motivation and ideas on SDE
- equivalent definitions of Brownian motion
- Transformations of Brownian motion,
- associated martingales
- finite dimensional distributions
- Existence and continuity. Kolmogorov Theorems.
- Markov property.

Third course

- discrete Stochastic integral
- stochastic integral for step processes.
- Quadratic variation of a standard Brownian motion.
- Itô Integral with respect to Brownian motion.

Fourth course

- first Itô formula
- Itô processes
- Doeblin-Itô formula.
- Integration by parts formula
- higher dimensions
- examples of SDE.

Stochastic processes

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probablity space and $(S, \mathcal{A})$ a measurable space. A stochastic process with values in the state space $(S, \mathcal{A})$ is a family of random variables $X = \{X_t : t \in J\}$, defined in $(\Omega, \mathfrak{F}, \mathbb{P})$ and with values in S. Usually J (the time parameter) is

- a finite or infinite subset of $\mathbb{N}$, and we say that the process is **discrete in time**.
- an interval of $\mathbb{R}^+$ of the form $[0, \infty)$ or $[a, b]$, with $0 \leq a < b$ and we say it is **continuous in time.**

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The most frequent case in this course will be $S = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$ i.e. the Borel sigma algebra in $\mathbb{R}$ and $J = [0, \infty)$ to $J = [0, T]$.
For each $\omega \in \Omega$ fixed, the function $t \to X_t(\omega)$ is called a **path** of the process X.
We can also regard a process as a function defined on the product space

$$X : J \times \Omega \to S$$

and write $X(t, \omega)$ instead of $X_t(\omega)$.

Stochastic processes

When J is an interval of $\mathbb{R}$, we say that : the process is a.s right continuous (resp. left continuous) if for almost all $\omega \in \Omega$ the path $t \to X_t(\omega)$ is right continuous, (resp. left continuous), A process is continuous if for almost all $\omega \in \Omega$ the path $t \to X_t(\omega)$ is continuous.

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There are several forms of equality amongst processes; two processes X and Y

- are equal if $X_t(\omega) = Y_t(\omega)$, for all t, ω.
- X is a modification of Y if for all $t \in J$, $\mathbb{P}(X_t = Y_t) = 1$
- X is indistinguishable from Y if $\mathbb{P}(X_t = Y_t, \text{for all } t \in J) = 1$

We can see (with an example) that even if two processes are a modification of each other, they might have different trajectoires. (Exercise).
Cleary If X is indistinguishable from Y, then X is a modification of Y.
For a reciprocal result we need some regularity of the paths:

Proposition 1

If X, Y are right continuous (or left continuous) and X is a modification of Y , then they are undistinguishable.

Proof: Let D be a countable dense subset of $J = [0, \infty)$ and N the complement of the set $\{X_t = Y_t, \text{for all } t \in D\}$. Then because of the countable subadditivity of $\mathbb{P}$ we have

$$\mathbb{P}(N) \leq \sum_{t \in D} \mathbb{P}(X_t \neq Y_t) = 0$$

Let $C \in \mathfrak{F}$ be the set where the paths of X and Y are right continuous. Then $\mathbb{P}(C) = 1$ and if $A := N \cup C^c$ then

$$\mathbb{P}(A) = \mathbb{P}(N \cup C^c) = 0.$$

The set $A^c = N^c \cap C$ has probability 1 and by the right continuity of the paths and the density of D we have $X_t = Y_t, \text{for all } t \in J$, so $\mathbb{P}(X_t = Y_t, \text{for all } t \in J) = \mathbb{P}(A^c) = 1$.

Stochastic processes

The **finite dimensional** distributions of a process are

$$\mathbb{P}(X_{t_i} \in B_1, X_{t_2} \in B_2 \ldots, X_{t_k} \in B_k)$$

with $k \in \mathbb{N}, t_i \in J, B_i \in \mathcal{A}, i = 1, 2, \ldots k$.
They are important, because in many occasions this is all we know about the process, and they are essential to construct a process (Kolmogorov consistency theorem).
It is clear that: if X is a modification or a version of Y then they have the same finite dimensional distributions (exercise).

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- we can also think of the processes X, Y defined in different probabilty spaces and in this case they are said to be **equivalent** if they have the same finite dimensional distributions.

A **filtration** $\{\mathfrak{F}_t : t \in J\}$ or $(\mathfrak{F}_t)_{t\in J}$ in $(\Omega, \mathfrak{F}, \mathbb{P})$ is a family of sub-sigma-algebras of $\mathfrak{F}$ such that $\mathfrak{F}_s \subset \mathfrak{F}_t, \;\; s < t, \;\; s, t \in J$. The probability space together with the filtration is denoted $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in J}, \mathbb{P})$.

Stochastic processes, measurable properties

A **filtration** $\{\mathfrak{F}_t : t \in J\}$ or $(\mathfrak{F}_t)_{t\in J}$ in $(\Omega, \mathfrak{F}, \mathbb{P})$ is a family of sub-sigma-algebras of $\mathfrak{F}$ such that $\mathfrak{F}_s \subset \mathfrak{F}_t, \;\; s < t, \;\; s, t \in J$. The probability space together with the filtration is denoted $(\Omega, \mathfrak{F}, (\mathfrak{F}_\mathrm{t})_{\mathrm{t}\in\mathrm{J}}, \mathbb{P})$.

We say that the process $\{X_t : t \in J\}$ is

- **measurable** if $X : J \times \Omega \to S$ is $\mathcal{B}(J) \otimes \mathfrak{F}$-measurable.
- **adapted** to the filtration $\{\mathfrak{F}_t : t \in J\}$ if for all $t \in J$, X_t is $\mathfrak{F}_t$-measurable.
- **progressively measurable** or simply **progressive** if for all $t > 0$ the function $(s, \omega) \to S$ from $[0, t] \times \Omega$ to S is $\mathcal{B}[0, t] \otimes \mathfrak{F}_t$ measurable.

Stochastic processes

These definitions are interesting in the continuous time setting. In the discrete time case, all we usually need is that the process is adapted in the space $(\Omega, \mathfrak{F}, (\mathfrak{F}_{\mathrm{n}})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$.

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The **canonical filtration** of a given process $\{X_t : t \in J\}$ is the filtration generated by the process, i.e, for each $t \in J$, $\mathfrak{F}_t$ is the sigma algebra generated by the family of random variables $\{X_s : s \leq t, s \in J\}$, which can also be written as

$$(\mathfrak{F}_t = \sigma(X_s : s \in J, s \leq t), \ \ t \in J.)$$

It is obvious that any process is adapted to its canonical filtration.

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Proof: We will do it in the case $J = [0, \infty)$. By definition of the product sigma algebra, for each $n \in \mathbb{N}$, the mapping $\phi_n : J \times \Omega \to \mathbb{R}$ defined as

$$(t, \omega) \to X_{\frac{[2^n t]}{2^n}}(\omega)$$

is $\mathcal{B}[0, \infty) \otimes \mathfrak{F}$- measurable for all $n \in \mathbb{N}$. Then by rigth contintuity we conclude.

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The same idea is used to prove the following result:

Proposition 3

** A right (or left) continuous adapted process is progressive.*

Stochastic processes

We will need several concepts related to the filtration:

- a **discrete stopping time** in $(\Omega, \mathfrak{F}, (\mathfrak{F}_\mathrm{n})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$ is a random variable $\tau : \Omega \to \mathbb{N} \cup \{+\infty\}$ such that $\{\tau \leq n\} \in \mathfrak{F}_n$ for all $n \in J \subset \mathbb{N}$.
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- a **stopping time**, with respect to given filtration $(\mathfrak{F}_t, \ t \in J)$ is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathfrak{F}_t$ for all $t \in J$.

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- a **stopping time**, with respect to given filtration $(\mathfrak{F}_t, \ \ t \in J)$ is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{\tau \leq t\} \in \mathfrak{F}_t$ for all $t \in J$.
- Given a stopping time τ and a process X, the mapping $X_\tau : \Omega \to S$ is defined as

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega).$$

Remarks: If X is measurable and τ is a finite stoping time , then X_τ is a random variable. A finer property is

Proposition 4

*** If X is progressively measurable and τ is a finite stopping time then X_τ is a random variable.*

Many examples of stopping times are related to the time a given process reaches a certain level, like in the ruin problems associated to the simple random walks.

- Random Walk (RW): if $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of IID random variables, the corresponding **random walk** is a discrete time process X defined as: $X_n := a + \sum_{i=1}^{n} \xi_i, \quad a \in \mathbb{R}$. Then if we define,

$$\tau = \begin{cases} \min\{n \in \mathbb{N} : X_n \geq 2a\}, & \text{if } \{n \in \mathbb{N} : X_n \geq 2a\} \neq \emptyset, \\ \infty \text{ if } \{n \in \mathbb{N} : X_n \geq 2a\} = \emptyset \end{cases} \tag{1}$$

τ is a discrete stopping time w.r.t the canonical filtration of the RW, since for all $m \in \mathbb{N}$,
$\{\tau \leq m\} = \bigcup_{i=1}^{m} \{X_i \geq 2a\} \in \mathfrak{F}_m$.
An example which is not a stopping time is
$\tau = \max\{n \in \mathbb{N} : X_n \geq 2a\}$.

- In fact, a similar result is true if we replace the R.W. by any adapted discrete time process.
- *If $X = (X_t : t \geq 0)$ is a process with continuous paths and A is a closed subset of $\mathbb{R}$ then **the hitting time** of a set A is defined as:

$$T_A = \begin{cases} \inf\{t \geq 0 : X_t \in A\}, & \text{if } \{t \geq 0 : X_t \in A\} \neq \emptyset, \\ \infty \text{ if } \{t \geq 0 : X_t \in A\} = \emptyset \end{cases} \tag{2}$$

It can be proved that it is a stopping time.

Martingales

A **martingale** defined in $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in J}, \mathbb{P})$ is a stochastic process $\{X_t : t \in J\}$ such that

- $\{X_t : t \in J\}$ is adapted to the given filtration,
- for all $t \in J$, $\mathbb{E}(|X_t|) < \infty$
- for all $s, t \in J$ such that $s < t$ the following condition is fulfilled:
$$\mathbb{E}(X_t|\mathfrak{F}_s) = X_s.$$

Martingales form a very important class of processes, as will be seen in the development of this course. A **submartingale** (resp. **supermartingale**) verifies the first two properties and $\mathbb{E}(X_t|\mathfrak{F}_s) \geq X_s$. (resp. $\mathbb{E}(X_t|\mathfrak{F}_s) \leq X_s$.)
We have the properties like $s \to E(X_s)$ is constant for a martingale because, for $s < t$,

$$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t|\mathfrak{F}_s)) = \mathbb{E}(X_s).$$

Martingales

Some examples are:

1. In the random walk if $\mathbb{E}(\xi_1) = 0$, then

$$X_n = a + \sum_{i=1}^{n} \xi_i$$

is a martingale with respect to the canonical filtration $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ and is called discrete time martingale.
If $\mathbb{E}(\xi_1) \geq 0$ (or $\mathbb{E}(\xi_1) \leq 0$) the corresponding random walk is a submartingale (supermartingale) with respect to the canonical filtration.

2. Any process $\{X_t : t \in [0, T]\}$ such that
   - $\mathbb{E}(X_t) = 0 \quad t \geq 0$,
   - $X_{t+s} - X_s$ and $\mathfrak{F}_s$ are independent, for all $s, t \geq 0$

   is a martingale with respect to the canonical filtration, $(\mathfrak{F}_t)_t$

To see this we simply write, in the first case,

$$\mathbb{E}(X_{n+m}|\mathfrak{F}_n) = \mathbb{E}\left[[X_n + \sum_{i=n+1}^{m} \xi_i] \ |\mathfrak{F}_n \right] = X_n.$$

We used the facts:

- the independence between $(\xi_i)_{i=n+1}^m$ and $\mathfrak{F}_n$,
- the knowledge that X_n is $\mathfrak{F}_n$-measurable,
- the hypothesis $\mathbb{E}(\xi_i) = 0$ for all $i \in \mathbb{N}$.

For the second example, observe that $\mathbb{E}(X_{t+s} - X_s|\mathfrak{F}_s) = \mathbb{E}(X_{t+s} - X_s) = 0$, because of the independence between $X_{t+s} - X_s$ and $\mathfrak{F}_s$.

Some important properties of martingales. Let $(X_n)_n$ be a discrete time martingale in $(\Omega, \mathfrak{F}, (\mathfrak{F}_\mathrm{n})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$.

Proposition 5

Optional sampling theorem or Doob stopping theorem: Let $(X_n)_n$ *be a discrete submartingale in* $(\Omega, \mathfrak{F}, (\mathfrak{F}_\mathrm{n})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$ *and* $S, T : \Omega \to \mathbb{N}$ *two discrete stopping times such that* $S \leq T$ *and bounded by* $k \in \mathbb{N}$*, then*

$$\mathbb{E}(X_T) \geq \mathbb{E}(X_S).$$

If X *is a martingale , we have an equality.*

Observe that for discrete time martingales, X_T is a random variable.

Idea of the proof: $\Omega = \bigcup_{j=0}^{k}(S=j)$ and it is a disjoint union; so we can write X_S as $X_S(\omega) = \sum_{j=0}^{k} X_j \mathbf{1}_{(S=j)}(\omega)$.
We shall only prove a special case since this is the case we will need: $T \equiv k$ and $S \leq k$.

$$X_T - X_S = \sum_{j=0}^{k}(X_k - X_j)\mathbf{1}_{(S=j)} = \sum_{j=0}^{k-1}(X_k - X_j)\mathbf{1}_{(S=j)}$$

and this sum has positive expectation (or is zero in the martingale case), since the set $(S=j) \in \mathfrak{F}_j$, we get

$$\mathbb{E}\left[\mathbf{1}_{(S=j)}(X_k - X_j)\right] \geq 0, \ \ j = 0, 1, \ldots (k-1).$$

Proposition 6

Let $(X)_{n=0}^{N}$, *be a discrete positive submartingale in* $(\Omega, \mathfrak{F}, (\mathfrak{F}_{\mathrm{n}})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$, *then for all* $\lambda > 0$

$$\mathbb{P}(\sup_n X_n \geq \lambda \leq \frac{1}{\lambda}\mathbb{E}(X_N.\mathbf{1}_{(\sup_n X_n \geq \lambda)}) \leq \frac{1}{\lambda}\mathbb{E}(|X_N|).$$

Proof:Observe that here $J = \{0, 1 \ldots N\}$. Let

$$S = \begin{cases} \inf\{n : X_n \geq \lambda\}, & \text{if } \{n : X_n \geq \lambda\} \neq \emptyset \\ N & \text{if } \{n : X_n \geq \lambda\} = \emptyset \end{cases} \tag{3}$$

S is a stopping time, $S \leq N$, so by (proposition 5),

$$\mathbb{E}(X_N) \geq \mathbb{E}(X_S) = \mathbb{E}(X_S.\mathbf{1}_{(\sup_n X_n \geq \lambda)}) + \mathbb{E}(X_S.\mathbf{1}_{(\sup_n X_n < \lambda)})$$

$$\geq \lambda\,\mathbb{P}(\sup_n X_n \geq \lambda) + \mathbb{E}(X_N.\mathbf{1}_{(\sup_n X_n < \lambda)})$$

The last inequalities are due to the definition of S, i.e. $X_S \geq \lambda$ on the set $(\sup_n X_n \geq \lambda)$ and $X_S = X_N$ on the set $(\sup_n X_n \leq \lambda)$.

For a discrete time martingale X, we then have (by Jensen Inequality) a positive submartingale $(|X_n|^p : n \in \{0, 1, 2 \dots N\})$ whenever $\mathbb{E}(|X_N|^p) < \infty$ and in any case we obtain:

Corollary 1

If X is a discrete time martingale indexed by the finite set $\{0, 1, 2 \dots N\}$ and if $p \geq 1$ then

$$\lambda^p \, \mathbb{P}\left[\sup_n |X_n| \geq \lambda\right] \leq \mathbb{E}(|X_N|^p), \ \ \lambda > 0$$

The former properties can be extended to the continuous time case:

Theorem 1

Doob's maximal inequality: Let $(M_t)_{t\geq 0}$ be a right continuous martingale and $p \geq 1$ then

$$\mathbb{P}(\sup_{0\leq s\leq t} |M_s| \geq \lambda) \leq \frac{1}{\lambda^p}\mathbb{E}(|M_t|^p), \qquad t \geq 0, \ \ \lambda > 0.$$

Proof. . Let D be a countable dense subset of the interval $[0,t]$. Because of the right continuity

$$\sup_{s\in D} |M_s| = \sup_{s\in[0,t]} |M_s|.$$

Let $D_n \subset D$ be an increasing sequence, such that D_n is finite and $D = \bigcup_n D_n$.
Since the result holds for each discrete submartingale indexed by a finite set D_n, $(|M_s|)_{s \in D_n}$, we have for all $n \in \mathbb{N}$

$$\mathbb{P}\left[\sup_{s \in D_n} |M_s| \geq \lambda\right] \leq \frac{1}{\lambda^p} \mathbb{E}(|M_t|^P).$$

Passing to the limit as $n \to \infty$ we conclude.

Gaussian random variables

Definition 1

The normal random variable $X : \Omega \to R$ with mean m and variance σ^2 ($X \approx \mathcal{N}(m, \sigma^2)$), has density

$$\frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right]$$

Some useful properties on gaussian or normal random variables:

a.- if $m = 0$, $\mathbb{E}(X^2) = \sigma^2$, $\mathbb{E}(X^4) = 3\sigma^4$,

b.- $\text{var}(X^2) = \mathbb{E}\left[X^2 - \sigma^2\right]^2 = 2\sigma^4$.

c.- $\psi(\lambda) = \mathbb{E}\left[\exp(\lambda X)\right] = \exp(\lambda m + \frac{\sigma^2\lambda^2}{2})$ (Laplace transform).

d.- $\Phi_X(u) = \mathbb{E}\left[\exp(iuX)\right] = \exp(imu - \frac{\sigma^2 u^2}{2})$
(Characteristic function.)
(Excercise).

The Brownian motion. This mathematical object is related to problems that evolve in time in very chaotic way, so in the beginning the XX century it seemed difficult to make any mathematical model to describe this. It has proven to be one of the most important processes in the theory and in many applications.

- In 1826-27 a biologist Robert Brown observed the movement of polen particles placed in water, and saw it was very irregular and erratic. He described it but could not explain why it moved;
- Bachelier in 1900 described the fluctuations in stock prices and found a mathematical model for them;
- later Einstein (re)-discovered it in 1905, when studying the movement of a diffusing particle.
- Smoluchowski found a description of BM as a limit of R.W.
- finally in 1923 Norbert Wiener gave a rigorous construction of the BM and studied many of its properties.
- Kolmogorov. P. Lévy.......

Definition 2

A real valued stochastic process $B = (B_t)_{t \geq 0}$ *defined in* $(\Omega, \mathfrak{F}, \mathbb{P})$ *is called the standard Brownian motion(BM) or Wiener process if*

1. $B_0 = 0$ *a.s.*
2. $B_{t+h} - B_t$ *has* $\mathcal{N}(0, h)$ *distribution, for* $0 \leq h$ *and* $0 \leq t$.
3. *For all* $n \in \mathbb{N}$ *and all times* $0 \leq t_1 < t_2 < \ldots < t_n$ *the random variables* $(B_{t_i} - B_{t_{i-1}})_{(i=1,2\ldots n)}$ *are independent.*

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The second property means that the process has **stationary increments** i.e. $B_{t+h} - B_t$ has the same distribution as $B_{s+h} - B_s$ for any $h, s, t \in [0, \infty)$ and $(B_{t+h} - B_t) \approx \mathcal{N}(0, h)$.

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Theorem 2

*(Kolmogorov)** If we have a process $(X_t)_{t\geq 0}$ such that for all $T>0$ there exist positive constants α, β, C with*

$$\mathbb{E}\left[|X_s - X_t|^{\alpha}\right] \leq C.|s-t|^{1+\beta} \quad s,t \in [0,T]$$

then the process admits a continuous version on $[0,T]$.

Brownian motion. Path continuity.

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As a consequence the BM has a continuous version, since the BM verifies the condition

$$\mathbb{E}_x[|B_t - B_s|^4] \leq C(t-s)^2.$$

A process B with values in $\mathbb{R}^n$, is an n-dimensional Brownian motion if it can be written as

$$B_t = (B_t^{(1)}, B_t^{(2)}, \ldots B_t^{(n)}), \;\; t \geq 0$$

where $(B_t^{(i)})_{i=1,2,\ldots n}$ are independent real valued BM.
It will also have a continuous version.

Another definition of Brownian motion is

Definition 3

A real valued stochastic process $B = (B_t)_{t \geq 0}$ *defined in* $(\Omega, \mathfrak{F}, \mathbb{P})$ *is called Brownian motion or Wiener process if*

A. *Is a Gaussian Process.*

B. $m(t) = \mathbb{E}(B_t) = 0, \quad t \geq 0,$

C. $cov(B_t, B_s) = C(s,t) = \min(s,t), \quad s,t \geq 0$

We now see one of the implications. The other one is left as an excercise.

If B verifies definition 2 then ,

A. It is a Gaussian process: given $n \in \mathbb{N}$, $0 \leq t_1 \leq t_2 \leq \ldots t_n$ for any $a_i \in \mathbb{R}, i = 1, 2, \ldots n$, then there exist $b_i \in \mathbb{R}, i = 1, 2, \ldots n$ verifying

$$\sum_{i=1}^{n} a_i B_{t_i} = \sum_{i=1}^{n} b_i (B_{t_i} - B_{t_{i-1}}).$$

B. $m(t) = 0, t \geq 0$ and

C. for $0 \leq s < t$

$$\operatorname{cov}(B_s, B_t) = \mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_t - B_s) + B_s^2) = s$$

since $\mathbb{E}(B_s)\,\mathbb{E}(B_t - B_s) = 0$ and $\mathbb{E}(B_s^2) = s$.

so it verifies definition 3.

Conversely, if it verifies definition 3

1. $B_0 = 0$ a.s because $\mathbb{E}(B_0^2) = 0$,
2. for $0 \le s \le t$, $(B_t - B_s)$ is gaussian and

   $$\mathbb{E}((B_t - B_s)^2) = \mathbb{E}(B_t^2 - 2B_tB_s + B_s^2)) = t - 2\min(s,t) + s = t - s.$$

   So $(B_t - B_s) \approx \mathcal{N}(0, t-s)$
3. to see the independence of the increments, it is enough to prove that the covariance matrix of the gaussian vector $(B_{t_i} - B_{t_{i-1}})_{i=1}^n$ is diagonal. This is a consequence of

   $$\mathbb{E}\left((B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})\right) = t_i - t_i - t_{i-1} + t_{i-1} = 0.$$

   for $t_i \le t_{j-1} < t_j$.

Proposition 7

Given a function $m : [0,T] \to R$ *and a symmetric positive definite function* $C : [0,T] \times [0,T] \to \mathbb{R}$ *there exists a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$ *and Gaussian process* $(X_t)_{t\in[0,T]}$ *defined on it such that it has mean function* m *and covariance function* C.

This result is based in Daniel-Kolmogorov existence theorem and on properties of Gaussian vectors.

Brownian motion

The existence of the BM is a consequence of the former proposition for Gaussian processes. This is done taking the mean function as $m(t) = 0, t \geq 0$ and the covariance function as $C(s,t) = \mathrm{cov}(B_s, B_t) = \min(s,t), \;\; s,t \geq 0$.

The existence of the BM is a consequence of the former proposition for Gaussian processes. This is done taking the mean function as $m(t) = 0, t \geq 0$ and the covariance function as $C(s,t) = \text{cov}(B_s, B_t) = \min(s,t), \;\; s,t \geq 0.$
All we must verify is that this function is positive definite (symmetry is immediate). So for any $n \in \mathbb{N}$, $s_i \in \mathbb{R}^+$ and $a_i \in \mathbb{R}, \;\; i = 1, 2 \ldots n,$

$$\sum_{1\leq i,j\leq n} a_i a_j C(s_i, s_j) = \sum_{1\leq i,j\leq n} a_i a_j \int_0^\infty \mathbf{1}_{[0,s_i)}(s)\mathbf{1}_{[0,s_j)}(s)ds =$$

$$\int_0^\infty \big(\sum_{i=1}^n a_i \mathbf{1}_{[0,s_i)}(s)\big)^2 ds \geq 0$$

Brownian motion

Some transformed processes of B real BM. The following processes are also real valued BM:

- Symmetry: $(Z_t = -B_t)_{t \geq 0}$
- Scaling: for any $c > 0$ the process $(B_t^c = \frac{1}{c} B_{c^2 t} : t \geq 0)$
- Translation by $h > 0$: the process $(Y_t = B_{t+h} - B_h)_{t \geq 0}$.
- reversion in time: for $T > 0$, the process $(R_t^T = B_T - B_{T-t})_{t \in [0,T]}$ is also a Brownian motion in $[0, T]$.
- $$Y_t := \begin{cases} tB_{\frac{1}{t}} \text{ if } t > 0 \\ 0 \text{ if } t = 0. \end{cases} \tag{4}$$

(Excercise:)

The following process associated to the Brownian motion $(B_t : t \geq 0)$ are martingales with respect to canonical filtration of the BM.

1. $(B_t : t \geq 0)$
2. $(B_t^2 - t : t \geq 0)$
3. For any $a \in \mathbb{R}$, $\left(M_t = \exp(aB_t - \frac{a^2}{2}t) : t \geq 0\right)$.

Brownian motion

Proofs: Let $0 \leq s < t$.

1. $\mathbb{E}(B_t - B_s|\mathfrak{F}_s) = \mathbb{E}(B_t - B_s) = 0$
2. $\mathbb{E}(B_t^2 - B_s^2 - (t-s)|\mathfrak{F}_s) =$
   $\mathbb{E}\left((B_t - B_s)^2 - (t-s) + 2B_s(B_t - B_s)|\mathfrak{F}_s\right) = 0$
3. $$\mathbb{E}\left(M_t|\mathfrak{F}_s\right) = M_s \text{ iff } \mathbb{E}\left(\frac{M_t}{M_s}|\mathfrak{F}_s\right) = 1$$

   and this last conditional expectation equals (using the properties of Gaussian r.v)
   $\mathbb{E}\left[\exp(a(B_t - B_s)) - \frac{a^2}{2}(t-s))|\mathfrak{F}_s\right] =$
   $\exp(-\frac{a^2}{2}(t-s))\,\mathbb{E}\left[\exp(a(B_t - B_s)\right] = 1$

Brownian motion

We also consider the BM **started at point** $y \in \mathbb{R}$. It is the translated process $(y + B_t : t \geq 0)$. We use the notations $\mathbb{E}_y(.),\ \mathbb{P}_y(.)$ for the expectation and the probability in this case.

We also consider the BM **started at point** $y \in \mathbb{R}$. It is the translated process $(y + B_t : t \geq 0)$. We use the notations $\mathbb{E}_y(.)$, $\mathbb{P}_y(.)$ for the expectation and the probability in this case. For $A \in \mathcal{B}(\mathbb{R})$ the definition of BM gives us the distribution of each B_t,

$$\mathbb{P}(B_t \in A) = \int_A p(x,0,t)dx = \frac{1}{(2\pi t)^{1/2}} \int_A \exp\left[-\frac{x^2}{2t}\right] dx,$$

$$\mathbb{P}_y(B_t \in A) = \int_A p(x,y,t)dx = \frac{1}{(2\pi t)^{1/2}} \int_A \exp\left[-\frac{(x-y)^2}{2t}\right] dx,$$

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and for any borel- measurable and bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_y(f(B_t)) = \int f(x) p(x, y, t) dx = \frac{1}{(2\pi t)^{1/2}} \int f(x) \exp\left[-\frac{(x-y)^2}{2t}\right] dx$$

Computation of joint probabilities.

Theorem 3

For $n \in \mathbb{N}, \;\; 0 \leq t_1 \leq t_2 \leq \ldots t_n$ *and* $f : \mathbb{R}^n \to \mathbb{R}$, *bounded and measurable,*

$$\mathbb{E}[f(B_{t_1}, B_{t_2}, \ldots B_{t_n})] = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f(x_1, x_2, \ldots x_n).$$

$$.p(x_1, 0, t_1)p(x_2, x_1, t_2 - t_1) \ldots p(x_n, x_{n-1}, t_n - t_{n-1})dx_n \ldots dx_2 dx_1$$

and

$$\mathbb{E}_x[f(B_{t_1}, B_{t_2}, \ldots B_{t_n})] = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f(x_1, x_2, \ldots x_n).$$

$$.p(x_1, x, t_1)p(x_2, x_1, t_2 - t_1) \ldots p(x_n, x_{n-1}, t_n - t_{n-1})dx_n \ldots dx_2 dx_1$$

Idea: there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$h(y_1, \dots, y_n) := f(T(y_1, \dots, y_n)) = f(y_1, y_1 + y_2, \dots, y_1 + y_2, \dots + y_n)$$

then define $Y_i = B_{t_i} - B_{t_{i-1}}, i = 1, 2, \dots n$ which are independent, so they have a joint density,
$p(y_1, 0, t_1)p(y_2, 0, t_2 - t_1) \dots p(y_n, 0, t_n - t_{n-1})$ and

$$\mathbb{E}[f(B_{t_1}, B_{t_2}, \dots B_{t_n})] = \mathbb{E}[h(Y_{t_1}, Y_{t_2}, \dots Y_{t_n})]$$

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} h(y_1, y_2, \dots y_n).$$

$$.p(y_1, 0, t_1)p(y_2, 0, t_2 - t_1) \dots p(y_n, 0, t_n - t_{n-1}) dy_n \dots dy_2\, dy_1$$

Since the jacobian of T equals 1, we can transform this integral in order to obtain the result.

Brownian motion: Markov property

Theorem 4

Let B be a standard Brownian motion. For a borel and bounded function, $f : \mathbb{R} \to \mathbb{R}$ and $0 \leq s \leq t$

1. $\mathbb{E}(f(B_t)|\mathfrak{F}_s) = \mathbb{E}(f(B_t)|B_s)$,
2. $\mathbb{E}\left[f(B_t)|\sigma(B_s)\right] = \frac{1}{(2\pi(t-s))^{1/2}} \int_{\mathbb{R}} f(x) \exp\left[-\frac{(x-B_s)^2}{2(t-s)}\right] dx$,
3. $\mathbb{E}(f(B_t)|B_s = y) = \frac{1}{(2\pi(t-s))^{1/2}} \int_{\mathbb{R}} f(x) \exp\left[-\frac{(x-y)^2}{2(t-s)}\right] dx$,

We will prove the second equality of the theorem: We call

$$\Psi(y) := \int_{\mathbb{R}} f(x)p(x,y,t-s)\,dx.$$

To prove the result we show that for all $C \in \sigma(B_s)$,

$$\mathbb{E}(f(B_t)\mathbf{1_C}) = \mathbb{E}(\Psi(B_s)\mathbf{1_C}),$$

i.e. $\mathbb{E}(f(B_t)|\sigma(B_s)) = \Psi(B_s)$.
But C is of the form $(B_s \in A)$ for some $A \in \mathcal{B}(\mathbb{R})$. Hence, the left hand side (by theorem 3) is equal to

$$\begin{aligned}\mathbb{E}\left[f(B_t)\mathbf{1}_{(B_s\in A)}\right] &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\mathbf{1}_A(y)p(y,0,s)p(y,x,t-s)\,dx\,dy\\ &= \int_A p(y,0,s)\Psi(y)\,dy\end{aligned}$$

The right hand side

$$\mathbb{E}(\Psi(B_s)\mathbf{1_C}) = \int_{\Omega} \mathbf{1}_A(B_s).\Psi(B_s)d\,\mathbb{P} = \int_A p(y,0,s)\Psi(y)\,dy$$

(the last equality is just the change of variable formula).

To prove 1. we define $F(x,y) := f(x+y), \;\; x,y \in \mathbb{R}$

$$\mathbb{E}\left(f(B_t)|\mathfrak{F}_s)\right) = \mathbb{E}\left(f(B_t - B_s + B_s)|\mathfrak{F}_s)\right) = \mathbb{E}\left(F(B_t - B_s, B_s)|\mathfrak{F}_s)\right)$$
$$= \frac{1}{(2\pi(t-s))^{1/2}} \int_R F(x, B_s) \exp\left[-\frac{x^2}{2(t-s)}\right] dx$$

We now define $\Phi(y) := \frac{1}{(2\pi(t-s))^{1/2}} \int_R F(x,y) \exp\left[-\frac{x^2}{2(t-s)}\right] dx$
and obtain,

$$\Phi(y) = \frac{1}{(2\pi(t-s))^{1/2}} \int_R f(x+y) \exp\left[-\frac{x^2}{2(t-s)}\right] dx$$
$$= \frac{1}{(2\pi(t-s))^{1/2}} \int_R f(x) \exp\left[-\frac{(x-y)^2}{2(t-s)}\right] dx$$

So

$$\mathbb{E}\left(f(B_t)|\mathfrak{F}_s)\right) = \Phi(B_s) = E\left[f(B_t)|B_s\right]$$

Stochastic Itô Integral.

Discrete case: integration w.r.t a Simple symmetric Random Walk. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent Bernoulli random variables, , i.e. $\mathbb{P}(\xi_1 = 1) = 1/2 = \mathbb{P}(\xi_1 = -1)$, then the corresponding **random walk**

$$X_n = \sum_{i=1}^{n} \xi_i, \qquad \Delta X_i = X_i - X_{i-1} = \xi_i$$

We can consider each ξ_i as the result of a game and we can try to find a good strategy for the game by betting each time a certain amount A_i but we have to make the decision before the next play, with just the information of what has happened until time $(i-1)$. i.e A_i must be $\sigma(X_k : k \leq i-1)$-measurable for each $i \in \mathbb{N}$, , and the capital up to time n, will be

$$Z_n = \sum_{i=1}^{n} A_i \xi_i = \sum_{i=1}^{n} A_i (X_i - X_{i-1}) = \sum_{i=1}^{n} A_i \Delta X_i$$

It is easy to see that

Proposition 8

- $(Z_n : n \in \mathbb{N})$ *is a martingale for the canonical filtration* $\mathfrak{F}_n = \sigma(X_k : k \leq n)$.
- $\mathbb{E}(Z_n) = 0, \;\; \mathrm{var}(Z_n) = \sum_{i=1}^n \mathbb{E}(A_i^2)$ *if* $A_n \in L_2(\Omega, \mathfrak{F}, \mathbb{P})$

Proof: It is a martingale: for any $n \in \mathbb{N}$,

$$\mathbb{E}(Z_n|\mathfrak{F}_{n-1}) = \mathbb{E}\left[(Z_{n-1} + A_n\xi_n)|\mathfrak{F}_{n-1}\right] = Z_{n-1},$$

because $\mathbb{E}(A_n\xi_n|\mathfrak{F}_{n-1}) = A_n\,\mathbb{E}(\xi_n) = 0$ since A_n is $\mathfrak{F}_{n-1}$-measurable and ξ_n is independent of $\mathfrak{F}_{n-1}$ and has mean 0.

Stochastic Itô Integral.

To compute its variance:

$$\mathbb{E}(Z_n^2)=\mathbb{E}(\sum_{i,j}^{n}A_iA_j\xi_i\xi_j)=\mathbb{E}(\sum_{i}^{n}A_i^2\xi_i^2)=\sum_{i}^{n}\mathbb{E}(A_i^2)\,\mathbb{E}(\xi_i^2)=\sum_{i}^{n}\mathbb{E}(A_i^2),$$

we used:

- for the second equality, if $i<j$,

$$\mathbb{E}(A_iA_j\xi_i\xi_j)=\mathbb{E}(\mathbb{E}(A_iA_j\xi_i\xi_j|\mathfrak{F}_{j-1}))=$$

$$=\mathbb{E}(A_iA_j\xi_i\,\mathbb{E}(\xi_j|\mathfrak{F}_{j-1}))=\mathbb{E}(A_iA_j\xi_i\,\mathbb{E}(\xi_j))=0,$$

  because of the measurability and independence assumed.
- for the third one, A_i depends only on $(\xi_k : k=1,2...i-1)$ and these are independent of ξ_i
- $\mathbb{E}(\xi_i^2)=1$.

Stochastic Itô Integral.

More generally, given a discrete Martingale $(X_n)_{n\in\mathbb{N}}$, in $(\Omega, \mathfrak{F}, (\mathfrak{F}_\mathrm{n})_{\mathrm{n}\in\mathrm{J}}, \mathbb{P})$ and a bounded positive process $(A_n)_n$ such that A_n is $\mathfrak{F}_{n-1}$ measurable, $n = 1, 2 \ldots$ we can define the stochastic integral (also called the martingale transform of A) as:

$$(A \cdot X)_n = Z_n = \sum_{i=1}^{n} A_i(X_i - X_{i-1}) = \sum_{i=1}^{n} A_i \Delta X_i.$$

It can be proved that it is also a martingale and we can compute its variance (left as exercise).

Stochastic Itô Integral.

We want to give sense to

- $Z_t = \int_0^t g(s)dB_s$, for $g : \mathbb{R} \to \mathbb{R}$,
- to $\int h(X_s)dB_s$ for a process $(X_s : s \geq 0)$.
- or to $\int H(s, X_s)dB_s$ for a process $(X_s : s \geq 0)$.

for convenient classes of functions g, h, H and processes X. The first one is rather easy to study (the Paley-Wiener integral, defined as $-\int_0^T g'(t)B_t dt$, for g differentiable and such that $g(0) = g(T) = 0$) but the second one, will prove to be quite tricky due to the great irregularity of the paths of the Brownian motion $t \to B_t$, as will be seen in the next slides.

Stochastic Itô Integral.

This concept is needed in order to develop a theory of stochastic differential equations (SDE) of the form:

$$dX_t = a(X_t, t)dt + b(X_t, t)dB_t$$

$$X_0 = x$$

which must be understood as

$$X_t = \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dB_s, \quad X_0 = x$$

So our task is to give a meaning to the last integral.

Stochastic Itô Integral.

A simple case: we will try to compute $\int_0^t BdB$: a **Riemann sum** for it would be

$$\sum_{\pi} B_{r_k}(B_{t_{k+1}} - B_{t_k})$$

where π is a partition: $\{0 = t_0 < t_1 < \dots t_m = t\}$ of $[0, t]$ and $r_k \in [t_k, t_{k+1}]$. Since (as will be seen later) the Brownian motion has paths with unbounded variation, we cannot expect to obtain a pathwise limit .

We will see that different choices of the intermediate point give rise to different limits. First we will study the variation of the brownian paths.

Quadratic variation of Brownian motion.

Definition 4

Let $(X_t : t \geq 0)$ *be a continuous stochastic process. We define its p-variation on* $[a, b] \subset [0, \infty)$ *as the limit in probability when* $|\pi_n| \to 0$ *(if it exist) of*

$$Q_n[a,b] := \sum_{\pi_n} |X_{t^n_{k+1}} - X_{t^n_k}|^p = \sum_{\pi_n} |\Delta X(t^n_k)|^p$$

where $\{\pi_n : n \in \mathbb{N}\}$ *a family of partitions of* $[a, b]$*, of the form* $\pi_n := \{a = t^n_0 < t^n_1 < \cdots < t^n_{m_n} = b\}$ *such that* $\lim_{n\to\infty} |\pi_n| = 0$*.*

Recall that $|\pi_n| = \max_k |t^n_{k+1} - t^n_k|$

When $p = 2$ it is called **quadratic variation** of X and for $p = 1$ the **total variation** of X in $[a, b]$;
(notation: $\Delta X(t_k^n) := X_{t_{k+1}^n} - X_{t_k^n}$).
We recall the fact:
(convergence in $L_2(\Omega, \mathfrak{F}, \mathbb{P})$) implies (convergence in probability.)

Proposition 9

The quadratic variation on $[0,t]$ of the Brownian motion, B is equal to t and is denoted $< B, B >_t$.

Let $\{\pi_n : n \in \mathbb{N}\}$ be a family of partitions of $[a,b]$, of the form $\pi_n := \{a = t_0^n < t_1^n < \cdots < t_{m_n}^n = b\}$, such that $\lim_{n\to\infty} |\pi_n| = 0$.

We will use the result: If $X \approx \mathcal{N}(0,\sigma^2)$ then $\mathrm{var}(X^2) = 2\sigma^4$.
So if $\Delta B(t_k^n) := B_{t_{k+1}^n} - B_{t_k^n}, \quad \Delta t_k^n := (t_{k+1}^n - t_k^n)$ then

$\Delta B(t_k^n) \approx \mathcal{N}(0, \Delta t_k^n)$ and $\mathrm{var}[(\Delta B(t_k^n))^2] = 2(\Delta t_k^n)^2$.

Stochastic Itô Integral.

We will prove that

$$\left[L_2 - \lim_{n\to\infty} Q_n\right] = \left[L_2 - \lim_{n\to\infty} \sum_{\pi_n} (\Delta B(t_k^n))^2\right] = (b-a).$$

$$\mathbb{E}[(Q_n - (b-a))^2] = \mathbb{E}\left[\Big(\sum_{\pi_n} \Delta B(t_k^n)^2 - \Delta t_k^n\Big)2\right] =$$

This is the variance of a sum of independent random variables, so it is equal to the sum of the variances:

$$= \sum_{\pi_n} \mathrm{var}\left[\Delta B(t_k^n)^2 - \Delta t_k^n\right] = 2\sum_{\pi_n} (\Delta t_k^n)^2 \leq 2|\pi^n|.(b-a),$$

and since $|\pi_n| \to 0$ as $n \to \infty$, the former goes to 0.
The case stated in the proposition is for $a = 0, b = t$.

Corollary 2

1.-If $1 \leq p < 2$ the p-variation of the BM is a.s. infinite.
2.- The total variation of the paths of BM is a.s infinite.

Idea: We will work the case $(a, b) = (0, t)$ and we suppose that B has finite p-variation for some $p \in [1, 2)$ and let $\delta = 2 - p > 0$,

$$\sum_{\pi_{n_k}} |B_{t^n_{k+1}} - B_{t^n_k}|^2 = \sum_{\pi_{n_k}} |B_{t^n_{k+1}} - B_{t^n_k}|^{2-\delta} |B_{t^n_{k+1}} - B_{t^n_k}|^{\delta}$$

$$\leq \sup_{\pi_{n_k}} |B_{t^n_{k+1}} - B_{t^n_k}|^{\delta} . \sum_{\pi_n} |B_{t^n_{k+1}} - B_{t^n_k}|^{p}.$$

This inequality leads to a contradiction.

Stochastic Itô Integral.

We return to the example $\int_0^t BdB$:

Lemma 1

Let $(\pi_n)_n$ be sequence of a partitions of $[0,t]$ with the same notations and $\lambda \in [0,1]$ fixed. We define

$$R_n(\lambda) = \sum_{\pi_n} B_{r_k^n} \Delta B(t_k^n)$$

with $r_k^n = (1-\lambda)t_k^n + \lambda t_{k+1}^n$. Then

$$L_2 - \lim_{n\to\infty} R_n(\lambda) = \frac{B_t^2}{2} + (\lambda - 1/2)t.$$

This means that the limit of the Riemann sum depends on the intermediate point r_k^n selected and there is a rule to choose it.

An easy (but long computation) shows that for a partition $\pi_n := \{0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t\}$, R_n can be written as $R_n = \frac{B_t^2}{2} + A_n + B_n + C_n$ with

- $A_n = -1/2 \sum_{\pi_n} (\Delta B(t_k^n))^2$, and $A_n \to -t/2$ in the L_2 norm.
- $B_n = \sum_{\pi_n} (B_{r_k^n} - B_{t_k^n})^2$, and $B_n \to \lambda t$ in the L_2 norm
- $C_n = \sum_{\pi_n} (B_{t_{k+1}^n} - B_{r_k^n})(B_{r_k^n} - B_{t_k^n})$ and using properties of the independence of the increments, we get $\mathbb{E}(C_n^2) \to 0$

Stochastic Itô Integral.

We will do in detail the case $\lambda = 0$, i.e. $r_k^n = t_k^n$:

$$R_n(0) = \sum_{\pi_n} B_{t_k^n} \Delta B(t_k^n) = \frac{1}{2} \sum_{\pi_n} \left[(B^2_{t^n_{k+1}} - B^2_{t^n_k}) - (B_{t^n_{k+1}} - B_{t^n_k})^2 \right]$$

the first term is a telescopic sum and is equal to $\frac{B_t^2}{2}$ and we already know that the second converges in L_2 to

$$-\frac{t}{2}$$

and this gives rise to the **Itô Stochastic Integral** , which is the one we will develop here and what we just computed is .

$$(B \cdot B)_t := \int_0^t B dB = \int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2}$$

The case $\lambda = 1/2$ according to the lemma is equal to:

$$R_n(1/2) \to \frac{B_t^2}{2},$$

which is also used and is called the **Stratonovich Integral**.
Observe that:$(\frac{B_t^2}{2} - \frac{t}{2},\ \ t \geq 0)$ is a martingale while the Stratonovich integral $\frac{B_t^2}{2}$, is not.

For technical reasons we will work with the completed canonical filtration for the BM, $(\mathfrak{F}_t)_t$.

Definition 5

Given a filtered space $(\Omega, \mathfrak{F}, (\mathfrak{F}_{\mathrm{t}})_{\mathrm{t}\in\mathrm{J}}, \mathbb{P})$*, and* $0 \leq a < b$ *the class* $\Gamma_2[a, b]$ *is the set of processes* $\{F : [0, \infty) \times \Omega \to \mathbb{R}\}$ *such that*

- F *is progressively measurable*
- $\mathbb{E}\left[\int_a^b F(s, \omega)^2 ds\right]$ *is finite.*

Stochastic Itô Integral.

Definition 6

A process G *defined in* $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \in J}, \mathbb{P})$ *is called* **elementary processes** *if it belongs to* $\Gamma_2[0,T]$ *and there exists a partition* $\pi = \{0 = t_0 < t_1 < \cdots < t_m = T\}$ *and random variables* $G_k : \Omega \to \mathbb{R}, \;\; k = 0, 1, 2 \ldots m-1$ *such that*

$$G(t,\omega) = G_k(\omega), \;\; \textit{for} \;\; t \in [t_k, t_{k+1}), \quad k = 0, 1, \ldots, m-1.$$

The set of elementary processes in $[0,T]$ *will be denoted* $\mathcal{E}[0,T]$.

The **Itô stochastic Integral** of an elementary process G (with the former notation) on the interval $[u,t] \subset [0,T]$ is defined as

$$\int_u^t GdB = G_{i-1}(B_{t_i} - B_u) + \sum_{k=i}^{n-1} G_k(B_{t_{k+1}} - B_{t_k}) + G_n(B_t - B_{t_n}),$$

where $i < n$ are such that
$u \in [t_{i-1}, t_i), \;\; t \in [t_n, t_{n+1}), \;\; i \leq (n+1) \leq m.$

Stochastic Itô Integral.

Observe

- $\mathbb{E}(G_k^2) < \infty$ and G_k is $\mathfrak{F}_{t_k}$-measurable, $k = 1, 2 \ldots m-1$ since the process is adapted (analogy with the betting strategy for RW).
- if $i = n$ i.e. if both points u, t belong to the same interval of the partition then $\int_u^t GdB = G_{i-1}(B_t - B_u)$
- $(G \cdot B)_t$ is an usual notation for the stochastic integral $\int_0^t GdB$.
- we can also write: $\int_u^t GdB = (G\mathbf{1}_{[u,t]} \cdot B)_T$
- $\int_0^T GdB$ is also the martingale transform of $(G_k)_k$ (seen as a discrete process) w.r.t. the martingale $(X_n = B_{t_n}, n = 1, 2, \ldots m)$

Notice that we could also have defined a elementary process without using the space $\Gamma_2[0, T]$.

Stochastic Itô Integral.

Proposition 10

For $G, G_1, G_2 \in \mathcal{E}[0,T]$, the stochastic integral

- *is additive:* $\int_u^t G dB = \int_u^s G dB + \int_s^t G dB.\ 0 \le u < s < t < T$
- *is linear:*
  $\int_0^t (aG_1 + bG_2) dB = a \int_0^t G_1 dB + b \int_0^t G_2 dB, \quad a, b \in \mathbb{R}$
- *verifies* $\mathbb{E}(\int_0^t G dB) = 0.$
- *the stochastic integral as a process:*
  $(Z_t := \int_0^t G dB : t \in [0,T])$ *is a continuous square integrable martingale.*

Proposition 11

For any elementary process G (with the notations of definition **??***) and $u, t \in [0,T], \;\; u < t$,*

$$\mathbb{E}\left[(\int_u^t G dB)^2 \right] = \mathbb{E}\left[\int_u^t G^2 ds \right]$$

Proposition 11

*For any elementary process G (with the notations of definition **??**) and $u, t \in [0, T], \;\; u < t$,*

$$\mathbb{E}\left[(\int_u^t G dB)^2 \right] = \mathbb{E}\left[\int_u^t G^2 ds \right]$$

We do it in the case $u = 0, t = T$ and use the notation $\Delta B_k := B(t_{k+1}) - B(t_k)$.

- $\mathbb{E}\left[\Delta B_k\right] = 0$ and
- $\mathbb{E}\left[(\Delta B_k)^2\right] = (t_{k+1} - t_k)$,
- $k \neq j, j < k$.

$$\mathbb{E}\left[G_k G_j \Delta B_k \Delta B_j\right] = \mathbb{E}\left[G_k G_j \Delta B_j\right] \mathbb{E}\left[\Delta B_k\right] = 0,$$

- $k = j$,

$$\mathbb{E}\left[G_k^2 (\Delta B_k)^2\right] = \mathbb{E}\left[G_k^2\right] \mathbb{E}\left[(\Delta B_k)^2\right] = \mathbb{E}(G_k^2)(t_{k+1} - t_k)$$

Now we use these results to compute:

$$\mathbb{E}\left[(\int_0^T GdB)^2\right] = \sum_{j,k} \mathbb{E}\left[G_k G_j \Delta B_k \Delta B_j\right] = \sum_k \mathbb{E}(G_k^2)(t_{k+1} - t_k)$$

which is equal to $\mathbb{E}\left[\int_0^T G^2 ds\right]$. This proves the Itô isometry for elementary processes.
For the case $u, t \in [0, T]$ we can do exactly the same with the modified partition $\pi = \{u = t_0 < t_1 < \cdots < t_n = t\}$ or else do it for the elementary processes $G\mathbf{1}_{[u,t]}$. □
Observe that $\mathbb{E}\left[\int_O^T G^2 ds\right]$ is the $L_2([0, T] \times \Omega)$-norm of G.

The class of elementary processes $\mathcal{E}[0,T]$ is a subset of the product space $L_2(([0,T]\times\Omega),\mathfrak{A},\mu)$ where $\mathfrak{A}$ is the product sigma algebra $\mathcal{B}([0,T])\otimes\mathfrak{F}$ and μ is the product measure $\lambda\times\mathbb{P}$, with λ the Lebesgue measure in $\mathcal{B}([0,T])$.

Theorem 5

The closure of $\mathcal{E}[0,T]$ with respect to the norm in the product space $L_2(([0,T]\times\Omega)$ is $\Gamma_2[0,T]$.

Idea of the proof: the measurability conditions are preserved since an L_2- limit admits a subsequence that is a.s. convergent.
Now , given a process X in $\Gamma_2[0,T]$ the method to find the sequence of elementary processes that converge in the product space to X relies on a deterministic lemma:

Lemma 2

Deterministic lemma: In the space $L_2[0,T]$ *of functions* $f:[0,T]\to\mathbb{R}$ *we define the operator* $P_n: L_2[0,T]\to L_2[0,T]$ *by*

$$P_n f(t)=\sum_{j=1}^{n}\left[\xi_{j-1}\mathbf{1}_{[t_{j-1},t_j)}\right],\quad \xi_{j-1}=\frac{n}{T}\int_{t_{j-2}}^{t_{j-1}}f(s)ds$$

with $(t_j)_{j=0}^{n}$ *the uniform partition of* $[0,T]$*:* $(t_j=\frac{jT}{n})$*. Then*

1- *for* ϕ *a step function in* $L_2[0,T]$, $\lim_n|P_n\phi-\phi|_2=0$,

2- *the set of step functions is a dense subset of* $L_2[0,T]$

3- *if* $\psi\in L_2[0,T]$ *then* $\lim_n|P_n\psi-\psi|_2=0$.

Lemma 3

Deterministic lemma: In the space $L_2[0,T]$ of functions $f:[0,T]\to\mathbb{R}$ we define the operator $P_n : L_2[0,T]\to L_2[0,T]$ by

$$P_n f(t) = \sum_{j=1}^{n}\left[\xi_{j-1}\mathbf{1}_{[t_{j-1},t_j)}\right],\ \ \xi_{j-1} = \frac{n}{T}\int_{t_{j-2}}^{t_{j-1}} f(s)ds$$

with $(t_j)_{j=0}^n$ the uniform partition of $[0,T]$: $(t_j = \frac{jT}{n})$. Then

1- *for ϕ a bounded continuous function in $[0,T]$,* $\lim_n |P_n\phi - \phi|_2 = 0$,

2- *the set of bounded continuous functions is a dense subset of* $L_2[0,T]$

3- *if $\psi \in L_2[0,T]$ then* $\lim_n |P_n\psi - \psi|_2 = 0$.

Given an element F of $\Gamma_2[0,T]$, we can suppose that it is bounded (truncation method). And then for each $\omega \in \Omega$ we construct $P_nF(t,\omega)$, and verify that it is an elementary process. It converges for each ω fixed to $F(\cdot,\omega)$ in the $L_2[0,T]$-norm, i.e.

$$\lim_n \int_0^T |P_nF(s,\omega) - F(s,\omega)|^2 ds = 0.$$

Since everything is bounded this is also true when taking expectation:

$$\lim_n \mathbb{E}\left[\int_0^T |P_nF(s,\omega) - F(s,\omega)|^2 ds\right] = 0. \tag{5}$$

We call $G_n = P_n F \in \Gamma_2[0,T]$. Thanks to equation (5), $(G_n)_n$ is a Cauchy sequence in the product space $L_2([0,T] \times \Omega)$ and thanks to the Itô isometry for elementary processes, for any $t \in [0,T]$, the sequence $(\int_0^t G_n dB)_n$ is a Cauchy sequence in $L_2(\Omega, \mathfrak{F}, \mathbb{P})$:

$$\mathbb{E}\left[|\int_0^t G_n dB - \int_0^t G_m dB|^2\right] = \mathbb{E}\left[|\int_0^t (G_n - G_m) dB|^2\right]$$
$$= \mathbb{E}\left[\int_0^t (G_n - G_m)^2 dt\right]$$

Stochastic Itô Integral.

Since $L_2(\Omega, \mathfrak{F}, \mathbb{P})$ is a complete metric space (even a Hilbert space), the $L_2(\Omega, \mathfrak{F}, \mathbb{P})$ limit of $(\int_0^t G_n dB)_n$ exists and is by definition **the stochastic integral** of F:

$$(F \cdot B)_t = \int_0^t F dB = L_2 - \lim_n \int_0^t G_n dB,$$

and the limit does not depend on the approximating sequence.

Corollary 3

The Itô Isometry in $\Gamma_2[0,T]$: For any process $F \in \Gamma_2[0,T]$,and $t \in [0,T]$,

$$\mathbb{E}\left[(\int_0^t FdB)^2 \right] = \mathbb{E}\left[\int_0^t F(s)^2 ds \right]$$

and if $F_n, F \in \Gamma_2[0,T]$ $n \in \mathbb{N}$ are such that

$$\lim_{n\to\infty} \mathbb{E}[\int_0^t |F_n - F|^2 dt] = 0$$

then $(\int_0^t F_n dB)_n$ converges in $L_2\left(\Omega, \mathfrak{F}, \mathbb{P}\right)$ to $\int_0^t FdB$.

The same properties we had in $\mathcal{E}[0,T]$ hold true in $\Gamma_2[0,T]$:

Theorem 6

If $F, G \in \Gamma_2[0,T], \;\; A \in \mathbb{R}$, *and* $0 \leq a < b < c \leq T$,

- *additivity* $\int_a^c F dB = \int_a^b F dB + \int_b^c F dB$ *a.s.*

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- *linearity* $\int_a^c (AF + G) dB = A \int_a^c F dB + \int_a^c G dB$
- $\mathbb{E}(\int_a^c F dB) = 0$
- $\int_0^t F dB$ *is* $\mathfrak{F}_t$ *measurable, for all* $t \in [0,T]$
- $((f \cdot B)_t = \int_0^t F dB : t \in [0,T])$ *is a continuous square integrable martingale.*

The proof is based in the fact that these results are true for elementary processes. The last one needs special attention:

Theorem 7

For $F \in \Gamma_2[0,T]$, the process defined by

$$(t,\omega) \to \int_0^t F dB, \quad t \in [0,T]$$

admits a continuous version and it is a square integrable martingale.

Proof:Let (G_n) be the sequence of elementary process that approximate F in $\Gamma_2[0,T]$ and we take any $t \in [0,T]$.

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Proof:Let (G_n) be the sequence of elementary process that approximate F in $\Gamma_2[0,T]$ and we take any $t \in [0,T]$.
Each $\int_0^t G_n dB$ is a continuous martingale, and so $\int_0^t G_n dB - \int_0^t G_m dB$ is also a continuous martingale for any $n, m \in \mathbb{N}$.

Thanks to the Doob maximal inequality for martingales (theorem 1) :

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq t\leq T}\left|\int_0^t G_n dB - \int_0^t G_m dB\right| > \epsilon\right) \\ \leq \frac{1}{\epsilon^2}\,\mathbb{E}\left(\left|\int_0^t G_n dB - \int_0^t G_m dB\right|^2\right) \\ = \frac{1}{\epsilon^2}\,\mathbb{E}\left(\int_0^t (G_n - G_m)^2 ds\right) \to 0, \quad n, m \to \infty.\end{aligned}$$

So we can find a subsequence (n_k) such that

$$\mathbb{P}(A_k) := \mathbb{P}\left[\sup_{0\le t\le T}\left|\int_0^t G_{n_{k+1}}dB - \int_0^t G_{n_k}dB\right| > 2^{-k}\right] \le 2^{-k}$$

Stochastic Itô Integral: the Itô integral as a process.

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And by the Borel Cantelli lemma

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This means that for almost all $\omega \in \Omega$ there exists $k_1(\omega)$ such that for $k \ge k_1$

$$\sup_{0\le t\le T}\left|(\int_0^t G_{n_{k+1}}dB - \int_0^t G_{n_k}dB)(\omega)\right| \le 2^{-k}, \quad k \ge k_1$$

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So a.s we get uniform convergence. Since $t \to \int_0^t G_n dB$ is continuous, the limit is also continuous. It is a martingale because it is the L_2-limit of martingales. It is clearly square integrable because of the Itô isometry.

As we have seen the computation of stochastic integrals is not easy. The Itô formula allows us to compute many more and it is fundamental in the theory, and is the main tool for the study of SDE. We will give the main ideas of two classical forms of proving the theorem.
The first approach to the simples Itô formula is using the Taylor expansion:

Itô Formulas

Theorem 8

Given $f : \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^2$, a real Brownian motion B, for any $t \in (0, T]$,

$$f(x + B_t) = f(x) + \int_0^t f'(x + B_s)dB_s + \frac{1}{2}\int_0^t f''(x + B_s)ds$$

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We will do the case when f, f' and f'' are bounded by a constant $M \in \mathbb{R}^+$ and are uniformly continuous. Let π_n be a partition of $[0, t]$ given by $\{0 = t_0^n \leq t_1^n, \ldots \leq t_{m_n}^n = t\}$. We can use the Taylor formula to obtain

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$$f(x + B_t) = f(x) + \sum_{\pi_n} f'(x + B_{t_j^n})[B_{t_{j+1}^n} - B_{t_j^n}]$$

$$+1/2 \sum_{\pi_n} f''(x + B_{t_j^n})[B_{t_{j+1}^n} - B_{t_j^n}]^2 + R_n$$

We have to analyze the error term R_n, given by :

$$R_n = \sum_{\pi_n} \left(f''(x+\xi_j^n) - f''(x+B_{t_j^n})\right)[B_{t_{j+1}^n} - B_{t_j^n}]^2$$

where ξ_n^j is a (random) point in the (non ordered) interval $(B_{t_j^n}(\omega), B_{t_{j+1}^n}(\omega))$.

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where ξ_n^j is a (random) point in the (non ordered) interval $(B_{t_j^n}(\omega), B_{t_{j+1}^n}(\omega))$.
We also use the notations:

$$A_n = \sum_{\pi_n} f''(x+B_{t_j^n})[B_{t_{j+1}^n} - B_{t_j^n}]^2, \qquad B_n = \sum_{\pi_n} f''(x+B_{t_j^n})[t_{j+1}^n - t_j^n]$$

$$L_n(j) = f''(x+B_{t_j^n})[(B_{t_{j+1}^n} - B_{t_j^n})^2 - (t_{j+1}^n - t_j^n)],$$

$$\Delta B(t_j^n) = (B_{t_{j+1}^n} - B_{t_j^n}), j = 1, 2 \ldots (m_n - 1)$$

Itô Formulas

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For the first one:

$$|R_n| \leq \sum_{\pi_n} |f''(x+\xi_j^n) - f''(x + B_{t_j^n})| [\Delta B(t_j^n)]^2$$

$$\leq \sup_i |f''(x+\xi_j^n) - f''(x + B_{t_j^n})| \sum_{\pi_n} [\Delta B(t_j^n]^2$$

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$$\leq \sup_i |f''(x + \xi_j^n) - f''(x + B_{t_j^n})| \sum_{\pi_n} [\Delta B(t_j^n]^2$$

Using the Cauchy -Schwarz inequality

$$[\mathbb{E}(R_n)]^2 \leq \mathbb{E}\left[\{\sup_j |f''(x+\xi_j^n) - f''(x+B_{t_j^n})|\}^2\right] \mathbb{E}\left[\{\sum_{\pi_n}[\Delta B(t_j^n)]^2\}^2\right]$$

The uniform continuity of $s \to f''(x + B_s)$, the DCT and the L_2-convergence of the quadratic variation allows us to conclude that R_n converges to 0 in $L_1(\Omega)$, and also in probability. Then by the generalized DCT we get convergence in $L_2(\omega)$.

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3.-is immediate from the definition of the usual Riemann integral. Then the convergence in L_2 is a consequence of the DCT.

We now prove 2 . We will see that

$$\mathbb{E}\left[(A_n - B_n)^2\right] \leq 2M^2 \sum_{\pi_n} (t^n_{j+1} - t^n_j)^2$$

Itô Formulas

$$\mathbb{E}\left[(A_n - B_n)^2\right] = \mathbb{E}\left[[\sum_{\pi_n} L_n(j)]^2\right] = \mathbb{E}\left[\sum_{j,k} L_n(j) L_n(k)\right] =$$

$$= \mathbb{E}\left[\sum_j L_n(j)^2\right] \leq M^2\, \mathbb{E}\left[\sum_{\pi_n} \left((B_{t^n_{j+1}} - B_{t^n_j})^2 - (t^n_{j+1} - t^n_j)\right)^2\right]$$

Itô Formulas

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$$= \mathbb{E}\left[\sum_j L_n(j)^2\right] \leq M^2\,\mathbb{E}\left[\sum_{\pi_n} \left((B_{t^n_{j+1}} - B_{t^n_j})^2 - (t^n_{j+1} - t^n_j)\right)^2\right]$$

$$= 2M^2 \sum_{\pi_n} (t^n_{j+1} - t^n_j)^2$$

For the last equality we used the fact that the if $Y \approx \mathcal{N}(0, t^n_{j+1} - t^n_j)$, then $\mathbb{E}(Y^2) = 2(t^n_{j+1} - t^n_j)^2$.

Itô Formulas

The same method can be used to prove a similar result for time dependent functions: $f : (\mathbb{R}^+) \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}_{1,2}$,

$$f(t, x+B_t) = f(0,x) + \int_0^t f'_x(s, x+B_s)dB_s + \frac{1}{2}\int_0^t f''_{xx}(s, x+B_s)ds +$$

$$+ \int_0^t f'_t(s, x+B_s)ds.$$

examples

- for $f(x) = x^m, \ \ m \geq 2$ then
  $(B_t)^m = \int_0^t mB_s^{m-1} dB_s + 1/2m(m-1) \int_0^t B_s^{m-2} ds.$
  So this tells us that
  $\int_0^t B_s^{m-1} dB_s = \frac{1}{m}(B_t)^m - \frac{1}{2}(m-1) \int_0^t B_s^{m-2} ds.$
- for $g(t,x) = tx$, $tB_t = \int_0^t s dB_s + \int_0^t B_s ds.$ This can be done , using the Itô formula in the time dependent case , or it can be proved directly.
- for $u(t,x) = \exp(\lambda x - \frac{\lambda^2 t}{2})$ and defining the process $Y_t = u(t, B_t), \ \ t \geq 0$ we get

$$\begin{cases} dY = \lambda Y dB \\ Y_0 = 1 \end{cases} \tag{6}$$

which is a stochastic differential equation.

Itô Processes

Definition 7 (Itô Processes.)

Given $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in J}, \mathbb{P})$ *, an Itô process is a process of the form*

$$X_t = X_0 + \int_0^t A(\omega, s)ds + \sum_{j=1}^{n} \int_0^t G_j(\omega, s)dB_s^j$$

with

- $G_j \in \Gamma_2[0, T]$ $j = 1, 2 \ldots n$*,*
- A *is a progressive integrable process,*
- $X_0 \in L_2(\Omega)$ *and* $\mathfrak{F}_0$*-measurable*
- $B_t = (B_t^{(1)}, B_t^{(2)}, \ldots B_t^{(n)})$ *an* $\mathbb{R}^n$ *-valued Brownian motion.*

In differential form this is written as :

$$dX_t = Ads + \sum_j G_j dB^{(j)}$$

This family of processes is a subspace of the set of continuous **semimartingales**, it is the sum of continuous martingales $\int_0^t A_j(\omega, s) dB_s$ and a bounded variation continuous process $\int_0^t b(\omega, s) ds$. We want to see if we have also an Itò formula for this family of processes, i.e. if we can compute $f(X_t)$ for functions f (with enough regularity conditions). Notice that $(f(X_t) : t \in [0, T])$ is a new process, and we will see that it will also be an Itô process.

Theorem 9

Suppose that X has a stochastic differential

$$dX = A\,dt + G\,dB$$

and that $U : \mathbb{R} \times [0,T] \to \mathbb{R}$ is of class $\mathcal{C}_{1,2}$, then $Y(t) := U(X(t),t)$ has stochastic differential given by

$$dU(X,t) = U_t\,dt + U_x\,dX + \frac{1}{2}U_{xx}G^2\,dt =$$

$$= (U_t + AU_x + \frac{1}{2}U_{xx}G^2)\,dt + U_x G\,dB$$

Integration by parts formula

The former theorem can be proved using Theorem 8 or can be proved using the Integration by parts formula:

Theorem 10

Suppose that X_1, X_2 have a stochastic differential

$$dX_1 = A_1\,dt + G_1\,dB$$

$$dX_2 = A_2\,dt + G_2\,dB$$

with $G_1, G_2 \in \Gamma_2$ and A_1, A_2 progressive and integrable. Then for $s < t$,

$$\int_s^t X_2\,dX_1 = X_1(r)X_2(r) - X_1(s)X_2(s) - \int_s^t X_1\,dX_2 - \int_s^r G_1G_2\,dt$$

This formula is proved first for $\mathfrak{F}_0$-measurable r.v, then for elementary processes and finally for processes in $\Gamma_2[0,T]$.

Definition 8

An $\mathfrak{M}^{n\times m}$-valued process , **G**$= (G_{i,j})_{i=1,\dots n, j=1,\dots m}$ *belongs to* $\Gamma_2^{n\times m}[0,T]$ *if each* $G_{i,j} \in \Gamma^2[0,T]$.
An $\mathbb{R}^n$ *valued process* **F**$= (F_i)_{i=1,\dots,n})$, *belongs to* $\Gamma_1^n[0,T]$ *if each component* F_i *is progressive and verifies* $\mathbb{E}\int_0^T |F_i(s)|ds$ *is finite.*

In this case we can also construct

$$\int_0^T \mathbf{G} d\mathbf{B}_t.$$

(here **B** is an m-dimensional Brownian motion.) We also have multidimensional Itô processes and Itô Chain rule.

SDE

We like to find solutions to stochastic differential equations of the form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_s$$

Some examples are the linear equations:

- $dX_t = -aX_t dt + \sigma dB_t$ The solution is the Ornstein-Uhlenbeck process

$$X_t = X_0 e^{-at} + \sigma \int_0^t e^{-a(t-s)} dB_s.$$

  Since it is the stochastic integral of a deterministic function, we see that is is a Gaussian process: the Riemann approximations of the stochastic integrals is gaussian, so is the limit. We can easily compute the mean function and the covariance matrix of this Gaussian process.

- The Vasicek model:$dX_t = (-aX_t + b)dt + \sigma dB_t$. has a solution given by

$$X_t = X_0 e^{-at} + b/a(1 - e^{-at})\sigma \int_0^t e^{-a(t-s)} dB_s.$$

- $dZ_t = Z_t \phi(t) dB_t$ with ϕ bounded and progressive. The solution is

$$Z_t = Z_0 \exp\left[\int_0^t \phi(s) dB_s - \frac{1}{2} \int \phi(s)^2 ds\right]$$

- $dZ_t = Z_t \left[\mu(t)dt + \phi(t)dB_t\right]$ where μ and ϕ are bounded and progressive and $Z_0 = 1$. The solution is

$$Z_t = \exp\left[\int_0^t \mu(s) ds + \int_0^t \phi(s) dB_s - \frac{1}{2} \int \phi(s)^2 ds\right]$$

What do we mean by a solution?? A solution X of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_s, \quad X_0 = X(0) \;\; t \in [0, T]$$

is a progressive process , b and σ progressive, such the $b \in L_1(\Omega, \mathfrak{F}, \mathbb{P})$ and $\sigma \in \Gamma_2[0, T]$ and for all times $t \in [0, T]$,

$$X_t = X_0 + \int_0^t b(t, X_t)dt + \int_0^t \sigma(t, X_t)dB_s.$$

Stratonovich Integral

$$\int_0^t H \bullet dB = \int_0^t H dB + \frac{1}{2} \int_0^t H_x dt$$

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