Introduction to PDEs

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What is a PDE?

An **ordinary differential equation (ODE)** is an equation which involves an unknown function in *one* variable and some of its derivatives.

A **partial differential equation (PDE)** is an equation which involves an unknown function in *several* variables and some of its partial derivatives. It can be written in the form

$$E\big(x, u(x), Du(x), \dots, D^k u(x)\big) = 0 \qquad \text{in } U$$

with

- U an open set in $\mathbb{R}^d$ (with $d \geq 2$),
- $k \in \mathbb{N}$ the *order* of the PDE,
- $u\colon U \to \mathbb{R}$ the unknown function (with derivatives $D^\ell u$),
- $E\colon U \times \mathbb{R} \times \ldots \times \mathbb{R}^{d^k} \to \mathbb{R}$ a given function.

What is a PDE good for?

PDEs can be used to describe a large variety of phenomena, such as from economy, physics, biology or social sciences. In these applications

- u usually represents the relevant quantity (e.g. the distribution of heat or of particles),
- E is given in terms of underlying laws (or interactions with external forces), according to which the quantity u evolves or behaves.

Understanding the way, a system behaves in principle, should allow for predictions of the systems, once suitable initial or boundary conditions are imposed. However, from the point of view of modeling, one might also compare the mathematical predictions with what one observes in reality – and see whether or not the model is “justified” and contains all relevant laws.

What are we interested in?

In general, we want to understand some properties of solutions to PDEs, such us

- Existence of solutions?
- Uniqueness of solutions (under suitable constraints, such as initial and boundary conditions)?
- How do solutions depend on the data (initial and boundary conditions?
- What are their qualitative properties (differentiability or, more generally, regularity, boundedness, behavior at infinity or close to singularities, explicit formulas etc.)?

Notice: In contrast to ODEs (Theorems of Peano and of Picard–Lindelöf), there is no general existence theory for PDEs (it depends crucially from the class of PDE under consideration).

What is a solution to a PDE?

The answer to the previous questions depend quite crucially from the notion of solution. The simplest one (used here) is:

Definition

A function $u\colon U \to \mathbb{R}$ is called a *classical solution* of the PDE

$$E\big(x, u(x), Du(x), \ldots, D^k u(x)\big) = 0 \qquad \text{in } U\,,$$

provided that $u \in C^k(U)$ holds and that this equation is satisfied everywhere in U.

Obviously, not all PDEs admit a classical solution, e.g.

$$|Du|^2 + 1 = 0\,.$$

What is a solution to a PDE?

Outlook: The notion of a classical solution is quite strong, and moreover, for general equations it is hard to prove the existence of classical solutions. If instead one employs a concept of “weak solutions”, then the existence of such solutions is guaranteed more easily (for example by methods from functional analysis). Here, the two requirements in the definition are relaxed as follows:

- weak solutions need not be of class C^k, but usually only differentiable in a weak sense (e.g. in a suitable Lebesgue or Sobolev space, depending on k),
- the equations needs to be satisfied only in a weak sense (e.g. “integrated against smooth, compactly supported test functions”).

Some examples of PDEs – the Laplace equation

$$\Delta u = \operatorname{div} Du = \sum_{i=1}^{d} D_i D_i u = 0 \qquad \text{in } U$$

This is the prototype of a linear, elliptic equation.

- The differential operator Δ is called the *Laplace operator*,
- classical solutions of the Laplace equation are called *harmonic functions*.

The Laplace equation and its inhomogeneous version (the *Poisson equation*) are employed in particular for the description of various physical phenomena, such as the stationary (i.e. time-independent) distribution of heat, electro-static and gravitational potentials.

Some examples of PDEs – Euler–Lagrange equations

$$\operatorname{div} D_z f(x, u, Du) = D_u f(x, u, Du) \qquad \text{in } U$$

These equations arise typically in the study of (sufficiently regular) energy densities $f\colon U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, while investigating critical points (or minimizers) of the energy functional

$$F[w] := \int_U f(x, w(x), Dw(x))\, dx$$

in some given class of functions $w\colon U \to \mathbb{R}$.

Some examples of PDEs – Euler–Lagrange equations

Derivation in the simplest case $f(x,u,z) \in C^2$:

Consider a minimizer u of $F[w] := \int_U f(x, w(x), Dw(x))\,dx$ among all functions in $C^2(U) \cap C^0(\overline{U})$ with boundary values u. Thus, we have

$$\frac{d}{d\varepsilon} F[\underbrace{u + \varepsilon\varphi}_{\text{“variation of } u\text{”}}] = \frac{d}{d\varepsilon} \int_U f\big(x, u(x) + \varepsilon\varphi(x), Du(x) + \varepsilon D\varphi(x)\big)\,dx = 0$$

for all $\varphi \in C_0^\infty(U)$. Interchanging integration and differentiation, we find with integration by parts

$$\begin{aligned} 0 &= \int_U \Big(D_u f(x, u(x), Du(x))\varphi(x) + D_z f(x, u(x), Du(x)) \cdot D\varphi(x) \Big)\,dx \\ &= \int_U \underbrace{\Big(D_u f(x, u(x), Du(x)) - \sum_{i=1}^d D_i D_{z_i} f(x, u(x), Du(x)) \Big)}_{=0 \text{ in } U \text{ by fundamental theorem of calculus } (\varphi \text{ is arbitrary!})} \varphi(x)\,dx\,. \end{aligned}$$

Some examples of PDEs – Minimal surface equation

$$\operatorname{div}\Big(\frac{Du}{(1+|Du|^2)^{1/2}}\Big) = 0 \qquad \text{in } U\,.$$

This equation represents the Euler–Lagrange equation of the area functional

$$A[w] = \int_U (1+|Dw(x)|^2)^{1/2}\,dx\,.$$

If $u \in C^2(U) \cap C^0(\overline{U})$ is a minimizer of A to given continuous boundary values, then the graph of u possesses the smallest area among all functions attaining these boundary values, and u further satisfies the minimal surface equation.

Some examples of PDEs – Heat equation

$$\partial_t u - \Delta u = 0 \qquad \text{in } \mathbb{R}^+ \times U$$

This equation is the prototype of a linear, parabolic equation and models general diffusion processes.

- **Diffusion** refers to the movement of particle in direction of the concentration gradient, from location with high concentration towards a location with lower concentration.

In particular, the distribution of heat follows this principle, and in this case $u(t,x)$ is interpreted as the temperature at point x in space and time t.

If an inhomogeneous term $f(t,x,u)$ is allowed (e.g. a heater), the equation is called a **reaction diffusion equation** and plays a crucial role for the modeling of chemical processes.

Some examples of PDEs – System of the Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + u \cdot Du &= -Dp \\ \operatorname{div} u &= 0 \end{cases} \qquad \text{in } \mathbb{R}^+ \times U\,.$$

This equation models the flow of incompressible fluids and is actually a *system* of PDEs in the unknowns of

- the fluid field $u\colon \mathbb{R}^+ \times U \to \mathbb{R}^d$,
- the pressure $p\colon \mathbb{R}^+ \times U \to \mathbb{R}$.

Some examples of PDEs – Transport equation

$$\partial_t u + b(t,x) \cdot Du = 0 \qquad \text{in } \mathbb{R}^+ \times U\,.$$

This equation models the evolution of a concentration u along a given velocity field $b\colon \mathbb{R}^+ \times U \to \mathbb{R}^d$. If $b \in \mathbb{R}^d$ is constant and $U = \mathbb{R}^d$, then initial values $u_0 \in C^1(\mathbb{R}^d)$ are transported according to the formula $u(t,x) \coloneqq u_0(x - bt)$.

Some examples of PDEs – Transport equation

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For $b = 1$ and $u(t,x)$ interpreted as the density of individuals of age x at time t, the aging process is described. The **renewal equation**

$$\begin{cases} \partial_t u + Du &= -du & \text{in } \mathbb{R}^+ \times \mathbb{R}^+\,, \\ u(t,0) &= \int_{\mathbb{R}^+} B(y) n(t,y)\, dx & \text{for } t \in \mathbb{R}^+\,, \end{cases}$$

includes also birth and death processes, with initial population density u_0, death rate $d\colon \mathbb{R}^+ \to \mathbb{R}_0^+$ and birth rate $B\colon \mathbb{R}^+ \to \mathbb{R}_0^+$.

Some examples of PDEs – Wave equation

$$\partial_{tt} u - \Delta u = 0 \qquad \text{in } \mathbb{R}^+ \times U\,.$$

This equation is the prototype of a linear, hyperbolic equation and describes oscillations in elastic media over space and time. For example, it models

- the displacement of a string in one space dimension,
- the vibrations of a membrane in two space dimensions,
- the propagation of waves, such as sound waves, light waves and water waves, in three space dimensions.

Part I:
The Laplace equation

1. Physical motivation
2. Examples of harmonic functions and fundamental solution
3. Mean value property
4. Consequences of the mean value property
5. Outlook: existence of solution to the Poisson equation

Physical motivation of the Laplace equation

The Laplace equation appears in a number of physical contexts, which describe a physical quantity u (e.g. concentration of particles or distribution of heat) in equilibrium. Mathematically, this means the net flux F (i.e. the concentration, which passes orthogonally to the flux through a unit area in one time unit) through the boundary for each (smooth) test volume $V \subset U$ is zero:

$$0 = \int_{\partial V} F \cdot \nu \, d\mathcal{H}^{d-1} \, .$$

Reminder ...

... of the generalization of the fundamental theorem of calculus to the multi-dimensional case.

Theorem (Gauss or divergence theorem)

Let U be a bounded open subset of $\mathbb{R}^d$ with a C^1-boundary ∂U. If F is a $C^0(\overline{U}, \mathbb{R}^d) \cap C^1(U, \mathbb{R}^d)$ vector field with $\operatorname{div} F \in L^1(U)$, then we have

$$\int_U \operatorname{div} F \, dx = \int_{\partial U} F \cdot \nu \, d\mathcal{H}^{d-1} \,,$$

where ν (=ν_U) denotes the outward pointing unit normal field of the boundary ∂U and $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff (or surface) measure.

Physical motivation of the Laplace equation

The Laplace equation describe an equilibrium state of a a physical quantity u. Mathematically, this means the net flux F through the boundary for each (smooth) test volume $V \subset U$ is zero:

$$0 = \int_{\partial V} F \cdot \nu \, d\mathcal{H}^{d-1} = \int_V \operatorname{div} F \, dx \, .$$

With V arbitrary, this means $\operatorname{div} F = 0$ in U.

In many situation it is reasonable to expect $F = -aDu$ for some $a > 0$ (negative sign: the flow is from regions of higher to regions of lower concentration), e.g.

- for diffusion of particles (with chemical concentration u) by Fick's law,
- for heat (with temperature distribution u) by Fourier's law of heat conduction.

Physical motivation of the Laplace equation

The Laplace equation describe an equilibrium state of a a physical quantity u. Mathematically, this means the net flux F through the boundary for each (smooth) test volume $V \subset U$ is zero:

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With V arbitrary, this means $\operatorname{div} F = 0$ in U.

In many situation it is reasonable to expect $F = -aDu$ for some $a > 0$ (negative sign: the flow is from regions of higher to regions of lower concentration). Thus, we end up with the Laplace equation for u:

$$\operatorname{div} F = -\operatorname{div}(aDu) = -a\Delta u = 0 \qquad \text{in } U \,.$$

Examples of harmonic functions

Definition

Classical solutions to the Laplace equation are called (C^2-)*harmonic*. If only one of the inequalities $\Delta u \geq 0$ or $\Delta u \leq 0$ is satisfied, then u is called (C^2-)*subharmonic* and (C^2-)*superharmonic*, respectively.

Examples of harmonic functions

(i) Every affine function is harmonic.

(ii) For $A \in \mathbb{R}^{d\times d}$ the function $u(x) := \sum_{1\le i,j\le d} A_{ij}x_ix_j$ is harmonic if and only if $\operatorname{Tr} A = 0$ holds.

(iii) Harmonic functions on $\mathbb{R}^2$ obtained by separation of variables are:

$$u(x_1,x_2) := \exp(ax_1)\sin(ax_2)\,,$$
$$u(x_1,x_2) := \exp(ax_1)\cos(ax_2) \quad \text{for } a \in \mathbb{R}\,.$$

Details: We look for a harmonic function of the form $u(x_1,x_2) = v(x_1)w(x_2)$ for C^2-functions v and w. This implies

$$v''(x_1)w(x_2) + v(x_1)w''(x_2) = 0\,.$$

Hence, up to multiplication with a constant or interchanging the roles of v and w, we find $v(x_1) = \exp(ax_1)$ und $w(x_2) = \sin(ax_2)$ or $w(x_2) = \cos(ax_2)$.

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$$u(x_1,x_2) \coloneqq \exp(ax_1)\cos(ax_2) \quad \text{for } a \in \mathbb{R}\,.$$

(iv) The real and imaginary part of holomorphic functions are harmonic.

Details: Let $h\colon \mathbb{R}^2 \cong \mathbb{C} \supset U \to \mathbb{C}$ be holonomic. Then $u_1(x_1,x_2) \coloneqq \operatorname{Re} h(x + \mathrm{i}\,y)$ and $u_2(x_1,x_2) \coloneqq \operatorname{Im} h(x + \mathrm{i}\,y)$ are smooth and satisfy the *Cauchy–Riemann equations* $D_1u_1 = D_2u_2$ and $D_2u_1 = -D_1u_2$. This implies

$$\Delta u_1 = D_1D_1u_1 + D_2D_2u_1 = D_1D_2u_2 - D_2D_1u_2 = 0 = \Delta u_2\,.$$

The fundamental solution

We now look at a very particular harmonic function in $\mathbb{R}^d \setminus \{0\}$ and require

- radial symmetry (i.e. $u(x) = v(|x|)$ for some function v),
- decays to zero as $|x| \to \infty$.

By harmonicity of u one obtains an ODE for v which can be solved explicitly.

Definition

The function $\Phi\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$, defined as

$$\Phi(x) = \begin{cases} -(2\pi)^{-1} \log |x| & \text{for } d = 2 \\ (d(d-2)\omega_d)^{-1} |x|^{2-d} & \text{for } d \geq 3 \end{cases}$$

for $x \in \mathbb{R}^d \setminus \{0\}$ (and with $\omega_d = |B_1(0)|$), is called the *fundamental solution of the Laplace equation*.

The mean value property

Even though the Laplace equations seems to be a condition only on (some sum) of the second order derivatives, harmonic functions enjoy a number of remarkable properties, the first one being the mean value property.

Notation: If A is a measurable set in $\mathbb{R}^d$ with $0 < \mathcal{L}^d(A) < \infty$ and $f \in L^1(A)$, we set

$$\fint_A f\,dx = \mathcal{L}^d(A)^{-1} \int_A f\,dx\,.$$

If M is a k-dimensional submanifold in $\mathbb{R}^d$ with $0 < \mathcal{H}^k(M) < \infty$ and $f \in L^1(M)$, then we set

$$\fint_M f\,d\mathcal{H}^k = \mathcal{H}^k(M)^{-1} \int_M f\,d\mathcal{H}^k\,.$$

The mean value property

Lemma

Let U be an open set in $\mathbb{R}^d$, $B_R(x_0) \subset U$ and $u \in C^2(U)$. Defining $\varphi\colon (0,R) \to \mathbb{R}$ via

$$\varphi(r) := \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \qquad \text{for } r \in (0,R)\,,$$

we have $\varphi(r) \to u(x_0)$ as $r \searrow 0$ and $\varphi'(r) = \frac{r}{d} \fint_{B_r(x_0)} \Delta u \, dx$.

Note: the number r/d arises from a comparison between volume and surface of balls. In fact, the application of the Gauss theorem to the ball $B_r = B_r(0) \subset \mathbb{R}^d$ (with $\nu(x) = x/|x|$) and the vector field $F(x) = x$ (with $\operatorname{div} F(x) = d$) yields

$$d\mathcal{L}^d(B_r) = \int_{B_r} \operatorname{div} F \, dx = \int_{\partial B_r} |x| \, d\mathcal{H}^{d-1} = r\mathcal{H}^{d-1}(\partial B_r)$$

for every $r > 0$.

The mean value property

Proof: We verify that

$$\varphi(r) := \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$$

- satisfies $\varphi(r) \to u(x_0)$ as $r \searrow 0$, by continuity of u:

$$\begin{aligned} |\varphi(r) - u(x_0)| &= \Big| \fint_{\partial B_r(x_0)} \big(u(x) - u(x_0)\big) \, d\mathcal{H}^{d-1}(x) \Big| \\ &\leq \fint_{\partial B_r(x_0)} |u(x) - u(x_0)| \, d\mathcal{H}^{d-1}(x) \\ &\leq \max_{x \in \partial B_r(x_0)} |u(x) - u(x_0)| \to 0 \quad \text{as } r \searrow 0\,. \end{aligned}$$

The mean value property

Proof: We verify that

$$\varphi(r) := \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$$

- satisfies $\varphi(r) \to u(x_0)$ as $r \searrow 0$, by continuity of u.
- is continuously differentiable, since we can first rewrite φ as a parameter-dependent integral

$$\varphi(r) = \fint_{\partial B_1(0)} u(x_0 + ry) \, d\mathcal{H}^{d-1}(y) \,,$$

then interchange the order of integration and differentiation

$$\begin{aligned}\varphi'(r) &= \fint_{\partial B_1(0)} Du(x_0 + ry) \cdot y \, d\mathcal{H}^{d-1}(y) \\ &= \fint_{\partial B_r(x_0)} Du(x) \cdot \underbrace{\frac{x - x_0}{r}}_{\text{outer unit normal for } B_r(x_0)} \, d\mathcal{H}^{d-1}(x) \,.\end{aligned}$$

The mean value property

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$$\varphi(r) := \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$$

- satisfies $\varphi(r) \to u(x_0)$ as $r \searrow 0$, by continuity of u
- is continuously differentiable, since we can first rewrite φ as a parameter-dependent integral

$$\varphi(r) = \fint_{\partial B_1(0)} u(x_0 + ry) \, d\mathcal{H}^{d-1}(y) \,,$$

  then interchange the order of integration and differentiation and apply the Gauss theorem

$$\begin{aligned} \varphi'(r) &= \frac{1}{\mathcal{H}^{d-1}(\partial B_r)} \int_{\partial B_r(x_0)} Du(x) \cdot \frac{x - x_0}{r} \, d\mathcal{H}^{d-1}(x) \\ &= \frac{1}{\mathcal{H}^{d-1}(\partial B_r)} \int_{B_r(x_0)} \operatorname{div} Du(x) \, dx = \frac{r}{d} \fint_{B_r(x_0)} \Delta u(x) \, dx \,. \end{aligned}$$

The mean value property

Lemma

Let U be an open set in $\mathbb{R}^d$, $B_R(x_0) \subset U$ and $u \in C^2(U)$. Defining $\varphi\colon (0,R) \to \mathbb{R}$ via

$$\varphi(r) := \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \qquad \text{for } r \in (0,R)\,,$$

we have $\varphi(r) \to u(x_0)$ as $r \searrow 0$ and $\varphi'(r) = \frac{r}{d} \fint_{B_r(x_0)} \Delta u \, dx$.

Corollary

Let U be an open set in $\mathbb{R}^d$, $B_r(x_0) \subset U$ and $u \in C^2(U)$. Then $\Delta u \geq / = / > 0$ in U implies

$$u(x_0) \leq / = / < \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \text{ and } u(x_0) \leq / = / < \fint_{B_r(x_0)} u \, dx\,.$$

The mean value property

Proof of the Corollary:

- The statements for "$\Delta u = 0$" follows from the statements for inequality "$\Delta u \geq 0$", applied to the functions u and $-u$.
- Now we assume $\Delta u \geq 0$. Then, in view of $\varphi'(r) = \frac{r}{d} \fint_{B_r(x_0)} \Delta u \, dx$, we observe that φ is monotone non-decreasing, and by continuity of φ we find

$$u(x_0) = \lim_{r \searrow 0} \varphi(r) \leq \varphi(r) = \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} .$$

  The second assertion follows in turn, after integration (via polar coordinates):

$$\begin{aligned} \mathcal{L}^d(B_r(x_0))u(x_0) &= \int_0^r \mathcal{H}^{d-1}(\partial B_\rho(x_0))u(x_0) \, d\rho \\ &\leq \int_0^r \int_{\partial B_\rho(x_0)} u \, d\mathcal{H}^{d-1} d\rho = \int_{B_r(x_0)} u \, dx . \end{aligned}$$

- The case $\Delta u > 0$ follows analogously (now φ is strictly increasing).

The mean value property

Theorem (Gauss)

Let U be an open set in $\mathbb{R}^d$ and $u \in C^2(U)$. The following properties are equivalent:

(i) u *is harmonic, i.e.* $\Delta u = 0$ *holds in* U,

(ii) u *satisfies the spherical mean value property, i.e.*

$$u(x_0) = \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \quad \text{for all } B_r(x_0) \Subset U \,,$$

(iii) u *satisfies the mean value property on balls, i.e.*

$$u(x_0) = \fint_{B_r(x_0)} u \, dx \quad \text{for all } B_r(x_0) \Subset U \,.$$

The mean value property

Theorem (Gauss)

Let U be an open set in $\mathbb{R}^d$ and $u \in C^2(U)$. The following properties are equivalent:

(i) *u is harmonic, i.e. $\Delta u = 0$ holds in U,*

(ii) *u satisfies the spherical mean value property,*

(iii) *u satisfies the mean value property on balls.*

Proof:

- (i) $\Rightarrow$ (ii), (iii): is the statement of the last Corollary.
- (ii) $\Rightarrow$ (iii): follows by integration (polar coordinates).
- (iii) $\Rightarrow$ (i): is proved by contradiction. If $\Delta u(x_0) \neq 0$ for some $x_0 \in U$, then, w.l.o.g. we have $\Delta u > 0$ in some ball $B_r(x_0) \subset U$. This implies strict inequality for the mean values, which is a contradiction to (iii).

The mean value property

Theorem (Gauss)

Let U be an open set in $\mathbb{R}^d$ and $u \in C^2(U)$. The following properties are equivalent:

(i) u *is harmonic, i.e.* $\Delta u = 0$ *holds in* U,

(ii) u *satisfies the spherical mean value property,*

(iii) u *satisfies the mean value property on balls*

Note: The mean value property does not hold on arbitrary sets (only on sets with spherical symmetry and such that the convex hull is a subset of U)!

E.g. $\exp(ax_1)\sin(ax_2)$ is harmonic in $\mathbb{R}^2$ for every $a \in \mathbb{R}$ and has vanishing average in every set of the form

$$\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \in (a_1, b_1), x_2 \in (a_2, a_2 + 2\pi)\}.$$

Consequences: maximum principles

Theorem

Let U be a bounded, open set in $\mathbb{R}^d$ and $u \in C^2(U) \cap C(\overline{U})$ a harmonic function. Then we have

(i) *weak maximum principle:* $\max_{\overline{U}} u = \max_{\partial U} u$,

(ii) *strong maximum principle: if U is connected and if there exists an interior point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u$, then u is constant.*

We further have the analogous minimum principles.

Consequences: maximum principles

Proof:

(ii) *Strong maximum principle:* consider an interior point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u =: M$. For each ball $B_r(x_0) \Subset U$, the mean value property on balls yields

$$M = u(x_0) = \fint_{B_r(x_0)} u\,dx \leq \fint_{B_r(x_0)} M\,dx = M\,,$$

implying $u \equiv M$ in $B_r(x_0)$. Hence, the set $U_M := \{x \in U\colon u(x) = M\}$ is open, and, as preimage of a closed set, relatively closed. Thus, because of $x_0 \in U_M$ (which excludes $U_M = \emptyset$), we must have $u \equiv M$ in U.

(i) *Weak maximum principle* follows from (ii): suppose (i) is false. Then we must have $\max_{\overline{U}} u > \max_{\partial U} u$ and we find an interior point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u$. On the connected component $U(x_0)$ of x_0 in U we get

$$\max_{\overline{U(x_0)}} u = \max_{\overline{U}} u > \max_{\partial U} u \geq \max_{\partial U(x_0)} u\,,$$

since $\partial U(x_0) \subset \partial U$. This contradicts the constancy of u in $U(x_0)$, by (ii).

Consequences: maximum principles

Note: Maximum principles do not rely on the mean values property and hold also for *linear elliptic PDEs* of the form

$$\sum_{i,j=1}^{d} a_{ij} D_{ij} u + \sum_{i=1}^{d} b_i D_i u \geq 0 \qquad \text{in } U\,,$$

where $(a)_{i,j=1,\dots,d}$ is a strictly positive definite matrix.

Idea of proof for the weak maximum principle:

- First assume strict inequality in the PDE: then, if u is a subsolution with maximum in an interior point $x_0 \in U$, then $Du = 0$ and $D^2 u \leq 0$ hold. Thus, $\sum_{i,j=1}^{d} a_{ij} D_{ij} u \leq 0$, which contradicts the assumption.
- In the general case consider $u_\varepsilon := u + \varepsilon \exp(\lambda x_1)$ (which, for λ large, is a strict subsolution). By the first step every u_ε satisfies the maximum principle, and the weak maximum principle for u then follows as $\varepsilon \searrow 0$.

Uniqueness and stability

The maximum principles has the following implications for solutions to the Poisson equation (so in particular for harmonic functions): Let U be bounded, open in $\mathbb{R}^d$, then we have

- **Uniqueness**: if $u, v \in C^2(U) \cap C(\overline{U})$ satisfy $\Delta u = \Delta v$ in U and $u = v$ on ∂U, then we have $u \equiv v$;

Alternatively, this can be seen from an energy method (involving L^2-norms): the integration by parts formula shows

$$0 = \int_U \Delta(u - v)(u - v)\, dx = -\int_U |Du - Dv|^2\, dx\,,$$

which implies $Du \equiv Dv$ in U and then, as $u \equiv v$ on ∂U, the claim $u \equiv v$ in U.

Uniqueness and stability

The maximum principles has the following implications for solutions to the Poisson equation (so in particular for harmonic functions): Let U be bounded, open in $\mathbb{R}^d$, then we have

- **Uniqueness**: if $u, v \in C^2(U) \cap C(\overline{U})$ satisfy $\Delta u = \Delta v$ in U and $u = v$ on ∂U, then we have $u \equiv v$;
- **Continuous dependence of the boundary data**: if $u, v \in C^2(U) \cap C(\overline{U})$ satisfy $\Delta u = \Delta v$ in U, then

  $$\max_{\overline{U}}(u-v) = \max_{\partial U}(u-v) \text{ and } \min_{\overline{U}}(u-v) = \min_{\partial U}(u-v)\,,$$

  and we then conclude

  $$\begin{aligned}\max_{\overline{U}} |u-v| &= \max\Big\{\max_{\overline{U}}(u-v), -\min_{\overline{U}}(u-v)\Big\} \\ &= \max\Big\{\max_{\partial U}(u-v), -\min_{\partial U}(u-v)\Big\} = \max_{\partial U} |u-v|\,.\end{aligned}$$

Consequences: Harnack's inequality

Theorem

Let U be an bounded, open set in $\mathbb{R}^d$ and consider an open connected subset $V \Subset U$. Then there exists a positive constant $c = c(U, V)$ such that

$$\sup_V u \leq c \inf_V u$$

for every non-negative, harmonic function u in U. In particular, all values of u in V are comparable in sense that

$$u(y) \leq cu(x) \leq c^2 u(y) \quad \text{for all } x, y \in V\,.$$

Consequences: Harnack's inequality

Proof:

- First consider $x_0, x_1 \in V$ with distance $|x_0 - x_1| < \operatorname{dist}(V, \partial U)/2 =: r$ (thus $B_r(x_0) \subset B_{2r}(x_1) \subset U$). In view of the mean value property on balls and the positivity of u we find
$$u(x_0) = \fint_{B_r(x_0)} u\,dx \leq 2^d \fint_{B_{2r}(x_1)} u\,dx = 2^d u(x_1)\,.$$
- For arbitrary $x, y \in V$ we consider a path $\varphi \in C([0,1], V)$ joining y and x (possible, since V is connected). We select a finite number of points on the path as follows: we cover $\overline{V}$ with finitely many balls $(B_{r/2}(z_i))_{0 \leq i < N}$ and choose the numbering and the points (x_i) on $\varphi([0,1])$
$$x_0 := y \in B_{r/2}(z_0)\,,$$
$$x_k := \varphi(t_k) \in B_{r/2}(z_k) \text{ s.t. } t_k := \sup\{t \in [0,1] \colon \varphi(t) \in B_{r/2}(z_{k-1})\}$$
for $k \geq 1$. Note that $|x_{k-1} - x_k| < r$ and each ball is chosen at most once (thus x is reached after at most N steps). This gives
$$u(y) \leq 2^{dN} u(x)\,,$$
and for Harnack we finally take the supremum or infimum, respectively.

Consequences: regularity

Even though the Laplace equation involves only second order derivatives, it turn out that harmonic functions are smooth:

Theorem

Let U be an open set in $\mathbb{R}^d$ and $u \in C(U)$ a function which satisfies the spherical mean value property, i.e.

$$u(x_0) = \fint_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \quad \text{for all } B_r(x_0) \Subset U \, .$$

Then u is of class $C^\infty(U)$ and harmonic in U.

Consequences: regularity

Proof: Consider the ε-regularization of u, defined as

$$u_\varepsilon(x) := \eta_\varepsilon * u(x) := \int_U \eta_\varepsilon(x-y)u(y)\,dy = \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,dy$$

for $\varepsilon < \operatorname{dist}(x, \partial U)$ and $\eta \in C_0^\infty(B_1(0))$ non-negative, radial with $\int_{\mathbb{R}^d} \eta\,dx = 1$ and $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(x/\varepsilon)$. This is a C^∞-function (differentiation of parameter dependent integrals).

We further have $u(x) = u_\varepsilon(x)$ (by spherical mean value property and the fact that η_ε is radial and normalized):

$$\begin{aligned} u_\varepsilon(x) &= \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)u(y)\,dy \\ &= \int_0^\varepsilon \int_{\partial B_r(x)} \eta_\varepsilon(x-y)u(y)\,d\mathcal{H}^{d-1}(y)\,dr \\ &= u(x)\int_0^\varepsilon \int_{\partial B_r(x)} \eta_\varepsilon(x-y)\,d\mathcal{H}^{d-1}(y)\,dr = u(x)\,. \end{aligned}$$

Consequences: regularity

Even though the Laplace equation involves only second order derivatives, it turns out that harmonic functions are smooth:

Theorem

Let U be an open set in $\mathbb{R}^d$ and $u \in C(U)$ a function which satisfies the spherical mean value property. Then u is of class $C^\infty(U)$ and harmonic in U.

Note: The regularity does not rely on the mean value property (which nevertheless allows for an elementary proof), but rather on the fact that every derivative of the solution satisfies an equation of the same type (here the derivatives satisfy even the same equation).

Outlook: existence of solutions

Problem: We want to find a solution $u \in C^2(U) \cap C^0(\overline{U})$ to

$$\begin{cases} -\Delta u &= f \quad \text{in } U \\ u &= g \quad \text{auf } \partial U \end{cases}$$

(for f, g and U sufficiently regular), i.e. we want to solve the Poisson equation with prescribed boundary values.

Since the equation is linear, we can split this into:

(a) Find a N_f to the Poisson equation $-\Delta N_f = f$ in U, without constraint on boundary conditions,

(b) find a harmonic function h with boundary values $g - N_f$ on ∂U,

then $u := N_f + h$ is a solution.

Existence of solutions - Part (a)

For f sufficiently regular, we want to find a solution to

$$-\Delta u = f \quad \text{in } U\,.$$

Idea: use the fundamental solution of the Laplace equation

$$\Phi(x) = \begin{cases} -(2\pi)^{-1}\log|x| & \text{for } d = 2\,, \\ (d(d-2)\omega_d)^{-1}|x|^{2-d} & \text{for } d \geq 3\,. \end{cases}$$

Properties:

- $\Phi, |D\Phi| \in L^1_{\text{loc}}(\mathbb{R}^d)$,
- $|D^2\Phi| \notin L^1_{\text{loc}}(\mathbb{R}^d)$ (singularity in $x = 0$!),
- $\Delta\Phi = 0$ in $\mathbb{R}^d \setminus \{0\}$. In fact, $\Delta\Phi$ acts like the distribution $-\delta_0$.

Existence of solutions - Part (a)

For f sufficiently regular, we want to find a solution to

$$-\Delta u = f \quad \text{in } U\,.$$

Theorem (Hölder, 1882)

For U a bounded, open set in $\mathbb{R}^d$ and $f \in C^{0,\alpha}(U) \cap L^\infty(U)$, the Newton potential N_f defined as

$$N_f(x) := (\Phi * f)(x) = \int_U \Phi(x-y) f(y)\, dy \quad \text{for } x \in \mathbb{R}^d$$

satisfies $N_f \in C^2(U) \cap C^1(\overline{U})$ and $-\Delta N_f = f$ in U.

Existence of solutions - Part (a)

Idea of proof: (for simplicity $U = \mathbb{R}^d$, $f \in C_0^2(\mathbb{R}^d)$, otherwise we need to work with cancellation effects in the Newton potential)

$$N_f(x) = \int_{\mathbb{R}^d} \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^d} \Phi(y) f(x-y)\, dy$$

- Differentiation under the integral leads to

$$D_j D_i N_f(x) = \int_{\mathbb{R}^d} \Phi(y) D_{x_j} D_{x_i} f(x-y)\, dy$$

- In order to calculate ΔN_f we cut out the singularity:

$$\Delta N_f(x) = \int_{B_\varepsilon(0)} \Phi(y) \Delta_y f(x-y)\, dy + \int_{\mathbb{R}^d \setminus B_\varepsilon(0)} \Phi(y) \Delta_y f(x-y)\, dy$$

  - The first term vanishes for $\varepsilon \to 0$ (explicit estimate for Φ, $f \in C^2$);
  - the second term is rewritten via Green's formula and harmonicity of Φ as two boundary integrals on $\partial B_\varepsilon(0)$, one with integrand $\Phi(y) D_y f(x-y) \cdot \nu$ (vanishing with $\varepsilon \to 0$), the other one with integrand $D\Phi(y) f(x-y) \cdot \nu$ (yielding $-f(x)$ for $\varepsilon \to 0$).

Existence of solutions - Part (b)

For g continuous, we want to find a solution to

$$\begin{cases} -\Delta u &= 0 \quad \text{in } U\,, \\ u &= g \quad \text{auf } \partial U\,. \end{cases}$$

Theorem (Perron, 1923)

Let U be a bounded, open, regular set in $\mathbb{R}^d$. Then for every $g \in C(\partial U)$ there exists a unique harmonic function $h \in C^2(U) \cap C(\overline{U})$ with $u = g$ on ∂U.

Poisson integral formula for balls

We first consider the case that U is a ball and construct in this case a so-called Green's function, which solves the Poisson problem, i.e. which provides a representation formula for solutions to the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } U\,, \\ u = g & \text{auf } \partial U\,. \end{cases}$$

Starting point is Green's representation formula

$$u(x) = -\int_U \Phi(x-y)\Delta u(y)\,dy + \int_{\partial U} \big(\Phi(x-y)Du(y) - u(y)D_y\Phi(x-y)\big)\cdot\nu(y)\,d\mathcal{H}^{d-1}(y)\,,$$

which is satisfied for any regular U and $u \in C^2(U)\cap C^1(\overline{U})$ with $\Delta u \in L^1(U)$ (cp. calculations for the Newton-potential).

Note: Solutions u to the Poisson equation can be written in terms of the known functions f, g and Φ, but we still need $D_\nu u$ on ∂U.

Poisson integral formula for balls

Definition

Let U be an open set in $\mathbb{R}^d$. A function $G\colon \{(x,y) \in U \times U : x \neq y\} \to \mathbb{R}$ is called *Green's function for* U if for all $x \in U$:

(i) $y \mapsto G(x,y) - \Phi(x-y)$ is of class $C^2(U) \cap C^1(\overline{U})$ and harmonic in U;

(ii) $y \mapsto G(x,y)$ vanishes on ∂U, i. e. we have $\lim_{y \to y_0} G(x,y) = 0$ for $y_0 \in \partial U$ (and also at infinity for U unbounded).

Note:

- $y \mapsto G(x,y) - \Phi(x-y)$ is called the *corrector function* and is unique (if existent) for U bounded.
- $y \mapsto G(x,y)$ has the same singularity in x as $\Phi(x-y)$;
- G is symmetric if U is regular and bounded (hence, also $x \mapsto G(x,y)$ is harmonic in $U \setminus \{y\}$).

Poisson integral formula for balls

Definition

Let U be an open set in $\mathbb{R}^d$. A function $G\colon \{(x,y) \in U \times U : x \neq y\} \to \mathbb{R}$ is called *Green's function for* U if for all $x \in U$:

(i) $y \mapsto G(x,y) - \Phi(x-y)$ is of class $C^2(U) \cap C^1(\overline{U})$ and harmonic in U;

(ii) $y \mapsto G(x,y)$ vanishes on ∂U, i. e. we have $\lim_{y \to y_0} G(x,y) = 0$ for $y_0 \in \partial U$ (and also at infinity for U unbounded).

Consequence: every solution u to the Poisson problem can now be written as

$$u(x) = \int_U G(x,y) f(y)\, dy - \int_{\partial U} D_y G(x,y) g(y) \cdot \nu(y)\, d\mathcal{H}^{d-1}(y)$$

(doesn't yet seem that much better, since G is unknown – but at least, once we know G, we can solve *any* boundary value problem on U).

Poisson integral formula for balls

In order to construct the Green's function for a ball $B_r(x_0)$ we apply a reflexion method. Idea: compensate a point charge in $x \in \mathbb{R}^d \setminus \partial B_r(x_0)$ by another point charge (of suitable magnitude) in the reflexion point

$$x^* := x_0 + r^2 \frac{x - x_0}{|x - x_0|^2} .$$

- With $|x^* - x_0||x - x_0| = r^2$ exactly one of the point x, x^* lies in $B_r(x_0)$;
- For points $y \in \partial B_r(x_0)$ one easily checks $|x^* - y|^2 = r^2 |x - x_0|^{-2} |y - x|^2$.

Similarly, as the shifted fundamental solution $y \mapsto \Phi(x - y)$ can be interpreted as the potential caused by a point charge in x, we now put a point charge in x^* for compensation and find:

Lemma

The Green's function for $B_r(x_0) \subset \mathbb{R}^d$ is given for $y \in B_r(x_0) \setminus \{x\}$ by

$$G(x, y) := \begin{cases} \Phi(x - y) - \Phi\left(\frac{|x - x_0|}{r}(y - x^*)\right) & \text{for } x \in B_r(x_0) \setminus \{x_0\} , \\ \Phi(x_0 - y) - \Phi(r e_1) & \text{for } x = x_0 . \end{cases}$$

Poisson integral formula for balls

Summary:

We have derived the Green's function on the ball

$$G(x,y) := \begin{cases} \Phi(x-y) - \Phi\big(\frac{|x-x_0|}{r}(y-x^*)\big) & \text{for } x \neq x_0 \\ \Phi(x_0-y) - \Phi(re_1) & \text{for } x = x_0 \end{cases}$$

for $x \neq y \in B_r(x_0)$. With the version of Green's representation formula we thus have established the desired representation of solutions to the Poisson problem in terms of the data f, g (and $B_r(x_0)$):

$$u(x) = \int_{B_r(x_0)} G(x,y) f(y)\, dy - \int_{\partial B_r(x_0)} \underbrace{D_y G(x,y) \cdot \nu(y)}_{= \frac{1}{d\omega_d} \frac{|x-x_0|^2 - r^2}{r|y-x|^d}} g(y)\, d\mathcal{H}^{d-1}(y)\,,$$

which is known as Poisson integral formula.

Note: Also the reverse implication is true, i.e. for f, g regular this formula defines the unique solution to the Poisson problem. In particular, every continuous function on a sphere can be extended to a harmonic function in the full ball (known as Schwarz Theorem).

Existence of solutions - Part (b)

Idea: semi-explicit construction, based on:

1. For balls $B_r(x_0)$, the function

$$h(x) = \int_{\partial B_r(x_0)} \frac{1}{d\omega_d} \frac{r^2 - |x - x_0|^2}{r|y - x|^d} g(y)\, d\mathcal{H}^{d-1}(y) \quad \text{for } x \in B_r(x_0)$$

   is harmonic with boundary values g (Poisson integral formula);

2. by Perron's method one can then find harmonic functions, starting from

$$\{u \text{ subharmonic with } u \leq g \text{ on } \partial U\}\,.$$

   - The supremum of these subharmonic functions provides a harmonic function (otherwise one could replace the resulting function on balls by its harmonic extension and find a contradiction);
   - under a suitable regularity condition on U, this limit also attains the prescribed boundary values.

Existence of solutions - Part (b)

Comments:

- Perron's method is still quite elementary and is easily extended to more general elliptic equations of second order, as it essentially relies on subsolutions (which always exist), the maximum principle and the solvability of the Dirichlet problem for arbitrary continuous boundary values;
- Conceptually similar is the balayage method (tracing back to Poincaré):
  - exploit U by a sequence of balls $(B_{r_k}(x_k))_{k\in\mathbb{N}}$,
  - define a sequence of subsolutions, starting from a fixed subsolution with boundary values g and replacing inductively the predecessor v_{k-1} on $B_{r_k}(x_k)$ by the harmonic extension,
  - show that this sequence converges to the desired harmonic function.

Outlook: probabilistic representation

Harmonic functions can be represented via the expectation of a Brownian motion, which is stopped at the boundary of the domain. This is a particular case of the **Feynman–Kac formula**.

Theorem

Consider D a bounded, open, regular set in $\mathbb{R}^d$, $g \in C(\partial U)$ and B_t a d-dimensional Brownian motion. Let

$$\tau = \tau_D := \inf \left\{ t > 0 \colon x + B_t \notin D \right\}$$

be the first exit time from D for a Brownian motion started in x. Then the unique harmonic function u on D with boundary values g is represented as

$$u(x) = \mathbb{E}[g(x + B_\tau)] \,.$$

Outlook: probabilistic representation

Idea: by Itô's formula we have (u is regular)

$$u(x+B_t) - u(x) = \int_0^t Du(x+B_s)\,dB_s + \frac{1}{2}\int_0^t \Delta u(x+B_s)\,ds$$

Then, by taking the expectation and using the solution property of u, we get

$$u(x) = \mathbb{E}[u(x)] = \mathbb{E}[u(x+B_\tau)] = \mathbb{E}[g(x+B_\tau)]\,.$$

Comments:

- The existence of the harmonic function (which is only true for U sufficiently regular) is due to Perron's theorem, this is why regularity of U is required;
- If $B_r(x_0) \subset U$, then we find

  $$u(x_0) = \mathbb{E}[u(x_0 + B_{\tau_{B_r(x_0)}})]\,,$$

  with τ now denoting the exit time from $B_r(x_0)$. Since the Brownian motion is isotropic in space, this can be interpreted as the spherical mean value property of harmonic function

  $$u(x_0) = \fint_{\partial B_r(x_0)} u\,d\mathcal{H}^{d-1}\,.$$

Part II:
The heat equation

1. Physical motivation
2. The fundamental solution
3. Mean value property
4. Existence of solutions (full space)

Physical motivation of the heat equation

The heat equation describes the change in time and space of a diffusive quantity u (e.g. concentration of particles, heat), which flows from regions of higher to regions of lower concentration and which thus evolves towards an equilibrium. Mathematically, this means the rate of change of u equals the negative net flux F through the boundary for each (smooth) test volume $V \subset U$:

$$\partial_t u = -\int_{\partial V} F \cdot \nu \, d\mathcal{H}^{d-1} = -\int_V \operatorname{div} F \, dx \,,$$

hence $\partial_t u = -\operatorname{div} F$ in $\mathbb{R}^+ \times U$. Assuming again $F = -aDu$ for some constant $a > 0$ on the flux, we find

$$\partial_t u = a\Delta u \qquad \text{in } \mathbb{R}^+ \times U \,,$$

which for $a = 1$ is the heat equation.

Examples of caloric functions

For studying time-dependent problems like the heat equation, one typically does not work on general subsets in $\mathbb{R}^{d+1}$, but rather on sets of the form $I \times U$ for a (possibly infinite) interval in $\mathbb{R}$. We shall use the notation:

- *parabolic cylinder* $U_T := (0, T] \times U \subset \mathbb{R}^{d+1}$,
- *parabolic boundary* $\partial_p U_T := (\{0\}) \times U) \cup ([0, T] \times \partial U) \subset \partial U_T$ (initial values on $\{0\}) \times U$, boundary values on $[0, T] \times \partial U$).

Since the heat equation involves second order space derivatives and first order time derivatives, it is convenient to work with the spaces

$$C_1^2(U_T) := \left\{ w \in C^1(U_T) \colon D_i D_j w \in C(U_T) \text{ for } i, j \in \{1, \ldots, d\} \right\}$$

for classical solutions.

Examples of caloric functions

Definition

Classical solutions to the heat equation are called (C_1^2-)*caloric*. If only one of the inequalities $\partial_t u - \Delta u \leq 0$ or $\partial_t u - \Delta u \geq 0$ is satisfied, then u is called (C_1^2-)*subcaloric* and (C_1^2-)*supercaloric*, respectively.

Examples of caloric functions

(i) Every harmonic function is caloric.

(ii) Caloric functions on $\mathbb{R}^+ \times \mathbb{R}$ obtained by separation of variables are:

$$u(t, x_1) := \exp(a^2 t)\big(c_1 \sinh(a x_1) + c_2 \cosh(a x_1)\big)\,,$$
$$u(t, x_1) := \exp(-a^2 t)\big(c_1 \sin(a x_1) + c_2 \cos(a x_1)\big)\,,$$

for $a, c_1, c_2 \in \mathbb{R}$. Also for $d > 1$ one obtains functions in this way (but then it involves the eigenvalue problem for the Laplace operator).

Details: We look for a caloric function of the form $u(t,x) = T(t)X(x)$ for C^2-functions v and w. This implies

$$T'(t)X(x_1) - T(t)X''(x_1) = 0\,,$$

which again leads to ODEs which can be solved explicitly.

The fundamental solution

Also in the case of the heat equation we look for a very particular caloric function, which satisfies

- radial symmetry in space (i.e. $u(t,x) = v(t,|x|)$ for some function v),
- invariance under suitable space-time dilatations arising from the structure of the PDE (i.e. $u(t,x) = t^{-n/2}\tilde{v}(t^{-1/2}x)$ for some function $\tilde{v}$),
- decay to zero as $t \to \infty$.

These requirements on a caloric function lead to an ODE which can be solved.

The fundamental solution

Definition

The function $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$, defined as

$$\Psi(t,x) = (4\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{|x|^2}{4t}\Big),$$

is called *fundamental solution of the heat equation* or *heat kernel*.

Note:

- By construction, we have $\partial_t \Psi - \Delta\Psi = 0$ in $\mathbb{R}^+ \times \mathbb{R}^d$;
- There is a singularity for $t = 0$: $\lim_{t\to 0} \Psi(t,0) = \infty$ and $\lim_{t\to 0} \Psi(t,x) = 0$ for $x \neq 0$;
- For $s,t > 0$ holds $\Psi(t,\cdot) * \Psi(s,\cdot) = \Psi(s+t,\cdot)$.

The heat ball

Before addressing a mean value-type property of caloric function, we first define specific sets in $\mathbb{R} \times \mathbb{R}^d$. We recall that the mean value property for harmonic functions holds on balls and spheres, which are very special in the sense that they are superlevel of level sets of the fundamental solution of the Laplace equation. This should motivate

Definition

Consider $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ and $r > 0$. We define the *heat ball* $W_r(t_0, x_0)$ at (t_0, x_0) as the subset

$$W_r(t_0, x_0) := \left\{(t, x) \colon t < t_0 \text{ and } \Psi(t_0 - t, x_0 - x) > r^{-d}\right\}$$

of $(-\infty, t_0) \times \mathbb{R}^d$.

Attention: $(t_0, x_0) \in \partial W_r(t_0, x_0)$!

The heat ball

Properties:

- *Monotonicity:* For $r < \tilde{r}$ there holds $W_r(t_0, x_0) \subset W_{\tilde{r}}(t_0, x_0)$;
- *Behavior under translations:* $W_r(t_0, x_0) = (t_0, x_0) + W_r(0, 0)$;
- *Parabolic scaling:* $(t, x) \in W_r(0, 0) \Leftrightarrow (r^{-2}t, r^{-1}x) \in W_1(0, 0)$;
- *Explicit representation:* for $t \in (-r^2/(4\pi), 0)$ we can rewrite the inequality in the definition of the heat ball as $\frac{|x|^2}{4t} > \log(r^{-d}(-4\pi t)^{\frac{d}{2}})$. Defining $b_r \colon \mathbb{R}^- \times \mathbb{R}^d \to \mathbb{R}$ as

  $$b_r(t, x) := \frac{|x|^2}{4t} + d \log r - \frac{d}{2} \log(-4\pi t)\,,$$

  we find

  $$W_r(0, 0) = \left\{(t, x) \in \mathbb{R}^- \times \mathbb{R}^d \colon b_r(t, x) > 0\right\} \quad \text{(bounded set)}\,,$$

  and similarly $\partial W_r(0, 0)$ is represented via $b_r(t, x) = 0$;
- *Weighted volume:* $\dfrac{1}{4r^d} \displaystyle\int_{W_r(0,0)} \frac{|x|^2}{t^2} d(t, x) = 1.$

The mean value property

Lemma

Let $u \in C_1^2(W_R(0,0))$. *Defining* $\psi\colon (0,R) \to \mathbb{R}$ *via*

$$\psi(r) := \frac{1}{4r^d} \int_{W_r(0,0)} u(t,x) \frac{|x|^2}{t^2} \, d(t,x) \quad \text{for } r \in (0,R)\,,$$

we have $\psi(r) \to u(0,0)$ *and*

$$\psi'(r) = \frac{d}{r^{d+1}} \int_{W_r(0,0)} \big[-\partial_t u(t,x) + \Delta u(t,x) \big] b_r(t,x) \, d(t,x)\,.$$

The mean value property

Proof: We verify that $\psi(r) \coloneqq \dfrac{1}{4r^d} \displaystyle\int_{W_r(0,0)} u(t,x) \frac{|x|^2}{t^2}\, d(t,x)$

- satisfies $\psi(r) \to u(0,0)$ as $r \searrow 0$, by continuity of u:

$$
\begin{aligned}
|\psi(r) - u(0,0)| &= \frac{1}{4r^d} \left| \int_{W_r(0,0)} \big(u(t,x) - u(0,0)\big) \frac{|x|^2}{t^2}\, d(t,x) \right| \\
&\le \frac{1}{4r^d} \int_{W_r(0,0)} |u(t,x) - u(0,0)| \frac{|x|^2}{t^2}\, d(t,x) \\
&\le \sup_{(t,x)\in W_r(0,0)} |u(t,x) - u(0,0)| \to 0
\end{aligned}
$$

The mean value property

Proof: We verify that $\psi(r) \coloneqq \frac{1}{4r^d} \int_{W_r(0,0)} u(t,x) \frac{|x|^2}{t^2} \, d(t,x)$

- satisfies $\psi(r) \to u(0,0)$ as $r \searrow 0$, by continuity of u;
- is continuously differentiable, since we can first rewrite ψ as a parameter-dependent integral

$$\psi(r) = \frac{1}{4} \int_{W_1(0,0)} u(r^2 s, ry) \frac{|y|^2}{s^2} \, d(s,y) \,.$$

then interchange the order of integration and differentiation

$$\begin{aligned} \psi'(r) &= \frac{1}{4} \int_{W_1(0,0)} \left[\partial_t u(r^2 s, ry) 2rs + Du(r^2 s, ry) \cdot y\right] \frac{|y|^2}{s^2} \, d(s,y) \\ &= \frac{1}{4r^{d+1}} \int_{W_r(0,0)} \left[\partial_t u(t,x) 2t + Du(t,x) \cdot x\right] \frac{|x|^2}{t^2} \, d(t,x) \,. \end{aligned}$$

The mean value property

Proof: We verify that $\psi(r) \coloneqq \frac{1}{4r^d} \int_{W_r(0,0)} u(t,x) \frac{|x|^2}{t^2} \, d(t,x)$

- satisfies $\psi(r) \to u(0,0)$ as $r \searrow 0$, by continuity of u
- is continuously differentiable with

$$\psi'(r) = \frac{1}{r^{d+1}} \int_{W_r(0,0)} \big[\partial_t u(t,x) \underbrace{\frac{|x|^2}{2t}}_{=Db_r(t,x)\cdot x} + Du(t,x) \cdot x \underbrace{\frac{|x|^2}{4t^2}}_{=-\partial_t b_r(t,x) x - dDb_r(t,x)} \big] \, d(t,x) \, .$$

Applying integration by parts, for the first term in x and for the second in t, and finally again in x (note that all boundary integrals vanish, due to $b_r(t,x) = 0$ on $\partial W_r(0,0)$), we get

$$\begin{aligned} \psi'(r) &= \frac{d}{r^{d+1}} \int_{W_r(0,0)} \big[- \partial_t u(t,x) b_r(t,x) - Du(t,x) \cdot Db_r(t,x) \big] \, d(t,x) \\ &= \frac{d}{r^{d+1}} \int_{W_r(0,0)} \big[- \partial_t u(t,x) + \Delta u(t,x) \big] b_r(t,x) \, d(t,x) \, . \end{aligned}$$

The mean value property

Corollary

Let U be an open set in $\mathbb{R}^d$, $T > 0$, $W_r(t_0, x_0) \Subset U_T$ und $u \in C_1^2(U_T)$. Then $\partial_t u - \Delta u \le / = / < 0$ in U_T implies

$$u(t_0, x_0) \le / = / < \frac{1}{4r^d} \int_{W_r(t_0,x_0)} u(t,x) \frac{|x - x_0|^2}{(t - t_0)^2} \, d(t,x) \, .$$

Moreover, we get the equivalence of

- u is caloric, i.e. $\partial_t u - \Delta u = 0$ in U_T,
- u satisfies the mean value property on heat balls, i.e.

$$u(t_0, x_0) = \frac{1}{4r^d} \int_{W_r(t_0,x_0)} u(t,x) \frac{|x - x_0|^2}{(t - t_0)^2} \, d(t,x)$$

  for all $W_r(t_0, x_0) \subset U_T$.

Consequences: maximum principles

Theorem

Let U be a bounded, open set in $\mathbb{R}^d$ and $u \in C_1^2(U_T) \cap C(\overline{U_T})$ a caloric function. Then we have

(i) *weak maximum principle:* $\max_{\overline{U_T}} u = \max_{\partial_p U_T} u$,

(ii) *strong maximum principle: if U is connected and if there exists an interior point $(t_0, x_0) \in U_T$ with $u(t_0, x_0) = \max_{\overline{U_T}} u$, then u is constant on U_{t_0}.*

We further have the analogous minimum principles.

Remark: The strong maximum principle does not state constancy on all of U_T, but only up to time t_0 (heat balls are defined "for past times").

Consequences: maximum principles

Theorem

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We further have the analogous minimum principles.

Also these maximum principles hold for more general parabolic PDEs, e.g. for

$$\partial_t u - \sum_{i,j=1}^{d} a_{ij} D_{ij} u + \sum_{i=1}^{d} b_i D_i u \leq 0 \qquad \text{in } U_T\,,$$

where $(a)_{i,j=1,\ldots,d}$ is a strictly positive definite matrix.

Consequences: maximum principles

Proof:

(i) follows from (ii) exactly as for harmonic functions, via contradiction.

(ii) We consider an interior point $(t_0, x_0) \in U_T$ with $u(t_0, x_0) = \max_{\overline{U_T}} u =: M$ und take $r > 0$ such that $W_r(t_0, x_0) \subset U_{t_0}$. From the mean value property for caloric functions, we see

$$M = u(t_0, x_0) \leq \frac{1}{4r^d} \int_{W_r(t_0,x_0)} u(t,x) \frac{|x - x_0|^2}{(t - t_0)^2} \, d(t,x) \leq M \, .$$

Hence, we have $u \equiv M$ in $W_r(t_0, x_0)$, *but*, since $W_r(t_0, x_0)$ is not a neighborhood of (t_0, x_0), this does not imply yet that $u^{-1}(M)$ is relatively open in U_{t_0}.

However, if (t_0, x_0) is joined to a point (t_1, x_1) with $t_1 < t_0$ by a straight line in U_{t_0}, then L_1 is connected in U_{t_0} and $u \equiv M$ on L_1 (as $u^{-1}(M) \cap L_1$ is both relatively open, see above, and closed, by continuity of u). This shows $u(t^*, x^*) = M$ for any $t* < t_0$ and $x^* \in U$, by joining (t_0, x_0) and (t^*, x^*) via finitely many such lines. Hence, by continuity of u we end up with $u \equiv M$ in U_{t_0}.

Consequences of the maximum principles

Let U be bounded, open in $\mathbb{R}^d$ and $T > 0$, then we have

- **Uniqueness** of solutions to the inhomogeneous heat equation: if $u, v \in C_1^2(U_T) \cap C(\overline{U_T})$ satisfy $\partial_t u - \Delta u = \partial_t v - \Delta v$ in U_T and $u = v$ on $\partial_p U_T$, then we have $u \equiv v$;

Alternatively, this can be seen from an energy method: we associate to a caloric function w the energy

$$e(t) := \int_U |w(t,x)|^2 \, dx \,.$$

Then, by integration by parts, we find

$$\frac{d}{dt} e(t) = 2 \int_U \partial_t w w \, dx = 2 \int_U \Delta w w \, dx = -2 \int_U |Dw|^2 \, dx \le 0 \,,$$

hence, the energy is non-increasing. For $w = v - u$ we have $e(0) = 0$, which implies $u \equiv v$ in U_T.

Consequences of the maximum principles

Let U be bounded, open in $\mathbb{R}^d$ and $T > 0$, then we have

- **Uniqueness** of solutions to the inhomogeneous heat equation: if $u, v \in C_1^2(U_T) \cap C(\overline{U_T})$ satisfy $\partial_t u - \Delta u = \partial_t v - \Delta v$ in U_T and $u = v$ on $\partial_p U_T$, then we have $u \equiv v$;
- **Infinite propagation speed:** if U is connected, the initial values on $\{0\} \times U$ are somewhere positive and the boundary values on $[0, T] \times \partial U$ are non-negative, then caloric functions are positive everywhere in U_T.
- **Long time behavior:** if $u \in C_1^2(U_T) \cap C(\overline{U_T})$ is a solution to

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } U_T \\ u = g & \text{on } [0, T] \times \partial U \end{cases}$$

  with regular, *time-independent* f and g, then u converges uniformly in x, as $t \to \infty$, to the stationary solution.

Existence of solutions in full space

Problem: We want to find a solution $u \in C_1^2([0,T] \times \mathbb{R}^d)$ to

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{in } [0,T] \times \mathbb{R}^d \\ u &= g \quad \text{on } \{0\} \times \mathbb{R}^d \end{cases}$$

(for f, g), i.e. we want to solve the inhomogeneous heat equation with given initial condition (but no boundary values, since we work in the full space).

Since the equation is linear, we can split this into:

(a) Find a solution u_h to the initial value problem for the heat equation,

(b) Find a special solution u_i to the inhomogeneous heat equation (with zero initial values),

then $u := u_h + u_i$ is the solution.

Existence of solutions in full space - Part (a)

For g bounded and continuous, we want to find a solution to the initial value problem

$$\begin{cases} \partial_t u - \Delta u &= 0 \quad \text{in } [0,T] \times \mathbb{R}^d\,, \\ u &= g \quad \text{on } \{0\} \times \mathbb{R}^d\,. \end{cases}$$

Idea: use the fundamental solution of the heat equation

$$\Psi(t,x) = (4\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{|x|^2}{4t}\Big)\,,$$

Properties:

- $\partial_t \Psi - \Delta \Psi = 0$ in $\mathbb{R}^+ \times \mathbb{R}^d$,
- normalization $\int_{\mathbb{R}^d} \Psi(t,x)\,dx = 1$ for all $t > 0$ (which can be interpreted as mass conservation),
- behavior for $t \searrow 0$: we have $\lim_{t\searrow 0} \Psi(t,x) = 0$ for $x \neq 0$ and $\lim_{t\searrow 0} \Psi(t,0) = \infty$. In fact, Ψ acts on $\{0\} \times \mathbb{R}^d$ like δ_0!

Existence of solutions in full space - Part (a)

For g bounded and continuous, we want to find a solution to the initial value problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } [0,T] \times \mathbb{R}^d\,, \\ u = g & \text{on } \{0\} \times \mathbb{R}^d\,. \end{cases}$$

Theorem

Let $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then the function $u_h\colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$, defined as

$$u_h(t,x) \coloneqq (\Psi(t,\cdot) * g)(x) = \int_{\mathbb{R}^d} \Psi(t, x-y) g(y)\, dy\,,$$

is of class $C_1^2((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ and satisfies the heat equation with initial values $u = g$ on $\{0\} \times \mathbb{R}^d$.

Existence of solutions in full space - Part (a)

$$u_h(t,x) := (\Psi(t,\cdot) * g)(x) = \int_{\mathbb{R}^d} \Psi(t,x-y)g(y)\,dy$$

Proof:

- Interior regularity of u_h follows from differentiation under the integral sign;
- In this way, we also see

$$\partial_t u_h(t,x) - \Delta u_h(t,x) = \int_{\mathbb{R}^d} \big(\partial_t \Psi(t,x-y) - \Delta_x \Psi(t,x-y)\big) g(y)\,dy = 0\,;$$

Existence of solutions in full space - Part (a)

$$u_h(t,x) := (\Psi(t,\cdot) * g)(x) = \int_{\mathbb{R}^d} \Psi(t,x-y)g(y)\,dy$$

Proof:

- Interior regularity of u_h follows from differentiation under the integral sign;
- In this way, we also see that u_h is caloric in $\mathbb{R}^+ \times \mathbb{R}^d$;
- Attainment of initial values: we show $\lim_{t\to 0, x\to x_0} u(t,x) = g(x_0)$ for all $x_0 \in \mathbb{R}^d$, using that on the one hand g varies little close to x_0 and that on the other hand $\Psi(0,t)$ is small away from 0. Indeed, we have

$$\begin{aligned} |u(t,x) - g(x_0)| &= \left| \int_{\mathbb{R}^d} \Psi(t,x-y)(g(y)-g(x_0))\,dy \right| \\ &\le \underbrace{\sup_{y\in B_\delta(x_0)} |g(y)-g(x_0)|}_{\text{arbitrarily small, for } \delta \text{ small}} + 2\|g\|_{L^\infty} \underbrace{\int_{\mathbb{R}^d\setminus B_\delta(x_0)} \Psi(t,x-y)\,dy}_{\substack{\text{arbitrarily small, once } \delta \text{ is fixed,} \\ \text{by choosing } t \text{ sufficiently small}}} . \end{aligned}$$

Existence of solutions in full space - Part (b)

For f sufficiently regular, we want to find a solution to

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } [0,T] \times \mathbb{R}^d\,, \\ u = 0 & \text{on } \{0\} \times \mathbb{R}^d\,. \end{cases}$$

Idea: Apply **Duhamel's principle** which allows to reduce the inhomogeneous problem to solving a family of initial value problems for the homogeneous equation. The candidate for the solution is

$$u_i(t,x) := \int_0^t \underbrace{\int_{\mathbb{R}^d} \Psi(t-s, x-y) f(s,y)\, dy}_{\text{caloric, with initial values } f(s,x) \text{ at time } t=s} \, ds\,.$$

Existence of solutions in full space - Part (b)

For f sufficiently regular, we want to find a solution to

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } [0,T] \times \mathbb{R}^d \,, \\ u = 0 & \text{on } \{0\} \times \mathbb{R}^d \,. \end{cases}$$

Theorem

Consider $f \in C_1^2([0,\infty) \times \mathbb{R}^d)$ with $\operatorname{spt}(f) \Subset [0,\infty) \times \mathbb{R}^d$. Then the function $u_i \colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$, defined as

$$u_i(t,x) := \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-y) f(s,y) \, dy \, ds \,,$$

is of class $C_1^2((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ and satisfies the equation $\partial_t u - \Delta u = f$ in $(0,\infty) \times \mathbb{R}^d$.

Existence of solutions in full space - Part (b)

$$u_i(t,x) := \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-y) f(s,y)\, dy\, ds$$
$$= \int_0^t \int_{\mathbb{R}^d} \Psi(s,y) f(t-s, x-y)\, dy\, ds$$

Proof:

- Interior regularity of u_i follows from differentiation under the integral sign, with

$$\partial_t u_i(t,x) = \int_0^t \int_{\mathbb{R}^d} \Psi(s,y) \partial_t f(t-s, x-y)\, dy\, ds$$
$$+ \int_{\mathbb{R}^d} \Psi(t,y) f(0, x-y)\, dy\,,$$
$$D_i D_j u_i(t,x) = \int_0^t \int_{\mathbb{R}^d} \Psi(s,y) D_{x_i} D_{x_j} f(t-s, x-y)\, dy\, ds\,;$$

Existence of solutions in full space - Part (b)

Proof:

- Interior regularity of u_i follows from differentiation under the integral sign, with

$$\begin{aligned}\partial_t u_i(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Psi(s,y)\partial_t f(t-s,x-y)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^d} \Psi(t,y) f(0,x-y)\,dy\,, \\ D_i D_j u_i(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Psi(s,y) D_{x_i} D_{x_j} f(t-s,x-y)\,dy\,ds\,;\end{aligned}$$

- u_i can be extended continuously by 0 for $t \searrow 0$ (integrability of Ψ, boundedness of f);
- solution property is shown by cutting out the singularity at $(0,0)$

$$\begin{aligned}\partial_t u_i(t,x) - \Delta u_i(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Psi(s,y)(\partial_t - \Delta_x) f(t-s,x-y)\,dy\,ds \\ &\quad + \int_{\mathbb{R}^d} \Psi(t,y) f(0,x-y)\,dy\end{aligned}$$

Existence of solutions in full space - Part (b)

Proof:

- Interior regularity of u_i follows from differentiation under the integral sign.
- u_i can be extended continuously by 0 for $t \searrow 0$ (integrability of Ψ, boundedness of f);
- solution property is shown by cutting out the singularity at $(0,0)$ and applying integration by parts formula

$$\begin{aligned}
\partial_t u_i(t,x) - \Delta u_i(t,x) &= \int_0^\varepsilon \int_{\mathbb{R}^d} \Psi(s,y)(\partial_t - \Delta_x) f(t-s,x-y)\,dy\,ds \\
&\quad + \int_\varepsilon^t \int_{\mathbb{R}^d} \Psi(s,y)(-\partial_s - \Delta_y) f(t-s,x-y)\,dy\,ds \\
&\quad + \int_{\mathbb{R}^d} \Psi(t,y) f(0,x-y)\,dy \\
&= C\varepsilon + \int_{\mathbb{R}^d} \Psi(\varepsilon,y) f(t-\varepsilon,x-y)\,dy \overset{\varepsilon\to 0}{\to} f(t,x)\,.
\end{aligned}$$

Existence of solutions in full space - Result

Theorem

Let $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $f \in C_1^2([0,\infty) \times \mathbb{R}^d)$ *with* $\operatorname{spt}(f) \Subset [0,\infty) \times \mathbb{R}^d$*. Then the function* $u\colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$*, defined as*

$$u(t,x) := \int_{\mathbb{R}^d} \Psi(t, x-y) g(y)\, dy + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-y) f(s,y)\, dy\, ds$$

is of class $C_1^2((0,\infty) \times \mathbb{R}^d) \cap C([0,\infty) \times \mathbb{R}^d)$ *and is a classical solution to*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u = g & \text{on } \{0\} \times \mathbb{R}^d\,. \end{cases}$$

Outlook: probabilistic representation

Similarly as for harmonic functions, also caloric functions can be represented via the expectation of a Brownian motion.

Theorem

Let $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and B_t a d-dimensional Brownian motion. Then the unique function u on $\mathbb{R}^+ \times \mathbb{R}^d$ solving

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^d\,, \\ u = g & \text{on } \{0\} \times \mathbb{R}^d\,. \end{cases}$$

is the function

$$u(t,x) = \mathbb{E}[g(x + B_t)]\,.$$

Outlook: probabilistic representation

Idea: by Itô's formula we have (u is regular)

$$\begin{aligned} u(t,x) - u(0, x+B_t) &= \int_0^t Du(t-s, x+B_s)\, dB_s \\ &\quad + \int_0^t \big[-\partial_t u(t-s, x+B_s) + \tfrac{1}{2}\Delta u(t-s, x+B_s) \big]\, ds\,. \end{aligned}$$

Then, by taking the expectation and using the solution property of u, we get

$$u(t,x) = \mathbb{E}[u(t,x)] = \mathbb{E}[u(0, x+B_t)] = \mathbb{E}[g(x+B_t)]\,.$$

Comment: Similarly to the representation of solutions via (stochastic) characteristics, this can be interpreted as following the Brownian motion backwards in time until it arrives at $t = 0$.

Part III:

Connection between PDEs and optimization problems

1. Introduction to the calculus of variations
2. Direct vs. indirect approach
3. Euler–Lagrange equations
4. Brachistochrone problem

Calculus of variations

The aim of the calculus of variations is to find optimal (to be specified) solutions to a (variational) problem and to describe their properties. As for PDEs, many problems arise from natural sciences, geometry or economy, and simple examples are

- what is the shortest connection between two points?
- what is the largest area to a fixed perimeter?
- what is the state of lowest energy?

In order to formulate such questions rigorously, one has to express the relevant quantities mathematically (such as distance, area, perimeter, energy), to determine the relations between the various quantities, and to choose a class of admissible “states” or “configurations” (incorporating also constraints), among which a solution should be determined.

Legendary: the brachistochrone problem

How the field of the calculus of variations was born: in 1696 Johann Bernoulli published the following challenge in a journal for his colleagues:

> *Given two points A and B in the vertical plane, with A not lower than B, find the curve on which a mass point, starting at rest from A and moving along the curve without friction and under constant gravity, moves to B in shortest time!*

Such curves are called brachistochrone curves.

More legend: solutions were sent by Isaac Newton, Jakob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital. It had taken Johann Bernoulli 2 weeks to solve the problem, while Newton came up with the solution after only one day.

Legendary: the brachistochrone problem

For the mathematical formulation, only a minimal knowledge of physics is required.

Let $A = (a, u_a), B = (b, u_b)$ be points in $\mathbb{R}^2$, with $a < b$ and $u_a > u_b$. If the mass point moves along the graph of a curve $u \in C^1([a, b])$, with time-dependence $(x(t), u(x(t)))$ for $t \in [0, T]$ such that $x(0) = a$ and $x(T) = b$, then conservation of energy yields the velocity of the mass point: for $t \in [0, T]$ it is given by

$$\frac{1}{2}v(t)^2 = \big(u_a - u(x(t))\big)g\,,$$

where g is the constant gravity.

Alternatively, $v(t)$ can be determined by differentiation (w.r.t. time) of the path length up to $x(t)$, that is $\int_a^{x(t)}(1 + |u'(s)|^2)^{\frac{1}{2}}\,ds$, which gives

$$v(t) = (1 + |u'(x(t))|^2)^{\frac{1}{2}}x'(t)\,.$$

Hence, we need to minimize

$$T = \int_0^T 1\,dt = \int_0^T \Big(\frac{1 + |u'(x(t))|^2}{2(u(x(t)) - u_a)g}\Big)^{\frac{1}{2}}x'(t)\,dt = \int_a^b \Big(\frac{1 + |u'(s)|^2}{2(u_a - u(s))g}\Big)^{\frac{1}{2}}\,ds\,.$$

Calculus of variations

In case of the brachistochrone problem we have a functional F (of variational form) and we want to determine a minimizer u among all C^1-functions which satisfy the boundary condition $u(a) = u_a$ and $u(b) = u_b$.

General problem: Given some vector space X and a functional $F\colon X \to \mathbb{R}$, determine a minimizer, i.e. $x \in X$ such that $F(x) \leq F(y)$ for all $y \in X$.

Typical examples for X are the spaces $C^1(U)$ of continuously differentiable functions or the Sobolev spaces $W^{1,p}(U)$, and for F the variational functional of the form

$$F[u] := \int_U f(x, u(x), Du(x))\, dx$$

for some given integrand f.

Direct vs. indirect approach

We now look at different approaches for finding minimizers. For comparison, let us first consider $f\colon X \to \mathbb{R}$ with X a set in $\mathbb{R}^d$. We might follow

1. a **indirect approach**: If $f \in C^1(X)$, then we can first determine the (interior) critical points $x \in X$ such that $Df(x) = 0$ and then see, which of the critical points is a minimizer (e.g. for f convex, all critical points are minimizers);
2. a **direct approach**: There always exists a minimizing sequence $(x_k)_{k\in\mathbb{N}}$ in X, i.e. with $\lim_{k\to\infty} f(x_k) = \inf_X f$.
   If $(x_k)_{k\in\mathbb{N}}$ is relatively compact in X, we find a convergent subsequence with limit $\bar{x}$, and if f is also lower semicontinuous, then $\bar{x}$ is a minimizer:

$$f(\bar{x}) \leq \lim_{k\to\infty} f(x_k) = \inf_X f\,.$$

Direct vs. indirect approach

For variational problems one can proceed similarly (but now X is usually infinite dimensional).

1. classical (indirect) approaches: trace back to Euler, Hamilton, Hilbert, Jacobi, Lagrange, Legendre, Hadamard, ... and consist in
   - find a necessary criterion for minimality,
   - determine criteria which allow to decide whether such a candidate is a minimizer;
2. semiclassical approaches: somewhere in between ...
3. direct method of the calculus of variations: traces back to Hilbert, Lebesgue, Tonelli, ... and consists in
   - Choose a space $Y \supset X$ and a topology such that we have compactness (to be able to select a convergent subsequence from a minimizing subsequence) and lower semicontinuity of F (which guarantees that the limit is a minimizer),
   - show that the minimizer belongs to X.

Euler–Lagrange equation

We consider variational functionals

$$F[u] := \int_U f(x, u(x), Du(x))\, dx\,.$$

Definition

We say that a function $u \in C^1(U)$ is a *minimizer of* F if

$$F[u] \le F[u+\varphi] \quad \text{for all } \varphi \in C_0^1(U)\,.$$

Euler–Lagrange equation

We have already derived a necessary criterion for minimality, the Euler–Lagrange equation in its simplest form:

Theorem

Let $f \in C^1(U \times \mathbb{R} \times \mathbb{R}^d)$. *Then every minimizer* $u \in C^1(U)$ *of* F *satisfies the weak Euler–Lagrange equation*

$$\int_U \left[D_z f(x,u,Du) \cdot D\varphi + D_u f(x,u,Du)\varphi \right] dx = 0$$

for all $\varphi \in C_0^1(U)$. *If in addition* $D_z f \in C^1(U \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R}^d)$ *and* $u \in C^2(U)$ *hold, then also the Euler–Lagrange equation holds:*

$$\operatorname{div} D_z f(x,u,Du) = D_u f(x,u,Du) \quad \text{in } U\,.$$

Euler–Lagrange equation

Comments:

- Not every solution to the (weak) Euler–Lagrange equation is a minimizer (e.g. a maximizer);
- If $(u, z) \mapsto f(x, u, z)$ is *convex* for all $x \in U$, then we have a sufficient criterion for minimality: every solution $u \in C^1(U)$ of the weak Euler–Lagrange equation is a minimizer;

Proof: If a function $g\colon \mathbb{R}^k \to \mathbb{R}$ is convex, we have

$$g(y) \geq g(y_0) + Dg(y_0) \cdot (y - y_0) \quad \text{for all } y, y_0 \in \mathbb{R}^k \,.$$

Thus, for all $\varphi \in C_0^1(U)$ and $x \in U$ we get

$$\begin{aligned} f(x, u + \varphi, Du + D\varphi) \geq f(x, u, Du) \\ + D_u f(x, u, Du) \cdot \varphi + D_z f(x, u, Du) \cdot D\varphi \,. \end{aligned}$$

Since u solves the weak Euler–Lagrange equation, integration over U shows the claim $F[u] \leq F[u + \varphi]$ for all $\varphi \in C_0^1(U)$.

Euler–Lagrange equation

Comments:

- Not every solution to the (weak) Euler–Lagrange equation is a minimizer (e.g. a maximizer);
- If $(u, z) \mapsto f(x, u, z)$ is *convex* for all $x \in U$, then we have a sufficient criterion for minimality: every solution $u \in C^1(U)$ of the weak Euler–Lagrange equation is a minimizer;
- If $(u, z) \mapsto f(x, u, z)$ is *strictly convex*, then the minimizer is also unique;
- The regularity assumption on u is often not reasonable;
- This is not an existence result. Existence of solutions to the (weak) Euler–Lagrange equation (not necessarily minimizers) can sometimes be proved by PDE-techniques.

Euler–Lagrange equation for $d = 1$

For $d = 1$ we have

$$\frac{d}{dx} D_z f(x, u, u') = D_u f(x, u, u')$$

in an interval $(a, b) \subset \mathbb{R}$, and in some special cases we can extract further information:

- for integrands of the form $f(x, u, z) \equiv f(z)$: here we have $(D_z f(u'))' = 0$ in (a, b), hence $D_z f(u') = c$ in (a, b) for some constant c and in particular, all straight lines are solutions to the Euler–Lagrange equation;
- for integrands of the form $f(x, u, z) \equiv f(u, z)$: here we compute

$$\begin{aligned}
&\tfrac{d}{dx}\big(D_z f(u, u')u' - f(u, u')\big) \\
&= \tfrac{d}{dx}\big(D_z f(u, u')\big)u' + D_z f(u, u')u'' - D_u f(u, u')u' - D_z f(u, u')u'' \\
&= \underbrace{\Big(\tfrac{d}{dx} D_z f(u, u') - D_u f(u, u')\Big)}_{=0} u' = 0\,,
\end{aligned}$$

  hence $D_z f(u, u')u' - f(u, u') = c$ in (a, b) for some constant c.

Back to the brachistochrone problem

We need to minimize

$$\int_a^b \Big(\frac{1+|u'(s)|^2}{2(u_a - u(s)u_a)g}\Big)^{\frac{1}{2}}\,ds$$

among all functions $u \in C^1([a,b])$ with $u(a) = u_a > u(b) = u_b$. The integrand of this functional is of the form

$$f(u,z) := \Big(\frac{1+|z|^2}{u_a - u}\Big)^{\frac{1}{2}}\,,$$

which, by constancy of $D_z f(u,u')u' - f(u,u')$ in (a,b), yields

$$\frac{|u'|^2}{[(1+|u'|^2)(u_a-u)]^{1/2}} - \Big(\frac{1+|u'|^2}{u_a-u}\Big)^{\frac{1}{2}} = c$$

and can be rewritten as

$$(u_a - u)(1+|u'|^2) = \tilde{c}\,.$$

This is a cycloid equation, and the constant $\tilde{c}$ is determined via the boundary condition. This is the solution to the brachistochrone problem.

Some references

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... and thank you for the attention

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