Deep factorisation and amplitudal reflection of the stable process[1]

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[1] Partly based on joint work with Victor Rivero and Batı Şengül

Stable processes

Definition

A Lévy process X is called (strictly) α-stable if it satisfies the scaling property

$$(cX_{c^{-\alpha}t})_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \quad c>0.$$

Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
The quantity $\rho = \mathrm{P}_0(X_t \geq 0) \in [0,1]$ will frequently appear as will $\hat{\rho} = 1-\rho$.

- The characteristic exponent $\Psi(\theta) := -t^{-1}\log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta\in\mathbb{R}.$$

- Assume jumps in both directions i.e. $\alpha\rho, \alpha\hat{\rho} \in (0,1)$

The Wiener–Hopf factorisation

- For a given characteristic exponent of a Lévy process, Ψ, there exist unique Bernstein functions, κ and $\hat{\kappa}$ such that, up to a multiplicative constant,

$$\Psi(\theta) = \hat{\kappa}(\mathrm{i}\theta)\kappa(-\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

- As Bernstein functions, κ and $\hat{\kappa}$ can be seen as the Laplace exponents of (killed) subordinators.
- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of X and of $-X$ respectively.

The Wiener–Hopf factorisation

- Explicit Wiener-Hopf factorisations are extremely rare!
- For the stable processes we are interested in we have

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0$$

  where $0 < \alpha\rho, \alpha\hat{\rho} < 1$.
- Hypergeometric Lévy processes are another recently discovered family of Lévy processes for which the factorisation are known explicitly: For appropriate parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)}.$$

Deep factorisation of the stable process

- Another factorisation also exists, which is more 'deeply' embedded in the stable process.
- Based around the representation of the stable process as a real-valued self-similar Markov process (rssMp):

An $\mathbb{R}$-valued regular strong Markov process $(X_t : t \geq 0)$ with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$, is a rssMp if, there is a stability index $\alpha > 0$ such that, for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

Markov additive processes (MAPs)

- $E = \{-1, 1\}$
- $(J(t))_{t \geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}, i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
    - $(\xi(t+s) - \xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
    - $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n \geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n \geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n \geq 1}$ are the jump times of J,

the process ξ has the representation

$$\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, -1\}$.
- Let

$$X_t = |x|e^{\xi(\tau(t))}J(\tau(t)) \qquad 0 \leq t < T_0,$$

  where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t|x|^{-\alpha}\right\}$$

  and

$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$

- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \geq 0)$ under P_x is P_{cx}.
- The converse (within a special class of rssMps) is also true.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\boldsymbol{Q} = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} **(when it exists)**.
- Write $G(z)$ for the 2×2 matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{F}(z) = \mathrm{diag}(\psi_1(z), \psi_{-1}(z)) + \boldsymbol{Q} \circ G(z),$$

(when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{F}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

(when it exists).

An α-stable process is a rssMp

- An α-stable process is a rssMp. Remarkably (thanks to work of Chaumont, Panti and Rivero) we can compute precisely its matrix exponent explicitly
- Denote the underlying MAP (ξ, J), we prefer to give the matrix exponent of (ξ, J) as follows:

$$\boldsymbol{F}(z) = \begin{bmatrix} -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

Ascending ladder MAP

- Observe the process (ξ, J) only at times of increase of new maxima of ξ. This gives a MAP, say $(H^+(t), J^+(t))_{t\geq 0}$, with the property that H is non-decreasing with the same range as the running maximum.
- Its exponent can be identified by $-\boldsymbol{\kappa}(-z)$, where

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \boldsymbol{\Lambda} \circ \boldsymbol{K}(\lambda), \qquad \lambda \geq 0.$$

- Here, for $i = 1, -1$, Φ_i are Bernstein functions (exponents of subordinators), $\boldsymbol{\Lambda} = (\Lambda_{i,j})_{i,j\in E}$ is the intensity matrix of J^+ and $\boldsymbol{K}(\lambda)_{i,j} = \mathbf{E}[\mathrm{e}^{-\lambda U^+_{i,j}}]$, where $U^+_{i,j} \geq 0$ are the additional discontinuities added to the path of ξ each time the chain J^+ switches from i to j, and $U^+_{i,i} := 0$, $i \in E$.

MAP WHF

Theorem

For $\theta \in \mathbb{R}$, up to a multiplicative factor,

$$-\boldsymbol{F}(\mathrm{i}\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \hat{\boldsymbol{\kappa}}(\mathrm{i}\theta)^{\mathsf{T}} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\kappa}(-\mathrm{i}\theta),$$

where $\boldsymbol{\Delta}_{\boldsymbol{\pi}} = diag(\boldsymbol{\pi})$, $\boldsymbol{\pi}$ is the stationary distribution of $\boldsymbol{Q}$, $\hat{\boldsymbol{\kappa}}$ plays the role of $\boldsymbol{\kappa}$, but for the dual MAP to (ξ, J).

The dual process, or time-reversed process is equal in law to the MAP with exponent

$$\hat{\boldsymbol{F}}(z) = \boldsymbol{\Delta}_{\pi}^{-1} \boldsymbol{F}(-z)^{\mathsf{T}} \boldsymbol{\Delta}_{\pi},$$

$\alpha \in (0,1]$

Define the family of Bernstein functions

$$\kappa_{q+i,p+j}(\lambda) := \int_0^\infty (1-\mathrm{e}^{-\lambda x})\frac{((q+i)\vee(p+j)-1)}{(1-\mathrm{e}^{-x})^{q+i}(1+\mathrm{e}^{-x})^{p+j}}\mathrm{e}^{-\alpha x}\mathrm{d}x,$$

where $q, p \in \{\alpha\rho, \alpha\hat{\rho}\}$ and $i, j \in \{0,1\}$ such that $q+p=\alpha$ and $i+j=1$.

Deep Factorisation $\alpha \in (0,1]$

Theorem

Fix $\alpha \in (0,1]$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\boldsymbol{\kappa}$, is given by

$$\begin{bmatrix} \kappa_{\alpha\rho+1,\alpha\hat\rho}(\lambda) + \frac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\kappa'_{\alpha\hat\rho,\alpha\rho+1}(0+) & -\frac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\frac{\kappa_{\alpha\hat\rho,\alpha\rho+1}(\lambda)}{\lambda} \\ -\frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\frac{\kappa_{\alpha\rho,\alpha\hat\rho+1}(\lambda)}{\lambda} & \kappa_{\alpha\hat\rho+1,\alpha\rho}(\lambda) + \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\kappa'_{\alpha\rho,\alpha\hat\rho+1}(0+) \end{bmatrix}$$

Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\boldsymbol{\kappa}}$ is given by

$$\begin{bmatrix} \kappa_{\alpha\hat\rho+1,\alpha\rho}(\lambda+1-\alpha) + \frac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\kappa'_{\alpha\rho,\alpha\hat\rho+1}(0+) & -\frac{\kappa_{\alpha\rho,\alpha\hat\rho+1}(\lambda+1-\alpha)}{\lambda+1-\alpha} \\ -\frac{\kappa_{\alpha\hat\rho,\alpha\rho+1}(\lambda+1-\alpha)}{\lambda+1-\alpha} & \kappa_{\alpha\rho+1,\alpha\hat\rho}(\lambda+1-\alpha) + \frac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\kappa'_{\alpha\hat\rho,\alpha\rho+1}(0+) \end{bmatrix}$$

$\alpha \in (1,2)$

Define the family of Bernstein functions by

$$\phi_{q+i,p+j}(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \left\{ \frac{((q+i) \vee (p+j) - 1)}{(1-e^{-u})^{q+i}(1+e^{-u})^{p+j}} - \frac{(\alpha - 1)}{2(1-e^{-u})^{q}(1+e^{-u})^{p}} \right\} e^{-u} du,$$

for $q, p \in \{\alpha\rho, \alpha\hat{\rho}\}$ and $i, j \in \{0, 1\}$ such that $q + p = \alpha$ and $i + j = 1$.

Deep Factorisation $\alpha \in (1,2)$

Theorem

Fix $\alpha \in (1,2)$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\boldsymbol{\kappa}$, is given by

$$\begin{bmatrix} \begin{matrix}\sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda+\alpha-1) \\ +\sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+)\end{matrix} & -\sin(\pi\alpha\hat{\rho})\dfrac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda+\alpha-1)}{\lambda+\alpha-1} \\ & \\ -\sin(\pi\alpha\rho)\dfrac{\phi_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda+\alpha-1)}{\lambda+\alpha-1} & \begin{matrix}\sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda+\alpha-1) \\ +\sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+)\end{matrix} \end{bmatrix}$$

for $\lambda \geq 0$.

Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\boldsymbol{\kappa}}$, is given by

$$\begin{bmatrix} \sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda)+\sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+) & -\sin(\pi\alpha\hat{\rho})\dfrac{\phi_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda)}{\lambda} \\ -\sin(\pi\alpha\rho)\dfrac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda)}{\lambda} & \sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda)+\sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+) \end{bmatrix}$$

for $\lambda \geq 0$.

Comments $\alpha \in (0,1]$

Recall that

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \begin{bmatrix} -\Lambda_{1,-1} & \Lambda_{1,-1}\int \mathrm{e}^{-\lambda x} F^{+}_{1,-1}(\mathrm{d}x) \\ \Lambda_{-1,1}\int \mathrm{e}^{-\lambda x} F^{+}_{-1,1}(\mathrm{d}x) & -\Lambda_{-1,1} \end{bmatrix}$$

In general, we can write

$$\Phi_i(\lambda) = \int_0^\infty (1-\mathrm{e}^{-\lambda x}) n_i(\varepsilon_\zeta \in \mathrm{d}x, J(\zeta) = i, \zeta < \infty),$$

where $\zeta = \inf\{s \geq 0 : \varepsilon(s) > 0\}$ for the canonical excursion ε of the Markov process $(\sup_{s\leq t}\xi_s - \xi_t, J_t)$, $t \geq 0$, from $(0,i)$, $i = \pm 1$. The measures n_i, $i = \pm 1$ are the excursion measures in the Cox process of excursions.

Comments $\alpha \in (0,1]$

Lemma

Let $T_a = \inf\{t > 0 : \xi(t) > a\}$. *Suppose that* $\limsup_{t\to\infty} \xi(t) = \infty$ *(i.e. the ladder height process* (H^+, J^+) *does not experience killing). Then for* $x > 0$ *we have up to a constant*

$$\lim_{a\to\infty} \mathbf{P}_{0,i}(\xi(T_a) - a \in \mathrm{d}x, J(T_a) = 1)$$
$$= \Big[\pi_1 n_1(\varepsilon(\zeta) > x, J(\zeta) = 1, \zeta < \infty) + \pi_{-1}\Lambda_{-1,1}(1 - F^+_{-1,1}(x))\Big]\mathrm{d}x.$$

- (π_{-1}, π_1) is easily derived by solving $\boldsymbol{\pi}\boldsymbol{Q} = 0$.
- We can work with the LHS in the above lemma e.g. via

$$\lim_{a\to\infty} \mathbf{P}_{0,1}(\xi(T_a) - a > u, J(T_a) = 1)$$
$$= \lim_{a\to\infty} \mathbb{P}_{e^{-a}}(X_{\tau_1^+ \wedge \tau_{-1}^-} > e^u;\ \tau_1^+ < \tau_{-1}^-).$$

Inverse Deep Factors

Would it be more natural to consider the factorisation

$$-\boldsymbol{F}(\mathrm{i}\theta)^{-1} = \boldsymbol{\kappa}(-\mathrm{i}\theta)^{-1}\boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}[\hat{\boldsymbol{\kappa}}(\mathrm{i}\theta)^{-1}]^{\mathsf{T}}\boldsymbol{\Delta}_{\boldsymbol{\pi}}$$

for computational purposes?

Matrix potential

This may be less of a computational burden since

$$\int_0^\infty e^{-\lambda x} \boldsymbol{U}_{i,j}(\mathrm{d}x) = [\boldsymbol{\kappa}(\lambda)^{-1}]_{i,j} \qquad \text{for each } i,j = \pm 1,$$

where

$$\boldsymbol{U}_{i,j}(\mathrm{d}x) = \mathbf{P}_{0,i}\left[\int_0^\infty \mathbf{1}_{(H_t^+ \leq x)} \mathrm{d}t\right].$$

Matrix potential

Moreover, we have, for example when $\alpha \in (1,2)$, that, for $y > 0$ and $m_\infty = \sup\{t \geq 0 : \sup_{s \leq t} \xi_s = \xi_t\}$

$$\boldsymbol{U}_{1,1}(y) = \mathbf{P}_{-y,1}(\sup_{s \geq 0} \xi_s \leq 0; J_{m_\infty} = 1) = \mathbb{P}_{e^{-y}}(X_{M_\infty} \in (0,1)),$$

where

$$M_\infty = \sup\{t \leq \tau^0 : |X_t| = \sup_{s \leq t} |X_s|\}$$

and $\tau^0 = \inf\{t > 0 : X_s = 0\}$.

Theorem ($\alpha \in (0,1)$)

Up to the multiplicative constant $2^{-\alpha}\Gamma(1-\alpha)^{-1}$ The potential density are given by the following.

$$\boldsymbol{u}(x) = \begin{pmatrix} \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat\rho} & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat\rho-1} \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(1-e^{-x})^{\alpha\hat\rho}(1+e^{-x})^{\alpha\rho-1} & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(1-e^{-x})^{\alpha\hat\rho-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$

and

$$\hat{\boldsymbol{u}}(x) = \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat\rho)}(e^{x}-1)^{\alpha\hat\rho-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat\rho)\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat\rho)}(e^{x}-1)^{\alpha\hat\rho}(e^{x}+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat\rho)}{\sin(\alpha\pi\hat\rho)\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat\rho-1} & \frac{\Gamma(1-\alpha\hat\rho)}{\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat\rho} \end{pmatrix}$$

Theorem ($\alpha = 1$)

$$\boldsymbol{u}(x) = \hat{\boldsymbol{u}}(x) = \begin{pmatrix} (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} & (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} \\ (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} & (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} \end{pmatrix}.$$

Theorem ($\alpha \in (1,2)$)

Up to the multiplicative constant $(\alpha-1)/2$,

$$\boldsymbol{u}(x) = \begin{pmatrix} (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}} & (1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ (1-e^{-x})^{\alpha\hat{\rho}}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$
$$- \frac{(\alpha-1)}{(\lambda+\alpha-1)} \begin{pmatrix} (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1} & (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1} \end{pmatrix}.$$

$$\hat{\boldsymbol{u}}(x) = \begin{pmatrix} (e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat{\rho}-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}} \end{pmatrix}$$
$$- \frac{(\alpha-1)}{(\lambda+\alpha-1)} \begin{pmatrix} (e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho-1} & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}-1} \end{pmatrix}.$$

Theorem ($\alpha \in (0,1)$)

Suppose that X is an α-stable process then we have that the factors $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{\kappa}}$ are given as follows. For $a, b, c \in \mathbb{R}$ define

$$\Psi(a,b,c) := \int_0^1 u^a (1-u)^b (1+u)^c \mathrm{d}u. \tag{1}$$

Then, up to the multiplicative constant $2^{-\alpha}\Gamma(1-\alpha)^{-1}$,

$$\boldsymbol{\kappa}^{-1}(\lambda) = \begin{pmatrix} \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}\Psi(\lambda-1, \alpha\rho-1, \alpha\hat{\rho}) & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}\Psi(\lambda-1, \alpha\rho, \alpha\hat{\rho}-1) \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}\Psi(\lambda-1, \alpha\hat{\rho}, \alpha\rho-1) & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}\Psi(\lambda-1, \alpha\hat{\rho}-1, \alpha\rho) \end{pmatrix}$$

and

$$\hat{\boldsymbol{\kappa}}^{-1}(\lambda) = \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}\Psi(\lambda-\alpha, \alpha\hat{\rho}-1, \alpha\rho) & \frac{\sin(\alpha\pi\hat{\rho})\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat{\rho})}\Psi(\lambda-\alpha, \alpha\hat{\rho}, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat{\rho})}{\sin(\alpha\pi\hat{\rho})\Gamma(\alpha\rho)}\Psi(\lambda-\alpha, \alpha\rho, \alpha\hat{\rho}-1) & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}\Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat{\rho}) \end{pmatrix}.$$

Theorem ($\alpha \in (1,2)$)

$$\boldsymbol{\kappa}^{-1}(\lambda) = \frac{\alpha-1}{2}\begin{pmatrix} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat{\rho}) & \Psi(\lambda-1, \alpha\rho, \alpha\hat{\rho}-1) \\ \Psi(\lambda-1, \alpha\hat{\rho}, \alpha\rho-1) & \Psi(\lambda-1, \alpha\hat{\rho}-1, \alpha\rho) \end{pmatrix} - \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat{\rho}-1) & \Psi(\lambda-1, \alpha\rho-1, \alpha\hat{\rho}-1) \\ \Psi(\lambda-1, \alpha\hat{\rho}-1, \alpha\rho-1) & \Psi(\lambda-1, \alpha\hat{\rho}-1, \alpha\rho-1) \end{pmatrix}.$$

and

$$\hat{\boldsymbol{\kappa}}^{-1}(\lambda) = \frac{\alpha-1}{2}\begin{pmatrix} \Psi(\lambda-\alpha, \alpha\hat{\rho}-1, \alpha\rho) & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}\Psi(\lambda-\alpha, \alpha\hat{\rho}, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}\Psi(\lambda-\alpha, \alpha\rho, \alpha\hat{\rho}-1) & \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat{\rho}) \end{pmatrix} - \frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix} \Psi(\lambda-\alpha, \alpha\hat{\rho}-1, \alpha\rho-1) & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}\Psi(\lambda-\alpha, \alpha\hat{\rho}-1, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}\Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat{\rho}-1) & \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat{\rho}-1) \end{pmatrix}.$$

Application

Corollary

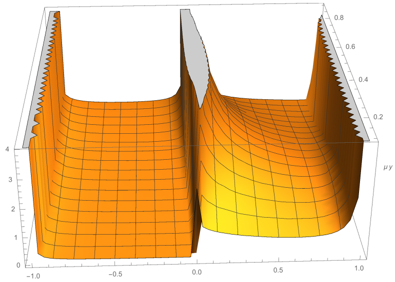
Suppose that $\alpha \in (0,1)$. *Let* $x \in (-1,1)$ *and let* R *be the process* $(X, \mathbb{P}_x)$ *reflected inside of the interval* $(-1,1)$, *i.e.*

$$R_t = \frac{X_t}{\max\{\sup_{s\le t}|X_s|, 1\}} \qquad t \ge 0.$$

Then R *has a stationary distribution* μ *given by*

$$\frac{\mu(\mathrm{d}y)}{\mathrm{d}y} = \frac{\pi_{\mathrm{sgn}(y)}}{y} \sum_{j=\pm 1} \hat{\boldsymbol{u}}_{\mathrm{sgn}(y),j}(-\log y)$$

where $\mathrm{sgn}(y)$ *is the sign of* y.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3
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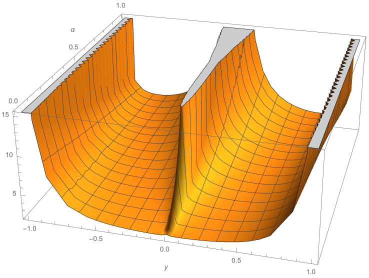
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α
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y

Thank you!