Exercise set 1*

Question 1

Suppose that X is a stable process in any dimension (including the case of a Brownian motion). Show that $|X|$ is a positive self-similar Markov process.

Solution Question 1

- Temporarily write $(X_t^{(x)}, t \geq 0)$ in place of $(X, \mathbb{P}_x)$
- Markov property of X tells us that, for $s, t \geq 0$,

  $$X_{t+s}^{(x)} = \tilde{X}_s^{(X_t^{(x)})},$$

  where $\tilde{X}^{(x)}$ is an independent copy of $X^{(x)}$.
- Isotropy implies that

  $$|X_{t+s}^{(x)}| = |\tilde{X}_s^{(y)}|_{y=X_t^{(x)}} =^d |\tilde{X}_s^{(z)}|_{z=(|X_t^{(x)}|,0,0\cdots,0)}$$

- Hence Markov property holds, strong Markov property (and Feller property) can be developed from this argument
- Self-similarity of $|X|$ follows directly from the self-similarity of X.

Question 2

Suppose that B is a one-dimensional Brownian motion. Prove that

$$\frac{B_t}{x}\mathbf{1}_{(\underline{B}_t>0)}, \qquad t \geq 0,$$

is a martingale, where $\underline{B}_t = \inf_{s\leq t} B_s$.

Solution Question 2

- Note that $(B_t, t \geq 0)$ is a martingale.
- Optional stopping implies that $(B_{t\wedge\tau_0^-}, t \geq 0)$ is a martingale, where $\tau_0^- = \inf\{t > 0 : B_t < 0\}$
- Since $B_{\tau_0^-} = 0$, it follows that $B_{t\wedge\tau_0^-} = B_t\mathbf{1}_{(t<\tau_0^-)}$, $t \geq 0$.
- Finally note that $\{\tau_0^- > t\} = \{\underline{B}_t > 0\}$

Question 3*

Suppose that X is a stable process with two-sided jumps

- Show that the range of the ascending ladder process H, say $\texttt{range}(H)$ has the property that it is equal in law to $c \times \texttt{range}(H)$.
- Hence show that, up to a multiplicative constant, the Laplace exponent of H satisfies $k(\lambda) = \lambda^{\alpha_1}$ for $\alpha_1 \in (0, 1)$ (and hence the ascending ladder height process is a stable subordinator).
- Use the fact that, up to a multiplicative constant

  $$\Psi(z) = |\theta|^\alpha(\mathrm{e}^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}) = \hat{\kappa}(\mathrm{i}z)\kappa(-\mathrm{i}z)$$

  to deduce that

  $$\kappa(\theta) = \theta^{\alpha\rho} \text{ and } \hat{\kappa}(\theta) = \theta^{\alpha\hat{\rho}}.$$

  and that $0 < \alpha\rho, \alpha\hat{\rho} < 1$
- What kind of process corresponds to the case that $\alpha\rho = 1$?

Solution Question 3

- Range of H agrees with the range of $\overline{X}$. Scaling of X applies to scaling of $\overline{X}$, in particular $(c\overline{X}_{c^{-\alpha}t}, t \geq 0)$ is equal in law to $(\overline{X}_t, t \geq 0)$, so that $c \times \texttt{range}\overline{X} = \texttt{range}(c\overline{X}) = \texttt{range}(\overline{X})$. And hence the same is true for H.

- The latter is equivalent to the condition that the Laplace exponent of H is proportional to that of cH, i.e $\kappa(z) = k_c \kappa(cz)$, for $z \geq 0$, where $k_c > 0$ is a constant that only depends on c. Since $\kappa(1)$ must be a constant, we see that $\kappa(z) = \kappa(cz)/\kappa(c)$. Hence, as κ is increasing, one can easily deduce that $k(\lambda) = \kappa(1)\lambda^{\alpha_1}$ for some $\alpha_1 \in [0,1]$. In other words, H is a stable subordinator with parameter α_1. We exclude the case $\alpha_1 = 0$ since it corresponds to the setting where the range of H is the empty set. A similar argument applied to $-X$ shows that the descending ladder height process must also belong to the class of stable subordinators.

- We therefore assume that (up to multiplicative constants) $\kappa(z) = z^{\alpha_1}$, $z \geq 0$, and $\hat{\kappa}(z) = z^{\alpha_2}$, $z \geq 0$, for some $\alpha_1, \alpha_2 \in (0,1]$. Appealing to the the stable process exponemt, we must choose the parameters α_1 and α_2 such that, for example, when $z > 0$,

$$z^{\alpha} \mathrm{e}^{\pi \mathrm{i} \alpha (\frac{1}{2} - \rho)} = z^{\alpha_1} \mathrm{e}^{-\frac{1}{2}\pi \mathrm{i} \alpha_1} \times z^{\alpha_2} \mathrm{e}^{\frac{1}{2}\pi \mathrm{i} \alpha_2}.$$

  Matching radial and angular parts, this is only possible if α_1 and α_2 satisfy

$$\begin{cases} \alpha_1 + \alpha_2 = \alpha, \\ \alpha_1 - \alpha_2 = -\alpha(1 - 2\rho), \end{cases}$$

  which gives us $\alpha_1 = \alpha\rho$ and $\alpha_2 = \alpha\hat{\rho}$. As X does not have monotone paths, it is necessarily the case that $0 < \alpha\rho \leq 1$ and $0 < \alpha\hat{\rho} \leq 1$. In conclusion, for $\theta \geq 0$,

$$\kappa(\theta) = \theta^{\alpha\rho} \text{ and } \hat{\kappa}(\theta) = \theta^{\alpha\hat{\rho}}.$$

- When $\alpha\rho = 1$, the ascending ladder height process is a pure linear drift. In that case, the range of the maximum process $\overline{X}$ is $[0, \infty)$. The process is spectrally negative.

Question 4

Suppose that (X, P_x), $x > 0$ is a positive self-similar Markov process and let $\zeta = \inf\{t > 0 : X_t = 0\}$ be the lifetime of X. Show that $\mathrm{P}_x(\zeta < \infty)$ does not depend on x and is either 0 for all $x > 0$ or 1 for all $x > 0$.

Solution Question 4

- We can appeal to the scaling property and write, for all $c > 0$,

$$\begin{aligned} \zeta^{(cx)} &= \inf\{t > 0 : X_t^{(cx)} = 0\} \\ &\stackrel{d}{=} c^{\alpha} \inf\{c^{-\alpha} t > 0 : cX_{c^{-\alpha}t}^{(x)} = 0\} \\ &= c^{\alpha} \zeta^{(x)}, \end{aligned}$$

  showing that $P_x(\zeta < \infty) = P_{cx}(\zeta < \infty)$, for all $x, c > 0$, as claimed. Note that this also shows that $x^{-\alpha}\zeta^{(x)}$ is independent of the value of x.

- Denote by $p \in [0,1]$ the common value of the probabilities $P_x(\zeta < \infty)$, $x > 0$. We shall now show that either $p = 0$ or $p = 1$. Thanks to the Markov property, we can now write, for all $x, t > 0$,

$$P_x(t < \zeta < \infty) = E_x(\mathbf{1}_{(t<\zeta)} P_{X_t}(\zeta < \infty)) = pP_x(t < \zeta),$$

  and, hence,

$$\begin{aligned} p &= P_x(\zeta \leq t) + P_x(t < \zeta < \infty) \\ &= P_x(\zeta \leq t) + p(1 - P_x(\zeta \leq t)) \\ &= p + (1-p)P_x(\zeta \leq t). \end{aligned}$$

  This forces us to conclude that either $p = 1$ or $P_x(\zeta \leq t) = 0$, for all $x, t > 0$. In other words, $p = 1$ or $p = 0$.

Question 5*

Suppose that X is a symmetric stable process in dimension one (in particular $\rho = 1/2$) and that the underlying Lévy process for $|X_t|\mathbf{1}_{(t<\tau^{\{0\}})}$, where $\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$, is written ξ. (Note the indicator is only needed when $\alpha \in (1, 2)$ as otherwise X does not hit the origin.) Show that (up to a multiplicative constant) its characteristic exponent is given by

$$\Psi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + \alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + 1))}{\Gamma(\frac{1}{2}(\mathrm{i}z + 1 - \alpha))}, \qquad z \in \mathbb{R}.$$

[Hint!] Think about what happens after X first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of X below the origin given a few slides back.

Solution Question 5

- The trick is to go back and think about the pssMp $(X_t\mathbf{1}_{\underline{X}_t>0}, t \geq 0)$. Previously we had identified the characteristics exponent of its underlying Lévy process (through the Lamperti transform) as killed Lévy process with exponent

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}, \tag{1}$$

  which contained a killing rate

$$q^* = \Psi^*(0) = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})}. \tag{2}$$

  (Note in our setting, $\rho = \hat{\rho} = 1/2$).

- Instead of killing the underlying ξ at rate q^* (i.e. the stable process goes negative), we want our process to regenerate in $(0, \infty)$, corresponding to our underlying Lévy process undergoing a jump at rate q^*. In other words, the Lévy process corresponding to $|X|$ can be written in the form

$$\xi = \xi^* + \xi^{**},$$

  where ξ^{**} is a compound Poisson process with arrival rate q^* and jump distribution F, which we are to determine. In particular, if Ψ is the characteristic exponent of the Lévy process that underlies $|X|$, then

$$\Psi(\theta) = (\Psi^*(\theta) - q^*) + q^* \int_{\mathbb{R}} (1 - \mathrm{e}^{\mathrm{i}\theta x})F(\mathrm{d}x) = \Psi^*(\theta) - q^* \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} F(\mathrm{d}x) \tag{3}$$

- The point of regeneration must correspond to precisely $-X_{\tau_0^-}$. In particular, referring to the Lamperti transform, we must have

$$\frac{-X_{\tau_0^-}}{X_{\tau_0^- -}} = \mathrm{e}^\Delta,$$

  where Δ has distribution F.

- We can use the previously given joint overshoot distribution

$$\begin{aligned}&\mathbb{P}_1(-X_{\tau_0^-} \in \mathrm{d}u,\, X_{\tau_0^- -} \in \mathrm{d}v)\\ &\quad= \frac{\sin(\alpha\pi/2)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha/2)^2} \left(\int_0^\infty \mathbf{1}_{(y \leq 1 \wedge v)} \frac{(1 - y)^{\frac{\alpha}{2} - 1}(v - y)^{\frac{\alpha}{2} - 1}}{(v + u)^{1+\alpha}} \mathrm{d}y \right) \mathrm{d}v\mathrm{d}u\end{aligned}$$

and note that (the point of issue x does not matter by scaling)

$$\begin{aligned}
&\int_{\mathbb{R}} e^{i\theta x} F(dx) \\
&= \mathbb{E}_x\left[\left(\frac{-X_{\tau_0^-}}{X_{\tau_0^- -}}\right)^{i\theta}\right] = \mathbb{E}_1\left[\left(\frac{-X_{\tau_0^-}}{X_{\tau_0^- -}}\right)^{i\theta}\right] \\
&= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\int_0^\infty\int_0^\infty\int_0^\infty \mathbf{1}_{(y\le 1\wedge v)}\frac{u^{i\theta}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{v^{i\theta}(v+u)^{1+\alpha}} du dv dy \\
&= \frac{\sin(\pi\alpha\hat{\rho})}{\pi}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}\int_0^1\int_y^\infty\int_0^\infty \frac{u^{i\theta}(1-y)^{\alpha\hat{\rho}-1}(v-y)^{\alpha\rho-1}}{v^{i\theta}(v+u)^{1+\alpha}}\, du\, dv\, dy \qquad (4)
\end{aligned}$$

Note $\rho = \hat{\rho} = 1/2$.

- For the innermost integral in (4), substituting $w = v/u$ and appealing to the integral representation of the beta function, we have

$$\int_0^\infty \frac{u^{i\theta}}{(u+v)^{1+\alpha}}\, du = v^{i\theta-\alpha}\int_0^\infty \frac{w^{i\theta}}{(1+w)^{1+\alpha}}\, dw = v^{i\theta-\alpha}\frac{\Gamma(i\theta+1)\Gamma(\alpha-i\theta)}{\Gamma(\alpha+1)}.$$

Substituting $z = v/y$, the next iterated integral in (4) becomes

$$\int_y^\infty v^{-\alpha}(v-y)^{\alpha\rho-1}\, dv = y^{-\alpha\hat{\rho}}\int_0^\infty \frac{z^{\alpha\rho-1}}{(1+z)^\alpha}\, dz = y^{-\alpha\hat{\rho}}\frac{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})}{\Gamma(\alpha)}.$$

Finally, it remains to calculate the resulting outer integral of (4),

$$\int_0^1 y^{-\alpha\hat{\rho}}(1-y)^{\alpha\hat{\rho}-1}\, dy = \Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho}).$$

Multiplying together these expressions and using the reflection identity for the gamma function[1] we obtain

$$\mathbb{E}_1\left[\left(-\frac{X_{\tau_0^-}}{X_{\tau_0^- -}}\right)^{i\theta}\right] = \frac{\Gamma(i\theta+1)\Gamma(\alpha-i\theta)}{\Gamma(\alpha)}. \qquad (5)$$

- Putting (1), (2), (3) and (5) together, together with the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we get (with $\rho = \hat{\rho} = 1/2$),

$$\begin{aligned}
\Psi(\theta) &= \frac{\Gamma(\alpha-i\theta)}{\Gamma(\frac{\alpha}{2}-i\theta)}\frac{\Gamma(1+i\theta)}{\Gamma(1-\frac{\alpha}{2}+i\theta)} - \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})}\frac{\Gamma(i\theta+1)\Gamma(\alpha-i\theta)}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha-i\theta)\Gamma(1+i\theta)}{\pi}\left(\sin(\frac{\pi\alpha}{2}-i\pi\theta)-\sin(\frac{\pi\alpha}{2})\right).
\end{aligned}$$

Next, recalling that

$$2\sin(\alpha)\sin(\beta) = \cos(\alpha-\beta)-\cos(\alpha+\beta),$$

we can write

$$\begin{aligned}
2\sin(\frac{\pi(\alpha-1)}{2}-i\frac{\pi\theta}{2})\sin(-i\frac{\pi\theta}{2}) &= \cos(\frac{\pi\alpha}{2}-\frac{\pi}{2})-\cos(\frac{\pi\alpha}{2}-\frac{\pi}{2}-\pi i\theta) \\
&= \sin(\frac{\pi\alpha}{2})-\sin(\frac{\pi\alpha}{2}-\pi i\theta).
\end{aligned}$$

At this point, things become nasty! We need to used the Legendre duplication formula,

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\Gamma(2z),$$

[1] $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$

the reflection formula again and the recursion formula, $\Gamma(1+z) = z\Gamma(z)$, to get

$$
\begin{aligned}
\Psi(\theta) &= -2\frac{\Gamma(\alpha-\mathrm{i}\theta)\Gamma(1+\mathrm{i}\theta)}{\pi}\sin(\frac{\pi(\alpha-1)}{2}-\mathrm{i}\frac{\pi\theta}{2})\sin(-\mathrm{i}\frac{\pi\theta}{2}) \\
&= -2\cdot 2^{-1+\alpha-\mathrm{i}\theta}\Gamma(\frac{\alpha}{2}-\mathrm{i}\frac{\theta}{2})\Gamma(\frac{\alpha}{2}+\frac{1}{2}-\mathrm{i}\frac{\theta}{2})\cdot 2^{-1+1+\mathrm{i}\theta}\Gamma(\frac{1}{2}+\mathrm{i}\frac{\theta}{2})\Gamma(1+\mathrm{i}\frac{\theta}{2}) \\
&\quad\times\frac{1}{\Gamma(\frac{\alpha}{2}-\frac{1}{2}-\mathrm{i}\frac{\theta}{2})\Gamma(\frac{3}{2}-\frac{\alpha}{2}+\mathrm{i}\frac{\theta}{2})}\cdot\frac{\pi}{\Gamma(-\mathrm{i}\frac{\theta}{2})\Gamma(1+\mathrm{i}\frac{\theta}{2})} \\
&= -\pi 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}\theta+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}\theta)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1))}{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1-\alpha))}\cdot\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2}-\mathrm{i}\frac{\theta}{2})\Gamma(1+\mathrm{i}\frac{\theta}{2})}{\Gamma(\frac{\alpha}{2}-\frac{1}{2}-\mathrm{i}\frac{\theta}{2})(\frac{1}{2}-\frac{\alpha}{2}+\mathrm{i}\frac{\theta}{2})\Gamma(1+\mathrm{i}\frac{\theta}{2})} \\
&= -\pi 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}\theta+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}\theta)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1))}{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1-\alpha))}\cdot\frac{\Gamma(\frac{\alpha}{2}-\frac{1}{2}-\mathrm{i}\frac{\theta}{2})(\frac{\alpha}{2}-\frac{1}{2}-\mathrm{i}\frac{\theta}{2})}{\Gamma(\frac{\alpha}{2}-\frac{1}{2}-\mathrm{i}\frac{\theta}{2})(\frac{1}{2}-\frac{\alpha}{2}+\mathrm{i}\frac{\theta}{2})} \\
&= \pi 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}\theta+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}\theta)}\frac{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1))}{\Gamma(\frac{1}{2}(\mathrm{i}\theta+1-\alpha))},
\end{aligned}
$$

which is the required exponent up to the multiplicative constant π.

Question 6*

Use the previous exercise to deduce that the MAP exponent underlying a stable process with two sided jumps is given by

$$
\begin{bmatrix}
-\frac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho}-z)\Gamma(1-\alpha\hat{\rho}+z)} & \frac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\
\frac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\frac{\Gamma(\alpha-z)\Gamma(1+z)}{\Gamma(\alpha\rho-z)\Gamma(1-\alpha\rho+z)}
\end{bmatrix},
$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

Solution Question 6

First recall that the matrix exponent takes the specific form

$$
\mathbf{\Psi}(z) = \begin{bmatrix} \psi_1(z) & 0 \\ 0 & \psi_{-1}(z) \end{bmatrix} + \begin{bmatrix} -Q_{1,-1} & Q_{1,-1} \\ Q_{-1,1} & -Q_{-1,1} \end{bmatrix} \circ \begin{bmatrix} 1 & \mathbf{E}[\mathrm{e}^{zU_{1,-1}}] \\ \mathbf{E}[\mathrm{e}^{zU_{-1,1}}] & 1 \end{bmatrix} \tag{6}
$$

We can borrow lots of ideas and calculations from Question 5. Until it first crosses the origin, X behaves like $X\mathbf{1}_{(\underline{X}>0)}$ and hence we can take

$$\psi_1(z) = -(\Psi^*(-\mathrm{i}z) - q^*).$$

Moreover the rate at which the change from the positive to the negative half-line occurs is rate q^*.

When the process jumps from the positive to negative half line, the process the chain maps from 1 to -1 transferring the left-limit $X_{\tau_0^- -}$ to $-X_{\tau_0^- -}$. However the actual positioning is $X_{\tau_0^-}$, so a multiplicative correction in the radial positioning of

$$\frac{|X_{\tau_0^-}|}{X_{\tau_0^- -}} = \mathrm{e}^{U_{1,-1}}.$$

From (5), we thus conclude that

$$\mathbf{E}[\mathrm{e}^{\mathrm{i}zU_{1,-1}}] = \frac{\Gamma(\mathrm{i}z+1)\Gamma(\alpha-\mathrm{i}z)}{\Gamma(\alpha)}$$

is needed.

By anti-symmetry (or otherwise noting that the behaviour on the negative half-line is that of $-X$ with the roles of ρ and $\hat\rho$ interchanged, written $\rho \leftrightarrow \hat\rho$). In conclusion we have everything we need to populate the matrices in (6). And hence,

$$\begin{aligned}
\mathbf{\Psi}(z) &= \begin{bmatrix} -(\Psi^*(-\mathrm{i}z) - q^*) & 0 \\ \bullet & \bullet \end{bmatrix} + \begin{bmatrix} -q^* & q^* \\ \bullet & \bullet \end{bmatrix} \circ \begin{bmatrix} 1 & \frac{\Gamma(z+1)\Gamma(\alpha-z)}{\Gamma(\alpha)} \\ \cdot & \bullet \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\Gamma(\alpha-z)}{\Gamma(\alpha\hat\rho-z)}\frac{\Gamma(1+z)}{\Gamma(1-\alpha\hat\rho+z)} & \frac{\Gamma(z+1)\Gamma(\alpha-z)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)} \\ \bullet & \bullet \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\Gamma(\alpha-z)}{\Gamma(\alpha\hat\rho-z)}\frac{\Gamma(1+z)}{\Gamma(1-\alpha\hat\rho+z)} & \frac{\Gamma(z+1)\Gamma(\alpha-z)}{\Gamma(\alpha\hat\rho)\Gamma(1-\alpha\hat\rho)} \\ \frac{\Gamma(z+1)\Gamma(\alpha-z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\frac{\Gamma(\alpha-z)}{\Gamma(\alpha\rho-z)}\frac{\Gamma(1+z)}{\Gamma(1-\alpha\rho+z)} \end{bmatrix}
\end{aligned}$$

as required.

Exercise Set 2

Question 1

Use the fact that the radial part of a d-dimensional ($d \geq 2$) isotropic stable process has MAP (ξ, Θ), for which the first component is a Lévy process with characteristic exponent given by

$$\Psi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+d))}{\Gamma(\frac{1}{2}(\mathrm{i}z+d-\alpha))}, \qquad z \in \mathbb{R}.$$

to deduce the following facts:

- Irrespective of its point of issue, we have $\lim_{t\to\infty} |X_t| = \infty$ almost surely.
- By considering the roots of Ψ show that

  $$\exp((\alpha-d)\xi_t), \qquad t \geq 0,$$

  is a martingale.
- Deduce that

  $$|X_t|^{\alpha-d}, \qquad t \geq 0,$$

  is a martingale.

Solution Question 1

- One can use the fact that $\mathbf{E}[\xi_1] = -\mathrm{i}\Psi'(0)$ to deduce that $\lim_{t\to\infty} \xi_t = \infty$ and hence, recalling that $|X|$ is a pssMp, and that only three types of behaviour are possible for pssMp, one has that $\lim_{t\to\infty} |X_t| = \infty$.
- If we write $\psi(\lambda) = -\Psi(-\mathrm{i}\lambda) = \log \mathbb{E}[\mathrm{e}^{\lambda X_1}]$ for the Laplace exponent of ξ, then it is well defined for $\lambda \in (-d, \alpha)$ with roots at $\lambda = 0$ and $\lambda = \alpha - d$.
- Note that

  $$\exp((\alpha-d)\xi_t), \qquad t \geq 0,$$

  is a martingale
- Recalling that $|X_t| = \exp(\xi_{\varphi_t})$ and that φ_t is an almost surely finite stopping time (because $\lim_{t\to\infty} \xi_t = \infty$) we can deduce that

  $$|X_t|^{\alpha-d}, \qquad t \geq 0,$$

  is a martingale (effectively invoking an Esscher transform to ψ).

Question 2

Remaining in d-dimensions ($d \geq 2$), recalling that

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\circ}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \qquad t \geq 0, x \neq 0,$$

show that under $\mathbb{P}^\circ$, X is absorbed continuously at the origin in an almost surely finite time.

Question 2 solution

Note that if (ξ°, Θ) is the MAP underlying $(X, \mathbb{P}^\circ)$, then it is still the case that ξ alone is a Lévy process because the change of measure $\mathbb{P}^\circ$ is written in terms of the radial component only. Moreover, the characteristic exponent of ξ° is given by

$$\Psi^\circ(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-\mathrm{i}z + d))}{\Gamma(-\frac{1}{2}(\mathrm{i}z + \alpha - d))} \frac{\Gamma(\frac{1}{2}(\mathrm{i}z + \alpha))}{\Gamma(\frac{1}{2}\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

and one can check that $\Psi^\circ(0) = 0$ and $\mathbf{E}^\circ[\xi_t] = -\mathrm{i}\Psi^{\circ\prime}(0) < 0$. Remember that there are only three categories of pssMp and $|X|$ must still be a pssMp because of isotropy of $(X, \mathbb{P}^\circ)$. So $|X|$ must fit the category of pssMp that is continuously absorbed at the origin.

Question 3*

Recall the following theorem.

Theorem. Define the function

$$g(x, y) = \pi^{-(d/2+1)}\,\Gamma(d/2)\,\sin(\pi\alpha/2)\frac{\left|1 - |x|^2\right|^{\alpha/2}}{\left|1 - |y|^2\right|^{\alpha/2}}|x - y|^{-d}$$

for $x, y \in \mathbb{R}^d \backslash \mathbb{S}_{d-1}$. Let

$$\tau^\oplus := \inf\{t > 0 : |X_t| < 1\} \text{ and } \tau_a^\ominus := \inf\{t > 0 : |X_t| > 1\}.$$

(i) Suppose that $|x| < 1$, then

$$\mathbb{P}_x(X_{\tau^\ominus} \in \mathrm{d}y) = g(x, y)\mathrm{d}y, \qquad |y| \geq 1.$$

(ii) Suppose that $|x| > 1$, then

$$\mathbb{P}_x(X_{\tau^\oplus} \in \mathrm{d}y,\, \tau^\oplus < \infty) = g(x, y)\mathrm{d}y, \qquad |y| \leq 1.$$

Prove (ii) (i.e. $|x| > 1$) from the identity in (i) (i.e. $|x| < 1$).

Solution Question 3

- Start by noting from the Riesz–Bogdan–Żak transform that, for $|x| > 1$,

  $$\mathbb{P}_x(X_{\tau^\oplus} \in D) = \mathbb{P}_{Kx}^\circ(KX_{\tau^\ominus} \in D),$$

  where $Kx = x/|x|^2$, $|Kx - Kz| = |x - z|/|x||z|$ and $KD = \{Kx : x \in D\}$.
- Noting that $\mathrm{d}(Kz) = |z|^{-2d}\mathrm{d}z$, we have

  $$\begin{aligned}
  &\mathbb{P}_x(X_{\tau^\oplus} \in D)\\
  &= \int_{KD} \frac{|y|^{\alpha-d}}{|Kx|^{\alpha-d}} g(Kx, y)\mathrm{d}y\\
  &= c_{\alpha,d} \int_{KD} |z|^{d-\alpha}|Kx|^{d-\alpha}\frac{\left|1 - |Kx|^2\right|^{\alpha/2}}{\left|1 - |y|^2\right|^{\alpha/2}}|Kx - y|^{-d}\mathrm{d}y\\
  &= c_{\alpha,d} \int_D |z|^{2d}\frac{\left|1 - |x|^2\right|^{\alpha/2}}{\left|1 - |z|^2\right|^{\alpha/2}}|x - z|^{-d}\mathrm{d}(Kz)\\
  &= c_{\alpha,d} \int_D \frac{\left|1 - |x|^2\right|^{\alpha/2}}{\left|1 - |z|^2\right|^{\alpha/2}}|x - z|^{-d}\mathrm{d}z
  \end{aligned}$$