Weak convergence and large deviation theory

- Large deviation principle
- Convergence in distribution
- The Bryc-Varadhan theorem
- Tightness and Prohorov's theorem
- Exponential tightness
- Tightness for processes
- (Exponential) tightness and results for finite dimensional distributions
- Conditions for (exponential) tightness

Joint work with Jin Feng

Second Lecture

Large deviation principle

(S,d) a (complete, separable) metric space.

X_n, $n=1,2,\ldots$ S-valued random variables

$\{X_n\}$ satisfies a *large deviation principle* (LDP) if there exists a lower semicontinuous function $I : S \to [0,\infty]$ such that for each open set A,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\{X_n \in A\} \geq -\inf_{x\in A} I(x)$$

and for each closed set B,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \in B\} \leq -\inf_{x\in B} I(x).$$

The rate function

I is called the *rate function* for the large deviation principle.

A rate function is *good* if for each $a \in [0, \infty)$, $\{x : I(x) \leq a\}$ is compact.

If I is a rate function for $\{X_n\}$, then

$$I_*(x) = \lim_{\epsilon \to 0} \inf_{y \in B_\epsilon(x)} I(y)$$

is also a rate function for $\{X_n\}$. I_* is lower semicontinuous.

If the large deviation principle holds with lower semicontinuous rate I function, then

$$I(x) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P\{X_n \in B_\epsilon(x)\} = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P\{X_n \in \overline{B}_\epsilon(x)\}$$

Convergence in distribution

$\{X_n\}$ *converges in distribution* to X if and only if for each $f \in C_b(S)$

$$\lim_{n\to\infty} E[f(X_n)] = E[f(X)]$$

Equivalent statements: Large deviation principle

$\{X_n\}$ satisfies an LDP with a good rate function if and only if $\{X_n\}$ is exponentially tight and

$$\Lambda(f) \equiv \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}]$$

for each $f \in C_b(S)$. Then

$$I(x) = \sup_{C_b(S)} \{f(x) - \Lambda(f)\}$$

and

$$\Lambda(f) = \sup_{x\in S}\{f(x) - I(x)\}$$

Bryc, Varadhan

Equivalent statements: Convergence in distribution

$\{X_n\}$ converges in distribution to X if and only if

$$\liminf_{n\to\infty} P\{X_n \in A\} \geq P\{X \in A\}, \text{ each open } A,$$

or equivalently

$$\limsup_{n\to\infty} P\{X_n \in B\} \leq P\{X \in B\}, \text{ each closed } B$$

LDP

Tightness

A sequence $\{X_n\}$ is *tight* if for each $\epsilon > 0$, there exists a compact set $K_\epsilon \subset S$ such that

$$\sup_n P\{X_n \notin K_\epsilon\} \leq \epsilon.$$

Prohorov's theorem

Theorem 1 *Suppose that $\{X_n\}$ is tight. Then there exists a subsequence $\{n(k)\}$ along which the sequence converges in distribution.*

Exponential tightness

$\{X_n\}$ is *exponentially tight* if for each $a > 0$, there exists a compact set $K_a \subset S$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin K_a\} \leq -a.$$

Analog of Prohorov's theorem

Theorem 2 *(Puhalskii, O'Brien and Vervaat, de Acosta) Suppose that* $\{X_n\}$ *is exponentially tight. Then there exists a subsequence* $\{n(k)\}$ *along which the large deviation principle holds with a good rate function.*

Stochastic processes in $D_E[0,\infty)$

(E,r) complete, separable metric space

$S = D_E[0,\infty)$

Modulus of continuity:

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(x(s),x(t))$$

where the infimum is over $\{t_i\}$ satisfying

$$0 = t_0 < t_1 < \cdots < t_{m-1} < T \le t_m$$

and $\min_{1\le i\le n}(t_i - t_{i-1}) > \delta$

X_n stochastic process with sample paths in $D_E[0,\infty)$

X_n adapted to $\{\mathcal{F}_t^n\}$: For each $t\ge 0$, $X_n(t)$ is $\mathcal{F}_t^n$-measurable.

Tightness in $D_E[0,\infty)$

Theorem 3 *(Skorohod) Suppose that for $t \in \mathcal{T}_0$, a dense subset of $[0,\infty)$, $\{X_n(t)\}$ is tight. Then $\{X_n\}$ is tight if and only if for each $\epsilon > 0$ and $T > 0$*

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P\{w'(X_n,\delta,T) > \epsilon\} = 0.$$

Theorem 4 *(Puhalskii) Suppose that for $t \in \mathcal{T}_0$, a dense subset of $[0,\infty)$, $\{X_n(t)\}$ is exponentially tight. Then $\{X_n\}$ is exponentially tight if and only if for each $\epsilon > 0$ and $T > 0$*

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{1}{n}\log P\{w'(X_n,\delta,T) > \epsilon\} = -\infty.$$

Identification of limit distribution

Theorem 5 *If* $\{X_n\}$ *is tight in* $D_E[0,\infty)$ *and*

$$(X_n(t_1),\ldots,X_n(t_k)) \Rightarrow (X(t_1),\ldots,X(t_k))$$

for $t_1,\ldots,t_k \in \mathcal{T}_0$, $\mathcal{T}_0$ *dense in* $[0,\infty)$, *then* $X_n \Rightarrow X$.

Identification of rate function

Theorem 6 *If* $\{X_n\}$ *is exponentially tight in* $D_E[0,\infty)$ *and for each* $0 \le t_1 < \cdots < t_m$, $\{(X_n(t_1),\ldots,X_n(t_m))\}$ *satisfies the large deviation principle in* E^m *with rate function* $I_{t_1,\ldots,t_m}$, *then* $\{X_n\}$ *satisfies the large deviation principle in* $D_E[0,\infty)$ *with good rate function*

$$I(x) = \sup_{\{t_i\}\subset\Delta_x^c} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)),$$

where Δ_x *is the set of discontinuities of* x.

Conditions for tightness

$S_0^n(T)$ collection of discrete $\{\mathcal{F}_t^n\}$-stopping times

$q(x,y) = 1 \wedge r(x,y)$

Suppose that for $t \in \mathcal{T}_0$, a dense subset of $[0,\infty)$, $\{X_n(t)\}$ is tight. Then the following are equivalent.

a) $\{X_n\}$ is tight in $D_E[0,\infty)$.

Conditions for tightness

b) For $T > 0$, there exist $\beta > 0$ and random variables $\gamma_n(\delta, T)$, $\delta > 0$, satisfying

$$\begin{aligned} E[q^\beta(X_n(t+u), X_n(t)) \wedge q^\beta(X_n(t), X_n(t-v))|\mathcal{F}_t^n] \\ \leq E[\gamma_n(\delta, T)|\mathcal{F}_t^n] \end{aligned} \tag{1}$$

for $0 \leq t \leq T$, $0 \leq u \leq \delta$, and $0 \leq v \leq t \wedge \delta$ such that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E[\gamma_n(\delta, T)] = 0$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E[q^\beta(X_n(\delta), X_n(0))] = 0. \tag{2}$$

Kurtz

Conditions for tightness

c) Condition (2) holds, and for each $T > 0$, there exists $\beta > 0$ such that

$$C_n(\delta, T) \equiv \sup_{\tau \in S_0^n(T)} \sup_{u \le \delta} E[\sup_{v \le \delta \wedge \tau} q^\beta(X_n(\tau + u), X_n(\tau)) \wedge q^\beta(X_n(\tau), X_n(\tau - v))]$$

satisfies $\lim_{\delta \to 0} \limsup_{n \to \infty} C_n(\delta, T) = 0$.

Aldous

Conditions for exponential tightness

$S_0^n(T)$ collection of discrete $\{\mathcal{F}_t^n\}$-stopping times

$q(x,y) = 1 \wedge r(x,y)$

Suppose that for $t \in \mathcal{T}_0$, a dense subset of $[0,\infty)$, $\{X_n(t)\}$ is exponentially tight. Then the following are equivalent.

a) $\{X_n\}$ is exponentially tight in $D_E[0,\infty)$.

Conditions for exponential tightness

b) For $T > 0$, there exist $\beta > 0$ and random variables $\gamma_n(\delta, \lambda, T)$, $\delta, \lambda > 0$, satisfying

$$E[e^{n\lambda q^\beta(X_n(t+u),X_n(t))\wedge q^\beta(X_n(t),X_n(t-v))}|\mathcal{F}_t^n] \leq E[e^{\gamma_n(\delta,\lambda,T)}|\mathcal{F}_t^n]$$

for $0 \leq t \leq T$, $0 \leq u \leq \delta$, and $0 \leq v \leq t \wedge \delta$ such that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log E[e^{\gamma_n(\delta,\lambda,T)}] = 0,$$

and

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log E[e^{n\lambda q^\beta(X_n(\delta),X_n(0))}] = 0. \tag{3}$$

Conditions for exponential tightness

c) Condition (3) holds, and for each $T > 0$, there exists $\beta > 0$ such that for each $\lambda > 0$

$$C_n(\delta, \lambda, T)$$
$$\equiv \sup_{\tau \in S_0^n(T)} \sup_{u \leq \delta} E[\sup_{v \leq \delta \wedge \tau} e^{n\lambda q^\beta(X_n(\tau+u), X_n(\tau)) \wedge q^\beta(X_n(\tau), X_n(\tau-v))}]$$

satisfies $\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log C_n(\delta, \lambda, T) = 0$.

Example

W standard Brownian motion

$X_n = \frac{1}{\sqrt{n}} W$

$$E[e^{n\lambda|X_n(t+u)-X_n(t)|}|\mathcal{F}_t^W] = E[e^{\lambda\sqrt{n}|W(t+u)-W(t)|}|\mathcal{F}_t^W] \leq 2e^{\frac{1}{2}n\lambda^2 u}$$

so

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log E[e^{\gamma_n(\delta,\lambda,T)}] = \lim_{\delta\to 0} \frac{1}{2}\lambda^2\delta = 0.$$

Equivalence to tightness for functions

Theorem 7 $\{X_n\}$ *is tight in* $D_E[0,\infty)$ *if and only if*

a) *(Compact containment condition) For each* $T > 0$ *and* $\epsilon > 0$, *there exists a compact* $K_{\epsilon,T} \subset E$ *such that*

$$\limsup_{n\to\infty} P(\exists t \leq T \ni X_n(t) \notin K_{\epsilon,T}) \leq \epsilon$$

b) *There exists a family of functions* $F \subset C(E)$ *that is closed under addition and separates points in* E *such that for each* $f \in F$, $\{f(X_n)\}$ *is tight in* $D_R[0,\infty)$.

Kurtz, Jakubowski

Equivalence to exponential tightness for functions

Theorem 8 *$\{X_n\}$ is exponentially tight in $D_E[0,\infty)$ if and only if*

a) *For each $T > 0$ and $a > 0$, there exists a compact $K_{a,T} \subset E$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\exists t \leq T \ni X_n(t) \notin K_{a,T}) \leq -a \tag{4}$$

b) *There exists a family of functions $F \subset C(E)$ that is closed under addition and separates points in E such that for each $f \in F$, $\{f(X_n)\}$ is exponentially tight in $D_R[0,\infty)$.*

Schied

Large Deviations for Markov Processes

- Martingale problems and semigroups
- Semigroup convergence and the LDP
- Control representation of the rate function
- Viscosity solutions and semigroup convergence
- Summary of method

Markov processes

$X_n = \{X_n(t), t \geq 0\}$ is a Markov process if

$$E[g(X_n(t+s))|\mathcal{F}_t^n] = E[g(X_n(t+s))|X_n(t)]$$

The *generator* of a Markov process determines its short time behavior

$$E[g(X_n(t+\Delta t)) - g(X_n(t))|\mathcal{F}_t] \approx A_n g(X_n(t))\Delta t$$

Martingale problems

X_n is a solution of the martingale problem for A_n if and only if

$$g(X_n(t)) - g(X_n(0)) - \int_0^t A_n g(X_n(s))ds \tag{5}$$

is an $\{\mathcal{F}_t^n\}$-martingale for each $g \in \mathcal{D}(A_n)$.

If g is bounded away from zero, (5) is a martingale if and only if

$$g(X_n(t)) \exp\{-\int_0^t \frac{A_n g(X_n(s))}{g(X_n(s))} ds\}$$

is a martingale. (You can always add a constant to g.)

Nonlinear generator

Define $\mathcal{D}(H_n) = \{f \in B(E) : e^{nf} \in \mathcal{D}(A_n)\}$ and set

$$H_n f = \frac{1}{n} e^{-nf} A_n e^{nf}.$$

Then

$$\exp\{nf(X_n(t)) - nf(X(0)) - \int_0^t nH_n f(X(s))ds\}$$

is a $\{\mathcal{F}_t^n\}$-martingale.

Tightness for solutions of MGPs

$$
\begin{aligned}
&E[f(X_n(t+u)) - f(X_n(t))|\mathcal{F}_t^n] \\
&\qquad = E[\int_t^{t+u} A_n f(X_n(s))ds|\mathcal{F}_t^n] \leq u\|A_n f\|
\end{aligned}
$$

For $\gamma_n(\delta, T) = \delta(\|A_n f^2\| + 2\|f\|\|A_n f\|)$ (see (1))

$$
\begin{aligned}
&E[(f(X_n(t+u)) - f(X_n(t)))^2|\mathcal{F}_t^n] \\
&\qquad = E[\int_t^{t+u} A_n f^2(X_n(s))ds|\mathcal{F}_t^n] \\
&\qquad\qquad -2f(X_n(t))E[\int_t^{t+u} A_n f(X_n(s))ds|\mathcal{F}_t^n] \\
&\qquad \leq u(\|A_n f^2\| + 2\|f\|\|A_n f\|) \leq \gamma_n(\delta, T)
\end{aligned}
$$

Exponential tightness

$$E[e^{n(\lambda f(X_n(t+u))-\lambda f(X_n(t))-\int_t^{t+u} H_n[\lambda f](X_n(s))ds}|\mathcal{F}_t^n] = 1$$

so

$$E[e^{n\lambda(f(X_n(t+u))-f(X_n(t)))}|\mathcal{F}_t^n] \leq e^{nu\|H_n\lambda f\|}$$

and

$$\gamma_n(\delta, \lambda, T) = \delta n(\|H_n[\lambda f]\| + \|H_n[-\lambda f]\|)$$

ET Conditions

The Markov process semigroup

Assume that the martingale problem for A_n is well-posed.

Define

$$T_n(t)f(x) = E[f(X_n(t))|X_n(0) = x]$$

By the Markov property

$$T_n(s)T_n(t)f(x) = T_n(t+s)f(x)$$

$$\lim_{t\to 0} \frac{T_n(t)f(x) - f(x)}{t} = A_n f(x)$$

Iterating the semigroup

For $0 \le t_1 \le t_2$,

$$\begin{aligned} E[f_1(X_n(t_1))f_2(X_n(t_2))|X_n(0)=x] \\ = T_n(t_1)(f_1 T_n(t_2-t_1)f_2)(x) \end{aligned}$$

and in general

$$\begin{aligned} E[f_1(X_n(t_1))\cdots f_k(X_n(t_k))|X_n(0)=x] \\ = E[f_1(X_n(t_1))\cdots f_{k-1}(X_n(t_{k-1})) \\ T_n(t_k-t_{k-1})f_k(X_n(t_{k-1}))|X_n(0)=x] \end{aligned}$$

Convergence of the semigroups implies convergence of the finite dimensional distributions.

A nonlinear semigroup (Fleming)

Assume that the martingale problem for A_n is well-posed.

Define

$$V_n(t)f(x) = \frac{1}{n} \log E_x[e^{nf(X_n(t))}]$$

By the Markov property

$$V_n(s)V_n(t)f(x) = V_n(t+s)f(x)$$

$$\lim_{t\to 0} \frac{V_n(t)f(x) - f(x)}{t} = \frac{1}{n} e^{-nf} A_n e^{nf}(x) = H_n f(x)$$

Exponential generator

Iterating the semigroup

For $0 \le t_1 \le t_2$, define

$$V_n(t_1, t_2, f_1, f_2)(x) = V_n(t_1)(f_1 + V_n(t_2 - t_1)f_2)(x)$$

and inductively

$$\begin{aligned} V_n(t_1, \ldots, t_k, f_1, \ldots, f_k)(x) \\ = V_n(t_1)(f_1 + V_n(t_2, \ldots, t_k, f_2, \ldots, f_k))(x). \end{aligned}$$

Then

$$\begin{aligned} E[e^{n(f_1(X_n(t_1)) + \cdots + f_k(X_n(t_k)))}] \\ = E[e^{nV_n(t_1, \ldots, t_k, f_1, \ldots, f_k)(X_n(0))}] \end{aligned}$$

By the Bryc-Varadhan result, convergence of semigroup should imply the finite dimensional LDP

Weaker conditions for the LDP

A collection of functions $D \subset C_b(S)$ *isolates points* in S, if for each $x \in S$, each $\epsilon > 0$, and each compact $K \subset S$, there exists $f \in D$ satisfying $|f(x)| < \epsilon$, $\sup_{y \in K} f(y) \leq 0$, and

$$\sup_{y \in K \cap B_\epsilon^c(x)} f(y) < -\frac{1}{\epsilon}.$$

A collection of functions $D \subset C_b(S)$ is *bounded above* if $\sup_{f \in D} \sup_y f(y) < \infty$.

A rate determining class

Proposition 9 *Suppose* $\{X_n\}$ *is exponentially tight, and let*

$$\Gamma = \{f \in C_b(S) : \Lambda(f) = \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}] \text{ exists}\}.$$

If $D \subset \Gamma$ *is bounded above and isolates points, then* $\Gamma = C_b(S)$ *and*

$$I(x) = \sup_{f \in D}\{f(x) - \Lambda(f)\}.$$

Semigroup convergence and the LDP

Suppose $D \subset C_b(E)$ contains a set that is bounded above and isolates points.

Suppose $X_n(0) = x$ and $\{X_n(t)\}$ is exponentially tight. If $V_n(t)f(x) \to V(t)f(x)$ for each $f \in D$, then $\{X_n(t)\}$ satisfies a LDP with rate function

$$I_t(y|x) = \sup_{f \in D}\{f(y) - V(t)f(x)\},$$

and hence

$$V(t)f(x) = \sup_{y}\{f(y) - I_t(y|x)\}$$

Think of $I_t(y|x)$ as the large deviation analog of a transition density.

Iterating the semigroup

Suppose D is closed under addition, $V(t) : D \to D$, $t \geq 0$, and $0 \leq t_1 \leq t_2$. Define

$$V(t_1, t_2, f_1, f_2)(x) = V(t_1)(f_1 + V(t_2 - t_1)f_2)(x)$$

and inductively

$$\begin{aligned} &V(t_1, \ldots, t_k, f_1, \ldots, f_k)(x) \\ &\qquad = V(t_1)(f_1 + V(t_2, \ldots, t_k, f_2, \ldots, f_k))(x) \end{aligned}$$

Semigroup convergence and the LDP

Theorem 10 *For each n, let $A_n \subset C_b(E) \times B(E)$, and suppose that existence and uniqueness holds for the $D_E[0,\infty)$-martingale problem for (A_n, μ) for each initial distribution $\mu \in \mathcal{P}(E)$.*

Let $D \subset C_b(E)$ be closed under addition and contain a set that is bounded above and isolates points, and suppose that there exists an operator semigroup $\{V(t)\}$ on D such that for each compact $K \subset E$

$$\sup_{x \in K} |V(t)f(x) - V_n(t)f(x)| \to 0, \quad f \in D.$$

Suppose that $\{X_n\}$ is exponentially tight, and that $\{X_n(0)\}$ satisfies a large deviation principle with good rate function I_0. Define

$$\Lambda_0(f) = \lim_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n(0))}], \quad f \in C_b(E).$$

a) For each $0 \le t_1 < \cdots < t_k$ and $f_1, \ldots, f_k \in D$,

$$\lim_{n\to\infty} \frac{1}{n} \log E[e^{nf_1(X_n(t_1))+\ldots+nf_k(X_n(t_k))}]$$
$$= \Lambda_0(V(t_1, \ldots, t_k, f_1, \ldots, f_k)).$$

Recall

$$E[e^{n(f_1(X_n(t_1))+\cdots+f_k(X_n(t_k)))}]$$
$$= E[e^{nV_n(t_1,\ldots,t_k,f_1,\ldots,f_k)(X_n(0))}]$$

b) For $0 \leq t_1 < \ldots < t_k$ $\{(X_n(t_1), \ldots, X_n(t_k))\}$ satisfies the large deviation principle with rate function

$$\begin{aligned} & I_{t_1,\ldots,t_k}(x_1, \ldots, x_k) \qquad (6) \\ & = \sup_{f_1,\ldots,f_k \in D \cap C_b(E)} \{f_1(x_1) + \ldots + f_k(x_k) \\ & \qquad -\Lambda_0(V(t_1, \ldots, t_k, f_1, \ldots, f_k))\} \\ & = \inf_{x_0 \in E} (I_0(x_0) + \sum_{i=1}^{k} I_{t_i - t_{i-1}}(x_i | x_{i-1})) \end{aligned}$$

c) $\{X_n\}$ satisfies the large deviation principle in $D_E[0, \infty)$ with rate function

$$\begin{aligned} I(x) &= \sup_{\{t_i\} \subset \Delta_x^c} I_{t_1,\ldots,t_k}(x(t_1), \ldots, x(t_k)) \\ &= \sup_{\{t_i\} \subset \Delta_x^c} (I_0(x(0)) + \sum_{i=1}^{k} I_{t_i - t_{i-1}}(x(t_i) | x(t_{i-1}))) \end{aligned}$$

Example: Freidlin and Wentzell small diffusion

Let X_n satisfying the Itô equation

$$X_n(t) = x + \frac{1}{\sqrt{n}} \int_0^t \sigma(X_n(s-))dW(s) + \int_0^t b(X_n(s))ds,$$

and define $a(x) = \sigma^T(x) \cdot \sigma(x)$. Then

$$A_n g(x) = \frac{1}{2n} \sum_{ij} a_{ij}(x)\partial_i\partial_j g(x) + \sum_i b_i(x)\partial_i g(x),$$

Take $\mathcal{D}(A_n)$ to be the collection of functions of the form $c + f$, $c \in R$ and $f \in C_c^2(R^d)$.

Convergence of the nonlinear generator

$$H_n f(x) = \frac{1}{2n}\sum_{ij} a_{ij}(x)\partial_{ij} f(x) + \frac{1}{2}\sum_{ij} a_{ij}(x)\partial_i f(x)\partial_j f(x) + \sum_i b_i(x)\partial_i f(x).$$

and $Hf = \lim_{n\to\infty} H_n f$ is

$$Hf(x) = \frac{1}{2}(\nabla f(x))^T \cdot a(x) \cdot \nabla f(x) + b(x) \cdot \nabla f(x).$$

A control problem

Let (E, r) and (U, q) be complete, separable metric spaces, and let $A : \mathcal{D}(A) \subset C_b(E) \to C(E \times U)$

Let H be as above, and suppose that that there is a nonnegative, lower semicontinuous function L on $E \times U$ such that

$$Hf(x) = \sup_{u \in U}(Af(x, u) - L(x, u)).$$

$\{V(t)\}$ should be the Nisio semigroup corresponding to an optimal control problem with "reward" function $-L$.

(cf. Book by Dupuis and Ellis)

Dynamics of control problem

Require

$$f(x(t)) - f(x(0)) - \int_{U\times[0,t]} Af(x(s),u)\lambda_s(du\times ds) = 0,$$

for each $f\in\mathcal{D}(A)$ and $t\geq 0$, where $x\in D_E[0,\infty)$ and $\lambda\in\mathcal{M}_m(U)$, the space of measures on $U\times[0,\infty)$ satisfying $\lambda(U\times[0,t])=t$.)

For each $x_0\in E$, we should have

$$\begin{aligned}&V(t)g(x_0)\\&\quad=\sup_{(x,\lambda)\in\mathcal{J}^t_{x_0}}\{g(x(t))-\int_{[0,t]\times U}L(x(s),u)\lambda(du\times ds)\}\end{aligned}$$

Representation theorem

Theorem 11 *Suppose* (E,r) *and* (U,q) *are complete, separable, metric spaces. Let* $A:\mathcal{D}(A)\subset C_b(E)\rightarrow C(E\times U)$ *and lower semicontinuous* $L(x,u)\geq 0$ *satisfy*

1. $\mathcal{D}(A)$ *is convergence determining.*
2. *For each* $x_0\in E$*, there exists* $(x,\lambda)\in\mathcal{J}$ *such that* $x(0)=x_0$ *and* $\int_{U\times[0,t]}L(x(s),u)\lambda(du\times ds)=0$, $t\geq 0$.
3. *For each* $f\in\mathcal{D}(A)$*, there exists a nondecreasing function* $\psi_f:[0,\infty)\rightarrow[0,\infty)$ *such that*

$$|Af(x,u)|\leq\psi_f(L(x,u)),\;(x,u)\in E\times U,$$

and $\lim_{r\rightarrow\infty}r^{-1}\psi_f(r)=0$.

4. *There exists a tightness function* Φ *on* $E\times U$, *such that* $\Phi(x,u)\leq L(x,u)$ *for* $(x,u)\in E\times U$.

Let $\{V(t)\}$ *be an LDP limit semigroup and satisfy the control identity. Then*

$$I(x)=I_0(x(0))+\inf_{\lambda:(x,\lambda)\in\mathcal{J}}\{\int_{U\times[0,\infty)}L(x(s),u)\lambda(du\times ds)\}.$$

Small diffusion

$$Hf(x) = \frac{1}{2}(\nabla f(x))^T \cdot a(x) \cdot \nabla f(x) + b(x) \cdot \nabla f(x)$$

For

$$Af(x,u) = u \cdot \nabla f(x)$$

and

$$L(x,u) = \frac{1}{2}(u - b(x)a(x)^{-1}(u - b(x)),$$

$$Hf(x) = \sup_{u \in \mathbb{R}^d}(Af(x,u) - L(x,u))$$

$$I(x) = \int_0^\infty \frac{1}{2}(\dot{x}(s) - b(x(s))a(x(s))^{-1}(\dot{x}(s) - b(x(s)))ds$$

Alternative representation

For

$$Af(x,u) = (u^T\sigma(x) + b(x))\nabla f(x)$$

and

$$L(x,u) = \frac{1}{2}|u|^2,$$

again

$$Hf(x) = \sup_{u\in\mathbb{R}^d}(Af(x,u) - L(x,u))$$

$$I(x) = \inf\{\int_0^\infty \frac{1}{2}|u(s)|^2 ds : \ \dot{x}(t) = u^T(t)\sigma(x(t)) + b(x(t))\}$$

Legendre transform approach

If $Hf(x) = H(x, \nabla f(x))$, where $H(x,p)$ is convex and continuous in p, then

$$L(x,u) = \sup_{p \in R^d} \{p \cdot u - H(x,p)\}$$

and

$$H(x,p) = \sup_{u \in R^d} \{p \cdot u - L(x,u)\},$$

so taking $Af(x,u) = u \cdot \nabla f(x)$,

$$Hf(x) = \sup_{u \in R^d} \{u \cdot \nabla f(x) - L(x,u)\}$$

Viscosity solutions

Let E be compact, $H \subset C(E) \times B(E)$, and $(f, g) \in H$ imply $(f + c, g) \in H$. Fix $h \in C(E)$ and $\alpha > 0$.

$\overline{f} \in B(E)$ is a *viscosity subsolution* of

$$f - \alpha H f = h \tag{7}$$

if and only if $\overline{f}$ is upper semicontinuous and for each $(f_0, g_0) \in H$ there exists $x_0 \in E$ satisfying $(\overline{f} - f_0)(x_0) = \sup_x(\overline{f}(x) - f_0(x))$ and

$$\frac{\overline{f}(x_0) - h(x_0)}{\alpha} \leq (g_0)^*(x_0)$$

or equivalently

$$\overline{f}(x_0) \leq \alpha (g_0)^*(x_0) + h(x_0)$$

$\underline{f} \in B(E)$ is a *viscosity supersolution* of (7) if and only if $\underline{f}$ is lower semicontinuous and for each $(f_0, g_0) \in H$ there exists $x_0 \in E$ satisfying $(f_0 - \underline{f})(x_0) = \sup_x (f_0(x) - \underline{f}(x))$ and

$$\frac{\underline{f}(x_0) - h(x_0)}{\alpha} \geq (g_0)_*(x_0)$$

or

$$\underline{f}(x_0) \geq \alpha (g_0)_*(x_0) + h(x_0)$$

A function $f \in C(E)$ is a *viscosity solution* of $f - \alpha H f = h$ if it is both a subsolution and a supersolution.

Comparison principle

The equation $f - \alpha H f = h$ satisfies a *comparison principle*, if $\overline{f}$ a viscosity subsolution and $\underline{f}$ a viscosity supersolution implies $\overline{f} \le \underline{f}$ on E.

Viscosity approach to semigroup convergence

Theorem 12 *Let (E,r) be a compact metric space, and for $n = 1,2,\ldots$, assume that the martingale problem for $A_n \subset B(E) \times B(E)$ is well-posed.*

Let

$$H_n f = \frac{1}{n} e^{-nf} A_n e^{nf}, \quad e^{nf} \in \mathcal{D}(A_n),$$

and let $H \subset C(E) \times B(E)$ with $\mathcal{D}(H)$ dense in $C(E)$. Suppose that for each $(f,g) \in H$, there exists $(f_n, g_n) \in H_n$ such that $\|f - f_n\| \to 0$ and $\|g - g_n\| \to 0$.

Fix $\alpha_0 > 0$. Suppose that for each $0 < \alpha < \alpha_0$, there exists a dense subset $D_\alpha \subset C(E)$ such that for each $h \in D_\alpha$, the comparison principle holds for

$$(I - \alpha H)f = h.$$

Then there exists $\{V(t)\}$ on $C(E)$ such that

$$\sup_x |V(t)f(x) - V_n(t)f(x)| \to 0, \quad f \in C(E).$$

If $\{X_n(0)\}$ satisfies a large deviation principle with a good rate function. Then $\{X_n\}$ is exponentially tight and satisfies a large deviation principle with rate function I given above (6).

Proof of a large deviation principle

1. Verify convergence of the sequence of operators H_n and derive the limit operator H. In general, convergence will be in the extended limit or graph sense.

2. Verify exponential tightness. Given the convergence of H_n, exponential tightness typically follows provided one can verify the exponential compact containment condition.

3. Verify the range condition or the comparison principle for the limiting operator H. The rate function is characterized by the limiting semigroup.

4. Construct a variational representation for H. This representation typically gives a more explicit representation of the rate function.

$\mathbb{R}^d$-valued processes

Let $a = \sigma\sigma^T$, and define

$$\begin{aligned} A_n f(x) \\ &= n\int_{R^d}(f(x+\frac{1}{n}z)-f(x)-\frac{1}{n}z\cdot\nabla f(x))\eta(x,dz) \\ &\quad +b(x)\cdot\nabla f(x)+\frac{1}{2n}\sum_{ij}a_{ij}(x)\partial_i\partial_j f(x) \end{aligned}$$

Nonlinear generator

The operator $H_nf = \frac{1}{n}e^{-nf}A_ne^{nf}$ is given by

$$\begin{aligned} H_nf(x) \;&=\; \int_{R^d}(e^{n(f(x+\frac{1}{n}z)-f(x))}-1-z\cdot\nabla f(x))\eta(x,dz) \\ &\quad +\frac{1}{2n}\sum_{ij}a_{ij}(x)\partial_i\partial_j f(x) \\ &\quad +\frac{1}{2}\sum_{ij}a_{ij}(x)\partial_i f(x)\partial_j f(x)+b(x)\cdot\nabla f(x) \end{aligned}$$

Limiting operator

$$
\begin{aligned}
Hf(x) &= \int_{R^d} (e^{\nabla f(x)\cdot z} - 1 - z\cdot\nabla f(x))\eta(x,dz) \\
&\qquad + \frac{1}{2}\sum_{ij} a_{ij}(x)\partial_i f(x)\partial_j f(x) + b(x)\cdot\nabla f(x)
\end{aligned}
$$

Note that H has the form

$$Hf(x) = H(x, \nabla f(x))$$

for

$$H(x,p) = \frac{1}{2}|\sigma^T(x)p|^2 + b(x)\cdot p + \int_{R^d} (e^{p\cdot z} - 1 - p\cdot z)\eta(x,dz)$$

pc

Gradient limit operators

Condition 13

1. *For each compact* $\Gamma \subset R^d$*, there exist* $\mu_m \to +\infty$ *and* $\omega : (0,\infty) \to [0,\infty]$ *such that* $\{(x_m, y_m)\} \subset \Gamma \times \Gamma$, $\mu_m |x_m - y_m|^2 \to 0$, *and*

$$\sup_m H_*(y_m, \mu_m(x_m - y_m)) < \infty$$

imply

$$\liminf_{m\to\infty} [\lambda H^*(x_m, \frac{\mu_m(x_m - y_m)}{\lambda}) - H_*(y_m, \mu_m(x_m - y_m))] \le \omega(\lambda)$$

and

$$\lim_{\epsilon\to 0} \inf_{|\lambda-1|\le\epsilon} \omega(\lambda) \le 0.$$

2. *If* $x_m \to \infty$ *and* $p_m \to 0$*, then* $\lim_{m\to\infty} H(x_m, p_m) = 0$.

Conditions for comparison principle

Lemma 14 *If Condition 13 is satisfied, then for* $h \in C(E)$ *and* $\alpha > 0$*, the comparison principle holds for*

$$(I - \alpha H)f = h.$$

Sufficient conditions

Lemma 15 *Suppose σ and b are bounded and Lipschitz and $\eta = 0$. Then Condition 13 holds with*

$$\omega(\lambda) = \begin{cases} 0 & \lambda > 1 \\ \infty & \lambda \leq 1. \end{cases}$$

If H is continuous and for each $x, p \in R^d$ $\lim_{r\to\infty} H(x, rp) = \infty$, then Condition 13.1 holds with

$$\omega(\lambda) = \begin{cases} 0 & \lambda = 1 \\ \infty & \lambda \neq 1. \end{cases}$$

If σ and b are bounded and

$$\lim_{|p|\to 0} \sup_x \int_{R^d} (e^{p\cdot z} - 1 - p \cdot z)\eta(x, dz) = 0,$$

then Condition 13.2 holds.

Diffusions with periodic coefficients (Baldi)

Let σ be periodic (for each $1 \leq i \leq d$, there is a period $p_i > 0$ such that $\sigma(y) = \sigma(y + p_i e_i)$ for all $y \in R^d$), and let X_n satisfy the Itô equation

$$dX_n(t) = \frac{1}{\sqrt{n}}\sigma(\alpha_n X_n(t))dW(t),$$

where $\alpha_n > 0$ and $\lim_{n\to\infty} n^{-1}\alpha_n = \infty$. Let $a = \sigma\sigma^T$. Then

$$A_n f(x) = \frac{1}{n}\sum_{ij} a_{ij}(\alpha_n x)\frac{\partial^2}{\partial x_i \partial x_j} f(x),$$

and

$$H_n f(x) = \frac{1}{2n}\sum_{ij} a_{ij}(\alpha_n x)\partial_{ij} f(x) + \frac{1}{2}\sum_{ij} a_{ij}(\alpha_n x)\partial_i f(x)\partial_j f(x).$$

Limit operator

Let $f_n(x) = f(x) + \epsilon_n h(x, \alpha_n x)$, where $\epsilon_n = n\alpha_n^{-2}$.

$\epsilon_n \alpha_n = n\alpha_n^{-1} \to 0$

If h has the same periods in y as the a_{ij} and

$$\frac{1}{2}\sum_{ij} a_{ij}(y)\left(\frac{\partial^2}{\partial y_i \partial y_j} h(x, y) + \partial_i f(x)\partial_j f(x)\right) = g(x)$$

for some g independent of y, then

$$\lim_{n\to\infty} H_n f_n(x, y) = g(x).$$

It follows that

$$g(x) = \frac{1}{2} \sum_{ij} \overline{a}_{ij} \partial_i f(x) \partial_j f(x),$$

where $\overline{a}_{ij}$ is the average of a_{ij} with respect to the stationary distribution for the diffusion on $[0, p_1] \times \cdots \times [0, p_d]$ whose generator is

$$A_0 f(y) = \frac{1}{2} \sum_{i,j} a_{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} f(y)$$

with periodic boundary conditions. In particular,

$$h(x, y) = \frac{1}{2} \sum_{ij} h_{ij}(y) \partial_i f(x) \partial_j f(x),$$

where h_{ij} satisfies

$$A_0 h_{ij}(y) = \overline{a}_{ij} - a_{ij}(y).$$