# RANDOM FRAGMENTATIONS

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Fragmentation is a natural phenomenon that can be observed in many sciences, at a great variety of scales. To give just a few examples, let us simply mention, the studies of stellar fragments in astrophysics, fractures and earthquakes in geophysics, breaking of crystals in crystallography, degradation of large polymer chains in chemistry, DNA fragmentation in biology, fission of atoms in nuclear physics, fragmentation of a hard drive in computer science, ...

The main purpose of this short course is to develop a mathematical model which may be used in situations where either phenomenon occurs randomly and repeatedly as time passes. For instance, we can think of the evolution of blocks of mineral in a crusher. In order to deal with models that can be studied mathematically, we are led to make hypotheses which may look at first sight somewhat stringent, but that are however, commonly assumed in applications. First, we suppose that the system has a memoryless evolution, that is its future only depends on its present state and not on its past. In particular, this excludes the possibility that an object might be more fragile (i.e. more likely to split) due to former shocks. Second, we assume that each fragment can be characterized by a real number that should be thought of as its size. This stops us from considering the spatial position of a fragment or further geometrical properties like its shape; physicists call such models *mean field*. Finally, we shall always suppose that the evolution of a given fragment does not depend on its environment, in the sense that fragments split independently of each other, or in other words, that the branching property is fulfilled.

Informally, imagine an object that falls apart randomly as time passes. The state of the system at some given time consists in the sequence of the sizes of the pieces, which are often called fragments or particles. Suppose that the evolution is Markovian and obeys

the following rules. First, different particles evolve independently of each other, that is the so-called branching property is fulfilled. Second, there is a parameter $\alpha \in \mathbb{R}$, which will be referred to as the index of self-similarity, such that each fragment with size $s$ is stable during an exponential time with parameter proportional to $s^\alpha$. In other words, a particle with size $s > 0$ has an exponential lifetime with mean $cs^{-\alpha}$, where $c > 0$ is some constant. At its death, this particle splits and there results a family of fragments, say with sizes $(s_i, i \in \mathbb{N})$, where the sequence of ratios $(s_i/s, i \in \mathbb{N})$ has the same distribution for all particles. The purpose of this course is to construct such self-similar fragmentation chains, to shed light on their genealogical structure, and to establish some of their fundamental properties.

# 1 Construction of fragmentation chains

In this section, we briefly present some basic elements on Markov chains and branching Markov chains in continuous time which are then used for the construction and the study of fragmentation chains. For convenience, we recall first some standard notation for sets of integers which will be used through this text without further reference.

The sets of positive integers, and respectively of integers, are denoted by

$$\mathbb{N} = \{1, 2, \ldots\}\,, \quad \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}\,,$$

and then

$$\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$$

designates the set of non-negative integers. When we shall need to consider infinity as an extended integer, we shall use the notation

$$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}\,, \quad \overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}\,.$$

## 1.1 Preliminaries on Markov chains

Let $(E, \mathrm{d})$ be a Polish space, that is a complete separable metric space, which we also endow with its Borel sigma-field. Consider a collection $(q(x, \cdot), x \in E)$ of finite measures on $E$ which is (weakly) measurable in the variable $x$, in the sense that for every Borel set $B \subseteq E$, the map $x \to q(x, B)$ is measurable. It is well-known that we can use the kernel $(q(x, \cdot), x \in E)$ as the jump rates of some Markov chain in continuous time. Let us briefly recall the main steps and refer, for example, to Norris [65] or Fristedt and Gray [40] for more details.

For every $x \in E$, we write $q(x) := q(x, E)$ for the total mass of the measure $q(x, \cdot)$, and we introduce the normalized probability measure on $E$ given by

$$\bar{q}(x, \cdot) = q(x, \cdot)/q(x)$$

with the convention that $\bar{q}(x, \cdot) = \delta_x$ is the Dirac point mass at $x$ when $q(x) = 0$. So $(\bar{q}(x, \cdot), x \in E)$ is a Markov kernel, that is a (weakly) measurable family of probability measures on $E$. We can think of $(\bar{q}(x, \cdot), x \in E)$ as the transition probabilities of a Markov

sequence [1] $Y = (Y(n), n \in \mathbb{Z}_+)$. That is, for every $n \in \mathbb{Z}_+$,

$$\mathbb{P}\left(Y(n+1) \in \cdot \mid Y(0), \ldots, Y(n)\right) \,=\, \bar{q}(Y(n), \cdot)\,.$$

Next, we shall transform the Markov sequence $Y$ into a Markov process $X = (X(t), t \geq 0)$ in continuous time which visits the same states as the sequence $Y$. More precisely, conditionally on the sequence $Y = (y_n, n \in \mathbb{Z}_+)$, we shall replace the unit waiting time at each step $y_n$ by an exponential variable with parameter $q(y_n)$ (thus depending on the state of $Y$ at this step), independently of the other steps. The construction is specified by the following procedure.

Let $\mathbf{e}_0, \mathbf{e}_1, \ldots$ be a sequence of i.i.d. standard exponential variables, which is independent of $Y$. We associate to every sample path of $Y$ the additive functional

$$A(n) \,:=\, \sum_{i=0}^{n} \mathbf{e}_i / q(Y(i))\,, \qquad n \in \mathbb{Z}_+\,,$$

which represents the instant at which $X$ jumps from the state $Y(n)$ to the state $Y(n+1)$. This procedure enables us to define $X(t)$ for any $t \geq 0$ if and only if the series $A(\infty) := \sum_{i=0}^{\infty} \mathbf{e}_i / q(Y(i))$ diverges. In this direction, we recall the following well-known fact.

**Lemma 1** *The conditions*

$$A(\infty) \,:=\, \sum_{i=0}^{\infty} \mathbf{e}_i / q(Y(i)) \,=\, \infty \qquad a.s. \tag{1}$$

*and*

$$\sum_{i=0}^{\infty} 1/q(Y(i)) \,=\, \infty \qquad a.s. \tag{2}$$

*are equivalent.*

*Proof* We shall prove a slightly stronger result. Let $(y_i, i \in \mathbb{Z}_+)$ be some sequence of points in $E$, which should be thought of as the sequence of the states visited by $Y$. On the one hand, the identity

$$\mathbb{E}\left(\sum_{i=0}^{\infty} \mathbf{e}_i / q(y_i)\right) \,=\, \sum_{i=0}^{\infty} 1/q(y_i)$$

[1] In the literature, Markov sequences are often called Markov chain in discrete time. However, in order to avoid a possible confusion with Markov chains in continuous time, which are the main object of interest in this section, we shall keep the terminology *chain* for processes in continuous time, and use *sequence* for processes in discrete time. We also mention that in the literature, Markov chains in continuous time generally concern only countable state spaces; some authors prefer to refer to pure-jump Markov processes in the case of a general topological state space. Nonetheless, we shall use here the name *chain* to underline the hold-jump structure (which will be described below), and keep the name *process* for continuous evolutions, that is situations where the process may not remain constant on arbitrarily small time-interval.

shows that if the series on the right-hand side converges, then $\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i) < \infty$ a.s. Conversely, taking the Laplace transform, we get

$$\begin{aligned} \mathbb{E}\left(\exp\left(-\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i)\right)\right) &= \prod_{i=0}^{\infty} \frac{q(y_i)}{1+q(y_i)} \\ &= \exp\left(-\sum_{i=0}^{\infty} \ln\left(1+1/q(y_i)\right)\right). \end{aligned}$$

If the series $\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i)$ converges with positive probability, then the right-hand side above has to be strictly positive and hence $\sum_{i=0}^{\infty} 1/q(y_i) < \infty$. Note from the first part of the proof that this forces the series $\sum_{i=0}^{\infty} \mathbf{e}_i/q(y_i)$ to converge with probability one, a fact that can also be observed directly from Kolmogorov's 0-1 law. □

Condition (2) is plainly fulfilled whenever

$$\sup_{x\in E} q(x) < \infty\,; \tag{3}$$

however, in general checking whether (2) holds can be tedious. Henceforth, taking (1) for granted, we may introduce the time-change

$$\alpha(t) \,=\, \min\left\{n \in \mathbb{Z}_+ : A(n) > t\right\}, \qquad t \geq 0;$$

one says that $\alpha(\cdot)$ is the right-continuous inverse of the additive functional $A(\cdot)$. Then we define a process in continuous time $X = (X(t), t \geq 0)$ by the identity

$$X(t) \,:=\, Y(\alpha(t)), \qquad t \geq 0\,.$$

This construction by random time-substitution can be rephrased in terms of a so-called *hold-jump* description: the states $x \in E$ with $q(x) = 0$ are *absorbing* for $X$, that is

$$\mathbb{P}(X(t) = x \text{ for all } t \geq 0 \mid X(0) = x) = 1\,,$$

and starting from some non-absorbing state $x \in E$ with $q(x) > 0$, the process $X$ stays at $x$ up to the holding time $\mathbf{e}_0/q(x)$ which has an exponential distribution with parameter $q(x)$, and then jumps [2] according to the probability distribution $\bar{q}(x, \cdot)$, independently of the holding time. It is easily seen from the absence of memory of exponential variables that $X$ enjoys the Markov property; one says that $X$ is a Markov chain (in continuous time). Note also that $X$ has right-continuous paths a.s.

The semigroup $(P_t, t \geq 0)$ of $X$ is the family of linear operators on the space of bounded measurable functions $f : E \to \mathbb{R}$ defined by

$$P_t f(x) \,:=\, \mathbb{E}(f(X(t) \mid X(0) = x), \qquad x \in E;$$

it satisfies the *Chapman-Kolmogorov equation*

$$P_t \circ P_s \,=\, P_{t+s}\,, \qquad t, s \geq 0\,.$$

[2] When $q(x, \{x\}) > 0$, the probability that process $X$ stays at the state $x$ after the exponential holding time is positive, so strictly speaking there may be no jump after this first holding time. However, this induces no difficulty whatsoever, and it is convenient not to distinguish this degenerate case.

It is easy to check from the hold-jump description that for every bounded measurable function $f : E \to \mathbb{R}$,

$$\begin{aligned} \mathtt{G}f(x) &:= \lim_{t\to 0+} \frac{1}{t}\mathbb{E}(f(X(t)) - f(X(0)) \mid X(0) = x) \\ &= \int_E (f(y) - f(x))\, q(x, dy)\,, \end{aligned} \tag{4}$$

which identifies the infinitesimal generator $\mathtt{G}$ of $X$. In particular, combining with the Chapman-Kolmogorov equation yields the classical *backward* equation

$$\frac{dP_t f(x)}{dt} = \mathtt{G}P_t f(x)\,, \qquad t \geq 0\,. \tag{5}$$

Further, when the function $\mathtt{G}f$ is bounded on $E$, we also have the *forward* equation

$$\frac{dP_t f(x)}{dt} = P_t \mathtt{G}f(x)\,, \qquad t \geq 0\,. \tag{6}$$

A well-known alternative characterization of the infinitesimal generator is that for every bounded measurable function $f : E \to \mathbb{R}$ such that $\mathtt{G}f$ is bounded, $\mathtt{G}f$ is the unique bounded measurable function $g : E \to \mathbb{R}$ for which the process

$$f(X(t)) - \int_0^t g(X(s))ds\,, \qquad t \geq 0$$

is a martingale under $\mathbb{P}(\cdot \mid X(0) = x)$ for every $x \in E$.

Either the construction of $X$ or (4), shows that the family $(q(x, \cdot), x \in E)$ can be thought of as the *jump rates* of $X$, and thus entirely characterizes the distribution of the Markov chain $X$. In the same vein, note also that when the space $E$ is discrete and $q(x, \{x\}) = 0$, the jump rates of the chain can be recovered from its one-dimensional distributions by

$$q(x, \{y\}) = \lim_{t\to 0+} \frac{1}{t}\mathbb{P}(X(t) = y \mid X(0) = x)\,, \qquad x \neq y. \tag{7}$$

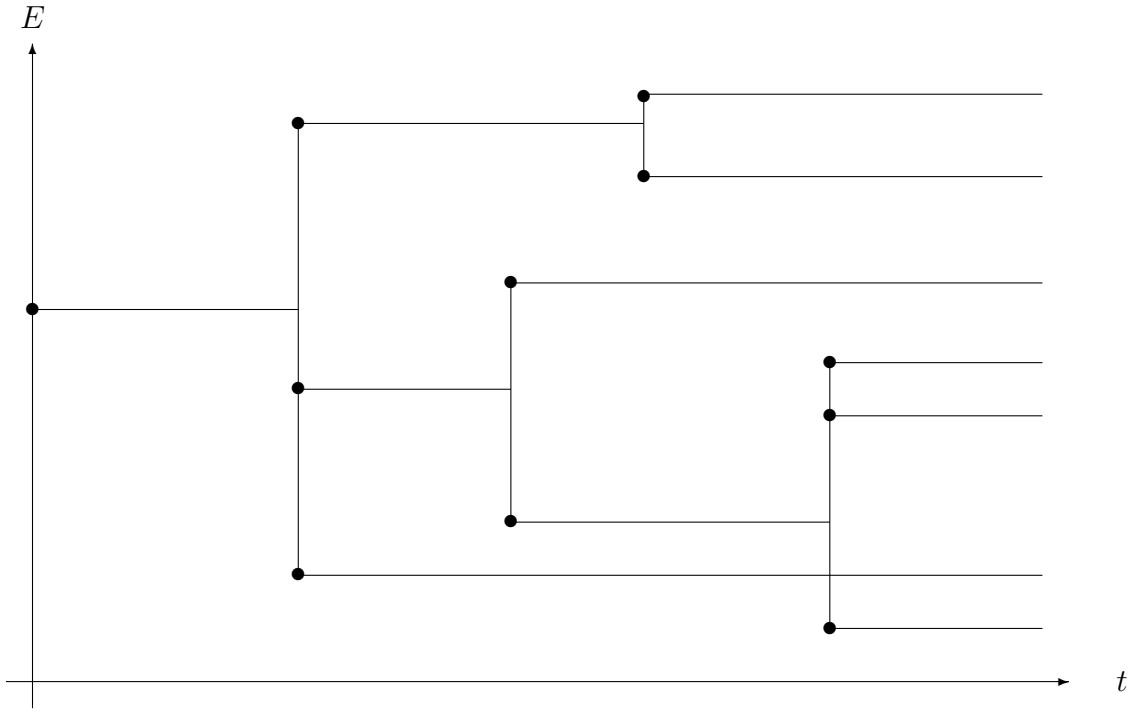
**Example** The so-called *compound Poisson processes* form one of the simplest and best known family of Markov chains in continuous time. Specifically, consider the special case when $E$ is some Euclidean space (or, more generally, some nice topological group) and the jump rates $(q(x, \cdot), x \in E)$ are translation invariant, that is for every $x \in E$, $q(x, \cdot)$ is the image of some given finite measure $\Lambda$ by the translation $y \to x + y$ (in particular $q(0, \cdot) = \Lambda$). The measure $\Lambda$ is known as the *Lévy measure* of the compound Poisson process. Plainly, the transition probabilities $(\bar{q}(x, \cdot), x \in E)$ are also translation invariant, and hence the Markov sequence $Y = (Y(n), n \in \mathbb{Z}_+)$ is a random walk with step distribution $\bar{\Lambda}(\cdot) := \Lambda(\cdot)/\Lambda(E)$. In other words, $Y(n) = Y(0) + \xi_1 + \cdots + \xi_n$, where the increments $\xi_i$ form a sequence of i.i.d. variables with law $\bar{\Lambda}(\cdot)$. Since $c := \Lambda(E) = q(x)$ does not depend on the state $x$, the continuous time Markov chain with jump rates $(q(x, \cdot), x \in E)$ can be obtained as the composition $X = Y \circ N$ where $N = (N_t, t \geq 0)$ is a Poisson process with intensity $c$ which is independent of the random walk $Y$. This construction triggers the name compound Poisson process for such Markov chains. More generally, it is easy to see that a Markov chain in continuous time can be constructed as a Markov sequence time-changed by an independent Poisson process if and only if the total jump rates are bounded, that is if and only if (3) holds.

## 1.2 Branching Markov chains

In this section, we shall consider an important class of Markov chains with values in a space of measures. First, we call a *finite point measure* on $E$ any measure of the type $m = \sum_{i=1}^{n} \delta_{x_i}$ where $n \in \mathbb{Z}_+$ and $x_i \in E$ ($x_i = x_j$ is allowed, and for $n = 0$, the measure $m = 0$ is trivial). We denote by $E_{\mathrm{p}}$ the space of finite point measures on $E$, endowed with the distance

$$\mathrm{dist}(m, m') := \sup \left| \int_E f(x)m(dx) - \int_E f(x)m'(dx) \right| ,$$

where the supremum is taken over the space of Lipschitz-continuous functions $f : E \to \mathbb{R}$ with $|f(x) - f(y)| \leq d(x, y)$ for every $x, y \in E$. It is easy to check that this distance is equivalent to the Prohorov metric on the space of finite measures on $E$, and that $(E_{\mathrm{p}}, \mathrm{dist})$ is a Polish space; see for example [27].




*Sample path of a branching Markov chain started from a single particle*

We shall think of the atoms of point measures as particles; see the picture above. Our goal is to construct a particle system whose dynamics can be described informally as follows. The system is non-interacting, that is different particles have independent evolutions. Each particle *branches*, that is it dies at some rate depending on its location, and at its death it is replaced by some random number of random particles, independently of the lifetime. More precisely, we consider a family $(\nu_x, x \in E)$ of finite measures on $E_{\mathrm{p}}$, which depends measurably on the variable $x$. A particle located at $x$ lives for an exponential time with parameter $\nu_x(E_{\mathrm{p}})$ (so it is immortal when $\nu_x(E_{\mathrm{p}}) = 0$). At its death, the particle is removed and replaced by new particles, say $y_1, \ldots, y_k$, where the finite point measure $\sum_{i=1}^{k} \delta_{y_i}$ is distributed according to the probability measure $\nu_x(\cdot)/\nu_x(E_{\mathrm{p}})$ on $E_{\mathrm{p}}$.

In order to construct such a particle system as a Markov chain on $E_{\mathrm{p}}$, we introduce some notation related to finite point measures. Given $m \in E_{\mathrm{p}}$, we denote by $\theta_m(\nu)$ the image of $\nu$ by the map $m' \to m + m'$. Then we associate to the family $(\nu_x, x \in E)$, a

measurable kernel $(q(m,\cdot), m \in E_{\rm p})$ of finite measures on $E_{\rm p}$ defined as follows. First, $q(0,\cdot) = 0$ and, second, if $m = \sum_{i=1}^{n} \delta_{x_i}$ with $n \geq 1$, we set

$$q(m,\cdot) \,:=\, \sum_{i=1}^{n} \theta_{m_i}(\nu_{x_i})\,, \tag{8}$$

where $m_i = \sum_{j\neq i} \delta_{x_j}$. The term $\theta_{m_i}(\nu_{x_i})$ in (8) corresponds to the rate at which the particle $x_i$ branches in the family $\{x_1, \ldots, x_n\}$. Note that the total mass $q(m) := q(m, E_{\rm p})$ is given by

$$q(m) \,=\, \sum_{i=1}^{n} \nu_{x_i}(E_{\rm p})\,,$$

and as a consequence, an exponential variable with parameter $q(m)$ can be thought of as the minimum of $n$ independent exponential variables with parameters $\nu_{x_1}(E_{\rm p}), \ldots, \nu_{x_n}(E_{\rm p})$. Observe also the important additivity property of this kernel, namely for every point measures $m, m' \in E_{\rm p}$ and every measurable functional $\Phi : E_{\rm p} \to \mathbb{R}_+$, there is the identity

$$\int_{E_{\rm p}} \Phi(m'+\mu) q(m, d\mu) + \int_{E_{\rm p}} \Phi(m+\mu) q(m', d\mu) \,=\, \int_{E_{\rm p}} \Phi(\mu) q(m+m', d\mu)\,. \tag{9}$$

We would like to use the kernel $(q(m,\cdot), m \in E_{\rm p})$ as jump rates of a Markov chain. In this direction, let us first describe the evolution of the Markov sequence $Y = (Y(i) : i \in \mathbb{Z}_+)$ with transition probabilities $\bar{q}(m,\cdot) = q(m,\cdot)/q(m)$ for $m \in E_{\rm p}$. As $q(0) = 0$, the measure 0 is an absorbing state for the sequence. Then, let the sequence start from some finite point measure $m = \sum_{i=1}^{n} \delta_{x_i} \neq 0$. The distribution of the next state of the sequence is obtained by picking an atom $x$ at random among $x_1, \ldots, x_n$, with probability proportional to $\nu_x(E_{\rm p})$, and replacing it by the atoms of a random point measure with law $\bar{q}(\delta_x, \cdot)$. Now, we should like to consider a continuous time Markov chain with jump rates $(q(m,\cdot), m \in E_{\rm p})$, so that we need conditions ensuring (2) (observe that the stronger condition (3) cannot hold).

**Lemma 2** *In the preceding notation,* (2) *is fulfilled whenever there is a finite constant* $c > 0$ *such that*

$$\int_{E_{\rm p}} \nu_x(dm) q(m) \,\leq\, c\nu_x\,(E_{\rm p})\,, \qquad \forall x \in E\,, \tag{10}$$

*where* $q(m) = q(m, E_{\rm p})$ *is the total jump rate from the state* $m \in E_{\rm p}$.

*Proof* Let the Markov sequence $Y$ start from some finite point measure; we have to check that $\sum_{i=0}^{\infty} 1/q(Y(i)) = \infty$ a.s. For every $n \in \mathbb{N}$, set

$$r(n) \,:=\, \sum_{i=1}^{k} \nu_{y_i}(E_{\rm p})\,,$$

where $y_1, \ldots, y_k$ denote the particles which are created at the $n$-th step of $Y$, and then $\sigma(n) = r(1) + \cdots + r(n)$. Plainly, we have

$$q(Y(n)) \leq q(Y(0)) + \sigma(n)\,,$$

as the right-hand side corresponds to the total jump rate from the point measure whose atoms are given by all the particles which are born before the $(n+1)$-th step or existed at time 0 (in other words the particles that have branched are not removed).

Condition (10) of the lemma states that the expectation of the total jump rate from a random point measure distributed according to $\bar{q}(x,\cdot)$ is bounded from above by $c$, for any $x \in E$. By conditioning on the particle that branches at the $n$-th step, we see that (10) ensures that $\mathbb{E}(r(n)) \leq c$, so that $\mathbb{E}(\sigma(n)) \leq cn$ for all $n$. Thus, by Fatou's lemma, we have

$$\liminf_{n\to\infty} \frac{q(Y(0)) + \sigma(n)}{n} \; < \; \infty \qquad \text{a.s.},$$

or equivalently

$$\limsup_{n\to\infty} \frac{n}{q(Y(0)) + \sigma(n)} \; > \; 0 \qquad \text{a.s.}$$

Since the random sequence $\sigma(\cdot)$ is increasing, this implies that

$$\sum_{n=1}^{\infty} 1/(q(Y(0)) + \sigma(n)) \; = \; \infty \qquad \text{a.s.}$$

and thus the series $\sum_{n=1}^{\infty} 1/q(Y(n))$ diverges a.s. □

From now on, we shall take the conditions of Lemma 2 for granted. The Markov chain in continuous time $X = (X(t), t \geq 0)$ on $E_{\mathrm{p}}$ associated with the family of jump rates $(q(m,\cdot), m \in E_{\mathrm{p}})$ is called a *branching Markov chain*. The measures $\nu_x$ $(x \in E)$ which are used to define the jump rates in (8) are referred to as the *branching rates* at locations $x \in E$. Standard properties of independent exponential variables show that its evolution indeed coincide with the dynamics of the non-interacting particle system which we aimed to construct. In particular, this makes the following important statement intuitively obvious.

**Proposition 1** (Branching property) *Let $X$ and $X'$ two independent versions of the same branching Markov chain, started respectively from two point measures $m$ and $m'$. Then $X + X'$ is a version of the branching Markov chain started from $m + m'$.*

*Proof* Let $T$ and $T'$ denote the first jump times of $X$ and $X'$, respectively. So $T$ and $T'$ are independent and follow the exponential distribution with parameters $q(m)$ and $q(m')$, respectively. Moreover, $X(T)$ and $X(T')$ are independent with respective laws $\bar{q}(m,\cdot)$ and $\bar{q}(m',\cdot)$, and are jointly independent of $T$ and $T'$. Thus the first jump time of $X + X'$ occurs at time $T \wedge T'$, which has the exponential distribution with parameter $q(m) + q(m') = q(m + m')$. Furthermore the value of $X + X'$ after this jump, namely $X(T \wedge T') + X'(T \wedge T')$, is independent of $T \wedge T'$, and has the law $\bar{q}(m + m', \cdot)$, since for every measurable functional $\Phi : E_{\mathrm{p}} \to \mathbb{R}_+$

$$\begin{aligned}
&\mathbb{E}(\Phi\left(X(T \wedge T') + X'(T \wedge T')\right)) \\
&\quad = \mathbb{P}(T < T')\mathbb{E}\left(\Phi(X(T) + m')\right) + \mathbb{P}(T' \leq T)\mathbb{E}\left(\Phi(m + X'(T'))\right) \\
&\quad = \frac{q(m)}{q(m+m')} \int_{E_{\mathrm{p}}} \Phi(\mu + m')\bar{q}(m, d\mu) + \frac{q(m')}{q(m+m')} \int_{E_{\mathrm{p}}} \Phi(m + \mu)\bar{q}(m', d\mu) \\
&\quad = \int_{E_{\mathrm{p}}} \Phi(\mu)\bar{q}(m + m', d\mu)\,,
\end{aligned}$$

where we used (9) in the third equality. This shows that the distribution of the time and location of the first jump of $X + X'$ is the same as that of the version of the branching Markov chain starting from $m + m'$. An application of the Markov property enables us to extend this identity to the following jumps, which completes the proof. □

**Example** *Branching random walks* (in continuous time) form a special class of branching Markov chains which have been introduced by Uchiyama [68]. Specifically, we now assume that $E$ is some Euclidean space, and we consider a finite measure $\nu$ on the space $E_{\rm p}$ of finite point measures on $E$. Next, to each site $x \in E$, we associate to $\nu$ its image by the translation of its atoms by $x$. That is, for every finite point measure $m = \sum_{i=1}^{n} \delta_{x_i}$, we denote the shifted measure by $m_x := \sum_{i=1}^{n} \delta_{x+x_i}$, and $\nu_x$ is the image of $\nu$ by this shift. The branching Markov chain with branching rates $(\nu_x, x \in E)$, is called a branching random walk with branching measure $\nu$. In this direction, observe that the condition (10) of Lemma 2 reduces to

$$\int_{E_{\rm p}} \nu(dm) m(E) \, < \, \infty \, .$$

It is easy to see that the process of total mass $(\langle X(t), 1\rangle, t \geq 0)$ is a classical branching process. More precisely, each particle lives for an exponential lifetime with parameter $\nu(E_{\rm p})$ and, at its death, it gives birth to a random number $Z$ of children, where the offspring distribution is specified by

$$\mathbb{P}(Z = n) \, = \, \frac{1}{\nu(E_{\rm p})} \int_{E_{\rm p}} \nu(dm) \mathbb{1}_{\{m(E)=n\}} \, .$$

## 1.3 Fragmentation chains

Throughout the rest of this chapter, we shall work with the state space of decreasing numerical sequences bounded from above by 1 and with limit 0:

$$\mathcal{S}^{\downarrow} := \{\mathbf{s} = (s_1, s_2, \ldots) : 1 \geq s_1 \geq s_2 \geq \cdots \geq 0 \text{ and } \lim s_i = 0\} \, .$$

We endow $\mathcal{S}^{\downarrow}$ with the uniform distance

$$d(\mathbf{s}, \mathbf{s}') \, := \, \max_{i \in \mathbb{N}} |s_i - s'_i| \, ,$$

which makes $\mathcal{S}^{\downarrow}$ a Polish space. We shall think of a sequence $\mathbf{s} \in \mathcal{S}^{\downarrow}$ as that of the sizes of the fragments resulting from the split of some block with unit size. No term of a sequence $\mathbf{s} \in \mathcal{S}^{\downarrow}$ exceeds 1; the total sum $\sum_{i=1}^{\infty} s_i$ of the series $\mathbf{s} \in \mathcal{S}^{\downarrow}$ may equal 1 (which corresponds to the so-called *conservative* situation), may be less than 1 (*dissipative case*), or even greater than 1 (this situation may occur for instance when the size of an object is the measure of its diameter). It is sometimes convenient to identify the sequence with a Radon point measure on $]0, \infty[$, $\sum_{i:s_i>0} \delta_{s_i}$. Observe that the latter is a finite point measure on $]0, \infty[$ if and only if $s_i = 0$ whenever $i$ is large enough.

We will be interested in a simple family of Markov process with càdlàg paths $X = (X(t), t \geq 0)$ and values in the space $\mathcal{S}^{\downarrow}$, called *fragmentation chains*. Informally, the evolution of $X = (X(t), t \geq 0)$ is given by a non-interacting particle system, in the sense that each particle in the system evolves independently of the others. The dynamics of each particle are the following. A particle lives for an exponential time with parameter

depending only on its size and, at the end of its life, it is replaced by a random cloud (possibly infinite) of smaller particles which is independent of the lifetime of the particle. Although this description bears obvious similarities with that for branching Markov chains, it is not completely clear that such a random evolution is well-defined, because we do not require the number of fragments (or particles) at some given time to be finite. As a consequence, the jump rates from a configuration $\mathbf{s} = (s_1, \ldots)$ with $s_i > 0$ for every $i \in \mathbb{N}$ may be (and indeed often are) unbounded. In particular, although a fragmentation chain is a Markov process, it is not necessarily a Markov chain in continuous time. However, we keep the terminology *chain* to underline the fact that each particle remains unchanged during some strictly positive time so, in some loose sense, the evolution of the system is discrete.

Let us now explain precisely the construction of this model. To start with, for every $x \in ]0,1]$, let $\nu_x$ be some finite measure on $\mathcal{S}^{\downarrow}$. We assume that this family depends in a measurable way on the variable $x$. As previously, the total mass $\nu_x(\mathcal{S}^{\downarrow})$ is the parameter of the exponential lifetime of the particle, and the probability law $\nu_x(\cdot)/\nu_x(\mathcal{S}^{\downarrow})$ is the law of the random cloud of particles resulting from the dislocation of $x$. We shall now discuss why the evolution of the system is well-defined under the following mild condition: we suppose henceforth first that

$$\nu_x(s_1 > x) \,=\, 0\,, \qquad x \geq 0 \tag{11}$$

(which means that a.s. the sizes of the fragments cannot be larger than that of the initial particle), and second that for every $\varepsilon > 0$, there exists some finite constant $c_\varepsilon$ such that for every $x > \varepsilon$,

$$\nu_x(\mathcal{S}^{\downarrow}) \,<\, c_\varepsilon \quad \text{and} \quad \int_{\mathcal{S}^{\downarrow}} \#\left\{i \in \mathbb{N} : s_i > \varepsilon\right\} \nu_x(d\mathbf{s}) \leq c_\varepsilon \nu_x(\mathcal{S}^{\downarrow})\,, \tag{12}$$

where $\#$ stands for the counting measure on $\mathbb{N}$. Observe that the second requirement in (12) is always fulfilled when the measure $\nu_x$ is conservative or dissipative, that is $\sum_{i=1}^{\infty} s_i \leq x$ for $\nu_x$ almost-every configuration $\mathbf{s}$, since then the bound $\#\left\{i : s_i > \varepsilon\right\} \leq x/\varepsilon$ holds $\nu_x(d\mathbf{s})$-a.e.

Call a sequence $\mathbf{s} = (s_1, \ldots) \in \mathcal{S}^{\downarrow}$ *finite* and write $\mathbf{s} \in \mathcal{S}^{\downarrow}_f$ if $s_j = 0$ for some large enough index $j$; clearly we may (and will) identify a finite sequence $\mathbf{s}$ as a finite point measure on $]0,1]$, $m = \sum \delta_{s_j}$, where the sum is taken over indices $j$ such that $s_j > 0$. Next, we introduce threshold operators which map $\mathcal{S}^{\downarrow}$ to the space of finite sequences. Specifically, for every $\varepsilon > 0$, we write $\varphi_\varepsilon : [0,1] \to [0,1]$ to be the function such that $\varphi_\varepsilon(x) = x$ if $x > \varepsilon$ and $\varphi_\varepsilon(x) = 0$ otherwise, and, by a slight abuse of notation, we still write $\varphi_\varepsilon : \mathcal{S}^{\downarrow} \to \mathcal{S}^{\downarrow}_f$ for its obvious extension to $\mathcal{S}^{\downarrow}$ (component by component). Plainly, the threshold operators form a projective system, that is $\varphi_\varepsilon \circ \varphi_\eta = \varphi_\eta \circ \varphi_\varepsilon = \varphi_\varepsilon$ for every $0 < \eta \leq \varepsilon$.

Let $\nu_x^\varepsilon$ be the image of $\nu_x$ by the threshold operator $\varphi_\varepsilon$; so we can think of $\nu_x^\varepsilon$ as a finite measure on the space of point measures on $]0,1]$. In this framework, it is easy to see that whenever (12) holds, the conditions of Lemma 2 are fulfilled, and thus we can construct a branching Markov chain in continuous time $X^\varepsilon = (X^\varepsilon(t), t \geq 0)$ with branching rates given by the family $(\nu_x^\varepsilon, x \in ]0,1])$ and started from an arbitrary finite sequence $\mathbf{s}$. This branching Markov chain takes values in the space of finite point measures on $]0,1]$, but we may also view it as a process with values in $\mathcal{S}^{\downarrow}_f$ (or even $\mathcal{S}^{\downarrow}$) by the preceding identification.

We shall now show that one can construct simultaneously the chains $X^\varepsilon$ for different values of the parameter $\varepsilon$ in a consistent way with respect to the threshold operators and, more precisely, that there exists a process $X$ in $\mathcal{S}^\downarrow$ such that its image by the threshold operator $\varphi_\varepsilon$ is a version of $X^\varepsilon$. Clearly, such process is unique in distribution and has the evolution that we wished.

**Lemma 3** *For every $\mathbf{s} \in \mathcal{S}^\downarrow$, there exists a unique (in law) process $X = (X(t), t \geq 0)$ with values in $\mathcal{S}^\downarrow$ such that for every $\varepsilon > 0$, $\varphi_\varepsilon(X)$ is a branching Markov chain with the same distribution as $X^\varepsilon$ started from $X^\varepsilon(0) = \varphi_\varepsilon(\mathbf{s})$.*

*Proof* Fix $0 < \eta < \varepsilon$, and let $X^\eta$ denote a version of the branching Markov chain with branching rates specified by the family $(\nu^\eta_x, x \in ]0,1])$ and started from the finite configuration $\varphi_\eta(\mathbf{s})$. Recall that our setting requires that the size of a child particle is never larger than that of its parent. As a consequence, for every $s, t \geq 0$, the conditional distribution of $\varphi_\varepsilon(X^\eta(t+s))$ given $X^\eta(t)$ only depends on $\varphi_\varepsilon(X^\eta(t))$. It follows that $\varphi_\varepsilon(X^\eta(\cdot))$ is a Markov process, more precisely a Markov chain in continuous time, and since $\nu^\varepsilon_x$ is the image of $\nu^\eta_x$ by the threshold operator $\varphi_\varepsilon$, the jump rates of $\varphi_\varepsilon(X^\eta(\cdot))$ are the same as those of $X^\varepsilon$ started from $X^\varepsilon(0) = \varphi_\varepsilon(\mathbf{s})$. Thus, the two processes have the same distribution.

This observation enables us to appeal to Kolmogorov's projective theorem, and we obtain the existence of a family of process $(X^\varepsilon(\cdot), \varepsilon > 0)$ such that for every $\varepsilon > \eta > 0$, $\varphi_\varepsilon(X^\eta(\cdot))$ has the same law as $X^\varepsilon(\cdot)$. Plainly, if we are given a family $(m^\varepsilon, \varepsilon > 0)$ of point measures on $]0,1]$ such that for every $\varepsilon > \eta > 0$, $m^\varepsilon$ is the image of $m^\eta$ by the threshold operator $\varphi_\varepsilon$, then there exists a unique sigma-finite measure $m$ on $]0,1]$ such that $m^\varepsilon$ is the image of $m$ by $\varphi_\varepsilon$. Thus the family of processes $(X^\varepsilon(\cdot), \varepsilon > 0)$ can be obtained as the images of the same process $X(\cdot)$ by the threshold operators. □

Let us now turn our attention to self-similarity. Suppose that $\nu$ is some finite measure on $\mathcal{S}^\downarrow$ such that

$$\int_{\mathcal{S}^\downarrow} \#\{i \in \mathbb{N} : s_i > \varepsilon\}\, \nu(d\mathbf{s}) \;<\; \infty \qquad \text{for all } \varepsilon > 0. \tag{13}$$

For every $x \in ]0,1]$, write $\nu_x$ for the image of $x^\alpha \nu$ by the dilation $\mathbf{s} \to x\mathbf{s}$. Then, the conditions (11) and (12) plainly hold. We are now able to introduce the following definition.

**Definition 1** (i) *Let $(\nu_x, 0 < x \leq 1)$ be a measurable kernel of finite measures on $\mathcal{S}^\downarrow$ which fulfils (11) and (12). The process $X = (X(t), t \geq 0)$ with values in $\mathcal{S}^\downarrow$ that has been constructed in Lemma 3 is called a* **fragmentation chain** *with dislocation rates $(\nu_x, 0 < x \leq 1)$.*

(ii) *Let $\nu$ be some finite measure on $\mathcal{S}^\downarrow$ such that (13) holds, and $\alpha \in \mathbb{R}$. For every $x \in ]0,1]$, let $\nu_x$ denote the image of $x^\alpha \nu$ by the dilation $\mathbf{s} \to x\mathbf{s}$. The fragmentation chain with dislocation rates $(\nu_x, 0 < x \leq 1)$ is called* **self-similar** *with* **index of self-similarity** *$\alpha$ and* **dislocation measure** *$\nu$.*

Throughout the rest of this chapter, $X$ will denote a self-similar fragmentation chain as defined above. Its law is entirely determined by the index of self-similarity $\alpha$, the dislocation measure $\nu$, and of course the initial configuration $X(0) \in \mathcal{S}^\downarrow$. The evolution of the process can be described as follows: a fragment with size $x$ lives for an exponential time

with parameter $x^\alpha \nu(\mathcal{S}^\downarrow)$, and then splits and gives rise to a family of smaller fragments distributed as $x\xi$, where $\xi$ has the law $\nu(\cdot)/\nu(\mathcal{S}^\downarrow)$.

It should be intuitively obvious that the behavior of a self-similar fragmentation depends crucially on its index of self-similarity. Informally, fragments get smaller as time passes; the rate of dislocations thus decreases when the index is positive, whereas it increases when $\alpha < 0$. In particular, we stress that for $\alpha < 0$, this description a priori makes sense only when the size $x$ is non-zero; however, by self-similarity, the children of a particle with size 0 all have size zero. Particles with size zero play no role, and the evolution is thus well-defined in all cases.

As for every $\varepsilon > 0$, the infinitesimal generator of the Markov chain $\varphi_\varepsilon(X)$, which is obtained from $X$ by discarding the fragments with size less than or equal to $\varepsilon$, is given in terms of its jump rates, we immediately derive from (4) explicit expressions for the infinitesimal generator $\mathtt{G}$ of a self-similar fragmentation chain with index $\alpha$ and dislocation measure $\nu$. Typically, consider a bounded measurable functional $\mathbf{f} : \mathcal{S}^\downarrow \to \mathbb{R}$ which only depends on fragments with size at least $\varepsilon$, in the sense that $\mathbf{f} = \mathbf{f} \circ \varphi_\varepsilon$. For every configurations $\mathbf{x}, \mathbf{s} \in \mathcal{S}^\downarrow$ and every integer $i$, introduce the notation $\mathrm{Frag}_i(\mathbf{x}, \mathbf{s})$ to designate the sequence obtained from $\mathbf{x}$ by removing its $i$-th term $x_i$, replacing it by the terms of the configuration $x_i\mathbf{s}$, and reordering all the terms in decreasing order. Then it is easily checked that in this situation, the infinitesimal generator $\mathtt{G}$ is given by

$$\mathtt{G}\mathbf{f}(\mathbf{x}) \;=\; \sum_{i=1}^{\infty} \mathbb{1}_{\{x_i>0\}} x_i^\alpha \int_{\mathcal{S}^\downarrow} (\mathbf{f}(\mathrm{Frag}_i(\mathbf{x}, \mathbf{s})) - \mathbf{f}(\mathbf{x}))\nu(d\mathbf{s})\,. \tag{14}$$

This expression has an interesting application to the so-called *fragmentation equation* which appears in various models in physics (see for example [22] and references therein).

**Corollary 1** *Assume that the dislocation measure $\nu$ is conservative or dissipative, that is $\nu(\mathbf{s} \in \mathcal{S}^\downarrow : \sum_{i=1}^\infty s_i > 1) = 0$. For every $t \geq 0$ and $0 < x \leq 1$, define a Radon measure $\mu_t^x$ on $]0, 1]$ by*

$$\langle \mu_t^x, f\rangle \;=\; \mathbb{E}_x\left(\sum_{i=1}^{\infty} f(X_i(t))\mathbb{1}_{\{X_i(t)>0\}}\right),$$

*where $f : ]0, 1] \to \mathbb{R}_+$ denotes a generic measurable function with compact support and $\langle \mu_t^x, f\rangle$ the integral of $f$ with respect to $\mu_t^x$. Then the family $(\mu_t^x)_{t\geq 0}$ solves the system of partial differential equations*

$$\frac{\partial}{\partial t}\langle \mu_t^x, f\rangle \;=\; \int_{]0,1]} \mu_t^x(dy) y^\alpha \int_{\mathcal{S}^\downarrow} \nu(d\mathbf{s}) \left(-f(y) + \sum_{i=1}^{\infty} f(ys_i)\right),$$

*with initial condition $\mu_0^x = \delta_x$.*

More generally, by linearity of the fragmentation, we obtain a solution $(\mu_t^m)_{t\geq 0}$ with initial condition $m$, where $m$ denotes an arbitrary Radon measure on $]0, 1]$, in the form

$$\mu_t^m \;=\; \int_{]0,1]} \mu_t^x m(dx)\,.$$

We refer to Haas [42] for a much deeper study of the applications of fragmentation processes to the fragmentation equation.

*Proof* Let $\varepsilon > 0$ such that $f = 0$ on $]0, \varepsilon]$. The dislocation measure $\nu$ being conservative or dissipative, the process of the total mass, $t \to \sum_{i=1}^{\infty} X_i(t)$ is non-increasing a.s.; in particular the number of fragments of size at least $\varepsilon$ in $X(t)$ is bounded from above by $x/\varepsilon$, $\mathbb{P}_x$-a.s. Define an additive functional $\mathbf{f} : \mathcal{S}^{\downarrow} \to \mathbb{R}$ by

$$\mathbf{f}(\mathbf{s}) \;=\; \sum_{i=1}^{\infty} f(s_i) \mathbb{1}_{\{s_i>0\}}\,, \qquad \mathbf{s} = (s_1, s_2, \ldots)\,.$$

Plainly the functional $\mathbf{f}$ only depends on fragments with size at least $\varepsilon$, but is not bounded on $\mathcal{S}^{\downarrow}$. However, the observations above show that $\mathbf{f}(X(t)) \leq \varepsilon^{-1} x \max f$ for all $t \geq 0$, $\mathbb{P}_x$-a.s., which enables us to work with the bounded functional $\tilde{\mathbf{f}} := \mathbf{f} \wedge (\varepsilon^{-1} x \max f)$. Specializing (14) yields that for every configuration $\mathbf{x} = (x_1, \ldots) \in \mathcal{S}^{\downarrow}$ which has at most $\lfloor x/\varepsilon \rfloor$ fragments of size greater than $\varepsilon$, we have

$$\mathtt{G}\tilde{\mathbf{f}}(\mathbf{x}) \;=\; \sum_{i=1}^{\infty} \mathbb{1}_{\{x_i>0\}} x_i^{\alpha} \int_{\mathcal{S}^{\downarrow}} (\mathbf{f}(x_i\mathbf{s}) - f(x_i)) \nu(d\mathbf{s})\,. \tag{15}$$

It is readily checked that our assumptions ensure that $\mathtt{G}\tilde{\mathbf{f}}$ is a bounded functional. Combining (15) with the forward equation (6) immediately yields the statement. □

Similarly, consider a measurable function $g : [0,1] \to ]0,1]$ with $g = 1$ on some neighborhood of 0, and define a multiplicative functional $\mathbf{g} : \mathcal{S}^{\downarrow} \to ]0, \infty[$ by

$$\mathbf{g}(\mathbf{s}) \;=\; \prod_{i=1}^{\infty} g(s_i)\,, \qquad \mathbf{s} = (s_1, s_2, \ldots)\,.$$

Plainly, $0 \leq \mathbf{g}(\mathbf{s}) \leq 1$ for every $\mathbf{s} \in \mathcal{S}^{\downarrow}$, for every $\mathbf{x} = (x_1, \ldots) \in \mathcal{S}^{\downarrow}$, (14) yields

$$\mathtt{G}\mathbf{g}(\mathbf{x}) \;=\; \sum_{i=1}^{\infty} \mathbb{1}_{\{x_i>0\}} x_i^{\alpha} \frac{\mathbf{g}(\mathbf{x})}{g(x_i)} \int_{\mathcal{S}^{\downarrow}} (\mathbf{g}(x_i\mathbf{s}) - g(x_i)) \nu(d\mathbf{s})\,.$$

Later in the text, it will be convenient to denote for every $x \in [0,1]$ by $\mathbb{P}_x$ the law of $X$ started from the configuration $(x, 0, \ldots)$. In other words, $\mathbb{P}_x$ is the distribution of the fragmentation chain when at the initial time, there is just one particle with size $x$. We shall further simply write $\mathbb{P} = \mathbb{P}_1$ when the size is $x = 1$. Quite often we shall work under the law $\mathbb{P}$; essentially this does not induce any loss of generality, thanks to the following basic properties.

**Proposition 2** (i) *Any fragmentation chain $X$ has the* **branching property**, *namely for every sequence $\mathbf{s} = (s_1, \ldots) \in \mathcal{S}^{\downarrow}$ and every $t \geq 0$, the distribution of $X(t)$ given that $X(0) = \mathbf{s}$ is the same as that of the decreasing rearrangement of the terms of independent random sequences $X^{(1)}(t), X^{(2)}(t), \ldots$, where for each $i \in \mathbb{N}$, $X^{(i)}(t)$ is distributed as $X(t)$ under $\mathbb{P}_{s_i}$.*

(ii) *Any self-similar fragmentation chain $X$ has the* **scaling property**, *namely for every $x \in [0,1]$, the distribution of the rescaled process $(xX(x^{\alpha}t), t \geq 0)$ under $\mathbb{P}_1$ is $\mathbb{P}_x$.*

We now conclude this section by discussing a few special cases and examples.

First, in the special case $\alpha = 0$, where the total dislocation rate of a fragment does not depend on its size, we say that the fragmentation chain is *homogeneous*. In this case, there is a natural connection with branching random walks (see Section 1.1.2). Specifically, let $\mu$ be some finite measure on the space of finite point measures on $]0, \infty[$, and let $Z$ denote a branching random walk in continuous time with branching measure $\mu$. Consider the process $X$ valued in $\mathcal{S}^{\downarrow}$ obtained by shifting the atoms of $Z$ by the exponential map $z \to \mathrm{e}^{-z}$, that is $X(t) = (\mathrm{e}^{-z_1}, \ldots, \mathrm{e}^{-z_k}, 0, \ldots)$, where $z_1, \ldots, z_k$ denote the atoms of the branching random walk at time $t$, ranked in increasing order. It should be plain that $X$ is a homogeneous fragmentation chain with dislocation measure $\nu$, where $\nu$ is the image of the branching measure $\mu$ by the exponential map $z \to \mathrm{e}^{-z}$. Conversely, consider a homogeneous fragmentation chain $X$ with dislocation measure $\nu$ charging only finite sequences. Then the process

$$Z^{(t)}(dx) := \sum \delta_{-\ln X_i(t)}(dx)\,,$$

where the sum is taken over the fragments with strictly positive size, is a branching random walk with branching measure the image of $\nu$ by the map $x \to -\ln x$. This elementary connection has a number of interesting consequences as it essentially reduces the study of the class of homogeneous fragmentations associated to a dislocation measure charging only finite configurations, to that of branching random walks on $]0, \infty[$, for which many results are known; see in particular the forthcoming Section 5 in this chapter.

Let us now present two simple examples of self-similar fragmentation chains with index $\alpha \neq 0$.

**Example** (Poissonian rain) Consider a Poisson point process with values in the unit interval and characteristic measure given by the Lebesgue measure on $]0, 1[$. In other words, we have a sequence $U_1, \ldots$ of i.i.d. uniform variables on $[0, 1]$; the times when they appear are the jump times of some independent Poisson process $(N_t, t \geq 0)$. Now, think of these Poissonian points as drops of rain, and consider for every time $t \geq 0$ the subset of $]0, 1[$ which is dry, that is to say $\vartheta(t) := ]0, 1[ \setminus \{U_i : i \leq N_t\}$. So $\vartheta(t)$ is a random open set which consists of $N_t + 1$ disjoint interval components. If we write $X_1(t) \geq \cdots \geq X_{N_t+1}(t)$ for the sequence of their lengths ranked in decreasing order and $X_i(t) = 0$ for $i > N_t + 1$, then it is easy to check that the process $X = (X(t), t \geq 0)$ is a (conservative) self-similar fragmentation chain with index $\alpha = 1$, with dislocation measure $\nu$ given by the distribution of $(1 - V, V, 0, \ldots)$ with $V$ uniformly distributed on $[0, 1/2]$.

The construction of the Poissonian rain can be extended to higher dimensions. For instance, in dimension 2, we start from a triangle $\mathbb{T}$ with unit area and consider a Poisson point process in $\mathbb{T}$ with Lebesgue intensity. That is we have a sequence $U_1, \ldots$ of i.i.d. uniform variables on $\mathbb{T}$, which we call atoms; the times when they appear are the jump times of some independent Poisson process with unit rate. Each time an atom of the point process arises, we split $\mathbb{T}$ into a sequence of smaller triangles and obtain a triangulation as follows. Specifically, at the instant when some atom $U$ occurs, $U$ belongs to some triangle, say $(A, B, C)$ of the current triangulation, and we split $(A, B, C)$ into $(U, B, C)$, $(A, U, C)$ and $(A, B, U)$. So at time $t$, we obtain a triangulation $\tau(t) = (\mathbb{T}_{1,n}, \ldots, \mathbb{T}_{2n+1,n})$ of $\mathbb{T}$, where $n = N(t)$ is the number of atoms that have occurred before time $t$. Denote by $X(t) = (X_1(t), X_2(t), \ldots)$ the mass-partition given by the ordered sequence of the areas of the triangles in the triangulation $\tau(t)$ and completed by an infinite sequence of 0. Then it is easy to check that $X = (X(t), t \geq 0)$ is a self-similar fragmentation with index $\alpha = 1$, and that its dislocation measure is the law of the decreasing rearrangement of a variable which

is uniformly distributed on the simplex $\{\mathbf{x} = (x_1, x_2, x_3) : x_i \geq 0 \text{ and } x_1 + x_2 + x_3 = 1\}$.

**Example** Let us briefly present a generalization of the Poissonian rain model above which has been considered by Baryshnikov and Gnedin concerning a packing problem related to communication networks; see [8] for much more on this example. As previously, let $U_1, \ldots$ be a sequence of i.i.d. uniform variables, and $\ell_1, \ell_2, \ldots$ a sequence of i.i.d. random variables, which is independent of the $U_i$. We throw successively the random intervals $[U_i, U_i + \ell_i]$ on $]0,1[$ at the jump times $T_1 < T_2 < \cdots$ of an independent rate 1 Poisson process $N = (N_t, t \geq 0)$. We construct a process of nested open subsets $\vartheta := (\vartheta(t), t \geq 0)$ as follows. The process starts from $\vartheta(0) :=]0,1[$ and remains constant except at times $T_1, T_2, \ldots$ where it may jump. Specifically, if $[U_i, U_i + \ell_i]$ is entirely contained in $\vartheta(T_i-)$, then $\vartheta(T_i) = \vartheta(T_i-)\backslash[U_i, U_i + \ell_i]$. Otherwise $\vartheta(T_i) = \vartheta(T_i-)$. Elementary properties of Poisson point processes easily imply that the process of the ranked lengths of the interval components of $\vartheta(\cdot)$ is a dissipative fragmentation chain. For instance, when the $\ell_i$ are uniformly distributed, it is easily checked that the fragmentation is self-similar with index $\alpha = 2$.

# 2 Genealogical structure

Throughout the rest of this chapter, we shall consider a self-similar fragmentation chain $X = (X(t), t \geq 0)$ with index $\alpha \in \mathbb{R}$ and dislocation measure $\nu$ which fulfills (13). In order to avoid uninteresting discussions of degenerate cases, we shall always implicitly assume that

$$\nu\left(s_i \in ]0,1[\right) > 0 \qquad \text{for some } i \in \mathbb{N}.$$

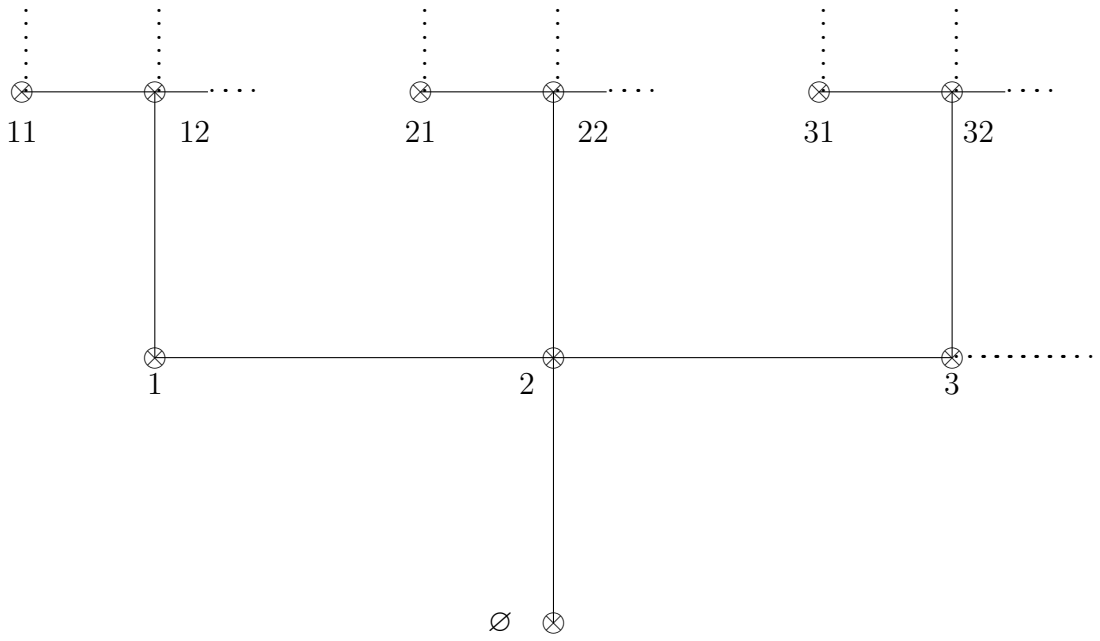
In this section, we point at a representation of fragmentation chains as random infinite marked trees. This representation can be viewed as a different parametrization of the process, in which the natural time is replaced by the generation of the different particles.

## 2.1 The tree of generations

We start by introducing some notation. We consider the infinite tree (see the figure below)

$$\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

with the convention $\mathbb{N}^0 = \{\varnothing\}$. We will refer to $\mathcal{U}$ as the *genealogical tree*; its elements are called *nodes* (or sometimes also *individuals*) and the distinguished node $\varnothing$ is the *root*. Nodes will be used to label the particles produced by a fragmentation chain.




*Genealogical tree* $\mathcal{U}$

For each $u = (u_1, \ldots, u_n) \in \mathcal{U}$, we call $n$ the *generation* of $u$ and write $|u| = n$, with the obvious convention $|\varnothing| = 0$. When $n \geq 1$ and $u = (u_1, \ldots, u_n)$, we call $u- := (u_1, \ldots, u_{n-1})$ the *parent* of $u$. Similarly, for every $i \in \mathbb{N}$ we write $ui = (u_1, \ldots, u_n, i) \in \mathbb{N}^{n+1}$ for the $i$-th *child* of $u$. Finally, any map from $\mathcal{U}$ to some (measurable) set is called a *mark* .

Now, consider a self-similar fragmentation chain $X = (X(t), t \geq 0)$ with index $\alpha \in \mathbb{R}$ and dislocation measure $\nu$. Suppose for simplicity that the process starts from a single fragment with size $x > 0$, that is we work under the law $\mathbb{P}_x$. We associate to each path of the process a mark on the infinite tree $\mathcal{U}$; roughly the mark at a node $u$ is the triple $(\xi_u, a_u, \zeta_u)$ where $\xi_u$ is the size, $a_u$ the birth-time and $\zeta_u$ the lifetime of the particle with label $u$. More precisely, the initial particle $x$ corresponds to the ancestor $\varnothing$ of the tree $\mathcal{U}$, and the mark at $\varnothing$ is the triple $(x, 0, \zeta_\varnothing)$ where $\zeta_\varnothing$ is the lifetime of the initial particle (in particular, $\zeta_\varnothing$ has the exponential law with parameter $x^\alpha \nu(\mathcal{S}^\downarrow)$). The nodes of the tree at the first generation are used as the labels of the particles arising at the first split. Again, the mark associated to each of the nodes $i \in \mathbb{N}^1$ at the first generation, is the triple $(\xi_i, a_i, \zeta_i)$, where $\xi_i$ is the size of the $i$-th child of the ancestor, $a_i = a_\varnothing + \zeta_\varnothing$ (the birth-time of a child particle coincides with the death-time of the parent), and $\zeta_i$ stands for the lifetime of the $i$-th child. And we iterate the same construction with each particle at each generation.

Clearly, the description of the dynamics of fragmentation implies that its genealogical representation also enjoys the branching property. Specifically, the distribution of the random mark can be described recursively as follows.

**Proposition 3** *There exist two independent families of i.i.d. variables indexed by the nodes of the genealogical tree,* $\left((\tilde{\xi}_{ui})_{i\in\mathbb{N}}, u \in \mathcal{U}\right)$ *and* $((\mathbf{e}_{ui})_{i\in\mathbb{N}}, u \in \mathcal{U})$*, where each* $(\tilde{\xi}_{ui})_{i\in\mathbb{N}}$ *is distributed according to the law* $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$*, and each* $(\mathbf{e}_{ui})_{i\in\mathbb{N}}$ *is a sequence of i.i.d. exponential variables with parameter* $\nu(\mathcal{S}^{\downarrow})$*, and such that the following holds:*

*Given the marks* $((\xi_v, a_v, \zeta_v), |v| \leq n)$ *of the first* $n$ *generations, the marks at generation* $n+1$ *are given by*

$$\xi_{ui} \;=\; \tilde{\xi}_{ui}\xi_u \;,\; a_{ui} \;=\; a_u + \zeta_u \;,\; \zeta_{ui} \;=\; \xi_{ui}^{-\alpha}\mathbf{e}_{ui}\,,$$

*where* $u = (u_1, \ldots, u_n)$ *and* $ui = (u_1, \ldots, u_n, i)$ *is the* $i$*-th child of* $u$*.*

The genealogical coding of a self-similar fragmentation chain yields an elementary transformation of the latter, which only affects the lifetime of particles, and enables us to change the index of self-similarity.

**Corollary 2** *Let* $(\xi, a, \zeta)$ *be the random mark on the genealogical tree induced by a self-similar fragmentation chain with index* $\alpha$ *and dislocation measure* $\nu$*. Fix* $\beta \in \mathbb{R}$ *and consider the random mark* $(\xi^{(\beta)}, a^{(\beta)}, \zeta^{(\beta)})$ *defined by*

$$\xi_u^{(\beta)} \;=\; \xi_u \;,\quad \zeta_u^{(\beta)} \;=\; \xi_u^{-\beta}\zeta_u \;,\quad a_u^{(\beta)} \;=\; \sum_{i=0}^{n-1} \zeta^{(\beta)}_{(u_1,\ldots,u_i)}\,,$$

*where* $u = (u_1, \ldots, u_n)$ *denotes a generic node. Then* $(\xi^{(\beta)}, a^{(\beta)}, \zeta^{(\beta)})$ *is distributed as the genealogical coding of a self-similar fragmentation chain with index* $\alpha + \beta$ *and dislocation measure* $\nu$*.*

*Proof* This is obvious from Proposition 3 as conditionally on the mass $\xi_u = \xi_u^{(\beta)}$ of the particle labeled by the node $u \in \mathcal{U}$, the lifetime $\zeta_u^{(\beta)}$ has an exponential distribution with parameter $\xi_u^{\alpha+\beta}\nu(\mathcal{S}^{\downarrow})$. □

Now to every node $u$ of the genealogical tree, we can associate the interval $I_u := [a_u, a_u + \zeta_u[$ during which the particle labeled by $u$ is alive. Putting the pieces together, we may express the fragmentation chain at time $t$ as the ranked sequence of the particles which are alive at time $t$:

**Proposition 4** *With probability one, for every* $t \geq 0$ *there is the identity between point measures on* $]0, \infty[$*:*

$$\sum_{i=1}^{\infty} \mathbb{1}_{\{X_i(t)>0\}}\delta_{X_i(t)} \;=\; \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t\in I_u\}}\delta_{\xi_u}\,.$$

*Proof* We have to check that all the fragments with positive size which are present at time $t$ have a finite generation, that is result from finitely many dislocations of the initial particle. In this direction, let us fix some arbitrarily small $\varepsilon > 0$, and consider the threshold operator $\varphi_\varepsilon$ which consists of removing all the fragments with size less than or equal to $\varepsilon$. Recall from Section 1.1.3 that $\varphi_\varepsilon(X)$ is a Markov chain, in particular the number of jumps accomplished by this chain before time $t$ is finite a.s., and this number obviously is an upper bound for the generation of all fragments with size greater than $\varepsilon$. □

## 2.2 Malthusian hypotheses and the intrinsic martingale

The purpose of this section is to introduce the so-called *intrinsic martingale* which is naturally induced by the tree representation of fragmentations. This martingale will play a crucial role when we investigate the asymptotic behavior of self-similar fragmentation chains.

We start by introducing the notation

$$\underline{p} := \inf\left\{ p > 0 : \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i^p \nu(d\mathbf{s}) < \infty \right\} \tag{16}$$

(with the convention $\inf \varnothing = \infty$) and then for every $p > \underline{p}$

$$\kappa(p) := \int_{\mathcal{S}^\downarrow} \left( 1 - \sum_{i=1}^{\infty} s_i^p \right) \nu(d\mathbf{s}) \, . \tag{17}$$

Note that $\kappa$ is always a continuous strictly increasing function on $]\underline{p}, \infty[$; $\kappa(\underline{p}+)$ may be finite or equal to $-\infty$. We stress that $\kappa$ and $\underline{p}$ depend on the dislocation measure $\nu$ but not on the parameter of self-similarity $\alpha$. Let us discuss a couple of examples:

First, recall that for the so-called Poissonian rain introduced in Section 1.1.3, the dislocation measure is the law of $(1 - V, V, 0, \ldots)$ where $V$ is a uniform random variable on $[0, 1/2]$. One readily finds that $\underline{p} = 0$ and $\kappa(p) = (p-1)/(p+1)$.

Second, consider the so-called uniform stick-breaking scheme (see the forthcoming Corollary **??** for much more on this). That is, cut the unit interval at a uniformly distributed random variable, keep the left portion, cut the right one at an independent uniformly distributed random variable, keep the left portion, and so on. The sequence of the lengths of the resulting intervals (ordered from the left to the right) is thus $U_1, (1 - U_1)U_2, (1 - U_1)(1 - U_2)U_3, \ldots$, where $U_1, U_2, \ldots$ are i.i.d. uniform variables; in particular the $p$-th moment of the $k$-th length is thus $(1 + p)^{-k}$. When the distribution of the sequence of the lengths ranked in decreasing order is used as the dislocation measure $\nu$ of some fragmentation chain, we get that $\underline{p} = 0$ and $\kappa(p) = 1 - 1/p$ for $p > 0$.

We now make the fundamental:

**Malthusian Hypotheses.** *There exists a (unique) solution $p^* \geq \underline{p}$ to the equation*

$$\kappa(p^*) \, = \, 0 \, ,$$

*which is called the* **Malthusian exponent**. *Furthermore the integral*

$$\int_{\mathcal{S}^\downarrow} \left( \sum_{i=1}^{\infty} s_i^{p^*} \right)^p \nu(d\mathbf{s}) \, ,$$

*is finite for some $p > 1$.*

Throughout the rest of this chapter, the Malthusian hypotheses will always be taken for granted.

In order to claim one of the basic results in this setting, it will be convenient to call by the name *extinction* the event that for some large enough $n \in \mathbb{N}$, all nodes $u$ at the

$n$-th generation have zero size, and by the name *non-extinction* the complementary event. Clearly, extinction may occur if and only if $\nu(s_1 = 0) > 0$ (i.e. a particle may disappear entirely after a dislocation). Moreover it follows from Proposition 3 that the number of fragments with positive size at the $n$-th generation

$$\#(n) := \#\left\{u \in \mathcal{U} : |u| = n \text{ and } \xi_u > 0\right\}, \qquad n \in \mathbb{Z}_+$$

is a Galton-Watson process with reproduction law given by the distribution of $\max\{i : s_i > 0\}$ under $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$. Combining the inequality

$$\sum_{i=1}^{\infty} s_i^{p^*} \leq \#\{i \in \mathbb{N} : s_i > 0\}, \qquad \text{whenever } 0 \leq \cdots \leq s_2 \leq s_1 \leq 1$$

and the Malthusian hypotheses, we see that this Galton-Watson process is super-critical. By a classical result, the probability of extinction is the unique solution $p \in [0, 1[$ to the equation $g(p) = p$, where $g$ is the moment generation function of the reproduction law. To summarize, the probability of non-extinction is always strictly positive, and equals one if and only if $\nu(s_1 = 0) = 0$.

For the sake of simplicity, we henceforth work under $\mathbb{P} = \mathbb{P}_1$, that is we assume that at the initial time there is a single fragment with size 1. Recall that this does not induce any significant loss of generality. We may now state.

**Theorem 1** *The process*

$$\mathcal{M}_n := \sum_{|u|=n} \xi_u^{p^*}, \qquad n \in \mathbb{Z}_+$$

*is a martingale which is bounded in $L^p$ for some $p > 1$, and in particular is uniformly integrable. Moreover, the terminal value $\mathcal{M}_\infty$ is strictly positive a.s. conditionally on non-extinction.*

Later in the text, we will refer to $(\mathcal{M}_n, n \in \mathbb{Z}_+)$ as the *intrinsic martingale*. Observe that in the important case when dislocations are conservative, in the sense that $\sum_{i=1}^{\infty} s_i = 1$, $\nu(d\mathbf{s})$-a.e., then the Malthusian hypotheses are automatically fulfilled with $p^* = 1$; further $\mathcal{M}_n = 1$ for all $n \in \mathbb{Z}_+$, and the statement in Theorem 1 is trivial.

*Proof* Denote by $\mathcal{G}_n$ the sigma field generated by $(\xi_u, |u| \leq n)$, so $(\mathcal{G}_n)$ is a filtration. It should be plain from the description of the dynamics of the random marks in Proposition 3 that for every $q > \underline{p}$,

$$\mathbb{E}\left(\sum_{|u|=n+1} \xi_u^q \mid \mathcal{G}_n\right) = \mathbb{E}\left(\sum_{|v|=n} \xi_v^q \sum_{i=1}^{\infty} \tilde{\xi}_{vi}^q \mid \mathcal{G}_n\right) = c(q) \sum_{|v|=n} \xi_v^q$$

where

$$c(q) = \int_{\mathcal{S}^{\downarrow}} \sum_{i=1}^{\infty} s_i^q \nu(d\mathbf{s})/\nu(\mathcal{S}^{\downarrow}) = 1 - \kappa(q)/\nu(\mathcal{S}^{\downarrow}).$$

In particular for the Malthusian exponent $q = p^*$, one has $c(p^*) = 1$ and the martingale property is proven.

In order to establish the boundedness of the martingale in $L^p(\mathbb{P})$, we shall use the following well-known consequence of the Burkholder-Davis-Gundy inequalities (see for

instance [57]). For every $p \in ]1,2]$, there exists a universal constant $c_p > 0$ such that for every martingale $(M_n, n \in \mathbb{Z}_+)$ with $M_0 = 0$, there is the inequality

$$\mathbb{E}\left(\sup_n |M_n|^p\right) \leq c_p \mathbb{E}\left(\sum_{n=0}^{\infty} |M_{n+1} - M_n|^p\right).$$

In particular, if $(\beta_i, i \in \mathbb{N})$ is a sequence of independent centered variables, then

$$\mathbb{E}\left(\left|\sum_{i=1}^{\infty} \beta_i\right|^p\right) \leq c_p \sum_{i=1}^{\infty} \mathbb{E}\left(|\beta_i|^p\right), \tag{18}$$

in the sense that whenever the right-hand side is finite, the series in the left-hand side converges a.s. and inequality (18) holds.

So all that we need is to check that the sum of the jumps of the intrinsic martingale raised to some power $p \in ]1,2]$ has a finite mean, that is

$$\mathbb{E}\left(\sum_{n=0}^{\infty} |\mathcal{M}_{n+1} - \mathcal{M}_n|^p\right) < \infty. \tag{19}$$

It this direction, we use Proposition 3 and express the $n$-th jump in the form

$$\mathcal{M}_{n+1} - \mathcal{M}_n = \sum_{|u|=n} \xi_u^{p*} \left(\left(\sum_{j=1}^{\infty} \tilde{\xi}_{u,j}^{p^*}\right) - 1\right)$$

where $(\tilde{\xi}_{u,\cdot}, |u| = n)$ is a family of i.i.d. variables with the law $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$, which is independent of $\mathcal{G}_n$. We raise this quantity to the power $p > 1$ and then take the conditional expectation given $\mathcal{G}_n$. By definition of the Malthusian exponent $p^*$, the variables $\left(\sum_{j=1}^{\infty} \tilde{\xi}_{u,j}^{p^*}\right) - 1$ are centered, so inequality (18) yields

$$\mathbb{E}\left(|\mathcal{M}_{n+1} - \mathcal{M}_n|^p \mid \mathcal{G}_n\right) \leq c_p b \sum_{|u|=n} \xi_u^{p^* p}$$

where

$$b := \int_{\mathcal{S}^{\downarrow}} \left|1 - \sum_{i=1}^{\infty} s_i^{p^*}\right|^p \nu(d\mathbf{s})/\nu(\mathcal{S}^{\downarrow})$$

is some finite constant, thanks to the second condition of the Malthusian hypotheses.

Now we know from the first part of the proof that

$$\mathbb{E}\left(\sum_{|u|=n} \xi_u^{p^* p}\right) = c(p^* p)^n = \left(1 - \kappa(pp^*)/\nu(\mathcal{S}^{\downarrow})\right)^n.$$

Since $p^* p > p^*$ (because $p^* > 0$) and $\kappa$ is strictly increasing, $\kappa(p^* p) > 0$ and thus $c(p^* p) < 1$. This yields (19) and completes the proof of the first part of the statement.

Finally, let us now check that $\mathcal{M}_\infty > 0$ a.s. conditionally on non-extinction. Write $\varrho = \mathbb{P}(\mathcal{M}_\infty = 0)$; the fact that $\mathbb{E}(\mathcal{M}_\infty) = 1$ ensures that $\varrho < 1$. On the other hand, an application of the branching property yields

$$\mathbb{E}\left(\varrho^{\#(n)}\right) = \varrho,$$

where $\#(n)$ is the number of fragments with positive size at the $n$-th generation. Clearly $(\#(n), n \in \mathbb{Z}_+)$ is a Galton-Watson process and it follows that $\varrho$ is its probability of extinction. Since $\mathcal{M}_\infty = 0$ on extinction, the two events coincide a.s. $\square$

The terminal value $\mathcal{M}_\infty$ of the intrinsic martingale will appear in many limit theorems for the fragmentation. In general its distribution is not known explicitly. However, it is straightforward from the branching property that there is the identity in law

$$\mathcal{M}_\infty \stackrel{(\mathrm{d})}{=} \sum_{j=1}^{\infty} \xi_j^{p^*} \mathcal{M}_\infty^{(j)} \tag{20}$$

where $\xi = (\xi_j, j \in \mathbb{N})$ has the law $\nu(\cdot)/\nu(\mathcal{S}^\downarrow)$, and $\mathcal{M}_\infty^{(j)}$ are independent copies of $\mathcal{M}_\infty$, also independent of $\xi$. It is known that under fairly general conditions, such an equation characterizes the law $\mathcal{M}_\infty$ uniquely, see for example [66, 59]. We also refer to Liu [58] for information of the tail behavior of the solution.

The intrinsic martingale $\mathcal{M}_n$ is indexed by the generations; it will also be convenient to consider its analog in continuous time, that is

$$\mathcal{M}(t) := \sum_{i=1}^{\infty} X_i^{p^*}(t) \; = \sum_{u \in \mathcal{U}} \mathbb{1}_{\{t \in I_u\}} \xi_u^{p^*} \,, \qquad t \geq 0 \,,$$

where in the right-hand side, $I_u$ denotes the life-interval of the particle indexed by the node $u$. It is straightforward to check that when the index of self-similarity is positive, $(\mathcal{M}(t), t \geq 0)$ is again a martingale in the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the fragmentation $(X(t), t \geq 0)$; and more precisely, we have the following.

**Proposition 5** *Assume that the index of self-similarity $\alpha$ is non-negative. Then*

$$\mathcal{M}(t) = \mathbb{E}\left(\mathcal{M}_\infty \mid \mathcal{F}_t\right) \,,$$

*where $\mathcal{M}_\infty$ is the terminal value of the intrinsic martingale $(\mathcal{M}_n, n \in \mathbb{N})$, and $(\mathcal{F}_t)_{t \geq 0}$ the natural filtration of $(X(t), t \geq 0)$. In particular $\mathcal{M}(t)$ converges in $L^p(\mathbb{P})$ to $\mathcal{M}_\infty$ for some $p > 1$.*

*Proof* We know that $\mathcal{M}_n$ converges in $L^p(\mathbb{P})$ to $\mathcal{M}_\infty$ as $n$ tends to $\infty$, so

$$\mathbb{E}\left(\mathcal{M}_\infty \mid \mathcal{F}_t\right) = \lim_{n \to \infty} \mathbb{E}\left(\mathcal{M}_n \mid \mathcal{F}_t\right) \,.$$

On the other hand, it is easy to deduce from the Markov property applied at time $t$ that

$$\mathbb{E}\left(\mathcal{M}_n \mid \mathcal{F}_t\right) = \sum_{i=1}^{\infty} X_i^{p^*}(t) \mathbb{1}_{\{G(X_i(t)) \leq n\}} + \sum_{|u|=n} \xi_u^{p^*} \mathbb{1}_{\{a_u + \zeta_u < t\}} \,,$$

where $G(x)$ stands for the generation of the particle $x$ (i.e. $G(\xi_u) = |u|$), and $a_u + \zeta_u$ is the instant when the particle corresponding to the node $u$ splits. We can express the latter quantity in the form

$$a_u + \zeta_u = x_0 \mathbf{e}_0 + x_1^{-\alpha} \mathbf{e}_1 + \cdots + x_{|u|}^{-\alpha} \mathbf{e}_{|u|}$$

where $\mathbf{e}_0, \ldots$ is a sequence of independent exponential variables with parameter $\nu(\mathcal{S}^\downarrow)$, which is also independent of $\xi_u$, and $x_i$ stands for the size of particle labeled by the ancestor of $u$ at the $i$-th generation. When the index of self-similarity is non-negative, $x_i^{-\alpha} \geq 1$ and hence for each fixed node $u \in \mathcal{U}$, $a_u + \zeta_u$ is bounded from below by the sum of $|u|+1$ independent exponential variables with parameter $\nu(\mathcal{S}^\downarrow)$ which are independent of $\xi_u$. It follows that

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{|u|=n} \xi_u^{p^*} \mathbb{1}_{\{a_u+\zeta_u<t\}}\right) = 0\,,$$

and we conclude that $\mathbb{E}\left(\mathcal{M}_\infty \mid \mathcal{F}_t\right) = \mathcal{M}(t)$. □

We stress that the statement *fails* when $\alpha < 0$; more precisely we shall see in Section 1.3 that then $\mathcal{M}(t) = 0$ whenever $t$ is sufficiently large.

## 2.3 A randomly tagged branch

Throughout this section, we shall implicitly suppose for simplicity that the fragmentation starts from a single fragment with unit size, that is we shall work under $\mathbb{P} = \mathbb{P}_1$. Our purpose is to introduce a fundamental tool which enables explicit calculations for the first moment of certain functionals of the fragmentation. Informally, the idea is to follow the evolution of a fragment picked at random; one of the issues being of course to specify the meaning of 'picking a fragment at random'.

In this direction, let us say that a *branch* of the genealogical tree $\mathcal{U}$ is an infinite sequence of positive integers $b = (i_1, \ldots)$, which we can think of as the line of ancestors of some leaf of the tree, in the sense that for each $n$, we associate to $b$ its ancestor $b_n := (i_1, \ldots, i_n)$ at the generation $n$. Following an original idea developed by Lyons, Pemantle and Peres [61], we shall enrich the probabilistic structure by distinguishing at random a branch, called the *tagged branch*.

Specifically, we consider a pair $(M, \beta)$ where $M : \mathcal{U} \to [0,1] \times \mathbb{R}_+ \times \mathbb{R}_+$ is a random mark on the genealogical tree and $\beta$ is a random branch of $\mathcal{U}$, whose joint distribution denoted by $\mathbb{P}^*$ is specified as follows. Let $\mathcal{H}_n$ stand for the space of bounded functionals $\Phi$ which depend on the mark $M$ and the branch $\beta$ only up to the $n$-th generation, that is such that $\Phi(M, \beta) = \Phi(M', \beta')$ if $\beta_n = \beta'_n$ and $M(u) = M'(u)$ whenever $|u| \leq n$. For such functionals, it will be convenient to use the slightly abusing notation $\Phi(M, \beta) = \Phi(M, \beta_n)$. It is immediately seen from the dynamics of the random mark $M$ described in Proposition 3 and the definition of the Malthusian exponent that

$$\mathbb{E}\left(\sum_{|u|=n}\Phi(M,u)\xi_u^{p*}\right) = \mathbb{E}\left(\sum_{|u|=n}\Phi(M,u)\xi_u^{p*}\sum_{i=1}^{\infty}\tilde{\xi}_{ui}^{p^*}\right)$$
$$= \mathbb{E}\left(\sum_{|v|=n+1}\Phi(M,v)\xi_v^{p*}\right),$$

for every $\Phi \in \mathcal{H}_n$. By Kolmogorov's consistency theorem, this allows us to define unambiguously a probability measure $\mathbb{P}^*$ viewed as the joint distribution of a random mark $M$ and a random branch $\beta$ by

$$\mathbb{E}^*\left(\Phi(M,\beta)\right) = \mathbb{E}\left(\sum_{|u|=n}\Phi(M,u)\xi_u^{p*}\right), \qquad \Phi \in \mathcal{H}_n. \tag{21}$$

Note that since the intrinsic martingale $(\mathcal{M}_n, n \in \mathbb{Z}_+)$ is uniformly integrable (cf. Theorem 1), the first marginal of $\mathbb{P}^*$ is absolutely continuous with respect to the law of the random mark $M$ under $\mathbb{P}$, with density $\mathcal{M}_\infty$.

The sizes of particles on the tagged branch will play an important role in the analysis of fragmentation chains. More precisely, recall that $a_{\beta_n}$ and $\zeta_{\beta_n}$ denote respectively the birth-time and lifetime of the particle labeled by tagged node $\beta_n$ (i.e. $\beta_n$ is the node of the tagged branch at the $n$-th generation). We write $\chi_n = \xi_{\beta_n}$ for the size of the particle corresponding to the node $\beta_n$, and $\chi(t)$ for the size of the tagged particle alive at time $t$, that is to say

$$\chi(t) = \chi_n \quad \text{if } a_{\beta_n} \le t < a_{\beta_n} + \zeta_{\beta_n},$$

and

$$\chi(t) = 0 \quad \text{if } t \ge a_{\beta_\infty} := \lim_{n\to\infty} a_{\beta_n}.$$

In this direction, we make the easy observation that $a_{\beta_\infty} = \infty$ $\mathbb{P}^*$-a.s. when the index of self-similarity $\alpha$ is non-negative (whereas $a_{\beta_\infty} < \infty$ $\mathbb{P}^*$-a.s. when $\alpha < 0$ as we shall see later on).

The following lemma shows that the first moment of additive functionals of the fragmentation are easily expressed in terms of the tagged particle.

**Lemma 4** *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Then we have for every $n \in \mathbb{N}$*

$$\mathbb{E}^*\left(f(\chi_n)\right) = \mathbb{E}\left(\sum_{|u|=n}\xi_u^{p^*}f(\xi_u)\right).$$

*Moreover, if $f(0) = 0$, then for every $t \ge 0$*

$$\mathbb{E}^*\left(f(\chi(t))\right) = \mathbb{E}\left(\sum_{i=1}^{\infty}X_i^{p^*}(t)f(X_i(t))\right).$$

*Proof* The identity (21) yields the first formula. The second also derives readily from (21) by conditioning on the generation $\gamma(t)$ of the tagged particle at time $t$. Indeed, we

have

$$\begin{aligned}\mathbb{E}^*\left(f(\chi(t))\right) &= \mathbb{E}^*\left(\sum_{n=0}^{\infty} f(\chi_n)\mathbb{1}_{\{\gamma(t)=n\}}\right)\\ &= \sum_{n=0}^{\infty}\mathbb{E}^*\left(f(\chi_n)\mathbb{1}_{\{a_{\beta_n}\le t<a_{\beta_n}+\zeta_{\beta_n}\}}\right)\\ &= \sum_{n=0}^{\infty}\mathbb{E}\left(\sum_{|u|=n} f(\xi_u)\xi_u^{p^*}\mathbb{1}_{\{a_u\le t<a_u+\zeta_u\}}\right)\\ &= \mathbb{E}\left(\sum_{i=1}^{\infty} X_i^{p^*}(t)f(X_i(t))\right),\end{aligned}$$

where the last identity stems from Proposition 4. □

Lemma 4 will be useful for computing first moments for fragmentations in combination with the following proposition.

**Proposition 6** *Under $\mathbb{P}^*$,*

$$S_n := -\ln\chi_n\,,\qquad n\in\mathbb{Z}_+$$

*is a random walk on $\mathbb{R}_+$ with step distribution*

$$\mathbb{P}\left(\ln\chi_n-\ln\chi_{n+1}\in dy\right) = \tilde{\nu}(dy)/\nu(\mathcal{S}^{\downarrow})\,,$$

*where the finite measure $\tilde{\nu}$ is defined by*

$$\int_{]0,\infty[} f(y)\tilde{\nu}(dy) = \int_{\mathcal{S}^{\downarrow}}\left(\sum_{i=1}^{\infty} x_i^{p^*} f\left(-\ln x_i\right)\right)\nu(d\mathbf{x})\,.$$

*Equivalently, the Laplace transform of the step distribution is given by*

$$\mathbb{E}^*\left(\mathrm{e}^{-pS_1}\right) = \mathbb{E}^*\left(\chi_1^p\right) = 1-\kappa(p+p^*)/\nu(\mathcal{S}^{\downarrow})\,,\qquad p\ge 0\,.$$

*Moreover, conditionally on $(\chi_n, n\in\mathbb{Z}_+)$ the sequence of the lifetimes $(\zeta_{\beta_0},\zeta_{\beta_1},\ldots)$ along the tagged branch is a sequence of independent exponential variables with respective parameters $\chi_0^{\alpha}\nu(\mathcal{S}^{\downarrow}),\chi_1^{\alpha}\nu(\mathcal{S}^{\downarrow}),\ldots$*

*Proof* Consider a functional $\Phi\in\mathcal{H}_n$. We see from (21) that for every $q\ge 0$

$$\begin{aligned}\mathbb{E}^*\left(\exp(-q(S_{n+1}-S_n))\Phi(M,\beta)\right) &= \mathbb{E}^*\left(\chi_{n+1}^q\chi_n^{-q}\Phi(M,\beta)\right)\\ &= \mathbb{E}\left(\sum_{|u|=n}\sum_{i=1}^{\infty}\tilde{\xi}_{ui}^q(\xi_u\tilde{\xi}_{ui})^{p^*}\Phi(M,u)\right),\end{aligned}$$

where each $(\tilde{\xi}_{ui})_{i\in\mathbb{N}}$ is a random variable in $\mathcal{S}^{\downarrow}$ with law $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$, which is independent of $\Phi(M,u)$. This shows that under $\mathbb{P}^*$, $S_n$ is a random walk with step distribution given by

$$\mathbb{E}^*(f(S_1)) = \int_{\mathcal{S}^{\downarrow}}\left(\sum_{i=1}^{\infty} f(-\ln s_i)s_i^{p^*}\right)\frac{\nu(d\mathbf{s})}{\nu(\mathcal{S}^{\downarrow})},$$

and thus establishes the first claim. The second is obvious. □

In particular, in the so-called homogeneous case when the index of self-similarity is $\alpha = 0$, Proposition 6 shows that the size of the tagged particle $(\chi(t), t \geq 0)$ can be expressed in the form $\chi(t) = \exp(-\eta_t)$, where

$$\eta_t \,=\, S \circ N_t \,, \qquad t \geq 0 \,,$$

with $N$ a Poisson process with parameter $\nu(\mathcal{S}^{\downarrow})$ which is independent of the random walk $S$. In other words, the tagged particle is the exponential of the opposite of the $\eta = S \circ N$.

In the case $\alpha \neq 0$, the process $(\chi(t), t \geq 0)$ is Markovian and enjoys an obvious scaling property. More precisely, an immediate check can be made that it can be expressed in the form

$$\chi(t) \,=\, \exp(-\eta_{\tau(t)}) \,, \qquad t \geq 0 \,,$$

where $\eta$ is the compound Poisson defined above and $\tau$ the time-change given as the inverse of the functional

$$t \to \int_0^t \exp(\alpha \eta_s) ds \,.$$

Such transformations of compound Poisson processes, and more generally of Lévy processes, were first considered by Lamperti [55], as the fundamental representation of self-similar Markov processes on $]0, \infty[$. We stress that the time-change $\tau(t)$ is finite for all $t \geq 0$ when $\alpha \geq 0$, whereas when $\alpha < 0$,

$$\tau(t) < \infty \Longleftrightarrow \, t < I := \int_0^\infty \exp(\alpha \eta_s) ds \,,$$

and then

$$\chi(t) \,=\, 0 \qquad \text{whenever } t \geq I \,.$$

More precisely, $I < \infty$ a.s. when $\alpha < 0$, and this random variable has the same distribution as $a_{\beta_\infty}$, the limit of the birth-time $a_{\beta_n}$ of the tagged particle at the $n$-th generation as $n \to \infty$.

These observations enable us in particular to compute the moments of power sums of self-similar fragmentations.

**Proposition 7** *We have for every $p \geq p^*$ and $t \geq 0$:*
(i) *in the homogeneous case $\alpha = 0$,*

$$\mathbb{E}\left( \sum_{i=1}^\infty X_i^p(t) \right) \,=\, \mathbb{E}^*(\chi(t)^{p-p^*}) \,=\, \exp(-t\kappa(p)) \,,$$

(ii) *in the case $\alpha > 0$ when the index of self-similarity is positive,*

$$\mathbb{E}\left( \sum_{i=1}^\infty X_i^p(t) \right) \,=\, \mathbb{E}^*(\chi(t)^{p-p^*}) \,=\, \sum_{n=0}^\infty \frac{(-t)^n}{n!} \Gamma(n,p) \,,$$

*where $\Gamma(0,p) = 1$ and for $n \geq 1$*

$$\Gamma(n,p) \,=\, \prod_{k=0}^{n-1} \kappa(p + \alpha k) \,.$$

*Proof* (i) Thanks to Proposition 6, we have whenever $p + p^* > \underline{p}$ that

$$\begin{aligned}
\mathbb{E}^*(\chi(t)^p) &= \mathbb{E}^*(\exp(-pS_{N_t})) \\
&= \exp(-t\nu(\mathcal{S}^\downarrow)) \sum_{k=0}^{\infty} \frac{(t\nu(\mathcal{S}^\downarrow))^k}{k!} \left( \int \mathrm{e}^{-py} \tilde{\nu}(dy)/\nu(\mathcal{S}^\downarrow) \right)^k \\
&= \exp(-t\kappa(p+p^*))\,.
\end{aligned}$$

In other words, the so-called Laplace exponent of the compound Poisson process $S \circ N = \eta$ is $\kappa(p^* + \cdot)$, and this characterizes its law. The stated formula now derives from Lemma 4.

(ii) We start by observing from Lemma 4 that for every sufficiently large $p$

$$\begin{aligned}
\int_0^\infty \mathbb{E}\left( \sum_{i=1}^\infty X_i^{p+p^*}(t) \right) dt &= \int_0^\infty \mathbb{E}^*\left(\chi(t)^p\right) dt \\
&= \mathbb{E}^*\left( \int_0^\infty \exp(-p\eta_{\tau(t)}) dt \right) \\
&= \mathbb{E}^*\left( \int_0^\infty \exp(-(p-\alpha)\eta_t) dt \right) \\
&= \int_0^\infty \exp(-t\kappa(p^*+p-\alpha)) dt \\
&= 1/\kappa(p^*+p-\alpha)\,,
\end{aligned}$$

where the fourth equality stems from (i). Next, we apply the self-similarity and branching properties of the fragmentation to see that for every $t \geq 0$

$$\begin{aligned}
\mathbb{E}\left( \int_t^\infty ds \sum_{i=1}^\infty X_i^p(s) \right) &= \mathbb{E}\left( \sum_{i=1}^\infty X_i^{p-\alpha}(t) \right) \mathbb{E}\left( \int_0^\infty ds \sum_{i=1}^\infty X_i^p(s) \right) \\
&= \mathbb{E}\left( \sum_{i=1}^\infty X_i^{p-\alpha}(t) \right) / \kappa(p-\alpha)\,,
\end{aligned}$$

where the second equality follows from the identity above. Taking the derivative in the variable $t$, we arrive at

$$\frac{d}{dt}\mathbb{E}\left( \sum_{i=1}^\infty X_i^{p-\alpha}(t) \right) = -\kappa(p-\alpha)\mathbb{E}\left( \sum_{i=1}^\infty X_i^p(t) \right)\,.$$

This equation implies that whenever $p > p^*$, the function $t \to \mathbb{E}\left(\sum_{i=1}^\infty X_i^p(t)\right)$ is completely monotone, and thus, by Bernstein's theorem, can be expressed as the Laplace transform of some measure $\mu$ on $\mathbb{R}_+$:

$$\mathbb{E}\left( \sum_{i=1}^\infty X_i^p(t) \right) = \int_{\mathbb{R}_+} \mathrm{e}^{-tx} \mu(dx)\,.$$

More precisely, as the moments of $\mu$ can be recovered from the derivatives of its Laplace transform, we get

$$\int_{\mathbb{R}_+} x^k \mu(dx) = \kappa(p) \ldots \kappa(p+(k-1)\alpha)\,.$$

By the series expansion of $\mathrm{e}^{-tx}$, we finally arrive at the expression

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i^p(t)\right) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \int_{\mathbb{R}_+} x^k \mu(dx)$$

which yields the desired formula. □

As a check, it may be interesting to recover the formulas in Proposition 7 using the expression (15) for the infinitesimal generator of the fragmentation chain applied to additive functionals and the backward equation.

# 3 Extinction and formation of dust for $\alpha < 0$

In this section, we shall again implicitly assume that at the initial time, the fragmentation chain starts from a single particle with unit size. The purpose of this section is to point to the following phenomenon. Under a natural and fairly general assumption, any self-similar fragmentation with a negative index of self-similarity almost surely reaches the absorbing state $(0, 0, \ldots)$ at a finite time (see Proposition 8 below). In the situation where the dislocation measure is conservative, that is when each time a particle splits, the sum of the sizes of the children particles is the same as the size of their parent, it is convenient to think of the size of a particle as a mass. So, when the index of self-similarity is negative, even if the dislocation measure is conservative, the initial mass is eventually reduced to dust (i.e. fragments of infinitesimal mass), and in the final section, we will investigate the phenomenon of formation of dust as time passes.

## 3.1 Extinction

Intuitively, when the index of self-similarity is negative, fragments with small sizes are subject to high splitting rates, and this makes them vanish entirely quickly. To make a rigorous statement, we shall suppose throughout this section that

$$\int_{\mathcal{S}^\downarrow} \#\left\{i \in \mathbb{N} : s_i = 1\right\} \nu(d\mathbf{s})/\nu(\mathcal{S}^\downarrow) \; < \; 1\,. \tag{22}$$

To explain the role of this condition, observe that the process $\#\left\{i \in \mathbb{N} : X_i(t) = 1\right\}$ that counts the number of fragments with unit size as time passes, is a branching process in continuous time, and (22) amounts to assuming that this branching process is sub-critical so that it becomes extinct a.s. Plainly, in the super-critical situation where

$$\int_{\mathcal{S}^\downarrow} \#\left\{i \in \mathbb{N} : s_i = 1\right\} \nu(d\mathbf{s})/\nu(\mathcal{S}^\downarrow) \; > \; 1\,,$$

this branching process has a positive probability of surviving forever, which clearly would impede the extinction of the fragmentation.

**Proposition 8** *Let (22) be fulfilled. Then the following assertions hold with probability one:*

(i) *For $\alpha < 0$, $X(t) = (0, \ldots)$ for all sufficiently large $t$.*

(ii) *When $\kappa(-\alpha) > 0$, for almost every $t > 0$*

$$\#\left\{j \in \mathbb{N} : X_j(t) > 0\right\} \, < \, \infty \, .$$

We stress that in general, no matter what the value of $\alpha$ is, there may exist random instants $t$ at which

$$\#\left\{j \in \mathbb{N} : X_j(t) > 0\right\} \, = \, \infty \, .$$

For instance when the dislocation measure fulfills

$$\nu\left(x_j = 0 \text{ for some } j \in \mathbb{N}\right) \, = \, 0 \, ,$$

then with probability one, each dislocation in the fragmentation produces infinitely many terms. This does not induce any contradiction with Proposition 8 (ii) when $\kappa(-\alpha) > 0$, because informally, as the index of self-similarity is negative, we know that fragments with small size vanish quickly.

*Proof* Recall that $\{\xi_u, |u| = k\}$ denotes the set of particles at the $k$-th generation. We get from Lemma 4 and Proposition 6 that

$$\mathbb{E}\left(\sum_{|u|=k} |\xi_u|^p\right) \, = \, \gamma(p)^k \, , \tag{23}$$

where

$$\gamma(p) = \int_{\mathcal{S}^\downarrow} \sum_{i=1}^{\infty} s_i^p \nu(d\mathbf{s})/\nu(\mathcal{S}^\downarrow) \, = \, \frac{\nu(\mathcal{S}^\downarrow) - \kappa(p)}{\nu(\mathcal{S}^\downarrow)} \, .$$

By monotone convergence, (22) yields

$$\lim_{p\to\infty} \kappa(p) \, = \, \int_{\mathcal{S}^\downarrow} \left(1 - \#\{i \in \mathbb{N} : s_i = 1\}\right) \nu(d\mathbf{s}) \, > \, 0 \, .$$

This enables us to choose $p$ sufficiently large such that $\kappa(p) > 0$, that is such that $\gamma(p) < 1$. In particular, the series

$$\sum_{k=1}^{\infty} \sum_{|u|=k} |\xi_u|^p$$

has a finite mean and thus converges a.s., and a fortiori

$$\lim_{k\to\infty} \max_{|u|=k} |\xi_u| \, = \, 0 \, . \tag{24}$$

Now suppose for simplicity that $\nu(\mathcal{S}^\downarrow) = 1$, pick $a > 0$ arbitrary and consider the event that for some generation $k$, there exists at least one particle $\xi_u$, $|u| = k$, with lifetime $\zeta_u > a/k^2$. Because each particle with size $x$ has a lifetime which is exponentially distributed with parameter $x^\alpha$, the probability of this event can be bounded from above by

$$\sum_{k=1}^{\infty} \mathbb{E}\left(\sum_{|u|=k} \exp\left(-ak^{-2}|\xi_u|^\alpha\right)\right) \, \leq \, c_p a^{-p} \sum_{k=1}^{\infty} k^{2p} \mathbb{E}\left(\sum_{|u|=k} |\xi_u|^{-\alpha p}\right) \, ,$$

where $c_p$ is some constant which depends only on $p$. Now use (23) and take $p$ sufficiently large, so that the right-hand side can be bounded from above by $c'_p a^{-p}$, where $c'_p$ is some constant which depends only on $p$ and $\nu$.

We see that provided that $a$ is chosen large enough, the probability that for all $k$ there are no particles of the $k$-th generation alive at time

$$t := \zeta_\varnothing + a \sum_{k=1}^{\infty} k^{-2} \,,$$

can be made as close to 1 as we wish. Recalling (24), this completes the proof of (i).

Next, suppose $\kappa(-\alpha) > 0$. In this situation, we have by (23) that

$$\mathbb{E}\left(\sum_{u\in\mathcal{U}} \zeta_u\right) \;=\; \mathbb{E}\left(\sum_{k=0}^{\infty} \sum_{|u|=k} \xi_u^{-\alpha}\right) \;<\; \infty\,.$$

So, if $I_u$ denotes the time-interval during which the particle with label $u$ is alive (so the length of $I_u$ is the lifetime $\zeta_u$ of this particle), we have

$$\mathbb{E}\left(\int_0^\infty dt \sum_{u\in\mathcal{U}} \mathbb{1}_{\{t\in I_u\}}\right) \;=\; \mathbb{E}\left(\sum_{u\in\mathcal{U}} \zeta_u\right) \;<\; \infty\,,$$

which implies that for almost every $t \geq 0$, there are only finitely many particles alive at time $t$. □

It would be interesting to have information on the distribution of the extinction time

$$T := \inf\left\{t \geq 0 : X(t) = (0, 0, \ldots)\right\} \,,$$

however, it does not seem possible to express this law in a closed form. Nonetheless, we point out that an application of the branching property at the first dislocation (cf. Proposition 3) yields the identity in distribution

$$T \overset{(\mathrm{d})}{=} \mathbf{e} + \max_{j\in\mathbb{N}} \xi_j^{-\alpha} T_j \,, \tag{25}$$

where $\mathbf{e}$ is a standard exponential variable, $(\xi_j, j \in \mathbb{N})$ is distributed according to $\nu(\cdot)/\nu(\mathcal{S}^\downarrow)$, $(T_j, j \in \mathbb{N})$ is a sequence of independent copies of $T$, and $\mathbf{e}$, $(\xi_j, j \in \mathbb{N})$ and $(T_j, j \in \mathbb{N})$ are independent. We refer to [2] for a survey of this type of equation in distribution.

## 3.2 Formation of dust

In this section, we focus on the case when the dislocation measure is conservative, that is when

$$\nu\left(\sum_{i=1}^{\infty} s_i \neq 1\right) \;=\; 0\,. \tag{26}$$

It is easy to deduce by iteration that for every $n \in \mathbb{N}$, the total mass of particles at the $n$-th generation is conserved, that is

$$\sum_{|u|=n} \xi_u \;=\; 1\,, \qquad \text{a.s.}$$

Turning our interest to the total mass of particles at time $t$, we introduce the quantity

$$\mathtt{D}(t) := 1 - \sum_{n=1}^{\infty} X_i(t)\,,$$

which can be viewed as the total mass of *dust*, that is of infinitesimal particles at time $t$. One could be tempted to believe that the assumption (26) would yield $\mathtt{D} \equiv 0$; indeed the argument in the proof of Proposition 5 shows that this holds when the index of self-similarity of the fragmentation is non-negative. However, Proposition 8 shows that for negative indices of self-similarity, $\mathtt{D}$ reaches 1 at a finite time a.s.

Furthermore, one readily sees that the process $\mathtt{D}$ increases and has right-continuous paths a.s. Indeed, let us fix $\varepsilon > 0$ and write for every $t \geq 0$

$$\mathtt{D}_\varepsilon(t) := 1 - \sum_{i=1}^{\infty} X_i(t) \mathbb{1}_{\{X_i(t)>\varepsilon\}}\,.$$

So $\mathtt{D}_\varepsilon(t)$ is a functional of the Markov chain $\varphi_\varepsilon(X(t))$, where $\varphi_\varepsilon : \mathcal{S}^\downarrow \to \mathcal{S}^\downarrow$ is the threshold operator which consists in removing fragments with size less than or equal to $\varepsilon$ (cf. Section 1.1.3). It is immediately seen that the process $\mathtt{D}_\varepsilon(\cdot)$ is increasing with right-continuous paths, and, by monotone convergence, that it decreases to $\mathtt{D}(\cdot)$ when $\varepsilon \downarrow 0$. As a consequence, the process $\mathtt{D}(\cdot)$ increases. Moreover, for every $t \geq 0$ we have

$$\mathtt{D}(t+) := \lim_{s \downarrow t} \mathtt{D}(s) \;=\; \inf_{s>t} \inf_{\varepsilon>0} \mathtt{D}_\varepsilon(s)\,,$$

and we can rewrite the right-hand side as

$$\inf_{\varepsilon>0} \inf_{s>t} \mathtt{D}_\varepsilon(s) \;=\; \lim_{\varepsilon \to 0} \mathtt{D}_\varepsilon(t) = \mathtt{D}(t)\,,$$

which shows that $\mathtt{D}(\cdot)$ is right-continuous. The following proposition gathers a couple of simple observations on this phenomenon of formation of dust.

**Proposition 9** *Suppose (26) holds and $\alpha < 0$. The following assertions hold with probability one:*

(i) $\mathtt{D}$ *is a continuous increasing process which reaches* 1 *in finite time.*

(ii) *If $\#(t) := \#\{i \in \mathbb{N} : X_i(t) > 0\}$ denotes the number of fragments with positive mass at time $t$, then*

$$\int_0^\infty \mathbb{1}_{\{\#(t)<\infty\}} d\mathtt{D}(t) \;=\; 0\,.$$

This statement again reflects the fact that, informally, dislocations occur faster and faster as time passes. Observe that it implies that almost-surely, there exist uncountable many times at which there are infinitely many fragments with positive size, which may be rather surprising (for instance in the case when dislocations are binary, that is produce exactly two fragments; see also Proposition 8).

*Proof* (i) We have already seen that $\mathtt{D}$ has right-continuous increasing paths that reach 1 a.s.; let us check the absence of jumps. In this direction, it is convenient to write $\partial\mathcal{U} = \mathbb{N}^{\mathbb{N}}$ for the boundary of the genealogical tree, that is the set of infinite sequences of integers.

An element $\ell \in \partial\mathcal{U}$ can be thought of as a leaf of the tree, and can also be identified with the branch connecting $\ell$ to the root $\varnothing$. For every $\ell, \ell' \in \partial\mathcal{U}$, we can define the distance $\mathtt{d}(\ell, \ell') = 2^{-g(\ell,\ell')}$, where $g(\ell, \ell')$ is the generation of the last common ancestor of $\ell$ and $\ell'$ (i.e. the sequences $\ell$ and $\ell'$ coincide up to the $g(\ell, \ell')$-th term and then differ).

We can use the random mark on $\mathcal{U}$ to define the length of the branch connecting a leaf $\ell$ to the root as

$$\lambda(\ell) := \sum_{n=0}^{\infty} \zeta_{\ell_n} ,$$

where $\ell_n$ is the node of the branch at the $n$-th generation (i.e. the sequence of the $n$ first terms of $\ell$), and $\zeta_{u_n}$ the lifetime of the fragment labeled by $u_n$. We stress that with probability one, $\lambda(\ell) < \infty$ for all leaves $\ell \in \partial\mathcal{U}$, see the proof of Proposition 8(i). On the other hand, the fact that dislocations preserve masses also enables us to endow $\partial\mathcal{U}$ with a natural (random) probability measure $\pi$: given a leaf $\ell \in \partial\mathcal{U}$ and an integer $n \in \mathbb{N}$, the ball $B(\ell, 2^{-n})$ centered at $\ell$ with radius $2^{-n}$ consists in the set of leaves $\ell'$ that have the same ancestor $\ell_n \in \mathbb{N}^n$ as $\ell$ at generation $n$, and we set $\pi(B(\ell, 2^{-n})) = \xi_{\ell_n}$, where $\xi_{\ell_n}$ is the size of the fragment with label $\ell_n$.

Recall from Proposition 4 that

$$\sum_{n=1}^{\infty} X_i(t) \;=\; \sum_{u\in\mathcal{U}} \xi_u \mathbb{1}_{\{t\in I_u\}}$$

where $I_u$ stands for the life-interval of the fragment labeled by $u$, and observe that in this setting the right-hand side can be expressed as $\pi(\{\lambda(\ell) > t\})$. Thus there is the identity

$$\mathtt{D}(t) \;=\; \pi(\{\lambda(\ell) \le t\}) , \tag{27}$$

and all that is needed now is to check that $\mathbb{P}$-a.s.

$$\pi \otimes \pi \left(\{\ell, \ell' \in \partial\mathcal{U} : \lambda(\ell') = \lambda(\ell)\}\right) \;=\; 0 . \tag{28}$$

Given the mark on the genealogical tree induced by the fragmentation, let us pick a leaf $L$ at random by the same procedure as for the randomly tagged branch in Section 1.2.3. The ancestor $L_0$ of $L$ at generation 0 is $\varnothing$, and for every $n \in \mathbb{N}$, given the ancestor $L_{n-1}$ of $L$ at generation $n-1$, the ancestor $L_n$ at generation $n$ is chosen at random among the children of $L_{n-1}$ with probability proportional to their sizes (i.e. by size-biased sampling). It should be plain from this construction that $L$ has law $\pi$. Next pick a second leaf $L'$ at random, independently of the first, and according to the same procedure, so that given the marks, $L$ and $L'$ are two independent variables distributed according to $\pi$. On the one hand, it follows from (24) that the random measure $\pi$ has no atoms, that is it does not charge $\{\ell\}$ for any $\ell \in \partial\mathcal{U}$, so $\pi(L = L') = 0$. Now by conditioning on the last common ancestor of $L$ and $L'$, we see that the lengths $\lambda(L)$ and $\lambda(L')$ are different $\pi$-a.s., which establishes (28).

(ii) As above, we work with a leaf $L$ picked at random according to $\pi$, and write $L_n$ for the ancestor of $L$ at generation $n$. For each $n \in \mathbb{N}$, consider the event $\Lambda_n$ that among the nodes at the $n$-th generation with parent $L_{n-1}$ (i.e. the brothers of $L_n$), at least one of them has a lifetime which is larger than the distance from $L_n$ to $L$. Observe that, by construction, the fragment corresponding to such a node is alive at time $\lambda(L)$,

and that its size cannot be zero, as nodes with zero size have lifetime $\zeta = 0$. From the description of the dynamics of the random marks in Proposition 3, it is readily seen that the probability of $\Lambda_n$ is bounded away from 0 as $n \to \infty$, and then, by Kolmogorov's 0-1 law, that $\limsup \Lambda_n$ has probability 1. This shows that at time $\lambda(L)$, there are infinitely many fragments with positive size, and by (27), this completes the proof. □

# 4 Some strong laws for $\alpha \geq 0$

In this section, we consider the asymptotic behavior of self-similar fragmentation chains, and shall establish some strong limit theorems for (a weighted version of) the empirical distribution of the fragments. Again, for the sake of simplicity, we assume implicitly that at the initial time the process starts from a single fragment with unit size; the general case follows easily. The approach relies crucially on an extension of the classical law of large numbers that we now present.

## 4.1 A variation of the law of large numbers

To start with, let us specify the setting. For each $t \geq 0$, let $\lambda(t) = (\lambda_i(t), i \in \mathbb{N})$ be a sequence of non-negative random variables such that for some fixed $p > 1$

$$\sup_{t \geq 0} \mathbb{E}\left(\left(\sum_{i=1}^{\infty} \lambda_i(t)\right)^p\right) < \infty \quad \text{and} \quad \lim_{t \to \infty} \mathbb{E}\left(\sum_{i=1}^{\infty} \lambda_i^p(t)\right) = 0\,.$$

Let also $(Y_i(t), i \in \mathbb{N})$ be a sequence of random variables which are independent conditionally on $\lambda(t)$. Finally, assume there is a sequence $(\bar{Y}_i, i \in \mathbb{N})$ of independent and identically distributed variables in $L^p(\mathbb{P})$, which is independent of $\lambda(t)$ for each fixed $t$, and such that $|Y_i(t)| \leq \bar{Y}_i$ for all $i \in \mathbb{N}$ and $t \geq 0$. We can now state:

**Lemma 5** *Under the preceding assumptions,*

$$\lim_{t \to \infty} \sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}(Y_i(t) \mid \lambda(t))) = 0 \qquad \textit{in } L^p(\mathbb{P}).$$

Before establishing this result, let us make the connection with the classical law of large numbers. Let $(Y_i, i \in \mathbb{N})$ be a sequence of i.i.d. variables with a finite $p$-th moment. We set $Y_i(t) = Y_i$ for all $t$, and $\lambda_i(t) = 1/t$ if $i \leq t$ and $\lambda_i(t) = 0$ otherwise. Then the assumptions above are clearly fulfilled, and an application of Lemma 5 gives

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{[t]} (Y_i - \mathbb{E}(Y_i)) = 0 \qquad \text{in } L^p(\mathbb{P}),$$

or equivalently $\lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} Y_i = \mathbb{E}(Y_1)$ in $L^p(\mathbb{P})$.

*Proof* Let $a > 0$ be an arbitrarily large real number. Introduce for every $i \in \mathbb{N}$ and $t \geq 0$ the truncated variables $Y_i(t,a) = \mathbb{1}_{\{|Y_i(t)|<a\}} Y_i(t)$. To start with, there is the upper bound

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - \mathbb{E}(Y_i(t) \mid \lambda(t)))\right| \;\leq\; & \left|\sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - Y_i(t,a))\right| \\ & + \left|\sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t,a) - \mathbb{E}(Y_i(t,a) \mid \lambda(t)))\right| \\ & + \left|\sum_{i=1}^{\infty} \lambda_i(t)\mathbb{E}(Y_i(t,a) - Y_i(t) \mid \lambda(t))\right| .\end{aligned}$$

Consider the first series in the right-hand side. The bounds $|Y_i(t) - Y_i(t,a)| \leq \mathbb{1}_{\{\bar{Y}_i > a\}} \bar{Y}_i$ and the independence of $(\bar{Y}_i, i \in \mathbb{N})$ and $\lambda(t)$ yield

$$\mathbb{E}\left(\left|\sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t) - Y_i(t,a))\right|^p\right) \leq \mathbb{E}(\mathbb{1}_{\{\bar{Y}_1 > a\}} \bar{Y}_1^p)\mathbb{E}\left(\left(\sum_{i=1}^{\infty} \lambda_i(t)\right)^p\right) ,$$

and the latter quantity converges to 0 as $a \to \infty$, uniformly for $t \geq 0$. The same argument also shows that

$$\lim_{a\to\infty} \sup_{t\geq 0} \mathbb{E}\left(\left|\sum_{i=1}^{\infty} \lambda_i(t)\mathbb{E}\left(Y_i(t) - Y_i(t,a) \mid \lambda(t)\right)\right|^p\right) \;=\; 0.$$

Finally, conditionally on $\lambda(t)$, the variables $Y_i(t,a) - \mathbb{E}(Y_i(t,a) \mid \lambda(t))$ are centered, independent, and bounded in absolute value by $a$. Thus, conditionally on $\lambda(t)$,

$$\sum_{i=1}^{n} \lambda_i(t)(Y_i(t,a) - \mathbb{E}(Y_i(t,a)) \mid \lambda(t)) , \qquad n \in \mathbb{N}$$

is a martingale bounded in $L^p$ and there exists a universal constant $c_p$ such that

$$\mathbb{E}\left(\left|\sum_{i=1}^{\infty} \lambda_i(t)(Y_i(t,a) - \mathbb{E}(Y_i(t,a) \mid \lambda(t))\right|^p \mid \lambda(t)\right) \leq c_p a^p \sum_{i=1}^{\infty} \lambda_i^p(t) .$$

Our assumptions ensure that the latter quantity converges to 0 as $t \to \infty$ in $L^1(\mathbb{P})$, so putting the pieces together this completes the proof of the statement. □

Let us now explain how we shall apply Lemma 5 in the rest of this section. We shall be interested in limit theorems involving functionals of the fragmentation of the type

$$A(t) := \sum_{i=1}^{\infty} X_i^{p^*}(t) g(X_i(t), t) ,$$

where $p^*$ is the Malthusian exponent and $g$ a certain measurable function. The first step of the analysis consists in considering this functional at time $t + s$ and applying the branching property of the fragmentation at time $t$. This yields an expression of the form

$$A(t+s) \;=\; \sum_{k=1}^{\infty} \lambda_k(t) Y_k(t,s)$$

where $\lambda_k(t) = X_k^{p^*}(t)$ and $Y_k(t,s)$ is a quantity depending of the fragmentation started at time $t$ from a single particle with size $X_k(t)$. More precisely, writing $\tilde{X}$ for the latter, we have

$$Y_k(t,s) \;=\; \sum_{j=1}^{\infty} \frac{\tilde{X}_j^{p^*}(s)}{X_k^{p^*}(t)} g(\tilde{X}_j(s), t+s)\,.$$

We then use Lemma 5 to reduce the study of the asymptotic behavior of the latter quantity as both $t, s \to \infty$ to that of

$$\sum_{k=1}^{\infty} \lambda_k(t) \mathbb{E}(Y_k(t,s) \mid X_k(t))\,.$$

Essentially, this reduction amounts to getting estimates for the first moment of additive functionals of the fragmentation, which are then obtained by limit theorems for the tagged particle (cf. Section 1.2.3). Roughly speaking, one shows that $\mathbb{E}(Y_k(t,s) \mid X_k(t)) \sim c$ as $s, t \to \infty$ for some constant $c$ depending on $g$, so, using Proposition 5, we can conclude that

$$A(t+s) \sim c \sum_{k=1}^{\infty} \lambda_k(t) \;=\; c \sum_{k=1}^{\infty} X_k^{p^*}(t) \sim c\mathcal{M}_\infty\,.$$

Of course, one has to check carefully the estimates above, which is somewhat technical.

## 4.2 The homogeneous case ($\alpha = 0$)

We suppose throughout this section that $X = (X(t), t \geq 0)$ is a homogeneous fragmentation (i.e. the index of self-similarity is $\alpha = 0$) for which the Malthusian hypotheses of Theorem 1 hold. We are interested in the asymptotic behavior of the empirical distribution of the fragments as time tends to infinity. The following pathwise limit theorem extends a result due to Kolmogorov [50] in the conservative case, which seems to have appeared in the very first probabilistic work on fragmentation. See also Asmussen and Kaplan [4] for a closely related result.

Introduce the first and second right-derivatives of $\kappa$ at the Malthusian exponent $p^*$, which are given in terms of the dislocation measure by

$$\kappa'(p^*) \;=\; -\int_{\mathcal{S}^\downarrow} \left( \sum_{i=1}^{\infty} x_i^{p^*} \ln x_i \right) \nu(d\mathbf{x})\;, \quad \kappa''(p^*) \;=\; -\int_{\mathcal{S}^\downarrow} \left( \sum_{i=1}^{\infty} x_i^{p^*} (\ln x_i)^2 \right) \nu(d\mathbf{x})\,.$$

Note that these quantities are finite since $p^* > \underline{p}$. Recall that $\mathcal{M}_\infty$ denotes the terminal value of the intrinsic martingale in Theorem 1.

**Theorem 2** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bounded function.*

(i) *We have*

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{-1} \ln X_i(t)) \;=\; \mathcal{M}_\infty f(-\kappa'(p^*))\,,$$

*in $L^p(\mathbb{P})$ for some $p > 1$.*

(ii) *Denote by* $\mathcal{N}(0,-\kappa''(p^*))$ *a centered Gaussian distribution with variance* $-\kappa''(p^*)$. *Then*

$$\lim_{t\to\infty}\sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{-1/2}(\ln X_i(t)+\kappa'(p^*)t)) \;=\; \mathcal{M}_\infty \mathbb{E}(f(\mathcal{N}(0,-\kappa''(p^*)))\,,$$

*in* $L^p(\mathbb{P})$ *for some* $p>1$.

Loosely speaking, the first part of the statement shows that most fragments decay exponentially fast with rate $\kappa'(p^*)$ as time tends to infinity, and the second part provides a sharper estimate of the second order.

*Proof* We shall apply Lemma 5 in the following situation. Let $f:\mathbb{R}_+\to[0,1]$ be a continuous function; we are interested in

$$\sum_{k=1}^{\infty} X_k^{p^*}(t+t^2) f\left((t+t^2)^{-1}\ln X_k(t+t^2)\right)\,.$$

By an application of the Markov property at time $t$ and self-similarity, we can re-express this variable in the form

$$\sum_{i=1}^{\infty}\lambda_i(t)Y_i(t)$$

where $\lambda_i(t)=X_i^{p^*}(t)$ and

$$Y_i(t)=\sum_{j=1}^{\infty} X_{ij}^{p^*}(t^2) f\left((t+t^2)^{-1}\ln(X_i(t)X_{ij}(t^2))\right)\,,$$

with $X_{1\cdot}, X_{2\cdot},\ldots$ a sequence of i.i.d. copies of $X$ which is independent of $X(t)=(X_1(t),\ldots)$.

It follows from Proposition 5 that the sequence $(\lambda_i(t), i\in\mathbb{N})$ fulfills the requirement of Lemma 5. Let us now consider the sequence $(Y_i(t), i\in\mathbb{N})$ conditionally on $X(t)$. Plainly, it is given by independent variables, each of which is bounded from above by

$$\bar{Y}_i := \sup_{s\geq 0}\sum_{j=1}^{\infty} X_{ij}^{p^*}(s)\,.$$

On the one hand, the $\bar{Y}_i$ are clearly i.i.d. On the other hand, because $\sum_{j=1}^{\infty} X_{ij}^{p^*}(s)$ is a martingale which is bounded in $L^p(\mathbb{P})$ for some $p>1$, it follows from Doob's inequality that its overall supremum belongs to $L^p(\mathbb{P})$.

Thus we may apply Lemma 5, which reduces the study to that of the asymptotic behavior of

$$\sum_{i=1}^{\infty}\lambda_i(t)\mathbb{E}(Y_i(t)\mid X(t))$$

as $t$ tends to $\infty$. In this direction, we thus compute the conditional expectation of $Y_i(t)$ given $X(t)$; we easily get on the event $\{X_i(t)=x\}$ that

$$\mathbb{E}(Y_i(t)\mid X(t)) = \mathbb{E}\left(\sum_{j=1}^{\infty} X_j^{p^*}(t^2) f((t+t^2)^{-1}(\ln X_j(t^2)+\ln x))\right).$$

Now recall the notion of tagged particle $(\chi(t), t \geq 0)$ in Section 1.2.3. In particular we know from Lemma 4 that there is the identity

$$\mathbb{E}\left(\sum_{j=1}^{\infty} X_j^{p^*}(t^2) f((t+t^2)^{-1}(\ln X_j(t^2) + \ln x))\right) \;=\; \mathbb{E}^*(f((t+t^2)^{-1}(\ln \chi(t^2) + \ln x)).$$

Moreover, recall from Proposition 6 that the process of the logarithm of size of the tagged particle is a compound Poisson process, $-\ln \chi(t) = S(N_t)$, where $S$ is a random walk with step distribution $\tilde{\nu}(\cdot)/\nu(\mathcal{S}^{\downarrow})$ and $N$ an independent Poisson process with parameter $\nu(\mathcal{S}^{\downarrow})$. In particular, $S(1)$ has finite mean $\kappa'(p^*)/\nu(\mathcal{S}^{\downarrow})$, and it follows from the law of large numbers that

$$\lim_{t\to\infty} \mathbb{E}^*(f((t+t^2)^{-1}(\ln \chi(t^2) + \ln x)) \;=\; f(-\kappa'(p^*))\,,$$

where the limit is uniform in $x$ such that, say, $-\ln x \leq t^{3/2}$. On the other hand, using again Lemma 4, we have

$$\mathbb{E}\left(\sum_{i=1}^{\infty} X_i^{p^*}(t) \mathbb{1}_{\{-\ln X_i(t)) > t^{3/2}\}}\right) = \mathbb{P}^*(-\ln \chi(t) > t^{3/2})\,,$$

and the latter quantity tends to 0 as $t \to \infty$.

Recall from Proposition 5 that $\sum_{i=1}^{\infty} \lambda_i(t)$ converges to $\mathcal{M}_\infty$ in $L^p(\mathbb{P})$. Putting the pieces together, we get that as $t \to \infty$

$$\sum_{i=1}^{\infty} \lambda_i(t) \mathbb{E}(Y_i(t) \mid X(t)) \sim f(-\kappa'(p^*)) \sum_{i=1}^{\infty} \lambda_i(t) \sim \mathcal{M}_\infty f(-\kappa'(p^*))\,.$$

(ii) The proof is similar; the arguments above are easily adapted to reduce the proof to asymptotics for the first moment

$$\mathbb{E}^*(f(t^{-1/2}(\ln \chi(t) + t\kappa'(p^*)))\,,$$

where $f$ is a continuous bounded function. We may then use the central limit theorem for the compound random walk $-\ln \chi(t) = S \circ N_t = \eta_t$ to show that the preceding quantity converges to $\mathbb{E}(f(\mathcal{N}(0, -\kappa''(p^*))))$ when $t \to \infty$. Details are left to the reader. □

## 4.3 The case $\alpha > 0$

We now suppose throughout this section that $X = (X(t), t \geq 0)$ is a self-similar fragmentation with index $\alpha > 0$, with a dislocation measure $\nu$. Again, we shall assume that the Malthusian hypotheses of Theorem 1 hold and we will be interested in the asymptotic behavior of the process as time tends to infinity. Because for positive indices of self-similarity, small fragments split more slowly than large fragments, we may expect some homogenization phenomenon.

We now state the main result of this section, which has been obtained first by Filippov [38] in the conservative case. Its extension to non-conservative self-similar fragmentations has been established recently by Bertoin and Gnedin [18] (see also the forthcoming Section 1.6 for further references related to this result). Roughly, it shows that most fragments decay like $t^{-1/\alpha}$ as time $t$ tends to infinity, which contrasts with the exponential decay in the homogeneous case (see Theorem 2).

**Theorem 3** *Suppose that $\alpha > 0$ and that the step distribution of the random walk $S_n = -\ln \chi_n$ is not arithmetic. Then for every bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}$*

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} X_i^{p^*}(t) f(t^{1/\alpha} X_i(t)) \;=\; \mathcal{M}_\infty \int_0^\infty f(y)\rho(dy)\,, \qquad \text{in } L^1(\mathbb{P}),$$

*where $\mathcal{M}_\infty$ is the terminal value of the intrinsic martingale and $\rho$ is a deterministic probability measure. More precisely, $\rho$ is determined by the moments*

$$\int_{]0,\infty[} y^{\alpha k}\rho(dy) \;=\; \frac{(k-1)!}{\alpha\kappa'(p^*)\,\kappa(p^*+\alpha)\cdots\kappa(p^*+(k-1)\alpha)} \quad \text{for } k \in \mathbb{N},$$

*(with the usual convention that the right-hand side above equals $1/(\alpha\kappa'(p^*))$ for $k = 1$).*

Before proving this result, let us consider a couple of examples. First, recall the Poissonian rain model of Section 1.1.3, for which $\alpha = 1$ and $\kappa(p) = (p-1)/(p+1)$. The dislocation measure is conservative, so $p^* = 1$ and the intrinsic martingale $\mathcal{M}$ is constant. We find

$$\int_{]0,\infty[} y^k \rho(dy) \;=\; \frac{(k-1)!}{\frac{1}{2}\times\frac{1}{3}\times\frac{2}{4}\times\cdots\times\frac{k-1}{k+1}} \;=\; (k+1)!\,,$$

and we conclude that $\rho$ has density $\rho(dy)/dy = y\mathrm{e}^{-y}$, a result which can also be checked directly by more elementary calculations for this specific case. More generally, for self-similar fragmentation chains with index $\alpha > 0$ and $\kappa(p) = (p-1)/(p+1)$, one gets that $\rho$ is the distribution of $Y^{1/\alpha}$ where $Y$ has the gamma$(2/\alpha)$ law; see [29].

Second, suppose that the dislocation measure $\nu$ is the distribution induced by the uniform stick-breaking scheme as described after equation (17). So $\kappa(p) = 1 - 1/p$, and the Malthusian exponent is $p^* = 1$. Moreover, the dislocation measure is conservative, and the intrinsic martingale is thus trivial, $\mathcal{M}_n \equiv 1$. Suppose further that the index of self-similarity is $\alpha = 1$. One then gets

$$\frac{(k-1)!}{\alpha\kappa'(p^*)\,\kappa(p^*+\alpha)\cdots\kappa(p^*+(k-1)\alpha)} \;=\; k!\,, \qquad k \in \mathbb{N}\,,$$

so the probability measure $\rho$ appearing in Theorem 3 is simply the standard exponential distribution.

Just as in the homogeneous case, Lemma 5 reduces the proof to the analysis of the so-called tagged particle; we shall only provide details on the latter aspect. Recall from Proposition 6 that the process $(\chi(t), t \geq 0)$ is a continuous time Markov chain, which enjoys an obvious scaling property. The classical renewal theory yields an important limit theorem for $\chi(t)$ as $t$ tends to infinity, which is due to Brennan and Durrett [29].

**Proposition 10** *Suppose that the step distribution of the random walk $S_n = -\ln \chi_n$ is not arithmetic. Then under $\mathbb{P}^*$, $t^{1/\alpha}\chi(t)$ converges in distribution as $t \to \infty$ towards some variable $Y_\alpha$ which can be expressed in the form*

$$Y_\alpha \;=\; \left(\sum_{n=0}^{\infty} \exp(-\alpha R_n)\mathbf{e}_n\right)^{1/\alpha},$$

*where* $(R_0, R_1, \ldots)$ *is a random walk with the same step distribution as* $S$ *and with initial distribution*

$$\frac{\nu(\mathcal{S}^{\downarrow})}{\kappa'(p^*)}\mathbb{P}^*(S_1 > y)dy\,,$$

*and* $\mathbf{e}_0, \mathbf{e}_1, \ldots$ *a sequence of i.i.d. exponential variables with parameter* $\nu(\mathcal{S}^{\downarrow})$*, which is independent of the random walk* $(R_0, R_1, \ldots)$*.*

This result can be proven by taking limits as $t \to \infty$ in the explicit moment calculations of Proposition 7(ii), using complex analysis and contour integral, see [18]. We develop below a more probabilistic approach.

*Proof* There is no loss of generality in assuming that the fragmentation starts from a single fragment with unit size. Write $T(y) := \min\{n \in \mathbb{N} : S_n > -\ln y\}$ for every $y \in ]0,1]$. We thus have

$$\mathbb{P}^*(\chi(t) < y) \;=\; \mathbb{P}^*\left(\sum_{n=0}^{T(y)} \exp(\alpha S_n)\mathbf{e}'_n \le t\right)\,,$$

where $\mathbf{e}'_0, \mathbf{e}'_1, \ldots$ is a sequence of i.i.d. exponential variables with parameter $\nu(\mathcal{S}^{\downarrow})$, which is independent of the random walk (so that $\zeta_n = \exp(\alpha S_n)\mathbf{e}'_n$ is the lifetime of the tagged particle at the $n$-th generation). Reversing the indices, we may re-express this quantity in the form

$$\mathbb{P}^*(\chi(t) < y) \;=\; \mathbb{P}^*\left(\sum_{n=0}^{T(y)} \exp(-\alpha R_n(y))\mathbf{e}_n \le y^\alpha t\right)\,,$$

where $R_n(y) := -\ln y - S_{T(y)-n}$ and $\mathbf{e}_0, \mathbf{e}_1, \ldots$ is a new sequence of i.i.d. exponential variables with parameter $\nu(\mathcal{S}^{\downarrow})$, which is again independent of $(R_n(y), n \in \mathbb{N})$. Rescaling thus gives

$$\mathbb{P}^*(t^{1/\alpha}\chi(t) < y) \;=\; \mathbb{P}^*\left(\sum_{n=0}^{T(t^{-1/\alpha}y)} \exp(-\alpha R_n(t^{-1/\alpha}y))\mathbf{e}_n \le y^\alpha\right)\,.$$

When $t$ tends to infinity, so does $T(t^{-1/\alpha}y)$, and, by renewal theory (see for example Chapter 10 in [41]), the sequence $(R_0(t^{-1/\alpha}y), R_1(t^{-1/\alpha}y), \ldots)$ converges in law to $(R_0, R_1, \ldots)$ where the latter is a random walk with the same step distribution as $S$, and the initial variable $R_0$ has the limiting law of the so-called age in a renewal process, that is to say

$$\mathbb{P}^*(R_0 \in dy) = \frac{\mathbb{P}^*(S_1 > y)}{\mathbb{E}(S_1)}\,dy\,.$$

This easily yields the statement. □

We now complete the proof of Theorem 3 by pointing out that the distribution of the limiting variable $Y_\alpha$ can be characterized by its moments as follows. Recall that the function $\kappa$ is defined in (17).

**Proposition 11** *For every integer* $k \ge 1$*, we have*

$$\mathbb{E}^*\left(Y_\alpha^{k\alpha}\right) \;=\; \frac{(k-1)!}{\alpha\kappa'(p^*)\kappa(p^*+\alpha)\cdots\kappa(p^*+(k-1)\alpha)}\,,$$

*and this determines uniquely the law of* $Y_\alpha$*.*

*Proof* It is convenient to combine Propositions 6 and 10 and re-express

$$Y_\alpha^\alpha \;=\; \exp(-\alpha R_0)\int_0^\infty \mathrm{e}^{-\alpha\Upsilon_t}dt\,, \tag{29}$$

where $\Upsilon = (\Upsilon_t, t \geq 0)$ is distributed as the increasing compound Poisson process $S \circ N$, where $N$ is a Poisson process with rate $\nu(\mathcal{S}^\downarrow)$ which is independent of $R_0$ and the random walk $S$ (recall the example at the end of Section 1.1.1). In particular, the Laplace transform of $\Upsilon_t$ is given by

$$\begin{aligned}\mathbb{E}^*\left(\mathrm{e}^{-q\Upsilon_t}\right) &= \exp\left(-t\nu(\mathcal{S}^\downarrow)\mathbb{E}(1-\mathrm{e}^{-qS_1})\right)\\ &= \exp\left(-t\int_{\mathcal{S}^\downarrow}\nu(d\mathbf{x})\sum_{i=1}^\infty\left(x_i^{p^*}-x_i^{p^*+q}\right)\right),\end{aligned}$$

and finally

$$\mathbb{E}^*\left(\mathrm{e}^{-q\Upsilon_t}\right) \;=\; \exp(-t\kappa(q+p^*))\,. \tag{30}$$

On the one hand, it is immediate that

$$\mathbb{E}^*\left(\exp(-qR_0)\right) \;=\; \frac{\kappa(q+p^*)}{q\kappa'(p^*)}\,, \qquad q>0\,. \tag{31}$$

On the other hand we shall check that

$$\mathbb{E}^*\left(\left(\int_0^\infty \exp\left(-\alpha\Upsilon_s\right)ds\right)^k\right) \;=\; \frac{k!}{\kappa(p^*+\alpha)\cdots\kappa(p^*+\alpha k)}\,, \qquad k=1,2,\ldots \tag{32}$$

For this purpose, set

$$I_t \;=\; \int_t^\infty \exp(-\alpha\Upsilon_s)ds$$

for every $t \geq 0$. On the one hand, for every positive real number $r > 0$, we have the identity

$$I_0^r - I_t^r \;=\; r\int_0^t \exp(-\alpha\Upsilon_s)I_s^{r-1}ds\,. \tag{33}$$

On the other hand, we may express $I_s$ in the form $I_s = \exp(-\alpha\Upsilon_s)I_0'$, where

$$I_0' \;=\; \int_0^\infty \exp(-\alpha\Upsilon_u')du \quad\text{and}\quad \Upsilon_u' = \Upsilon_{s+u} - \Upsilon_s\,. \tag{34}$$

From the independence and stationarity of the increments of the Lévy process, we see that $I_0'$ has the same law as $I_0 = I$ and is independent of $\Upsilon_s$. Plugging this into (33) and taking expectations, we get using (30) that

$$\begin{aligned}\mathbb{E}^*\left(I^r\right)\left(1-\exp(-t\kappa(p^*+\alpha r))\right) &= r\int_0^t \exp(-s\kappa(p^*+\alpha r))\mathbb{E}^*(I^{r-1})\,ds\\ &= \frac{r}{\kappa(p^*+\alpha r)}\left(1-\mathrm{e}^{-t\kappa(p^*+\alpha r)}\right)\mathbb{E}^*\left(I^{r-1}\right)\,.\end{aligned}$$

Finally

$$\mathbb{E}^*\left(I^r\right) \;=\; \frac{r}{\kappa(p^*+\alpha r)}\mathbb{E}^*\left(I^{r-1}\right)\,,$$

and since $\mathbb{E}^*(I^0) = 1$, we get the formula (32) by iteration, taking $r = k \in \mathbb{N}$. Combining (29), (31) and (32) completes the proof of the first statement.

Finally, as $\lim_{q\to\infty}\kappa(q) = \nu(\mathcal{S}^\downarrow)$, we see that $Y_\alpha^\alpha$ possesses exponential moments of any order less than $\nu(\mathcal{S}^\downarrow)$, and therefore is determined by its entire moments. □

## 4.4 Another strong law via renewal theory

Finally, we turn our interest to a question which is motivated by the mining industry. Specifically, we are concerned with the stage during which mineral blocks are crushed in mills to produce a thin powder. Mineral grains are screened during the process, so that when they become smaller than the diameter of the mesh of a thin grid, they are removed from the mill.

We use a fragmentation chain to model the crushing process, and we are interested in the distribution of the small mineral grains that go across the grid. In other words, we would like to get information about the distribution of the state of a fragmentation chain when we stop each particle at the instant when it becomes smaller than some small parameter $\varepsilon > 0$. Clearly, this does not depend on the index of self-similarity $\alpha$, but only on the dislocation measure $\nu$. More precisely, we use the genealogical tree representation and consider

$$\varphi_\varepsilon(da) := \sum_{u\in\mathcal{U},u\neq\varnothing} \mathbb{1}_{\{\xi_{u-}\geq\varepsilon,\xi_u<\varepsilon\}}\xi_u^{p^*}\delta_{\xi_u/\varepsilon}(da)\,,$$

where $u-$ stands for the parent of $u$. So $\varphi_\varepsilon$ is a random finite measure on $]0,1[$, which can be viewed as a weighted version of the empirical measure of the particles taken at the instant when they become smaller than $\varepsilon$ and rescaled. Henceforth, we shall suppose that the fragmentation is not arithmetic as in Proposition 10. Recall also that $\mathcal{M}_\infty$ denotes the terminal value of the intrinsic martingale.

**Proposition 12** *As $\varepsilon \to 0$, $\varphi_\varepsilon$ converges in probability to $\mathcal{M}_\infty\varphi$, where $\varphi$ is a deterministic probability measure on $[0,1]$ given by*

$$\varphi(da) = \left(\int_{\mathcal{S}^\downarrow}\sum_{i=1}^{\infty}\mathbb{1}_{\{s_i<a\}}s_i^{p^*}\nu(d\mathbf{s})\right)\frac{da}{a\kappa'(p^*)}\,.$$

*Proof* We start by considering the quantities

$$\sum_{u\in\mathcal{U},u\neq\varnothing}\mathbb{1}_{\{\xi_{u-}\geq\varepsilon,\xi_u<\varepsilon\}}\xi_u^{p^*}\,,\qquad \varepsilon>0.$$

Write $\mathcal{G}_\varepsilon$ for the sigma-field generated by the variables $(\mathbb{1}_{\{\xi_{u-}\geq\varepsilon\}}\xi_u, u-\in\mathcal{U})$, so $(\mathcal{G}_\varepsilon)_{\varepsilon>0}$ is a reversed filtration. An easy application of the branching property of the marked tree, similar to that in Proposition 5, implies that

$$\sum_{u\in\mathcal{U},u\neq\varnothing}\mathbb{1}_{\{\xi_{u-}\geq\varepsilon,\xi_u<\varepsilon\}}\xi_u^{p^*} = \mathbb{E}\left(\mathcal{M}_\infty \mid \mathcal{G}_\varepsilon\right),$$

and it follows that

$$\lim_{\varepsilon\to0}\sum_{u\in\mathcal{U},u\neq\varnothing}\mathbb{1}_{\{\xi_{u-}\geq\varepsilon,\xi_u<\varepsilon\}}\xi_u^{p^*} = \mathcal{M}_\infty \qquad \text{in } L^p(\mathbb{P})$$

for some $p>1$.

Next, we turn our attention to first moment estimates. Let $f:[0,1]\to\mathbb{R}$ be a continuous function with support in $]0,1[$, and consider

$$\langle\varphi_\varepsilon, f\rangle = \sum_{u\in\mathcal{U},u\neq\varnothing}\mathbb{1}_{\{\xi_{u-}\geq\varepsilon,\xi_u<\varepsilon\}}\xi_u^{p^*}f(\xi_u/\varepsilon).$$

Fix $\eta > 0$ and work under $\mathbb{P}_\eta$, that is when at the initial time there is a unique fragment with size $\eta$. We compute the expectation of this variable by conditioning on the mark of the parent $u- = v$ of $u$ and applying the branching property. We get

$$\begin{aligned}\mathbb{E}_\eta\left(\langle\varphi_\varepsilon, f\rangle\right) &= \mathbb{E}_\eta\left(\sum_{v\in\mathcal{U}} \mathbb{1}_{\{\xi_v\geq\varepsilon\}}\xi_v^{p^*}\sum_{i=1}^{\infty}\mathbb{1}_{\{\xi_v\tilde{\xi}_i<\varepsilon\}}\tilde{\xi}_i^{p^*}f(\xi_v\tilde{\xi}_i/\varepsilon)\right)\\ &= \mathbb{E}_\eta\left(\sum_{v\in\mathcal{U}} \mathbb{1}_{\{\ln\xi_v\geq\ln\varepsilon\}}\xi_v^{p^*}\sum_{i=1}^{\infty}\mathbb{1}_{\{\tilde{\xi}_i<\varepsilon/\xi_v\}}\tilde{\xi}_i^{p^*}f(\xi_v\tilde{\xi}_i/\varepsilon)\right),\end{aligned}$$

where $(\tilde{\xi}_i)_{i\in\mathbb{N}}$ has the law $\nu(\cdot)/\nu(\mathcal{S}^\downarrow)$ and is independent of $\xi_v$. Integrating with respect to the latter gives

$$\mathbb{E}_\eta\left(\langle\varphi_\varepsilon, f\rangle\right) = \mathbb{E}_\eta\left(\sum_{n=0}^{\infty}\sum_{|v|=n}\mathbb{1}_{\{\ln\xi_v\geq\ln\varepsilon\}}\xi_v^{p^*}h(\ln\xi_v-\ln\varepsilon)\right),$$

where for $a\geq 0$

$$h(a) = \mathbb{E}\left(\sum_{i=1}^{\infty}\mathbb{1}_{\{\tilde{\xi}_i<\mathrm{e}^{-a}\}}\tilde{\xi}_i^{p^*}f\left(\tilde{\xi}_i\mathrm{e}^a\right)\right).$$

Now we can evaluate this expression using the tagged branch and the random walk $S_n = -\ln\chi_n$, thanks to Lemma 4:

$$\mathbb{E}_\eta\left(\langle\varphi_\varepsilon, f\rangle\right) = \sum_{n=0}^{\infty}\mathbb{E}^*\left(\mathbb{1}_{\{S_n\leq\ln\eta-\ln\varepsilon\}}h(\ln\eta-S_n-\ln\epsilon)\right).$$

Our assumptions enable us to apply the renewal theorem to the renewal process $S_n$, and we get the estimate

$$\lim_{\varepsilon\to 0}\mathbb{E}_\eta\left(\langle\varphi_\varepsilon, f\rangle\right) = \frac{1}{\mathbb{E}^*(S_1)}\int_0^\infty h(a)da,$$

where the convergence is uniform in $\eta$ as long as $\varepsilon/\eta\to 0$.

Then using the extension of the law of large numbers stated in Lemma 5 in a similar way as in the proof of Theorem 2 (again technical details are left to the reader), we can check that

$$\lim_{\varepsilon\to 0}\langle\varphi_\varepsilon, f\rangle = \frac{\mathcal{M}_\infty}{\mathbb{E}^*(S_1)}\int_0^\infty h(a)da \qquad \text{in } L^1(\mathbb{P}).$$

Finally, we already know that $\mathbb{E}^*(S_1) = \kappa'(p^*)/\nu(\mathcal{S}^\downarrow)$, and on the other hand, an easy computation gives

$$\int_0^\infty h(a)da = \int_0^1 f(b)b^{-1}\mathbb{E}\left(\sum_{i=1}^{\infty}\mathbb{1}_{\{\tilde{\xi}_i<b\}}\tilde{\xi}_i^{p^*}\right)db,$$

which completes the proof. □

# 5 Additive martingales (homogeneous case $\alpha = 0$)

In this section, we consider a homogeneous fragmentation chain (i.e. self-similar with index $\alpha = 0$), say $X = (X(t), t \geq 0)$ with dislocation measure $\nu$. The starting point of our study lies in the observation that one can associate to $X$ a whole family of natural martingales, which includes the intrinsic martingale as a special case. We shall specify which of these martingales are uniformly integrable, and derive further information on the asymptotic behavior of homogeneous fragmentation chains. In this direction, recall from Section 1.1.3 that when $\nu$ only charges finite sequences, there is a simple connection with branching random walks in continuous time; and the properties that we shall establish here essentially rephrase parts of the folklore of the theory of branching random walks in the setting of homogeneous fragmentation chains.

Recall also that the Malthusian hypotheses are enforced. It will be convenient to introduce the following terminology: we say that the fragmentation becomes extinct if $T := \inf\{t \geq 0 : X(t) = (0, \ldots)\}$ is finite, and that it survives forever otherwise. Plainly, the fragmentation survives forever a.s. whenever $\nu(s_1 = 0) = 0$, and the Malthusian hypotheses imply that the probability of extinction is always strictly less than 1; see Proposition 5.

## 5.1 Convergence of additive martingales

A crucial fact for the study of homogeneous fragmentation is that there is a simple formula for the moments of the process. Indeed, we know from Proposition 7(i) and self-similarity that for every $p > \underline{p}$ and $t \geq 0$,

$$\mathbb{E}_x\left(\sum_{i=1}^{\infty} X_i^p(t)\right) \;=\; x^p \exp(-t\kappa(p))\,.$$

It follows immediately from the branching property and scaling that:

**Corollary 3** *For every $p > \underline{p}$, the process*

$$M(p,t) := \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t)$$

*is a non-negative martingale which converges a.s.*

In order to investigate the asymptotic behavior of homogeneous fragmentation chains, it is crucial to know if the limit of the martingale above is strictly positive or zero. A first step in the analysis is the following elementary lemma.

**Lemma 6** *The function $p \to \kappa(p)/p$ reaches its maximum at a unique location $\bar{p} > p^*$, which is the unique solution to the equation*

$$p\kappa'(p) = \kappa(p)\,.$$

*More precisely, the function $p \to \kappa(p)/p$ increases on $]\underline{p}, \bar{p}[$ and decreases on $]\bar{p}, \infty[$, and the value of its maximum is $\kappa'(\bar{p}) = \kappa(\bar{p})/\bar{p}$.*

*Proof* We first point out that the function $\kappa$ is concave and increasing. It follows that

$$\text{the function } p \to p\kappa'(p) - \kappa(p) \text{ decreases on } ]\underline{p}, \infty[. \tag{35}$$

Indeed, this function has derivative $p\kappa''(p)$, which is negative since $\kappa$ is concave. Recall that $\kappa(p^*) = 0$ by the definition of the Malthusian exponent $p^*$. On the other hand, it is obvious that $\lim_{q\to\infty} \kappa(q)/q = 0$, hence the function $p \to \kappa(p)/p$ has the same limit at $p^*$ and at $\infty$, so it reaches its overall maximum at a unique point $\bar{p} > p^*$. In particular, we deduce from (35) that the derivative of $p \to \kappa(p)/p$ is positive on $]\underline{p}, \bar{p}[$ and negative on $]\bar{p}, \infty[$. Finally, the derivative must be zero at $\bar{p}$, which implies that the overall maximum is given by $\kappa'(\bar{p}) = \kappa(\bar{p})/\bar{p}$. □

We may now state the main result of this section.

**Theorem 4** *Assume that there exists some constants $a, b > 0$ such that*

$$\nu\left(\sum_{i=1}^{\infty} s_i^a > b\right) \;=\; 0\,. \tag{36}$$

*Then for every $p \in ]\underline{p}, \bar{p}[$, the martingale $M(p,\cdot)$ is bounded in $L^q(\mathbb{P})$ for some $q > 1$. Moreover its terminal value is strictly positive conditionally on non-extinction.*

*Proof* The proof uses the same route as that of Theorem 1. Recall that all that we need is to check that the sum of the jumps of the martingale raised to some power $q > 1$ has a finite mean, that is

$$\mathbb{E}\left(\sum_{t>0} |M(p,t) - M(p,t-)|^q\right) \;<\; \infty\,. \tag{37}$$

It is convenient to re-express the left-hand side in terms of the generation of the fragments. Specifically, denote by $\xi_1^{(k)}, \ldots$ the fragments of the $k$-th generation, and by $T_i^{(k)}$ the instant when $\xi_i^{(k)}$ splits. The jump of $M(p,t)$ at time $T_i^{(k)}$ is

$$\exp(\kappa(p)T_i^{(k)})|\xi_i^{(k)}|^p \left(\sum_{j=1}^{\infty} \tilde{\xi}_j^p - 1\right)$$

where $\tilde{\xi}$ is independent of $T_i^{(k)}$ and $\xi_i^{(k)}$ and has the law $\nu(\cdot)/\nu(\mathcal{S}^{\downarrow})$. The conditional expectation of this quantity raised to the power $q$, given the splitting time $T_i^{(k)}$ and $\xi_i^{(k)}$ is

$$c\exp(q\kappa(p)T_i^{(k)})|\xi_i^{(k)}|^{pq}\,,$$

where

$$c := \int_{\mathcal{S}^{\downarrow}} \left|1 - \sum_{i=1}^{\infty} s_i^p\right|^q \nu(d\mathbf{s})/\nu(\mathcal{S}^{\downarrow})\,.$$

We point out that $c < \infty$; indeed (36) and Jensen's inequality yield

$$\begin{aligned}\left|\sum_{i=1}^{\infty} s_i^p\right|^q &= b^q \left|\sum_{i=1}^{\infty} \frac{s_i^a}{b} s_i^{p-a}\right|^q \\ &\le b^{q-1} \sum_{i=1}^{\infty} s_i^{q(p-a)+a}\,,\end{aligned}$$

which yields the claim since $q(p-a)+a > \underline{p}$ provided that $q$ is chosen sufficiently close to 1.

On the one hand, we know that $T_i^{(k)}$ is the sum of $k+1$ independent exponential variables with parameter $\nu(\mathcal{S}^\downarrow)$; in other words it has the gamma distribution with parameters $k+1$ and $\nu(\mathcal{S}^\downarrow)$. In particular,

$$\mathbb{E}\left(\exp(q\kappa(p)T_i^{(k)})\right) \;=\; \left(\frac{\nu(\mathcal{S}^\downarrow)}{\nu(\mathcal{S}^\downarrow)-q\kappa(p)}\right)^{k+1}.$$

On the other hand, we have already seen in (23) that

$$\mathbb{E}\left(\sum_{i=1}^{\infty}|\xi_i^{(k)}|^{pq}\right) \;=\; \left(\frac{\nu(\mathcal{S}^\downarrow)-\kappa(pq)}{\nu(\mathcal{S}^\downarrow)}\right)^{k}.$$

Because $p < \bar{p}$, thanks to Lemma 6 we may choose $q > 1$ small enough so that $q\kappa(p) < \kappa(pq)$, and then the series

$$\sum_{k=0}^{\infty}\mathbb{E}\left(\sum_{i=1}^{\infty}\exp(q\kappa(p)T_i^{(k)})|\xi_i^{(k)}|^{pq}\right)$$

converges, which completes the proof of (37).

Finally, checking that the terminal value $M(p,\infty)$ is strictly positive conditionally on non-extinction follows the same argument as in the proof of Theorem 1. □

## 5.2 Some applications

In this section, we develop some applications of the preceding theorem to the asymptotic behavior of homogeneous fragmentation. First, we consider the largest fragment.

**Corollary 4** *The assumptions are the same as in Theorem 4. Then, conditionally on non-extinction, it holds with probability one that*

$$\lim_{t\to\infty}\frac{1}{t}\ln X_1(t) \;=\; -\kappa'(\bar{p}) \;=\; -\frac{\kappa(\bar{p})}{\bar{p}}.$$

In other words, the largest fragment $X_1(t)$ decays exponentially fast as $t\to\infty$, with rate $\kappa'(\bar{p})$. Observe that this is smaller than $\kappa'(p^*)$ (because $p^* < \bar{p}$ and $\kappa'$ is decreasing), which is the exponential rate of decay of a typical fragment; see Theorem 2(i).

*Proof* For every $p > \underline{p}$, we have

$$\exp(t\kappa(p))X_1^p(t) \le \exp(t\kappa(p))\sum_{i=1}^{\infty}X_i^p(t)$$

and the right-hand side remains bounded as $t$ tends to infinity. Hence

$$\limsup_{t\to\infty}\frac{1}{t}\ln X_1(t) \;\le\; -\frac{\kappa(p)}{p},$$

and optimizing over $p$ yields

$$\limsup_{t\to\infty} \frac{1}{t} \ln X_1(t) \ \leq \ -\frac{\kappa(\bar{p})}{\bar{p}} \, .$$

On the other hand, for every $p \in ]\underline{p}, \bar{p}[$ and $\varepsilon > 0$ sufficiently small, we have the lower bound

$$\exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t) \leq X_1^{\varepsilon}(t) \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^{p-\varepsilon}(t) \, .$$

We know that both limits

$$\lim_{t\to\infty} \exp(t\kappa(p)) \sum_{i=1}^{\infty} X_i^p(t) \quad \text{and} \quad \lim_{t\to\infty} \exp(t\kappa(p-\varepsilon)) \sum_{i=1}^{\infty} X_i^{p-\varepsilon}(t)$$

are finite and strictly positive a.s. conditionally on non-extinction, and we deduce that

$$\liminf_{t\to\infty} \frac{1}{t} \ln X_1(t) \ \geq \ -\frac{\kappa(p) - \kappa(p-\varepsilon)}{\varepsilon} \, .$$

We take the limit of the right-hand side as $\varepsilon \to 0+$ and then as $p$ tends to $\bar{p}$ to conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \ln X_1(t) \ \geq \ -\kappa'(\bar{p}) \, .$$

Now, this quantity coincides with $-\kappa(\bar{p})/\bar{p}$, as we know from Lemma 6. □

We point out that the argument of the proof above also shows that the martingale $M(p,t)$ converges to 0 a.s. (and a fortiori is not uniformly integrable) for $p > \bar{p}$. In fact, it can even be shown that the same remains true for $p = \bar{p}$.

Finally we conclude this section by an application to the asymptotic behavior of homogeneous fragmentations which is easily deduced from the martingales that we considered and classical large deviations techniques. Again, we shall focus for the sake of simplicity on the case when the dislocation measure $\nu$ is conservative. Further, it will be convenient here to represent the random sequence $X(t) = (X_1(t), \ldots)$ by the empirical distribution,

$$\rho_t(dy) \ := \ \sum_{i=1}^{\infty} \delta_{\frac{1}{t}\ln X_i(t)}(dy) \, . \tag{38}$$

Define the convex decreasing function $\Lambda$ on $]\underline{p}, \infty[$ by

$$\Lambda(p) \ = \ \begin{cases} -\kappa(p) & \text{if } \underline{p} < p < \bar{p} \, , \\ -p\kappa'(\bar{p}) & \text{if } p \geq \bar{p} \, . \end{cases}$$

**Corollary 5** *The assumptions are the same as in Theorem 4. It holds a.s. conditionally on non-extinction that*

$$\lim_{t\to\infty} \frac{1}{t} \ln \int_{\mathbb{R}} \mathrm{e}^{tpy} \rho_t(dy) \ = \ \Lambda(p)$$

*for every $p > \underline{p}$.*

*Proof* Let us first prove the statement for a fixed $p > \underline{p}$. Observe that

$$\int_{\mathbb{R}} \mathrm{e}^{tpy} \rho_t(dy) \;=\; \sum_{i=1}^{\infty} X_i^p(t)\,.$$

The case $\underline{p} < p < \overline{p}$ follows from Theorem 4, so suppose that $\bar{p} \leq p$. We use the bounds

$$X_1^p(t) \;\leq\; \sum_{i=1}^{\infty} X_i^p(t) \;\leq X_1^{p-\bar{p}}(t) \sum_{i=1}^{\infty} X_i^{\bar{p}}(t)\,.$$

Recall first from Corollary 4 that $\ln X_1(t) \sim -t\kappa'(\bar{p})$ as $t \to \infty$, and then from Lemma 6 and Corollary 3 that $\mathrm{e}^{t\bar{p}\kappa'(\bar{p})} \sum_{i=1}^{\infty} X_i^{\bar{p}}(t)$ is a martingale which converges a.s. It follows immediately that

$$\lim_{t\to\infty} \frac{1}{t} \ln \sum_{i=1}^{\infty} X_i^p(t) \;=\; \lim_{t\to\infty} \frac{1}{t} \ln \int_{\mathbb{R}} \mathrm{e}^{tpy} \rho_t(dy) \;=\; -p\kappa'(\bar{p}) \;=\; \Lambda(p) \qquad \text{a.s.}$$

The limit above holds a.s. simultaneously for every rational number $p > \underline{p}$, and by an immediate monotonicity argument, the proof is complete. □

Pathwise large deviation estimates for the family of random measures $(\rho_t, t \geq 0)$ follow from Corollary 5. Introducing the Fenchel-Legendre transform of $\Lambda$,

$$\Lambda^*(a) \;=\; \sup_{p>\underline{p}} \left(ap - \Lambda(p)\right)\,,$$

the classical duality for the Fenchel-Legendre transform (see for example [31]) yields the identity

$$\Lambda^*(a) \;=\; \kappa(p) - p\kappa'(p)\,, \qquad \text{for every } p \in ]\underline{p}, \bar{p}[ \text{ and } a = -\kappa'(p).$$

Note also that $\Lambda^*(a) = \infty$ for every $a > -\kappa'(\bar{p})$ and that $\Lambda^*$ is left-continuous at $-\kappa'(\bar{p})$.

**Corollary 6** *The assumptions are the same as in Theorem 4. The following holds a.s. conditionally on non-extinction:*

(i) *For any closed set $F \subseteq ]\underline{p}, \infty[$,*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \rho_t(F) \;\leq\; -\inf\left\{\Lambda^*(a), a \in F\right\}\,.$$

(ii) *For any open set $G \subseteq \mathbb{R}$,*

$$\liminf_{t\to\infty} \frac{1}{t} \ln \rho_t(G) \;\geq\; -\inf\left\{\Lambda^*(a), a > -\kappa'(\underline{p}+) \text{ and } a \in G\right\}\,.$$

(iii) *If moreover $\kappa'(\underline{p}+) = \infty$, then $(\rho_t, t \geq 0)$ satisfy the Large Deviation Principle with the good convex rate function $\Lambda^*$ (see for instance [31] for the terminology) .*

*Proof* We aim to apply the Gärtner-Ellis theorem (see Section 2.3 in Dembo and Zeitouni [31]). The fundamental condition on the behavior of the Laplace transform of $\rho_t$ (see Assumption 2.3.2 in [31]), is the conclusion of Corollary 5. Note that the assumption

$\underline{p} < 1$ ensures that 0 belongs to the interior of the domain of $\Lambda$. According to Lemma 2.3.9 in [31], every $x \in ]-\kappa'(\underline{p}+), -\kappa'(\bar{p})[$ is a so-called exposed point of the Fenchel-Legendre transform $\Lambda^*$, and since $\Lambda^*(a) = \infty$ for every $a > -\kappa'(\bar{p})$, statements (i) and (ii) of Corollary 6 merely rephrase Theorem 2.3.6 in [31]. The last statement follows from the first two and Lemma 2.3.9 in [31]. □

One can interpret Corollary 6 as a multi-scale limit theorem for numbers of fragments. Specifically, it is immediately seen that when the hypothesis of Corollary 6 is fulfilled, then we have for every $a > -\kappa'(\underline{p}+)$ that

$$\lim_{\varepsilon \to 0+} \lim_{t \to \infty} \frac{1}{t} \ln \# \left\{ i \in \mathbb{N} : \mathrm{e}^{(a-\varepsilon)t} \leq X_i(t) \leq \mathrm{e}^{(a+\varepsilon)t} \right\} = -\Lambda^*(a) \, .$$

Observe that this quantity equals $-\infty$ for $a > -\kappa'(\bar{p})$, in agreement with Corollary 3. On the other hand, taking $a = -\kappa'(p)$ for some $p \in ]\underline{p}, \bar{p}[$, we see that, roughly speaking, the number of fragments of size approximately $\mathrm{e}^{-\kappa'(p)t}$ at time $t$ is approximately $\exp\left(t(p\kappa'(p) - \kappa(p))\right)$ when $t$ is large. We refer to Berestycki [12] for further developments in this vein, related to the so-called multifractal spectrum of homogeneous fragmentations (see also Krell [53] for a related work).

# 6 Comments

It seems that theoretical works on random fragmentation chains have been motivated initially by the study of the crushing of blocks of mineral in the mining industry. [3] In particular, the first significant probabilistic contribution in this field was due to Kolmogorov [50] himself in 1941, who provided an explanation for the statistical observation that the logarithms of the sizes of mineral grains are often normally distributed. In this direction, Kolmogorov introduced a version of homogeneous fragmentation chains in discrete time as follows. At the initial time consider a mass, say $m > 0$. At time 1, this mass is broken randomly, which produces smaller masses, say $m\xi_1 \geq m\xi_2, \geq \cdots \geq 0$ where $\xi = (\xi_1, \ldots)$ has a fixed distribution with $\sum_{i=1}^{\infty} \xi_i = 1$ a.s. (in other words, the dislocation is conservative). The next steps consist of independent iterations, that is each mass that results from the previous step is broken independently of each other and according to the same law. Kolmogorov established in this framework the discrete time analog of Theorem 2 (note that then the intrinsic martingale $\mathcal{M}$ is trivial since the dislocations are conservative). Nowadays, such a result should be viewed as a special case of the central limit theorem for branching random walks, see for example [4] and [24].

Further important developments in this vein were made in 1961 by one of Kolmogorov's students, Filippov [38], who considered self-similar fragmentation chains as described in the present chapter. In particular, Filippov proved the version of Theorem 3 in the conservative case. This result was then re-discovered independently by Brennan and Durrett [29] for the binary and conservative dislocations; more precisely these authors obtained a strong limit theorem in this setting, with almost-sure convergence instead of convergence in probability. Some special cases of these mathematical results have also appeared in the literature in physics, see for example [10, 52] and references therein. Baryshnikov

[3] In this direction it may be interesting to mention that a significant proportion of the energy consumption in the world is used for particle size reduction in the mining industry.

and Gnedin [8] studied instances of dissipative self-similar fragmentation chains and obtained the convergence of the mean measures associated with the empirical distribution of fragments. The present statement of Theorem 3 for general non-conservative dislocation measures was established quite recently in [18].

As it has been briefly mentioned in the Introduction, fragmentation phenomena occur frequently in physics and have thus motivated a variety of works in this field (see for example [1, 10, 22, 51, 52] and the references therein). Self-similar fragmentation chains, as introduced in this chapter, also arise in different areas such as analysis of algorithms (for example quick-search [30], recursive trees [33], ...), degradation of polymer chains (see [9], [29], ...), packing problems in communication networks [8], ... In these applications, the interests concern both deterministic and statistical aspects of fragmentation. The former are often considered via so-called *fragmentation equations* that are meant to describe the evolution of the density of particles in media where particles break at certain rates; a simple example of such a fragmentation equation is given in Corollary 1. More general systems where the self-similarity assumption is dropped and further physical phenomena such as coagulation of particles or spatial motions can be incorporated, have been studied intensively in the literature, see for instance [36, 37, 56] and references therein. We stress that stochastic models of fragmentations provide an efficient tool for establishing the existence and studying properties of pure fragmentation equations; see in particular [39] and [42].

The genealogical structure of self-similar fragmentation as described in Proposition 3 has arisen first in the context of so-called multiplicative cascades and random fractal constructions; see the pioneering works of Mandelbrot [62], Kahane and Peyrière [49], Mauldin and Williams [63], and also Barral [7] and Liu [58] for further references. Of course, it plays also an important part in branching processes, see for example Athreya and Ney [5], Jagers [47], and Haccou, Jagers and Vatutin [46]. The so-called Malthusian hypotheses, [4] the intrinsic martingale, and techniques based on randomly tagged branches, are classical cornerstones of these theories.

The phenomenon of formation of dust in self-similar fragmentation chains with negative indices of self-similarity was first observed by Filippov [38]. In the setting of (deterministic) fragmentation equations (cf. Corollary 1) with conservative dislocation measures, it corresponds to a phenomenon of loss of mass or shattering; see [3] and [6]. The sharper results about a.s. extinction such as stated in Proposition 8 appear in [16]; see also [39], [42] and [48]. We refer to the recent work of Haas [43] for several deep results about the regularity of the formation of dust in self-similar fragmentations with negative indices and conservative dislocation measures, and to Wagner [69] for the study of this phenomenon for more general processes.

Lemma 5, which provides a key technical step to the limit theorems in Section 1.4, is due to Nerman [64], who used it to develop a renewal theory for supercritical general branching processes. The results of Nerman play a crucial role in the asymptotic analysis of the fragmentation energy (a problem motivated by the mining industry) in [19], from which Proposition 12 is an excerpt.

Additive martingales are one of the most powerful tools for the study of branching random walks, both in discrete time (see the well-known article by Biggins [23]) and in

[4] We stress that hypotheses that we make in Section 1.2.2 are slightly stronger than the usual $L \log L$ conditions, but they are easier to handle for the applications we have in mind.

continuous time (cf. in particular [25, 54, 68]). The approach to convergence which is used in Section 1.5.1 relies on a standard application of stochastic calculus for purely discontinuous martingales in continuous time. Alternatively, one can also establish Theorem 4 by adapting the conceptual proof of Lyons *et al.* [61] (see also Lyons [60]) which is based on a clever argument of change of probability measures. We further point out that Biggins [25] has obtained a stronger limit theorem for additive martingales associated to branching random walks, in which convergence holds uniformly in the parameter $p$, and this reinforcement yields precise large deviation estimates for the empirical measure of a branching random walk. Such results can be shifted to homogeneous fragmentation chains by time-discretization techniques, see [21]. In a somewhat different direction, one can obtain precise large deviation estimates for the probability of presence of abnormally large fragments as time goes to infinity. See [20, 21], which extend earlier results in this vein by Rouault [67] for branching random walks in the sub-critical region.

Finally, fragmentation chains are clearly Markov processes of transitive type, in the sense that they never return to a state they visited before. Nonetheless, combining fragmentation either with random coagulation, or with immigration of particles, may produce recurrent processes. We refer to the book by Whittle [70], and to the recent works of Ben-Naim and Krapivsky [10], Berestycki [13], Diaconis *et al.* [32], Durrett *et al.* [34], Erlihson and Granovsky [35] and Haas [44], for some studies of such models.

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