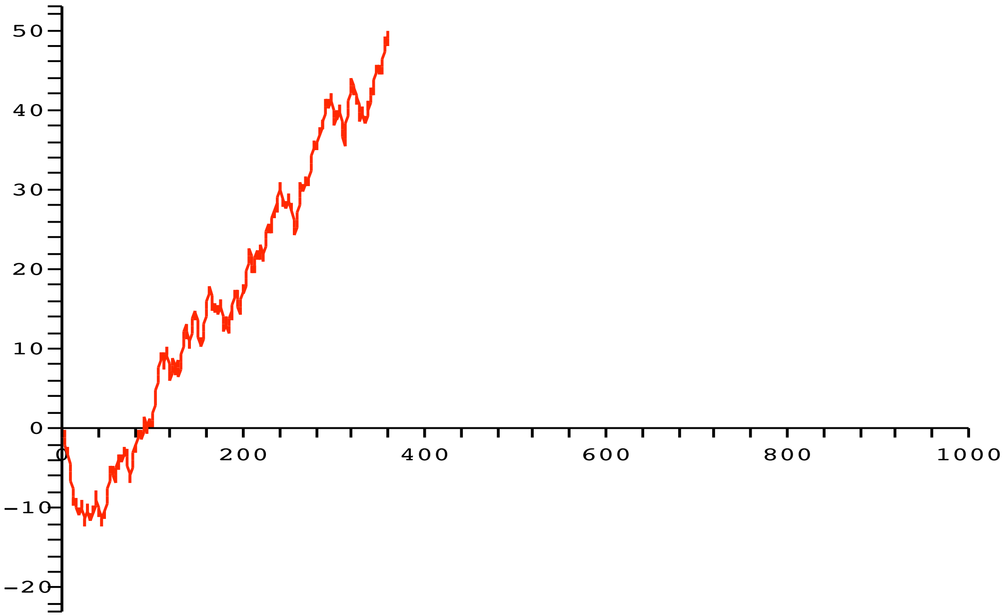
# *Path Decomposition of Markov Processes*

Götz Kersting

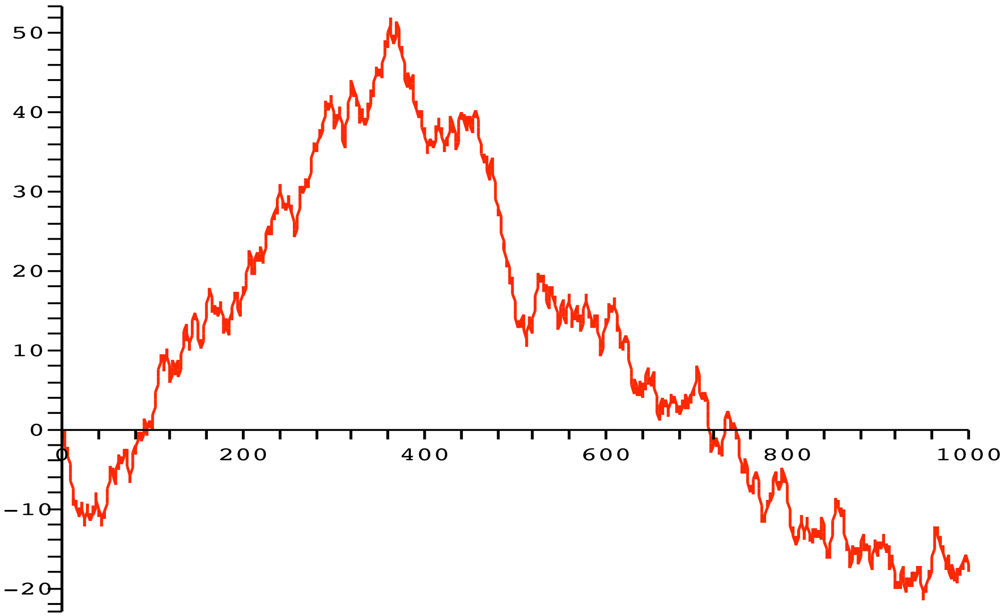
University of Frankfurt/Main

joint work with Kaya Memisoglu, Jim Pitman

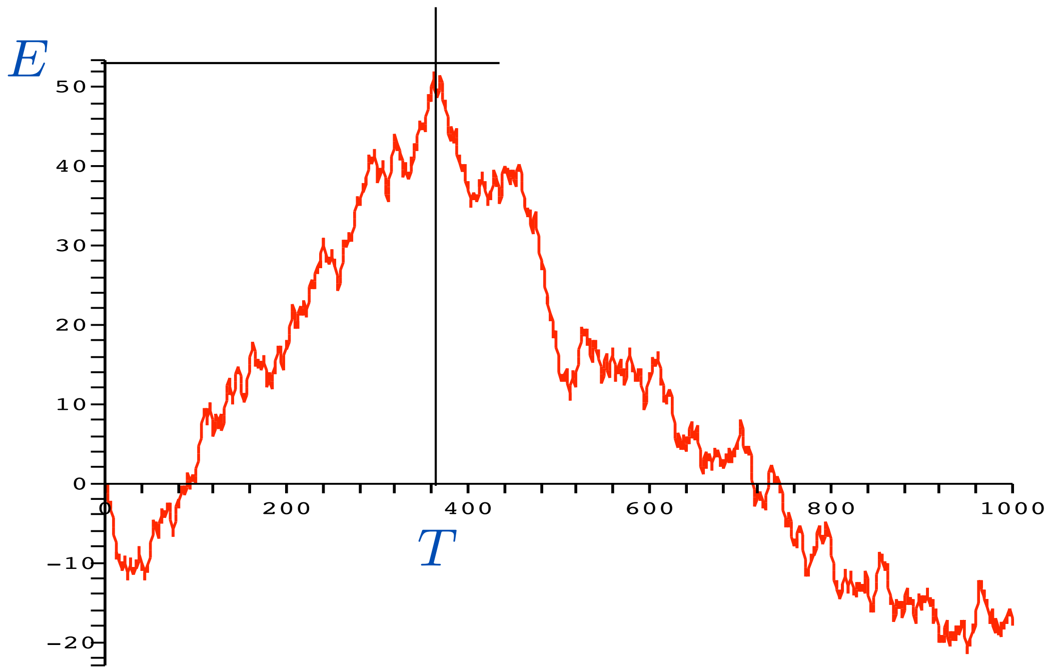
*A Brownian path with positive drift*



*from a path with negative drift*



*Notation:*



*First half of David Williams' theorem.*

**Theorem.** *Let* $X = (X_t)$ *be a BM starting at 0 with negative drift, say* $-b$*, and let*

$$T \ := \ \text{sup}\{t \geq 0 : X_s < X_t \text{ for all } s < t\}$$

*be the moment, when it takes its maximum.*

*Also let* $X' = (X'_t)$ *be a BM starting at 0 with positive drift* $b$ *and let* $E'$ *be an independent exponential random variable with expectation* $1/2b$*. Define the hitting time of* $E'$

$$\tau' \ := \ \text{inf}\{t \geq 0 : X'_t = E'\}.$$

*Then*

$$(X_t)_{t<T} \quad \text{and} \quad (X'_t)_{t<\tau'}$$

*are equal in distribution.*

*A generalization.*

Let the stochastic process $X$ with values in $S \subset \mathbb{R}^d$ start at $X_0 = x$ and obey the equation

$$dX \;=\; dW + b(X)\,dt \;,$$

where $W$ is a $d$-dimensional standard BM. Let

$$h : S \to \mathbb{R}^+$$

be *harmonic*, i.e. solve the equation

$$\nabla h \cdot b + \frac{1}{2}\Delta h \;=\; 0 \;.$$

Let

$$T \;:=\; \sup\{t \geq 0 : h(X_s) < h(X_t) \text{ for all } s < t\}$$

be the moment, when $h(X_t)$ takes its maximum for the first time.

*Continuation.*

Also consider the process $X'$ given by

$$dX' \;=\; dW + \left[b(X') + \frac{1}{h(X')}\nabla h(X')\right] dt$$

and the hitting time

$$\tau' \;:=\; \inf\left\{t \geq 0 : h(X'_t) \;=\; \frac{h(x)}{U}\right\} ,$$

where $U$ is an independent r.v. with uniform distribution in $[0, 1]$.

**Theorem.**

$$(X)_{t<T} \quad \textit{and} \quad (X'_t)_{t<\tau'}$$

*are equal in distribution.*

*The second half of the process.*

Also consider the process

$$dX'' \; = \; dW + \left[ b(X'') - \frac{1}{m - h(X'')} \nabla h(X'') \right] dt \; .$$

**Theorem.** *Given* $h(X_T) = m$ *and* $X_0'' = X_T$

$$(X_{t+T})_{t \geq 0} \quad \textit{and} \quad (X_t'')_{t \geq 0}$$

*are equal in distribution.*

**Thus:**

$X$ is first pushed into the direction, where $h$ takes its supremum, and then with a sudden kick into the opposite direction.

*Doob-transforms.*

Now let $X = (X_t)_{t<\zeta}$ denote a strong Markov process with lifetime $\zeta$, right continuous paths in a locally compact state space $S$ with countable base and probabilities $\mathbf{P}_x$. For convenience let $\zeta = \infty$ $\mathbf{P}_x$-$a.s.$

Further let

$$h : S \to \mathbb{R}^+$$

be such that $h(X_t)$ is cadlag. The Doob-transform is the collection of measures given by

$$\mathbf{Q}_x\{A\} \;:=\; \frac{1}{h(x)}\mathbf{E}_x[h(X_t); A] \quad \text{with} \quad A \in \sigma(X_x, s \leq t) \ ,$$

provided that $h$ is an exzessive function.

*Harmonic functions.*

$h$ is called *harmonic*, if it fulfils for all $t, C$ the mean value property

$$h(x) \;=\; \mathbf{E}_x[h(X_{t\wedge\sigma(C)})]\ ,$$

where $\sigma(C)$ denotes the exit time of $X$ from the compact subset $C \subset S$. Let $\partial$ denote a coffin state.

**Proposition.** *Let $h$ be excessive. Then the following statements are equivalent:*

*i) $h$ is harmonic,*

*ii) $X_{\zeta-} = \partial$ $\mathbf{Q}_x$-$a.s.$ on the event $\zeta < \infty$ for all $x$.*

**Thus:**

$h$ is harmonic, iff killing of $X$ cannot occur by a jump to $\partial$ under $\mathbf{Q}_x$.

*Processes with continuous paths.*

Again let

$$T \;:=\; \sup\{t \geq 0 : h(X_s) < h(X_t) \text{ for all } s < t\}$$

and

$$\tau \;:=\; \inf\left\{t \geq 0 : h(X_t) = \frac{h(X_0)}{U}\right\}$$

with independent $U$, uniform in $[0,1]$.

**Theorem.** *Let $X$ have continuous paths (or more generally $h(X)$ upwards skipfree), then*

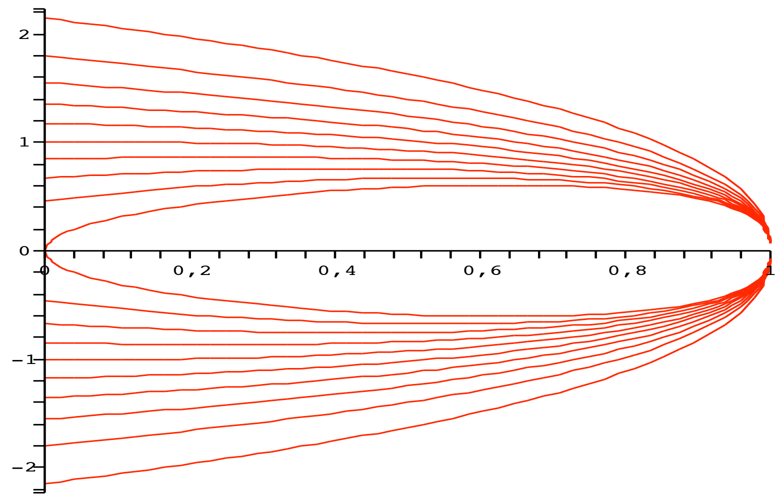
$$\mathcal{L}_{\mathbf{P}_x}\Big[(X_t)_{t<T}\Big] \;=\; \mathcal{L}_{\mathbf{Q}_x}\Big[(X_t)_{t<\tau}\Big] \;.$$

In particular, $T$ coinsides in distribution with a hitting time.

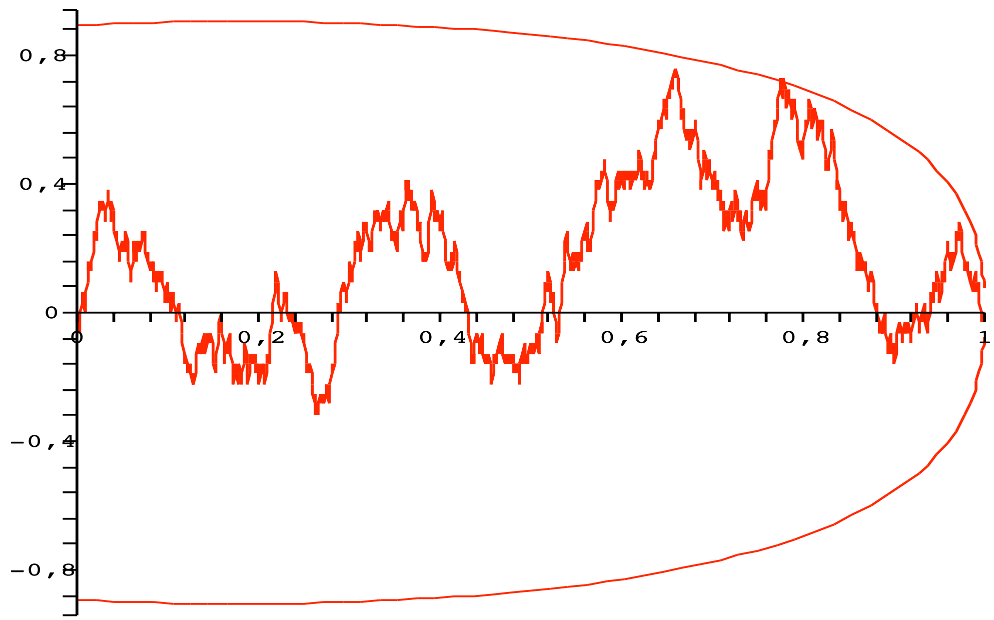
*Example: Brownian Bridge (space-time harmonic function).*

$$h(x,t) \;:=\; \sqrt{1-t}\,\exp\left(x^2/2(1-t)\right)$$

Levellines:



Choose a random levelline according to $h(x,t) = 1/U$. Start with a standard BM, till it hits the line.



*Markov chains.*

Let $(X_n)$ be a discrete time Markov chain with general state space $S$ and transition kernel $P(x, dy)$, and let

$$h : S \to \mathbb{R}^+$$

be harmonic, i.e. $Ph = h$. Then the $h$-transform is given by the kernel

$$Q(x, dy) \ := \ \frac{1}{h(x)} P(x, dy) h(y) \ .$$

Matters seem easier.

Why not replace $\tau$ here by

$$\tau_w \;:=\; \min\Big\{n \geq 0 : h(X_n) \geq \frac{h(x)}{U}\Big\} \text{ ?}$$

But: $\tau_w \;=\; \tau_{wrong}$ !

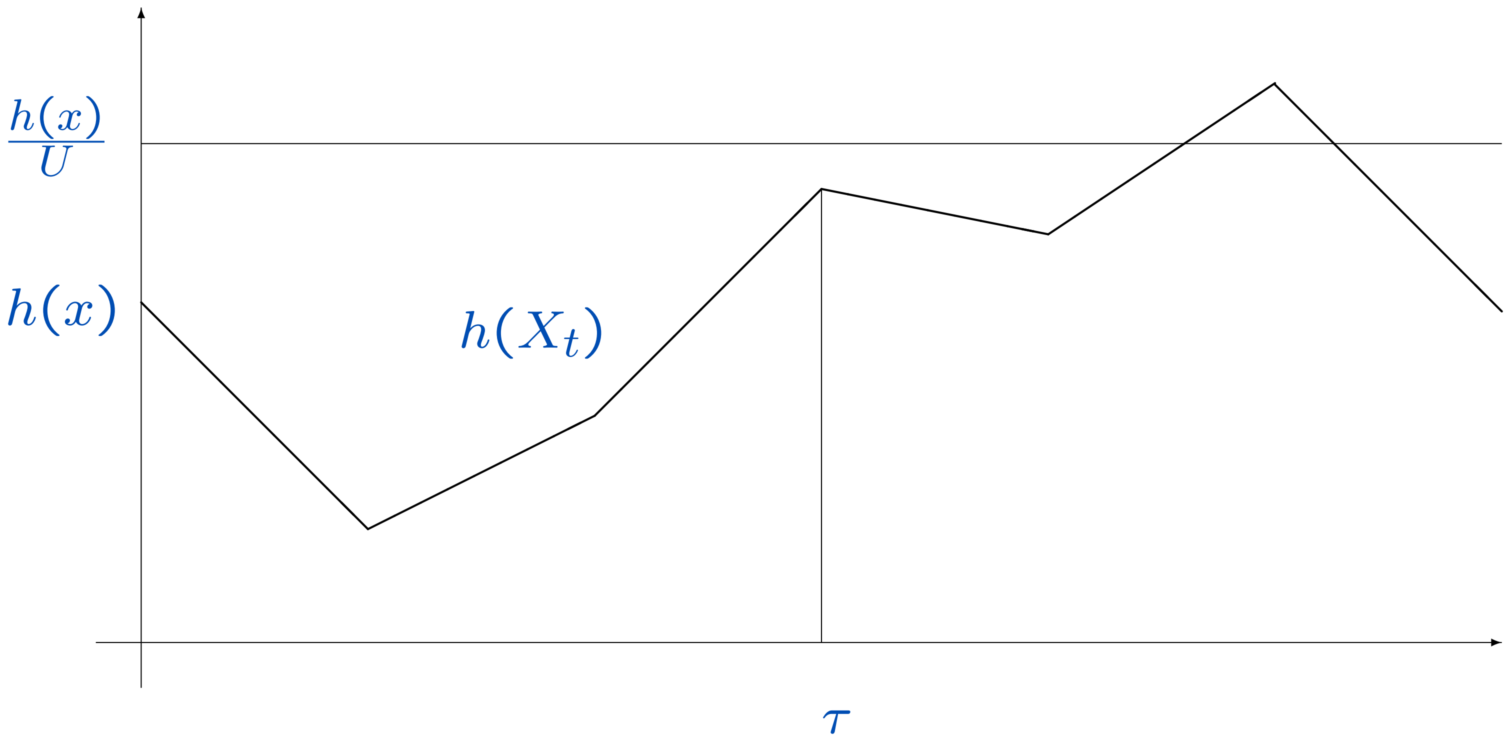
Namely with this choice:

$$\mathbf{Q}_x\Big\{h(X_{\tau_w}) \geq y\Big\} \;\geq\; \mathbf{Q}_x\Big\{\frac{h(x)}{U} \geq y\Big\} \;=\; \frac{h(x)}{y}$$

whereas by Doob's inequality

$$\mathbf{P}_x\{h(X_T) \geq y\} \;=\; \mathbf{P}_x\{h(X_{\tau_w}) \geq y\} \;\leq\; \frac{h(x)}{y} \;.$$

*The right choice:*



Thus choose $\tau$ as the moment, when $h(X_n)$ reaches its maximum (for the first time), before $h(x)/U$ is surpassed,

$$\tau \;:=\; \max\Big\{n \geq 0 : h(X_m) < h(X_n) < \frac{h(x)}{U} \text{ for all } m < n\Big\}$$

Then

**Theorem.** *For a Markov chain*

$$\mathcal{L}_{\mathbf{P}_x}\Big[(X_n)_{n \leq T}\Big] \;=\; \mathcal{L}_{\mathbf{Q}_x}\Big[(X_n)_{n \leq \tau}\Big] \;.$$

*The general result for cadlag paths.*

Here we have to consider

$$T \;:=\; \sup\{t \geq 0 : h(X_s) < h(X_t) \vee h(X_{t-}) \text{ for all } s < t\}$$

and,

$$\tau \;:=\; \sup\left\{t \geq 0 : \\ h(X_s) < h(X_t) \vee h(X_{t-}) < \frac{h(X_0)}{U} \text{ for all } s < t\right\}$$

This is the time of last maximum, before $h(X_0)/U$ is surpassed. Note that in contrast to $T$ the value of $\tau$ may be settled in finite time.

*Millar's theorem*

**Theorem.** Given $(X_t)_{t \le T}$ and given that $h(X_T) \vee h(X_{T-}) = m$, the process $(X_{T+t})_{t>0}$ is strong Markov under $\mathbf{P}_x$. Its marginal distributions form an entrance law on $\{x \in S : h(x) \le m\}$ with respect to the transition kernel

$$Q_t^m(x, dy) \;:=\; \mathbf{P}_x\{X_t \in dy | \sup_s h(X_s) \le m\}.$$

The statement seems obvious, but the proof is profound.