Deep factorisation of the stable process

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Stable processes

Definition

A Lévy process X is called (strictly) α-stable if it satisfies the scaling property

$$(cX_{c^{-\alpha}t})_{t\geq 0}\Big|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \quad c > 0.$$

Necessarily $\alpha \in (0,2]$. [$\alpha = 2 \to$ BM, exclude this.]
The quantity $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$.

- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi \mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

- Assume jumps in both directions.

The Wiener–Hopf factorisation

- For a given characteristic exponent of a Lévy process, Ψ, there exist unique Bernstein functions, κ and $\hat{\kappa}$ such that, up to a multiplicative constant,

$$\Psi(\theta) = \hat{\kappa}(\mathrm{i}\theta)\kappa(-\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

- As Bernstein functions, κ and $\hat{\kappa}$ can be seen as the Laplace exponents of (killed) subordinators.
- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of X and of $-X$ respectively.

The Wiener–Hopf factorisation

- Explicit Wiener-Hopf factorisations are extremely rare!
- For the stable processes we are interested in we have

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0$$

  where $0 < \alpha\rho, \alpha\hat{\rho} < 1$.
- Hypergeometric Lévy processes are another recently discovered family of Lévy processes for which the factorisation are known explicitly: For appropriate parameters $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$

$$\Psi(z) = \frac{\Gamma(1-\beta+\gamma-\mathrm{i}z)}{\Gamma(1-\beta-\mathrm{i}z)} \frac{\Gamma(\hat{\beta}+\hat{\gamma}+\mathrm{i}z)}{\Gamma(\hat{\beta}+\mathrm{i}z)}.$$

Deep factorisation of the stable process

- Another factorisation also exists, which is more 'deeply' embedded in the stable process.
- Based around the representation of the stable process as a real-valued self-similar Markov process (rssMp):

An $\mathbb{R}$-valued regular strong Markov process $(X_t : t \geq 0)$ with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$, is a rssMp if, there is a stability index $\alpha > 0$ such that, for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}$, $i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J,

the process ξ has the representation

$$\xi(t) = \mathbb{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.

rssMps, MAPs, Lamperti-Kiu (Chaumont, Panti, Rivero)

- Take the statespace of the MAP to be $E = \{1, -1\}$.
- Let

$$X_t = x e^{\xi(\tau(t))} J(\tau(t)) \qquad 0 \leq t < T_0,$$

  where

$$\tau(t) = \inf\left\{ s > 0 : \int_0^s \exp(\alpha\xi(u)) \mathrm{d}u > t|x|^{-\alpha} \right\}$$

  and

$$T_0 = |x|^{-\alpha} \int_0^\infty e^{\alpha\xi(u)} \mathrm{d}u.$$

- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \geq 0)$ under P_x is P_{cx}.
- The converse (within a special class of rssMps) is also true.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $Q = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let
$$F(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \circ G(z),$$
(when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by
$$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{F(z)t}\right)_{i,j}, \qquad i, j \in E,$$
(when it exists).

An α-stable process is a rssMp

- An α-stable process is a rssMp. Remarkably (thanks to work of Chaumont, Panti and Rivero) we can compute precisely its matrix exponent explicitly
- Denote the underlying MAP (ξ, J), we prefer to give the matrix exponent of (ξ, J) as follows:

$$F(z) = \begin{bmatrix} -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

Ascending ladder MAP

- Observe the process (ξ, J) only at times of increase of new maxima of ξ. This gives a MAP, say $(H^+(t), J^+(t))_{t\geq 0}$, with the property that H is non-decreasing with the same range as the running maximum.
- Its exponent can be identified by $-\boldsymbol{\kappa}(-z)$, where

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \cdots, \Phi_N(\lambda)) - \boldsymbol{\Lambda} \circ K(\lambda), \qquad \lambda \geq 0.$$

- Here, for $i = 1, \cdots, N$, Φ_i are Bernstein functions (exponents of subordinators), $\boldsymbol{\Lambda} = (\Lambda_{i,j})_{i,j\in E}$ is the intensity matrix of J^+ and $K(\lambda)_{i,j} = \mathbf{E}[\mathrm{e}^{-\lambda U^+_{i,j}}]$, where $U^+_{i,j} \geq 0$ are the additional discontinuities added to the path of ξ each time the chain J^+ switches from i to j, and $U^+_{i,i} := 0$, $i \in E$.

MAP WHF

Theorem

For $\theta \in \mathbb{R}$, up to a multiplicative factor,

$$-F(\mathrm{i}\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1}\hat{\boldsymbol{\kappa}}(\mathrm{i}\theta)^{\mathsf{T}}\boldsymbol{\Delta}_{\boldsymbol{\pi}}\boldsymbol{\kappa}(-\mathrm{i}\theta),$$

where $\boldsymbol{\Delta}_{\boldsymbol{\pi}} = diag(\boldsymbol{\pi})$, $\boldsymbol{\pi}$ is the stationary distribution of Q, $\hat{\boldsymbol{\kappa}}$ plays the role of $\boldsymbol{\kappa}$, but for the dual MAP to (ξ, J).

The dual process, or time-reversed process is equal in law to the MAP with exponent

$$\hat{F}(z) = \boldsymbol{\Delta}_{\pi}^{-1}F(-z)^{\mathsf{T}}\boldsymbol{\Delta}_{\pi},$$

$\alpha \in (0,1]$

Define the family of Bernstein functions

$$\kappa_{q+i,p+j}(\lambda) := \int_0^\infty (1 - e^{-\lambda x}) \frac{((q+i) \vee (p+j) - 1)}{(1-e^{-x})^{q+i}(1+e^{-x})^{p+j}} e^{-\alpha x} dx,$$

where $q, p \in \{\alpha\rho, \alpha\hat{\rho}\}$ and $i, j \in \{0, 1\}$ such that $q + p = \alpha$ and $i + j = 1$.

Deep Factorisation $\alpha \in (0,1]$

Theorem

Fix $\alpha \in (0,1]$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\boldsymbol{\kappa}$, is given by

$$\begin{bmatrix} \kappa_{\alpha\rho+1,\alpha\hat\rho}(\lambda) + \dfrac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\kappa'_{\alpha\hat\rho,\alpha\rho+1}(0+) & -\dfrac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\dfrac{\kappa_{\alpha\hat\rho,\alpha\rho+1}(\lambda)}{\lambda} \\ -\dfrac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\dfrac{\kappa_{\alpha\rho,\alpha\hat\rho+1}(\lambda)}{\lambda} & \kappa_{\alpha\hat\rho+1,\alpha\rho}(\lambda) + \dfrac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\kappa'_{\alpha\rho,\alpha\hat\rho+1}(0+) \end{bmatrix}$$

Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\boldsymbol{\kappa}}$ is given by

$$\begin{bmatrix} \kappa_{\alpha\hat\rho+1,\alpha\rho}(\lambda+1-\alpha) + \dfrac{\sin(\pi\alpha\rho)}{\sin(\pi\alpha\hat\rho)}\kappa'_{\alpha\rho,\alpha\hat\rho+1}(0+) & -\dfrac{\kappa_{\alpha\rho,\alpha\hat\rho+1}(\lambda+1-\alpha)}{\lambda+1-\alpha} \\ -\dfrac{\kappa_{\alpha\hat\rho,\alpha\rho+1}(\lambda+1-\alpha)}{\lambda+1-\alpha} & \kappa_{\alpha\rho+1,\alpha\hat\rho}(\lambda+1-\alpha) + \dfrac{\sin(\pi\alpha\hat\rho)}{\sin(\pi\alpha\rho)}\kappa'_{\alpha\hat\rho,\alpha\rho+1}(0+) \end{bmatrix}$$

$\alpha \in (1,2)$

Define the family of Bernstein functions by

$$\phi_{q+i,p+j}(\lambda) = \int_0^\infty (1-\mathrm{e}^{-\lambda u}) \left\{ \frac{((q+i)\vee(p+j)-1)}{(1-\mathrm{e}^{-u})^{q+i}(1+\mathrm{e}^{-u})^{p+j}} - \frac{(\alpha-1)}{2(1-\mathrm{e}^{-u})^{q}(1+\mathrm{e}^{-u})^{p}} \right\} \mathrm{e}^{-u} \mathrm{d}u,$$

for $q,p \in \{\alpha\rho, \alpha\hat{\rho}\}$ and $i,j \in \{0,1\}$ such that $q+p=\alpha$ and $i+j=1$.

Deep Factorisation $\alpha \in (1,2)$

Theorem

Fix $\alpha \in (1,2)$. Up to a multiplicative constant, the ascending ladder MAP exponent, $\boldsymbol{\kappa}$, is given by

$$\begin{bmatrix} \begin{array}{c}\sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda+\alpha-1)\\ +\sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+)\end{array} & -\sin(\pi\alpha\hat{\rho})\dfrac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda+\alpha-1)}{\lambda+\alpha-1} \\ & \\ -\sin(\pi\alpha\rho)\dfrac{\phi_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda+\alpha-1)}{\lambda+\alpha-1} & \begin{array}{c}\sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda+\alpha-1)\\ +\sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+)\end{array} \end{bmatrix}$$

for $\lambda \geq 0$.

Up to a multiplicative constant, the dual ascending ladder MAP exponent, $\hat{\boldsymbol{\kappa}}$, is given by

$$\begin{bmatrix} \sin(\pi\alpha\hat{\rho})\phi_{\alpha\hat{\rho}+1,\alpha\rho}(\lambda)+\sin(\pi\alpha\hat{\rho})\phi'_{\alpha\rho,\alpha\hat{\rho}+1}(0+) & -\sin(\pi\alpha\hat{\rho})\dfrac{\phi_{\alpha\rho,\alpha\hat{\rho}+1}(\lambda)}{\lambda} \\ -\sin(\pi\alpha\rho)\dfrac{\phi_{\alpha\hat{\rho},\alpha\rho+1}(\lambda)}{\lambda} & \sin(\pi\alpha\rho)\phi_{\alpha\rho+1,\alpha\hat{\rho}}(\lambda)+\sin(\pi\alpha\rho)\phi'_{\alpha\hat{\rho},\alpha\rho+1}(0+) \end{bmatrix}$$

for $\lambda \geq 0$.

Tools: 1

Recall that

$$\boldsymbol{\kappa}(\lambda) = \mathrm{diag}(\Phi_1(\lambda), \Phi_{-1}(\lambda)) - \begin{bmatrix} -\Lambda_{1,-1} & \Lambda_{1,-1}\int \mathrm{e}^{-\lambda x} F^+_{1,-1}(\mathrm{d}x) \\ \Lambda_{-1,1}\int \mathrm{e}^{-\lambda x} F^+_{-1,1}(\mathrm{d}x) & -\Lambda_{-1,1} \end{bmatrix}$$

In general, we can write

$$\Phi_i(\lambda) = n_i(\zeta = \infty) + \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) n_i(\varepsilon_\zeta \in \mathrm{d}x, J(\zeta) = i, \zeta < \infty),$$

where $\zeta = \inf\{s \geq 0 : \varepsilon(s) > 0\}$ for the canonical excursion ε of ξ from its maximum.

Tools: 1

Lemma

Let $T_a = \inf\{t > 0 : \xi(t) > a\}$*. Suppose that* $\limsup_{t\to\infty} \xi(t) = \infty$ *(i.e. the ladder height process* (H^+, J^+) *does not experience killing). Then for* $x > 0$ *we have up to a constant*

$$\lim_{a\to\infty} \mathbf{P}_{0,i}(\xi(T_a) - a \in \mathrm{d}x, J(T_a) = 1)$$
$$= \Big[\pi_1 n_1(\varepsilon(\zeta) > x, J(\zeta) = 1, \zeta < \infty) + \pi_{-1}\Lambda_{-1,1}(1 - F^+_{-1,1}(x))\Big]\mathrm{d}x.$$

- (π_{-1}, π_1) is easily derived by solving $\boldsymbol{\pi} Q = 0$.
- We can work with the LHS in the above lemma e.g. via

$$\lim_{a\to\infty} \mathbf{P}_{0,1}(\xi(T_a) - a > u, J(T_a) = 1)$$
$$= \lim_{a\to\infty} \mathbb{P}_{e^{-a}}(X_{\tau_1^+ \wedge \tau_{-1}^-} > e^u;\ \tau_1^+ < \tau_{-1}^-).$$

Tools: 2

- The problem with applying the Markov additive renewal in the case that $\alpha \in (1,2)$ is that (H^+, J^+) does experience killing.
- It turns out that $\det F(z) = 0$ has a root at $z = \alpha - 1$. Moreover the exponent of a MAP (Esscher transform of F)

$$F^\circ(z) = \boldsymbol{\Delta}_{\boldsymbol{\pi}^\circ}^{-1} F(z + \alpha - 1) \boldsymbol{\Delta}_{\boldsymbol{\pi}^\circ},$$

  where $\boldsymbol{\pi}^\circ = (\sin(\pi\alpha\hat{\rho}), \sin(\pi\alpha\rho))$ is the stationary distribution of $F^\circ(0)$.
- And $\boldsymbol{\kappa}^\circ(\lambda) = \boldsymbol{\Delta}_{\boldsymbol{\pi}^\circ}^{-1} \boldsymbol{\kappa}(\lambda - \alpha + 1) \boldsymbol{\Delta}_{\boldsymbol{\pi}^\circ}$ does not experience killing.
- However, in order to use Markov additive renewal theory to compute $\boldsymbol{\kappa}^\circ$, need to know something about the rssMp to which the MAP with exponent F° corresponds.

Riesz-Bogdan-Zak transform

Theorem (Riesz–Bogdan–Zak transform)

Suppose that X is a stable process as outlined in the introduction. Define

$$\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} \mathrm{d}u > t\}, \qquad t \geq 0.$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$, $(-1/X_{\eta(t)})_{t\geq 0}$ under $\mathbb{P}_x$ is equal in law to $(X, \mathbb{P}^\circ_{-1/x})$, where

$$\left.\frac{\mathrm{d}\mathbb{P}^\circ_x}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \left(\frac{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(X_t)}{\sin(\pi\alpha\rho) + \sin(\pi\alpha\hat{\rho}) - (\sin(\pi\alpha\rho) - \sin(\pi\alpha\hat{\rho}))\mathrm{sgn}(x)}\right) \left|\frac{X_t}{x}\right|^{\alpha-1} \mathbf{1}_{(t<\tau^{\{0\}})}$$

and $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. Moreover, the process $(X, \mathbb{P}^\circ_x)$, $x \in \mathbb{R}\backslash\{0\}$ is a self-similar Markov process with underlying MAP via the Lamperti-Kiu transform given by $F^\circ(z)$.

$$
\begin{array}{ccc}
(X, \mathbb{P}) & \overset{\text{Riesz-Bogdan-Zak}}{\longleftrightarrow} \atop \text{Doob } h\text{-transform} & (X, \mathbb{P}^\circ) \\
\uparrow & & \uparrow \\
\text{Lamperti-Kiu} & & \text{Lamperti-Kiu} \\
\downarrow & & \downarrow \\
\text{MAP: } F(z) & \overset{\text{Esscher}}{\longleftrightarrow} & \text{MAP: } F^\circ(z)
\end{array}
$$

Computing $\Phi_1^\circ(\lambda)$ from $\boldsymbol{\kappa}^\circ(\lambda)$

If we write $\overline{X}_t = \sup_{s\leq t} X_s$ and $\underline{X}_t = \inf_{s\leq t} X_s$, $t \geq 0$, then we also have

$$
\begin{aligned}
&\pi_1^\circ n_1^\circ(\varepsilon(\zeta) > u, J(\zeta) = 1, \zeta < \infty) \\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}u} \lim_{x\to 0} \mathbb{P}_x^\circ \left(X_{\tau_1^+} > \mathrm{e}^u, \overline{X}_{\tau_1^+ -} > |\underline{X}_{\tau_1^+ -}|, \tau_1^+ < \tau_{-1}^- \right) \\
&\quad = -\lim_{x\to 0} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^1 \mathbb{P}_x^\circ (X_{\tau_1^+} > \mathrm{e}^u, \overline{X}_{\tau_1^+ -} \in \mathrm{d}z, \tau_1^+ < \tau_{-z}^-) \\
&\quad = -\lim_{x\to 0} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}y} \frac{\mathrm{d}}{\mathrm{d}u} \mathbb{P}_x^\circ (X_{\tau_1^+} > \mathrm{e}^u, \overline{X}_{\tau_1^+ -} \leq y, \tau_1^+ < \tau_{-z}^-) \Big|_{y=z} \mathrm{d}z \\
&\quad = -\lim_{x\to 0} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}y} \frac{\mathrm{d}}{\mathrm{d}u} \mathbb{P}_x^\circ (X_{\tau_y^+} > \mathrm{e}^u, \tau_y^+ < \tau_{-z}^-) \Big|_{y=z} \mathrm{d}z
\end{aligned}
$$

Computing $\Phi_1^\circ(\lambda)$ from $\boldsymbol{\kappa}^\circ(\lambda)$

For $0 < x < y < 1$ and $u > 0$,

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}u}\lim_{x\to 0}\mathbb{P}_x^\circ\left(X_{\tau_y^+} > \mathrm{e}^u, \tau_y^+ < \tau_{-z}^-\right)\\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}u}\lim_{x\to 0}\mathbb{P}_{-1/x}(X_{\tau(-1/y,1/z)} \in (-\mathrm{e}^{-u}, 0))\\
&\quad = -\frac{\mathrm{d}}{\mathrm{d}u}\lim_{x\to 0}\hat{\mathbb{P}}_{1/x}(X_{\tau(-1/z,1/y)} \in (0, \mathrm{e}^{-u}))\\
&\quad = \hat{p}_{\pm\infty}\left(\frac{2yz\mathrm{e}^{-u} - z + y}{y+z}\right)\frac{2yz}{y+z}\mathrm{e}^{-u},
\end{aligned}
$$

where

$$
\hat{p}_{\pm\infty}(y) = 2^{\alpha-1}\frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha\hat{\rho})\Gamma(1-\alpha\rho)}(1+y)^{-\alpha\hat{\rho}}(1-y)^{-\alpha\rho}
$$

was computed recently in a paper by K. Pardo & Watson (2014).

Computing $\Phi_1^\circ(\lambda)$ from $\boldsymbol{\kappa}^\circ(\lambda)$

Putting the pieces together, we have, up to a constant

$$
\begin{aligned}
\Phi_1^\circ(\lambda) &= \int_0^\infty (1 - \mathrm{e}^{-\lambda x}) n_1^\circ(\varepsilon_\zeta \in \mathrm{d}x, J(\zeta) = 1, \zeta < \infty) \\
&= \lambda \int_0^\infty \mathrm{e}^{-\lambda x} n_1^\circ(\varepsilon_\zeta > x, J(\zeta) = 1, \zeta < \infty) \\
&= \phi_{\alpha\rho+1, \alpha\hat{\rho}}(\lambda)
\end{aligned}
$$

Thank you!