

# Self-similar Markov processes

## Part I: One dimension

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A more thorough set of lecture notes can be found here:

<https://arxiv.org/abs/1707.04343>

Other related material found here

<https://arxiv.org/abs/1511.06356>

<https://arxiv.org/abs/1706.09924>

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## §1. Quick review of Lévy processes

## (KILLED) LÉVY PROCESS

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- ▶ Process is entirely characterised by its one-dimensional transitions, which are coded by the Lévy–Khinchine formula

$$\mathbf{E}[e^{i\theta \cdot \xi_t}] = e^{-\Psi(\theta)t}, \quad \theta \in \mathbb{R}^d,$$

where,

$$\Psi(\theta) = q + i\mathbf{a} \cdot \theta + \frac{1}{2}\theta \cdot \mathbf{A}\theta + \int_{\mathbb{R}^d} (1 - e^{i\theta \cdot x} + i(\theta \cdot x)\mathbf{1}_{(|x|<1)})\Pi(dx),$$

where  $\mathbf{a} \in \mathbb{R}$ ,  $\mathbf{A}$  is a  $d \times d$  Gaussian covariance matrix and  $\Pi$  is a measure satisfying  $\int_{\mathbb{R}^d} (1 \wedge |x|^2)\Pi(dx) < \infty$ . Think of  $\Pi$  as the intensity of jumps in the sense of

$$\mathbf{P}(X \text{ has jump at time } t \text{ of size } dx) = \Pi(dx)dt + o(dt).$$

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- ▶ In one dimension the path of a Lévy process can be monotone, in which case it is called a *subordinator* and we work with the Laplace exponent

$$\mathbf{E}[e^{-\lambda \xi_t}] = e^{-\Phi(\lambda)t}, \quad t \geq 0$$

where

$$\Phi(\lambda) = q + \delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\Upsilon(dx), \quad \lambda \geq 0.$$

## LÉVY PROCESS: ONE DIMENSION

Two examples in one dimension:

- ▶ **Stable subordinator** ( $\xi_t, t \geq 0$ ) is a subordinator which satisfies the additional scaling property: For  $c > 0$

under  $\mathbb{P}$ , the law of  $(c\xi_{c^{-\alpha}t}, t \geq 0)$  is equal to  $\mathbb{P}$ ,

where  $\alpha \in (0, 1)$ . We have

$$\Phi(\lambda) = \lambda^\alpha, \quad \lambda \geq 0, \quad \text{and} \quad \Pi(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} dx, \quad x > 0.$$

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- ▶ **Hypgeometric Lévy process:** For  $\beta \leq 1, \gamma \in (0, 1), \hat{\beta} \geq 0, \hat{\gamma} \in (0, 1)$

$$\Psi(\theta) = \frac{\Gamma(1-\beta+\gamma-i\theta)}{\Gamma(1-\beta-i\theta)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+i\theta)}{\Gamma(\hat{\beta}+i\theta)} \quad \theta \in \mathbb{R}.$$

The Lévy measure has a density with respect to Lebesgue measure which is given by

$$\pi(x) = \begin{cases} -\frac{\Gamma(\eta)}{\Gamma(\eta-\hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1+\gamma, \eta; \eta-\hat{\gamma}; e^{-x}), & \text{if } x > 0, \\ -\frac{\Gamma(\eta)}{\Gamma(\eta-\gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1+\hat{\gamma}, \eta; \eta-\gamma; e^x), & \text{if } x < 0, \end{cases}$$

where  $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$ .



## LÉVY PROCESS: ONE DIMENSION

- If  $\xi$  has a characteristic exponent  $\Psi$  then necessarily

$$\Psi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \quad \theta \in \mathbb{R}.$$

where  $\kappa$  and  $\hat{\kappa}$  are Bernstein functions, e.g.

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- ▶ The factorisation has a physical interpretation:
  - ▶ range of the  $\kappa$ -subordinator agrees with the range of  $\sup_{s \leq t} \xi_s, t \geq 0$
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- ▶ We have already seen the hypergeometric example

$$\Psi(\theta) = \frac{\Gamma(1 - \beta + \gamma - i\theta)}{\Gamma(1 - \beta - i\theta)} \times \frac{\Gamma(\hat{\beta} + \hat{\gamma} + i\theta)}{\Gamma(\hat{\beta} + i\theta)} \quad \theta \in \mathbb{R}.$$

## HITTING POINTS

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### Theorem (Kesten (1969)/Bretagnolle (1971))

Suppose that  $\xi$  is not a compound Poisson process. Then  $\xi$  can hit points if and only if

$$\int_{\mathbb{R}} \operatorname{Re} \left( \frac{1}{1 + \Psi(z)} \right) dz < \infty.$$

If the Kesten-Bretagnolle integral test is satisfied, then

$$\mathbb{P}(\tau^{\{x\}} < \infty) = \frac{u(x)}{u(0)},$$

where  $\tau^{\{x\}} = \inf\{t > 0 : \xi_t = x\}$ , providing we can compute the inversion

$$u(x) = \int_{c+i\mathbb{R}} \frac{e^{-zx}}{\Psi(-iz)} dz$$

for some  $c \in \mathbb{R}$ .

## §2. Self-similar Markov processes



## SELF-SIMILAR MARKOV PROCESSES (SSMP)

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### Definition

A regular strong Markov process  $(Z_t : t \geq 0)$  on  $\mathbb{R}^d$ , with probabilities  $\mathbb{P}_x, x \in \mathbb{R}^d$ , is a rssMp if there exists an index  $\alpha \in (0, \infty)$  such that for all  $c > 0$  and  $x \in \mathbb{R}^d$ ,

$$(cZ_{tc-\alpha} : t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (Z_t : t \geq 0) \text{ under } \mathbb{P}_{cx}.$$


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## SOME OF YOUR BEST FRIENDS ARE SSMP

- Write  $\mathcal{N}_d(\mathbf{0}, \Sigma)$  for the Normal distribution with mean  $\mathbf{0} \in \mathbb{R}^d$  and correlation (matrix)  $\Sigma$ . The moment generating function of  $X_t \sim \mathcal{N}_d(\mathbf{0}, \Sigma t)$  satisfies, for  $\theta \in \mathbb{R}^d$ ,

$$\mathbf{E}[e^{\theta \cdot X_t}] = e^{t\theta^T \Sigma \theta / 2} = e^{(c^{-2}t)(c\theta)^T \Sigma (c\theta) / 2} = E[e^{\theta \cdot cX_c - 2t}].$$

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- ▶ Thinking about the stationary and independent increments of Brownian motion, this can be used to show that  $\mathbb{R}^d$ -**Brownian motion**: is a ssMp with  $\alpha = 2$ .

## SOME OF YOUR BEST FRIENDS ARE SSMP

Suppose that  $(X_t : t \geq 0)$  is an  $\mathbb{R}$ -Brownian motion:

- ▶ Write  $\underline{X}_t := \inf_{s \leq t} X_s$ . Then  $(X_t, \underline{X}_t), t \geq 0$  is a Markov process.

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- ▶ For  $c > 0$  and  $\alpha = 2$ ,

$$\left( \begin{array}{c} c\underline{X}_{c^{-\alpha}t} \\ cX_{c^{-\alpha}t} \end{array} \right) = \left( \begin{array}{c} c \inf_{s \leq c^{-\alpha}t} X_s \\ cX_{c^{-\alpha}t} \end{array} \right) = \left( \begin{array}{c} \inf_{u \leq t} cX_{c^{-\alpha}u} \\ cX_{c^{-\alpha}t} \end{array} \right), \quad t \geq 0,$$

and the latter is equal in law to  $(X, \underline{X})$ , because of the scaling property of  $X$ .

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- ▶ Markov process  $Z_t := X_t - (-x \wedge \underline{X}_t), t \geq 0$  is also a ssMp on  $[0, \infty)$  issued from  $x > 0$  with index 2.

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- ▶  $Z_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}, t \geq 0$  is also a ssMp, again on  $[0, \infty)$ .

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Suppose that  $(X_t : t \geq 0)$  is an  $\mathbb{R}^d$ -Brownian motion:

- ▶ Consider  $Z_t := |X_t|, t \geq 0$ . Because of rotational invariance, it is a Markov process.
- ▶ Again the self-similarity (index 2) of Brownian motion, transfers to the case of  $|X|$ . Note again, this is a ssMp on  $[0, \infty)$ .



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- ▶ Note that  $|X_t|, t \geq 0$  is a Bessel- $d$  process. It turns out that all Bessel processes, *and* all squared Bessel processes are self-similar on  $[0, \infty)$ . One can check this by e.g. considering scaling properties of their transition semi-groups.

## SOME OF YOUR BEST FRIENDS ARE SSMP

Suppose that  $(X_t : t \geq 0)$  is an  $\mathbb{R}^d$ -Brownian motion:

- ▶ Note when  $d = 3$ ,  $|X_t|, t \geq 0$  is also equal in law to a Brownian motion conditioned to stay positive: i.e if we define, for a 1- $d$  Brownian motion  $(B_t : t \geq 0)$ ,

$$\mathbb{P}_x^\uparrow(A) = \lim_{s \rightarrow \infty} \mathbb{P}_x(A | \underline{B}_{t+s} > 0) = \mathbb{E}_x \left[ \frac{B_t}{x} \mathbf{1}_{(B_t > 0)} \mathbf{1}_A \right]$$

where  $A \in \sigma\{B_t : u \leq t\}$ , then

$(|X_t|, t \geq 0)$  with  $|X_0| = x$  is equal in law to  $(B, \mathbb{P}_x^\uparrow)$ .

## SOME OF THE BEST FRIENDS OF YOUR BEST FRIENDS ARE SSMP

- ▶ All of the previous examples have in common that their paths are continuous. Is this a necessary condition?

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§2.  
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§3.  
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Exercises.  
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- ▶ All of the previous examples are functional transforms of Brownian motion and have made use of the scaling and Markov properties and (in some cases) isotropic distributional invariance.
- ▶ If we replace Brownian motion by an  $\alpha$ -stable process, a Lévy process that has scale invariance, then all of the functional transforms still produce new examples of self-similar Markov processes.

## $\alpha$ -STABLE PROCESS

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### Definition

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- ▶ Assume jumps in both directions ( $0 < \alpha\rho, \alpha\hat{\rho} < 1$ ), so that the Lévy **density** takes the form

$$\frac{\Gamma(1 + \alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} (\sin(\pi\alpha\rho) \mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho}) \mathbf{1}_{\{x<0\}})$$

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- ▶ which by stationary and independent increments is equivalent to saying  $(cX_{c^{-\alpha}t}, t \geq 0) \stackrel{d}{=} (X_t, t \geq 0)$  when  $X_0 = 0$ ,
- ▶ or equivalently is equivalent to saying  $(cX_{c^{-\alpha}t}^{(x)}, t \geq 0) \stackrel{d}{=} (X_t^{(cx)}, t \geq 0)$ , where we have indicated the point of issue as an additional index.

## STABLE PROCESS PATH PROPERTIES

| index                            | jumps | path                | recurrence/transience  |
|----------------------------------|-------|---------------------|--|
| $\alpha \in (0, 1)$              |       |                     | transient  |
| $\rho = 0$                       | –     | monotone decreasing | $\lim_{t \rightarrow \infty} X_t = -\infty$  |
| $\rho = 1$                       | +     | monotone increasing | $\lim_{t \rightarrow \infty} X_t = \infty$   |
| $\rho \in (0, 1)$                | +, –  | bounded variation   | $\lim_{t \rightarrow \infty}  X_t  = \infty$   |
| $\alpha = 1$                     |       |                     | recurrent  |
| $\rho = \frac{1}{2}$             | +, –  | unbounded variation | $\limsup_{t \rightarrow \infty}  X_t  = \infty,$<br>$\liminf_{t \rightarrow \infty}  X_t  = 0$   |
| $\alpha \in (1, 2)$              |       |                     | recurrent  |
| $\alpha\rho = 1$                 | –     | unbounded variation | $\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R},$<br>– $\liminf_{t \rightarrow \infty} X_t = \limsup_{t \rightarrow \infty} X_t = \infty$ |
| $\alpha\rho = \alpha - 1$        | +     | unbounded variation | $\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R},$<br>– $\liminf_{t \rightarrow \infty} X_t = \limsup_{t \rightarrow \infty} X_t = \infty$ |
| $\alpha\rho \in (\alpha - 1, 1)$ | +, –  | unbounded variation | $\mathbb{P}_x(\tau^{\{0\}} < \infty) = 1, x \in \mathbb{R},$<br>– $\liminf_{t \rightarrow \infty} X_t = \limsup_{t \rightarrow \infty} X_t = \infty$ |

## YOUR NEW FRIENDS

Suppose  $X = (X_t : t \geq 0)$  is within the assumed class of  $\alpha$ -stable processes in one-dimension and let  $\underline{X}_t = \inf_{s \leq t} X_s$ .

Your new friends are:

- ▶  $Z = X$
- ▶  $Z = X - (-x \wedge \underline{X}), x > 0.$
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- ▶  $Z = |X|$  providing  $\rho = 1/2$



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- ▶  $Z = |X|$  providing  $\rho = 1/2$
- ▶ What about  $Z = "X \text{ conditioned to stay positive}"$ ?

## CONDITIONED $\alpha$ -STABLE PROCESSES

- Recall that each Lévy processes,  $\xi = \{\xi_t : t \geq 0\}$ , enjoys the Wiener-Hopf factorisation i.e. up to a multiplicative constant,  $\Psi_\xi(\theta) := t^{-1} \log \mathbb{E}[e^{i\theta\xi_t}]$  respects the factorisation

$$\Psi_\xi(\theta) = \kappa(-i\theta)\hat{\kappa}(i\theta), \quad \theta \in \mathbb{R},$$

where  $\kappa$  and  $\hat{\kappa}$  are Bernstein functions. That is e.g.  $\kappa$  takes the form

$$\kappa(\lambda) = q + a\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\nu(dx), \quad \lambda \geq 0$$

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- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of  $\xi$  and of  $-\xi$  respectively.
- In the case of  $\alpha$ -stable processes, up to a multiplicative constant,

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \quad \lambda \geq 0.$$

## CONDITIONED $\alpha$ -STABLE PROCESSES

- Associated to the descending ladder subordinator  $\hat{\kappa}$  is its potential measure  $\hat{U}$ , which satisfies

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$$\mathbb{P}_x^\uparrow(A) = \lim_{s \rightarrow \infty} \mathbb{P}_x(A | \underline{X}_{t+s} > 0) = \mathbb{E}_x \left[ \frac{\hat{U}(X_t)}{\hat{U}(x)} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)} \right]$$

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- In the  $\alpha$ -stable case  $\hat{U}(x) \propto x^{\alpha\hat{\rho}}$   
[Note in the excluded case that  $\alpha = 2$  and  $\rho = 1/2$ , i.e. Brownian motion,  $\hat{U}(x) = x$ .]

## CONDITIONED $\alpha$ -STABLE PROCESSES

- For  $c, x > 0, t \geq 0$  and appropriately bounded, measurable and non-negative  $f$ , we can write,

$$\begin{aligned}
 & \mathbb{E}_x^\uparrow[f(\{cX_{c-\alpha_s} : s \leq t\})] \\
 &= \mathbb{E} \left[ f(\{cX_{c-\alpha_s}^{(x)} : s \leq t\}) \frac{(X_{c-\alpha_t}^{(x)})^{\alpha\hat{\rho}}}{x^{\alpha\hat{\rho}}} \mathbf{1}_{(X_{c-\alpha_t}^{(x)} \geq 0)} \right] \\
 &= \mathbb{E} \left[ f(\{X_s^{(cx)} : s \leq t\}) \frac{(X_t^{(cx)})^{\alpha\hat{\rho}}}{(cx)^{\alpha\hat{\rho}}} \mathbf{1}_{(X_t^{(cx)} \geq 0)} \right] \\
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- ▶ This also makes the process  $(X, \mathbb{P}_x^\uparrow), x > 0$ , a self-similar Markov process on  $[0, \infty)$ .
- ▶ Unlike the case of Brownian motion, the conditioned stable process does not have the law of the radial part of a 3-dimensional stable process (the analogue to the Brownian case).

§1.  
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§2.  
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§3.  
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§4.  
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§5.  
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§6.  
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Exercises.  
○○○

### §3. Lamperti Transform

## NOTATION

- ▶ Use  $\xi := \{\xi_t : t \geq 0\}$  to denote a Lévy process which is killed and sent to the cemetery state  $-\infty$  at an independent and exponentially distributed random time,  $\mathbf{e}_q$ , with rate in  $q \in [0, \infty)$ . The characteristic exponent of  $\xi$  is thus written

$$-\log \mathbf{E}(e^{i\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy-Khintchine}$$

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- ▶ Define the associated integrated exponential Lévy process

$$I_t = \int_0^t e^{\alpha\xi_s} ds, \quad t \geq 0. \quad (1)$$

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- ▶ Also interested in the inverse process of  $I$ :

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \quad t \geq 0. \quad (2)$$

As usual, we work with the convention  $\inf \emptyset = \infty$ .

## LAMPERTI TRANSFORM FOR POSITIVE ssMP

### Theorem (Part (i))

Fix  $\alpha > 0$ . If  $Z^{(x)}$ ,  $x > 0$ , is a positive self-similar Markov process with index of self-similarity  $\alpha$ , then up to absorption at the origin, it can be represented as follows. For  $x > 0$ ,

$$Z_t^{(x)} \mathbf{1}_{(t < \zeta^{(x)})} = x \exp\{\xi_{\varphi(x^{-\alpha}t)}\}, \quad t \geq 0,$$

where  $\zeta^{(x)} = \inf\{t > 0 : Z_t^{(x)} = 0\}$  and either

- (1)  $\zeta^{(x)} = \infty$  almost surely for all  $x > 0$ , in which case  $\xi$  is a Lévy process satisfying  $\limsup_{t \uparrow \infty} \xi_t = \infty$ ,
- (2)  $\zeta^{(x)} < \infty$  and  $Z_{\zeta^{(x)}-}^{(x)} = 0$  almost surely for all  $x > 0$ , in which case  $\xi$  is a Lévy process satisfying  $\lim_{t \uparrow \infty} \xi_t = -\infty$ , or
- (3)  $\zeta^{(x)} < \infty$  and  $Z_{\zeta^{(x)}-}^{(x)} > 0$  almost surely for all  $x > 0$ , in which case  $\xi$  is a Lévy process killed at an independent and exponentially distributed random time.

In all cases, we may identify  $\zeta^{(x)} = x^\alpha I_\infty$ .

## LAMPERTI TRANSFORM FOR POSITIVE ssMP

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### Theorem (Part (ii))

Conversely, suppose that  $\xi$  is a given (killed) Lévy process. For each  $x > 0$ , define

$$Z_t^{(x)} = x \exp\{\xi_{\varphi(x-\alpha t)}\} \mathbf{1}_{(t < x^\alpha I_\infty)}, \quad t \geq 0.$$

Then  $Z^{(x)}$  defines a positive self-similar Markov process, up to its absorption time  $\zeta^{(x)} = x^\alpha I_\infty$ , with index  $\alpha$ .

---



## LAMPERTI TRANSFORM FOR POSITIVE ssMp

 $(Z, \mathbb{P}_x)_{x>0}$  pssMp

$$Z_t = \exp(\xi_{S(t)}),$$

 $S$  a random time-change $\leftrightarrow$  $(\xi, \mathbf{P}_y)_{y \in \mathbb{R}}$  killed Lévy

$$\xi_s = \log(Z_{T(s)}),$$

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$\left. \begin{array}{l} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array} \right\}$

$\leftrightarrow$

$\left\{ \begin{array}{l} \xi \rightarrow \infty \text{ or } \xi \text{ oscillates} \\ \xi \rightarrow -\infty \\ \xi \text{ is killed} \end{array} \right.$

## §4. Positive self-similar Markov processes

## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

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- ▶ Write  $\xi^* = \{\xi_t^* : t \geq 0\}$  for the underlying Lévy process and denote its killing rate by  $q^*$ .
- ▶ Let’s try and decode the characteristics of  $\xi^*$ .

## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- ▶ We know that the  $\alpha$ -stable process experiences downward jumps at rate

$$\frac{\Gamma(1 + \alpha)}{\pi} \sin(\pi\alpha\hat{\rho}) \frac{1}{|x|^{1+\alpha}} dx, \quad x < 0.$$



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- ▶ Given that we know the value of  $Z_{t-}^*$ , on  $\{\underline{X}_t > 0\}$ , the stable process will pass over the origin at rate

$$\frac{\Gamma(1 + \alpha)}{\pi} \sin(\pi\alpha\hat{\rho}) \left( \int_{Z_{t-}^*}^{\infty} \frac{1}{|x|^{1+\alpha}} dx \right) = \frac{\Gamma(1 + \alpha)}{\alpha\pi} \sin(\pi\alpha\hat{\rho}) (Z_{t-}^*)^{-\alpha}.$$

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- ▶ On the other hand, the Lamperti transform says that on  $\{t < \zeta\}$ , as a pssMp,  $Z$  is sent to the origin at rate

$$q^* \frac{d}{dt} \varphi(t) = q^* e^{-\alpha \xi_{\varphi}^*(t)} = q^* (Z_t^*)^{-\alpha}.$$

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- ▶ Comparing gives us

$$q^* = \Gamma(\alpha) \sin(\pi\alpha\hat{\rho}) / \pi = \frac{\Gamma(\alpha)}{\Gamma(\alpha\hat{\rho}) \Gamma(1 - \alpha\hat{\rho})}.$$

## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- ▶ Referring again to the Lamperti transform, we know that, under  $\mathbb{P}_1$  (so that  $\xi_0^* = 0$  almost surely),

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- ▶ This motivates the computation

$$\mathbb{E}_1[(Z_{\zeta^-}^*)^{i\theta}] = \mathbb{E}_0[e^{i\theta \xi_{\mathbf{e}_{q^*}}^*}] = \frac{q^*}{(\Psi^*(z) - q^*) + q^*}, \quad \theta \in \mathbb{R},$$

where  $\Psi^*$  is the characteristic exponent of  $\xi^*$ .

## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

Remembering the “overshoot-undershoot” distributional law at first passage (well known in the literature for Lévy processes c.f. the quintuple law - Chapter 7 of my book) and deduce that, for all  $v \in [0, 1]$ ,

$$\begin{aligned} & \mathbb{P}_1(X_{\tau_0^-} \in dv) \\ &= \hat{\mathbb{P}}_0(1 - X_{\tau_1^+} \in dv) \\ &= \frac{\sin(\alpha \hat{\rho} \pi)}{\pi} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha \rho) \Gamma(\alpha \hat{\rho})} \left( \int_0^\infty \int_0^\infty \mathbf{1}_{(y \leq 1 \wedge v)} \frac{(1-y)^{\alpha \hat{\rho} - 1} (v-y)^{\alpha \rho - 1}}{(v+u)^{1+\alpha}} du dy \right) dv \\ &= \frac{\sin(\alpha \hat{\rho} \pi)}{\pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha \rho) \Gamma(\alpha \hat{\rho})} \left( \int_0^1 \mathbf{1}_{(y \leq v)} v^{-\alpha} (1-y)^{\alpha \hat{\rho} - 1} (v-y)^{\alpha \rho - 1} dy \right) dv, \end{aligned}$$

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where  $\hat{\mathbb{P}}_0$  is the law of  $-X$  issued from 0.

**Note:** more generally (which you will need for an exercise later):

$$\begin{aligned} & \mathbb{P}_1(-X_{\tau_0^-} \in du, X_{\tau_0^-} \in dv) \\ &= \frac{\sin(\alpha\hat{\rho}\pi)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \left( \int_0^\infty \mathbf{1}_{(y \leq 1 \wedge v)} \frac{(1-y)^{\alpha\hat{\rho}-1} (v-y)^{\alpha\rho-1}}{(v+u)^{1+\alpha}} dy \right) dv du \end{aligned}$$

## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

We are led to the conclusion that

$$\begin{aligned} & \frac{q^*}{\Psi^*(\theta)} \\ &= \frac{\sin(\alpha\hat{\rho}\pi)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^1 (1-y)^{\alpha\hat{\rho}-1} \int_0^\infty \mathbf{1}_{(y\leq v)} v^{i\theta-\alpha\hat{\rho}-1} \left(1-\frac{y}{v}\right)^{\alpha\rho-1} dv dy \\ &= \frac{\sin(\alpha\hat{\rho}\pi)}{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})} \int_0^1 (1-y)^{\alpha\hat{\rho}-1} y^{i\theta-\alpha\hat{\rho}} dy \frac{\Gamma(\alpha\hat{\rho}-i\theta)\Gamma(\alpha\rho)}{\Gamma(\alpha-i\theta)} \\ &= \frac{\Gamma(\alpha\hat{\rho}-i\theta)\Gamma(\alpha\rho)\Gamma(1-\alpha\hat{\rho}+i\theta)\Gamma(\alpha\hat{\rho})\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})\Gamma(\alpha\hat{\rho})\Gamma(1+i\theta)\Gamma(\alpha-i\theta)}, \end{aligned}$$

where in the first equality Fubini's Theorem has been used, in the second equality a straightforward substitution  $w = y/v$  has been used for the inner integral on the preceding line together with the classical beta integral and, finally, in the third equality, the Beta integral has been used for a second time. Inserting the respective values for the constants  $q^*$  and  $K$ , we come to rest at the following result:



## STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

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### Theorem

For the pssMp constructed by killing a stable process on first entry to  $(-\infty, 0)$ , the underlying killed Lévy process,  $\xi^*$ , that appears through the Lamperti transform has characteristic exponent given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - iz)}{\Gamma(\alpha\hat{\rho} - iz)} \frac{\Gamma(1 + iz)}{\Gamma(1 - \alpha\hat{\rho} + iz)}, \quad z \in \mathbb{R}.$$


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## STABLE PROCESSES CONDITIONED TO STAY POSITIVE

- Use the Lamperti representation of the  $\alpha$ -stable process  $X$  to write, for  $A \in \sigma(X_u : u \leq t)$ ,

$$\mathbb{P}_x^\uparrow(A) = \mathbb{E}_x \left[ \frac{X_t^{\alpha \hat{\rho}}}{x^{\alpha \hat{\rho}}} \mathbf{1}_{(X_t > 0)} \mathbf{1}_{(A)} \right] = \mathbb{E}_0 \left[ e^{\alpha \hat{\rho} \xi_\tau^*} \mathbf{1}_{(\tau < e_{\hat{\rho}^*})} \mathbf{1}_{(A)} \right],$$

where  $\tau = \varphi(x^{-\alpha} t)$  is a stopping time in the natural filtration of  $\xi^*$ .

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### Theorem

The underlying Lévy process,  $\xi^\uparrow$ , that appears through the Lamperti transform applied to  $(X, \mathbb{P}_x^\uparrow)$ ,  $x > 0$ , has characteristic exponent given by

$$\Psi^\uparrow(z) = \frac{\Gamma(\alpha\rho - iz)}{\Gamma(-iz)} \frac{\Gamma(1 + \alpha\hat{\rho} + iz)}{\Gamma(1 + iz)}, \quad z \in \mathbb{R}.$$

- ▶ In particular  $\Psi^\uparrow(z) = \Psi^*(z - i\alpha\hat{\rho})$ ,  $z \in \mathbb{R}$  so that  $\Psi^\uparrow(0) = 0$  (i.e. no killing!)
- ▶ One can also check by hand that  $\Psi^{\uparrow\prime}(0+) = \mathbb{E}_0[\xi_1^\uparrow] > 0$  so that  $\lim_{t \rightarrow \infty} \xi_t^\uparrow = \infty$ .

## DID YOU SPOT THE OTHER ROOT?

- ▶ In essence, the case of the stable process conditioned to stay positive boils down to an Esscher transform in the underlying (Lamperti-transformed) Lévy process.
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- ▶ However, there is another root of the equation

$$\Psi^*(z) = \frac{\Gamma(\alpha - iz)}{\Gamma(\alpha\hat{\rho} - iz)} \frac{\Gamma(1 + iz)}{\Gamma(1 - \alpha\hat{\rho} + iz)} = 0,$$

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- ▶ And this means that

$$e^{(1-\alpha\hat{\rho})\xi^*}, \quad t \geq 0,$$

is a unit-mean Martingale, which can also be used to construct an Esscher transform:

$$\Psi^\downarrow(z) = \Psi^*(z - i(1 - \alpha\hat{\rho})) = \Psi^\downarrow(z) = \frac{\Gamma(1 + \alpha\rho - iz)}{\Gamma(1 - iz)} \frac{\Gamma(iz + \alpha\hat{\rho})}{\Gamma(iz)}.$$

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- ▶ The choice of notation is pre-emptive since we can also check that  $\Psi^\downarrow(0) = 0$  and  $\Psi^{\downarrow\prime}(0) < 0$  so that if  $\xi^\downarrow$  is a Lévy process with characteristic exponent  $\Psi^\downarrow$ , then  $\lim_{t \rightarrow \infty} \xi_t^\downarrow = -\infty$ .



## REVERSE ENGINEERING

- What now happens if we define for  $A \in \sigma(X_u : u \leq t)$ ,

$$\mathbb{P}_x^\downarrow(A) = \mathbb{E}_0 \left[ e^{(1-\alpha\hat{\rho})\xi_\tau^*} \mathbf{1}_{(\tau < \mathbf{e}_{q^*})} \mathbf{1}_{(A)} \right] = \mathbb{E}_x \left[ \frac{X_t^{(1-\alpha\hat{\rho})}}{x^{(1-\alpha\hat{\rho})}} \mathbf{1}_{(\underline{X}_t > 0)} \mathbf{1}_{(A)} \right],$$

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- ▶ In the same way we checked that  $(X, \mathbb{P}_x^\uparrow)$ ,  $x > 0$ , is a pssMp, we can also check that  $(X, \mathbb{P}_x^\downarrow)$ ,  $x > 0$  is a pssMp.

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- ▶ In the same way we checked that  $(X, \mathbb{P}_x^\uparrow)$ ,  $x > 0$ , is a pssMp, we can also check that  $(X, \mathbb{P}_x^\downarrow)$ ,  $x > 0$  is a pssMp.
- ▶ In an appropriate sense, it turns out that  $(X, \mathbb{P}_x^\downarrow)$ ,  $x > 0$  is the law of a stable process conditioned to continuously approach the origin from above.

## $\xi^*$ , $\xi^\uparrow$ AND $\xi^\downarrow$

- ▶ The three examples of pssMp offer quite striking underlying Lévy processes
- ▶ Is this exceptional?

## CENSORED STABLE PROCESSES

- ▶ Start with  $X$ , the stable process.
- ▶ Let  $A_t = \int_0^t \mathbf{1}_{(X_t > 0)} dt$ .
- ▶ Let  $\gamma$  be the right-inverse of  $A$ , and put  $\check{Z}_t := X_{\gamma(t)}$ .
- ▶ Finally, make zero an absorbing state:  $Z_t = \check{Z}_t \mathbf{1}_{(t < T_0)}$  where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

Note  $T_0 < \infty$  a.s. if and only if  $\alpha \in (1, 2)$  and otherwise  $T_0 = \infty$  a.s.

- ▶ This is the **censored stable process**.

# CENSORED STABLE PROCESSES

## Theorem

Suppose that the underlying Lévy process for the censored stable process is denoted by  $\tilde{\xi}$ . Then  $\tilde{\xi}$  is equal in law to  $\xi^{**} \oplus \xi^C$ , with

- ▶  $\xi^{**}$  equal in law to  $\xi^*$  with the killing removed,
- ▶  $\xi^C$  a compound Poisson process with jump rate  $q^* = \Gamma(\alpha)\sin(\pi\alpha\hat{\rho})/\pi$ .

Moreover, the characteristic exponent of  $\tilde{\xi}$  is given by

$$\tilde{\Psi}(z) = \frac{\Gamma(\alpha\rho - iz)}{\Gamma(-iz)} \frac{\Gamma(1 - \alpha\rho + iz)}{\Gamma(1 - \alpha + iz)}, \quad z \in \mathbb{R}.$$

## THE RADIAL PART OF A STABLE PROCESS

- ▶ Suppose that  $X$  is a symmetric stable process, i.e  $\rho = 1/2$ .
- ▶ We know that  $|X|$  is a pssMp.

### Theorem

Suppose that the underlying Lévy process for  $|X|$  is written  $\xi$ , then its characteristic exponent is given by

$$\Psi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-iz + \alpha))}{\Gamma(-\frac{1}{2}iz)} \frac{\Gamma(\frac{1}{2}(iz + 1))}{\Gamma(\frac{1}{2}(iz + 1 - \alpha))}, \quad z \in \mathbb{R}.$$

## HYPERGEOMETRIC LÉVY PROCESSES (REMINDER)

### Definition (and Theorem)

For  $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$  in

$$\left\{ \beta \leq 2, \gamma, \hat{\gamma} \in (0, 1) \hat{\beta} \geq -1, \text{ and } 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0 \right\}$$

there exists a (killed) Lévy process, henceforth referred to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1 - \beta + \gamma - iz)}{\Gamma(1 - \beta - iz)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + iz)}{\Gamma(\hat{\beta} + iz)} \quad z \in \mathbb{R}.$$

The Lévy measure of  $Y$  has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\frac{\Gamma(\eta)}{\Gamma(\eta - \hat{\gamma})\Gamma(-\gamma)} e^{-(1-\beta+\gamma)x} {}_2F_1(1 + \gamma, \eta; \eta - \hat{\gamma}; e^{-x}), & \text{if } x > 0, \\ -\frac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\Gamma(-\hat{\gamma})} e^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1(1 + \hat{\gamma}, \eta; \eta - \gamma; e^x), & \text{if } x < 0, \end{cases}$$

where  $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$ , for  $|z| < 1$ ,  ${}_2F_1(a, b; c; z) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$ .



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Exercises.  
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## §5. Entrance Laws

## STARTING FROM ZERO

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- ▶ This should ring alarm bells when we look at the Lamperti transform

$$Z_t^{(x)} \mathbf{1}_{(t < \zeta(x))} = x \exp\{\xi_{\varphi(x-\alpha t)}\} = \exp\{\xi_{\varphi(x-\alpha t)} + \log x\}, \quad t \geq 0,$$

- ▶ On the one hand  $\log x \downarrow -\infty$ , which is the point of issue of  $\xi$ , but

$$\varphi(x^{-\alpha t}) = \inf\{s > 0 : \int_0^s e^{\alpha(\xi_u + \log x)} du > t\},$$

meaning that we are sampling the Lévy process over a longer and longer time horizon.

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- ▶ We know that some of our new friends have no problem using the origin as an entrance point, but also a point of recurrence, e.g.  $X - \underline{X}$ , where  $X$  is an  $\alpha$ -stable process (or Brownian motion).

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$$\mathbb{P}_0(Z_t \in dy) := \lim_{x \downarrow 0} \mathbb{P}_x(Z_t \in dy), \quad t, y > 0.$$



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- ▶ In that case, for any sequence of times  $0 < t_1 \leq t_2 \leq \dots \leq t_n < \infty$  and  $y_1, \dots, y_n \in (0, \infty)$ ,  $n \in \mathbb{N}$ , the Markov property gives us

$$\begin{aligned} & \mathbb{P}_0(Z_{t_1} \in dy_1, \dots, Z_{t_n} \in dy_n) \\ &:= \lim_{x \downarrow 0} \mathbb{P}_x(Z_{t_1} \in dy_1, \dots, Z_{t_n} \in dy_n) \\ &= \lim_{x \downarrow 0} \mathbb{P}_x(Z_{t_1} \in dy_1) \mathbb{P}_{y_1}(Z_{t_2-t_1} \in dy_2, \dots, Z_{t_n-t_2} \in dy_n) \\ &= \mathbb{P}_0(Z_{t_1} \in dy_1) \mathbb{P}_{y_1}(Z_{t_2-t_1} \in dy_2, \dots, Z_{t_n-t_2} \in dy_n). \end{aligned}$$

When the limit exists, it implies the existence of  $\mathbb{P}_0$  as limit of  $\mathbb{P}_x$  as  $x \downarrow 0$ , in the sense of convergence of finite-dimensional distributions.

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$$\mathbb{E}_0[f(Z_s : s \leq t)] := \lim_{x \rightarrow 0} \mathbb{E}_x[f(Z_s : s \leq t)]$$

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- ▶ The right setting to discuss *distributional convergence* is with respect to so-called *Skorokhod topology*.
- ▶ **ROUGHLY:** There is a metric on cadlag path space which does a better job of measuring how “close” two paths are than e.g. the uniform functional metric.
- ▶ This metric induces a topology (the Skorokhod topology). From this topology, we build a measurable space around the space of cadlag paths.
- ▶ Think of  $\mathbb{P}_x, x > 0$  as a family of measures on this space and we want weak convergence “ $\mathbb{P}_0 := \lim_{x \rightarrow 0} \mathbb{P}_x$ ” on this space.

## STARTING FROM ZERO

### Theorem

Suppose that  $(\xi, \mathbf{P}_x)$ ,  $x \in \mathbb{R}$  is the Lévy process (not a compound Poisson process) underlying the pssMp  $(Z, \mathbb{P}_x)$ ,  $x > 0$ . The limit  $\mathbb{P}_0 := \lim_{x \rightarrow 0} \mathbb{P}_x$  exists in the sense of convergence with respect to the Skorokhod topology if and only if  $\mathbf{E}_0(H_1^+) < \infty$  ( $H^+$  is the ascending ladder process of  $\xi$ ). Under the assumption that  $\mathbf{E}(\xi_1) > 0$ , for any positive measurable function  $f$  and  $t > 0$ ,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{-\alpha \hat{\mathbf{E}}_0(\xi_1)} \hat{\mathbf{E}}_0 \left( \frac{1}{I_\infty} f \left( \left( \frac{t}{I_\infty} \right)^{1/\alpha} \right) \right),$$

where  $I_\infty = \int_0^\infty e^{\alpha \xi_t} dt$  and  $(\xi, \hat{\mathbf{P}}_0)$  is equal in law to  $(-\xi, \mathbf{P}_0)$ .

## RECURRENT EXTENSION

- ▶ The previous construction has insisted that  $Z$  is a  $pssMp$  with  $\zeta = \infty$  a.s. But what about the case that  $\zeta < \infty$  a.s.

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- ▶ A cadlag strong Markov process,  $\vec{Z} := \{\vec{Z}_t : t \geq 0\}$  with probabilities  $\{\vec{P}_x, x \geq 0\}$ , is a *recurrent extension* of  $Z$  if, for each  $x > 0$ , the origin is not an absorbing state  $\vec{P}_x$ -almost surely and  $\{\vec{Z}_{t \wedge \vec{\zeta}} : t \geq 0\}$  under  $\vec{P}_x$  has the same law as  $(Z, P_x)$ , where

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$$\vec{\zeta} = \inf\{t > 0 : \vec{Z}_t = 0\}.$$

## Theorem

If  $\zeta < \infty$  a.s. and  $X_{\zeta-} = 0$ , then there exists a unique recurrent extension of  $Z$  which leaves 0 continuously if and only if there exists a  $\beta \in (0, \alpha)$  such

$$\mathbf{E}_0(e^{\beta \xi_1}) = 1.$$

Here, as usual,  $\alpha$  is the index of self-similarity.

## §6. Real valued self-similar Markov processes

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Exercises.  
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► So far we only spoke about  $[0, \infty)$ .

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- ▶ What can we say about  $\mathbb{R}$ -valued self-similar Markov processes.
- ▶ This requires us to first investigate Markov Additive (Lévy) Processes

## MARKOV ADDITIVE PROCESSES (MAPs)

- ▶  $E$  is a finite state space
- ▶  $(J(t))_{t \geq 0}$  is a continuous-time, irreducible Markov chain on  $E$
- ▶ process  $(\xi, J)$  in  $\mathbb{R} \times E$  is called a *Markov additive process (MAP)* with probabilities  $\mathbf{P}_{x,i}$ ,  $x \in \mathbb{R}$ ,  $i \in E$ , if, for any  $i \in E$ ,  $s, t \geq 0$ : Given  $\{J(t) = i\}$ ,  
 $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$  with law  $\mathbf{P}_{0,i}$ .

## PATHWISE DESCRIPTION OF A MAP

The pair  $(\xi, J)$  is a Markov additive process if and only if, for each  $i, j \in E$ ,

- ▶ there exist a sequence of iid Lévy processes  $(\xi_i^n)_{n \geq 0}$
- ▶ and a sequence of iid random variables  $(U_{ij}^n)_{n \geq 0}$ , independent of the chain  $J$ ,
- ▶ such that if  $T_0 = 0$  and  $(T_n)_{n \geq 1}$  are the jump times of  $J$ , the process  $\xi$  has the representation

$$\xi(t) = \mathbf{1}_{(n > 0)}(\xi(T_n -) + U_{J(T_n -), J(T_n)}^n) + \xi_{J(T_n)}^n(t - T_n),$$

for  $t \in [T_n, T_{n+1})$ ,  $n \geq 0$ .

## CHARACTERISTICS OF A MAP

- ▶ Denote the transition rate matrix of the chain  $J$  by  $\mathbf{Q} = (q_{ij})_{i,j \in E}$ .
- ▶ For each  $i \in E$ , the Laplace exponent of the Lévy process  $\xi_i$  will be written  $\psi_i$  (when it exists).
- ▶ For each pair of  $i, j \in E$  with  $i \neq j$ , define the Laplace transform  $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$  of the jump distribution  $U_{ij}$  (when it exists).
- ▶ Otherwise define  $U_{i,i} \equiv 0$ , for each  $i \in E$ .
- ▶ Write  $G(z)$  for the  $N \times N$  matrix whose  $(i, j)$ th element is  $G_{ij}(z)$ .
- ▶ Let

$$\Psi(z) = \text{diag}(\psi_1(z), \dots, \psi_N(z)) + \mathbf{Q} \circ G(z),$$

(when it exists), where  $\circ$  indicates elementwise multiplication.

- ▶ The matrix exponent of the MAP  $(\xi, J)$  is given by

$$\mathbf{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = (e^{\Psi(z)t})_{i,j}, \quad i, j \in E,$$

(when it exists).



## DUAL MAP

- ▶ Thanks to irreducibility, the Markov chain  $J$  necessarily has a stationary distribution. We denote it by the vector  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_N)$ .
- ▶ Each MAP has a dual process, also a MAP, with probabilities  $\hat{\mathbf{P}}_{x,i}$ ,  $x \in \mathbb{R}$ ,  $i \in E$ , determined by the dual characteristic matrix exponent (when it exists),

$$\hat{\Psi}(z) := \text{diag}(-\Psi_1(-z), \dots, -\Psi_N(-z)) + \hat{Q} \circ G(-z)^T,$$

where  $\hat{Q}$  is the time-reversed Markov chain  $J$ ,

$$\hat{q}_{i,j} = \frac{\pi_j}{\pi_i} q_{j,i}, \quad i, j \in E.$$

Note that the latter can also be written  $\hat{Q} = \Delta_{\boldsymbol{\pi}}^{-1} Q^T \Delta_{\boldsymbol{\pi}}$ , where  $\Delta_{\boldsymbol{\pi}} = \text{diag}(\boldsymbol{\pi})$ .

- ▶ When it exists,

$$\hat{\Psi}(z) = \Delta_{\boldsymbol{\pi}}^{-1} \Psi(-z)^T \Delta_{\boldsymbol{\pi}},$$

showing that

$$\pi_i \hat{\mathbf{E}}_{0,i} \left[ e^{iz\xi_t}, J_t = j \right] = \pi_j \mathbf{E}_{0,j} \left[ e^{-iz\xi_t}, J_t = i \right].$$

### Lemma

The time-reversed process  $((\xi_{(t-s)-} - \xi_t, J_{(t-s)-}), s \leq t)$  under  $\mathbf{P}_{0,\boldsymbol{\pi}}$  is equal in law to  $((\xi_s, J_s), s \leq t)$  under  $\hat{\mathbf{P}}_{0,\boldsymbol{\pi}}$ .

# LAMPERTI-KIU TRANSFORM

- ▶ Take  $J$  to be irreducible on  $E = \{1, -1\}$ .

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$$Z_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \quad 0 \leq t < T_0,$$

where

$$\tau(t) = \inf \left\{ s > 0 : \int_0^s \exp(\alpha\xi(u))du > t \right\}$$

and

$$T_0 = |x|^{-\alpha} \int_0^\infty e^{\alpha\xi(u)} du.$$

- ▶ Then  $Z_t$  is a real-valued self-similar Markov process in the sense that the law of  $(cZ_{tc^{-\alpha}} : t \geq 0)$  under  $\mathbb{P}_x$  is  $\mathbb{P}_{cx}$ .

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- ▶ The converse (within a special class of rsmMps) is also true.

## ENTRANCE AT ZERO

- ▶ Given the Lamperti-Kiu representation

$$Z_t = e^{\xi(\tau(|x|^{-\alpha}t)) + \log |x|} J(\tau(|x|^{-\alpha}t)) \quad 0 \leq t < T_0,$$

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- ▶ We need to construct a stationary version of the pair  $(\xi, J)$  which is indexed by  $\mathbb{R}$  and pinned at space-time point  $(-\infty, \infty)$ .
- ▶ Just like the theory of Lévy processes, by observing the range of the process  $(\xi_t, J_t)$   $t \geq 0$ , **only** at the points of its new suprema, we see a process  $(H_t^+, J_t^+)$ ,  $t \geq 0$ , which is also a MAP, where  $H^+$  is has increasing paths.

## ENTRANCE AT ZERO

### Theorem

Assume that  $Z$  is a conservative real self-similar Markov process. Moreover, suppose that the MAP  $((\xi, \Theta), \mathbf{P})$ , associated with  $Z$  through the Lamperti-Kiu transform, is such that  $\xi$  is not concentrated on a lattice and its ascending ladder height process  $H$  which satisfies  $\mathbf{E}_{0,\pi}(H_1) < \infty$ . Then  $\mathbb{P}_0 := \lim_{x \downarrow 0} \mathbb{P}_x$  exists, in the sense of convergence of on the Skorokhod space, under which  $Z$  leaves the origin continuously. Conversely, if  $\mathbf{E}_{0,\pi}(H_1) = \infty$ , then this limit does not exist. Under the additional assumption that  $\mathbf{E}_{0,\pi}(\xi_1) > 0$ , for any positive measurable function  $f$  and  $t > 0$ ,

$$\mathbb{E}_0(f(Z_t)) = \frac{1}{-\alpha \hat{\mathbf{E}}_{0,\pi}(\xi_1)} \sum_{i=\pm 1} \pi_i \hat{\mathbf{E}}_{0,i} \left( \frac{1}{I_\infty} f \left( i \left( \frac{t}{I_\infty} \right)^{1/\alpha} \right) \right), \quad (3)$$

where  $I_\infty = \int_0^\infty \exp\{\alpha \xi_s\} ds$ , and  $\hat{\mathbf{E}}_{x,i}$ ,  $x \in \mathbb{R}$ ,  $i = \pm 1$ .



## AN $\alpha$ -STABLE PROCESS IS A rSSMp

- ▶ An  $\alpha$ -stable process up to absorption in the origin is a rSSMp.
- ▶ When  $\alpha \in (0, 1]$ , the process never hits the origin a.s.

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- ▶ The matrix exponent of the underlying MAP is given by:

$$\begin{bmatrix} -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{bmatrix},$$

for  $\text{Re}(z) \in (-1, \alpha)$ . Note a matrix  $A$  in this context is arranged with the ordering

$$\begin{pmatrix} A_{1,1} & A_{1,-1} \\ A_{-1,1} & A_{-1,-1} \end{pmatrix}.$$

# ESSCHER TRANSFORM FOR MAPs

- ▶ If  $\Psi(z)$  is well defined then it has a real simple eigenvalue  $\chi(z)$ , which is larger than the real part of all its other eigenvalues.
- ▶ Furthermore, the corresponding right-eigenvector  $\mathbf{v}(z) = (v_1(z), \dots, v_N(z))$  has strictly positive entries and may be normalised such that  $\pi \cdot \mathbf{v}(z) = 1$ .

## Theorem

Let  $\mathcal{G}_t = \sigma\{(\xi(s), J(s)) : s \leq t\}$ ,  $t \geq 0$ , and

$$M_t := e^{\gamma \xi(t) - \chi(\gamma)t} \frac{v_{J(t)}(\gamma)}{v_i(\gamma)}, \quad t \geq 0,$$

for some  $\gamma \in \mathbb{R}$  such that  $\chi(\gamma)$  is defined. Then,  $M_t$ ,  $t \geq 0$ , is a unit-mean martingale. Moreover, under the change of measure

$$d\mathbf{P}_{0,i}^\gamma \Big|_{\mathcal{G}_t} = M_t d\mathbf{P}_{0,i} \Big|_{\mathcal{G}_t}, \quad t \geq 0,$$

the process  $(\xi, J)$  remains in the class of MAPs with new exponent given by

$$\Psi_\gamma(z) = \Delta_v(\gamma)^{-1} \Psi(z + \gamma) \Delta_v(\gamma) - \chi(\gamma) \mathbf{I}.$$

Here,  $\mathbf{I}$  is the identity matrix and  $\Delta_v(\gamma) = \text{diag}(v(\gamma))$ .

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Exercises.  
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## ESSCHER AND DRIFT

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- ▶ With all of the above

$$\lim_{t \rightarrow \infty} \frac{\xi_t}{t} = \chi'(0) \quad \text{a.s.}$$

# ESSCHER AND THE STABLE-MAP

- For the MAP that underlies the stable process  $D = (-1, \alpha)$ , it can be checked that  $\det \Psi(\alpha - 1) = 0$  i.e.  $\chi(\alpha - 1) = 0$ , which makes

$$\Psi^\circ(z) = \Delta^{-1} \Psi(z + \alpha - 1) \Delta$$

$$= \begin{bmatrix} -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\rho-z)\Gamma(\alpha\rho+z)} & \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} \\ \frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} & -\frac{\Gamma(1-z)\Gamma(\alpha+z)}{\Gamma(1-\alpha\hat{\rho}-z)\Gamma(\alpha\hat{\rho}+z)} \end{bmatrix},$$

where  $\Delta = \text{diag}(\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$ .



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- ▶ When  $\alpha \in (1, 2)$ ,  $\chi'(0) < 0$  (because the stable process touches the origin a.s.) and  $\Psi^\circ(z)$ -MAP drifts to  $+\infty$ .

## RIESZ-BOGDAN-ZAK TRANSFORM

### Theorem (Riesz–Bogdan–Zak transform)

Suppose that  $X$  is an  $\alpha$ -stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}, \quad t \geq 0.$$

Then, for all  $x \in \mathbb{R} \setminus \{0\}$ ,  $(-1/X_{\eta(t)})_{t \geq 0}$  under  $\mathbb{P}_x$  is equal in law to  $(X, \mathbb{P}_{-1/x}^\circ)$ , where

$$\frac{d\mathbb{P}_x^\circ}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \left( \frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\operatorname{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\operatorname{sgn}(x)} \right) \left| \frac{X_t}{x} \right|^{\alpha-1} \mathbf{1}_{(t < \tau\{0\})}$$

and  $\mathcal{F}_t := \sigma(X_s : s \leq t)$ ,  $t \geq 0$ . Moreover, the process  $(X, \mathbb{P}_x^\circ)$ ,  $x \in \mathbb{R} \setminus \{0\}$  is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by  $\Psi^\circ(z)$ .

## WHAT IS THE $\Psi^\circ$ -MAP?

Thinking of the affect on the long term behaviour of the underlying MAP of the Esscher transform

- ▶ When  $\alpha \in (0, 1)$ ,  $(X, \mathbb{P}_x^\circ)$ ,  $x \neq 0$  has the law of the the stable process conditioned to absorb continuously at the origin in the sense,

$$\mathbb{P}_y^\circ(A) = \lim_{a \rightarrow 0} \mathbb{P}_y(A, t < T_0 \mid \tau_{(-a, a)} < \infty),$$

for  $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$ ,

$\tau_{(-a, a)} = \inf\{t > 0 : |X_t| < a\}$  and  $T_0 = \inf\{t > 0 : X_t = 0\}$ .

- ▶ When  $\alpha \in (1, 2)$ ,  $(X, \mathbb{P}_x^\circ)$ ,  $x \neq 0$  has the law of the stable process conditioned to avoid the origin in the sense

$$\mathbb{P}_y^\circ(A) = \lim_{s \rightarrow \infty} \mathbb{P}_y(A \mid T_0 > t + s),$$

for  $A \in \mathcal{F}_t = \sigma(X_s, s \leq t)$  and  $T_0 = \inf\{t > 0 : X_t = 0\}$ .

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Exercises.

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## §Exercise Set 1

## EXERCISES

1. Suppose that  $X$  is a stable process in any dimension (including the case of a Brownian motion). Show that  $|X|$  is a positive self-similar Markov process.
2. Suppose that  $B$  is a one-dimensional Brownian motion. Prove that

$$\frac{B_t}{x} \mathbf{1}_{(B_t > 0)}, \quad t \geq 0,$$

is a martingale, where  $\underline{B}_t = \inf_{s \leq t} B_s$ .

3. Suppose that  $X$  is a stable process with two-sided jumps
  - ▶ Show that the range of the ascending ladder process  $H$ , say  $\text{range}(H)$  has the property that it is equal in law to  $c \times \text{range}(H)$ .
  - ▶ Hence show that, up to a multiplicative constant, the Laplace exponent of  $H$  satisfies  $k(\lambda) = \lambda^{\alpha_1}$  for  $\alpha_1 \in (0, 1)$  (and hence the ascending ladder height process is a stable subordinator).
  - ▶ Use the fact that, up to a multiplicative constant

$$\Psi(z) = |\theta|^\alpha (e^{\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{(\theta > 0)} + e^{-\pi i \alpha (\frac{1}{2} - \rho)} \mathbf{1}_{(\theta < 0)}) = \hat{\kappa}(iz) \kappa(-iz)$$

to deduce that

$$\kappa(\theta) = \theta^{\alpha \rho} \text{ and } \hat{\kappa}(\theta) = \theta^{\alpha \hat{\rho}}.$$

and that  $0 < \alpha \rho, \alpha \hat{\rho} < 1$

- ▶ What kind of process corresponds to the case that  $\alpha \rho = 1$ ?

# EXERCISES

- 4. Suppose that  $(X, P_x), x > 0$  is a positive self-similar Markov process and let  $\zeta = \inf\{t > 0 : X_t = 0\}$  be the lifetime of  $X$ . Show that  $P_x(\zeta < \infty)$  does not depend on  $x$  and is either 0 for all  $x > 0$  or 1 for all  $x > 0$ .
- 5. Suppose that  $X$  is a symmetric stable process in dimension one (in particular  $\rho = 1/2$ ) and that the underlying Lévy process for  $|X_t|\mathbf{1}_{(t < \tau_{\{0\}})}$ , where  $\tau^{\{0\}} = \inf\{t > 0 : X_t = 0\}$ , is written  $\xi$ . (Note the indicator is only needed when  $\alpha \in (1, 2)$  as otherwise  $X$  does not hit the origin.) Show that (up to a multiplicative constant) its characteristic exponent is given by

$$\Psi(z) = 2^\alpha \frac{\Gamma(\frac{1}{2}(-iz + \alpha))}{\Gamma(-\frac{1}{2}iz)} \frac{\Gamma(\frac{1}{2}(iz + 1))}{\Gamma(\frac{1}{2}(iz + 1 - \alpha))}, \quad z \in \mathbb{R}.$$

[Hint!] Think about what happens after  $X$  first crosses the origin and apply the Markov property as well as symmetry. You will need to use the law of the overshoot of  $X$  below the origin given a few slides back.

## EXERCISES

6. Use the previous exercise to deduce that the MAP exponent underlying a stable process with two sided jumps is given by

$$\left[ \begin{array}{cc} -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\frac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{array} \right],$$

for  $\text{Re}(z) \in (-1, \alpha)$ .