Attraction to and repulsion from patches on the hypersphere and hyperplane for isotropic d-dimensional α-stable processes with index in $\alpha \in (0,1]$ and $d \geq 2$.

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ISOTROPIC α-STABLE PROCESS IN DIMENSION $d \geq 2$

For $d \geq 2$, let $X := (X_t : t \geq 0)$ be a d-dimensional isotropic stable process.

- X has stationary and independent increments (it is a Lévy process)
- Characteristic exponent $\Psi(\theta) = -\log \mathbb{E}_0(\mathrm{e}^{\mathrm{i}\theta \cdot X_1})$ satisfies

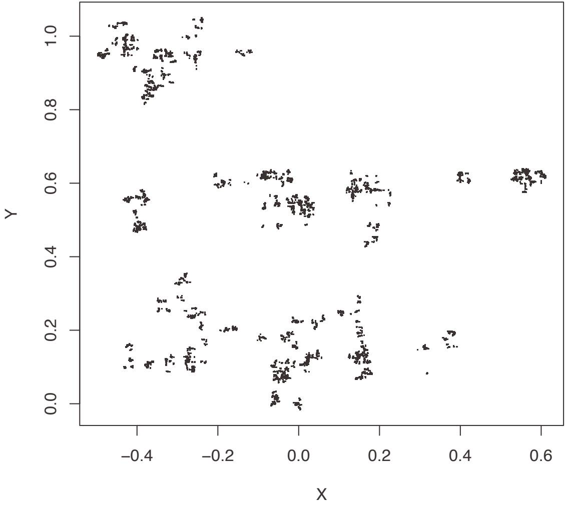
$$\Psi(\theta) = |\theta|^\alpha, \qquad \theta \in \mathbb{R}.$$

- Necessarily, $\alpha \in (0, 2]$, we exclude 2 as it pertains to the setting of a Brownian motion.
- Associated Lévy measure satisfies, for $B \in \mathcal{B}(\mathbb{R}^d)$,

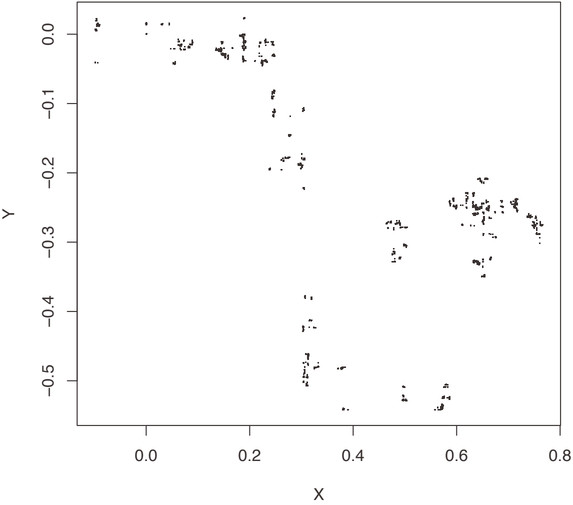
$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} \mathrm{d}y.$$

- X is Markovian with probabilities denoted by $\mathbb{P}_x$, $x \in \mathbb{R}^d$

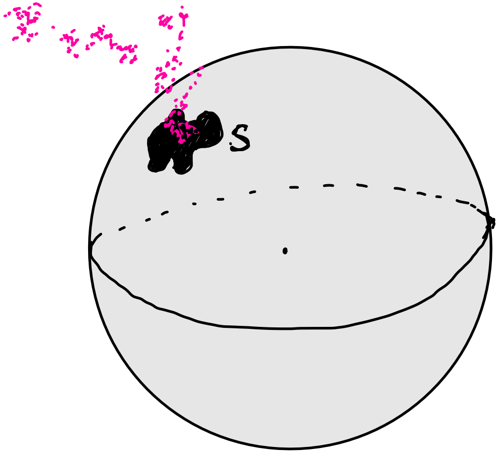
SAMPLE PATH, $\alpha = 1.2$



SAMPLE PATH, $\alpha = 0.9$



Conditioning to hit a patch on a unit sphere from outside

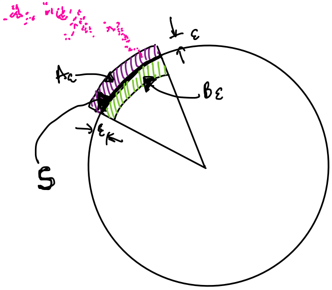


CONDITIONING TO CONTINUOUSLY HIT $S \subseteq \mathbb{S}^{d-1}$ FROM OUTSIDE

- Recall $d \geq 2$, the process $(X, \mathbb{P})$ is transient in the sense that $\lim_{t\to\infty} |X_t| = \infty$ almost surely.
- Define

$$\underline{G}(t) := \sup\{s \leq t \colon |X_s| = \inf_{u \leq s} |X_u|\}, \qquad t \geq 0,$$

- Transience of $(X, \mathbb{P})$ means $\underline{G}(\infty) := \lim_{t\to\infty} \underline{G}(t)$ describes the point of closest reach to the origin in the range of X.
- $A_\varepsilon = \{r\theta : r \in (1, 1+\varepsilon), \theta \in S\}$ and $B_\varepsilon = \{r\theta : r \in (1-\varepsilon, 1), \theta \in S\}$, for $0 < \varepsilon < 1$

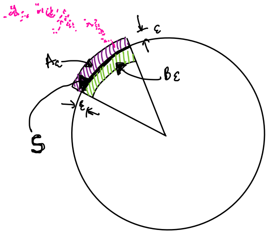


CONDITIONING TO CONTINUOUSLY HIT $S \subseteq \mathbb{S}^{d-1}$ FROM OUTSIDE

- We are interested in the asymptotic conditioning

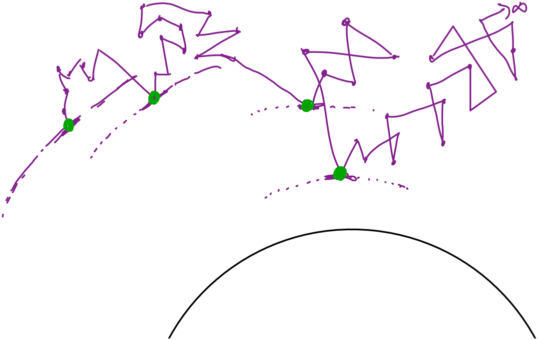
$$\mathbb{P}^S_x(A,\, t < \zeta) = \lim_{\varepsilon \to 0} \mathbb{P}_x(A,\, t < \tau_1^{\oplus} | C^S_\varepsilon), \qquad A \in \sigma(\xi_u : u \leq t),$$

where $\tau_1^{\oplus} = \inf\{t > 0 : |X_t| < 1\}$ and $C^S_\varepsilon := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$.



- Works equally well if we replace $C^S_\varepsilon := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$ by $C^S_\varepsilon = \{X_{\tau_1^{\oplus}} \in B_\varepsilon\}$, or indeed $C^S_\varepsilon = \{X_{\tau_1^{\oplus}-} \in A_\varepsilon\}$

Point of closest reach[1]



Recent work: For $|x| > |z| > 0$,

$$\mathbb{P}_x(X_{\underline{G}(\infty)} \in \mathrm{d}\, z) = \pi^{-d/2} \frac{\Gamma\left(d/2\right)^2}{\Gamma\left((d-\alpha)/2\right)\Gamma\left(\alpha/2\right)} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^\alpha} |x - z|^{-d}\, \mathrm{d}\, z,$$

[1] K. Rivero, Satitkanitkul 2020

Conditioning to continuously hit $S \subseteq \mathbb{S}^{d-1}$ from outside

- Remember $C^S_\varepsilon := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$, switch to generalised polar coordinates and estimate
$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-d}\mathbb{P}_x(C^S_\varepsilon) = c_{\alpha,d}\int_S (|x|^2-1)^{\alpha/2}|x-\theta|^{-d}\sigma_1(\mathrm{d}\,\theta),$$
where $c_{\alpha,d}$ does not depend on x or S and σ_1 is the unit surface measure on $\mathbb{S}^{d-1}$.
- Use
$$\mathbb{P}_x(A, t<\tau^\oplus_\beta | C^S_\varepsilon) = \mathbb{E}_x\left[\mathbf{1}_{\{A,t<\tau^\oplus_\beta\}}\frac{\mathbb{P}_{X_t}(C^S_\varepsilon)}{\mathbb{P}_x(C^S_\varepsilon)}\right], \qquad A\in\sigma(\xi_u : u\le t),$$
pass the limit through the expectation on the RHS (carefully with DCT!) to get
$$\left.\frac{d\mathbb{P}^S_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{(t<\tau^\oplus_1)}\frac{M_S(X_t)}{M_S(x)}, \qquad \text{if } x\in\bar{\mathbb{B}}^c_d$$

with
$$M_S(x) = \begin{cases} \int_S |\theta-x|^{-d}||x|^2-1|^{\alpha/2}\sigma_1(\mathrm{d}\theta) & \text{if } \sigma_1(S)>0 \\ |\vartheta-x|^{-d}||x|^2-1|^{\alpha/2} & \text{if } S=\{\vartheta\}, \end{cases}$$
which is a superharmonic function.

Decoupling hitting position

- Suppose ζ is the lifetime of $(X, \mathbb{P}^S)$. Let S' be an open subset of S. Then for any $x \in \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$, we have

$$\mathbb{P}^S_x(X_{\zeta-} \in S') = \frac{\int_{S'} |\theta - x|^{-d} \sigma_1(\mathrm{d}\theta)}{\int_S |\theta - x|^{-d} \sigma_1(\mathrm{d}\theta)},$$

- Hence, for $\theta \in S$,

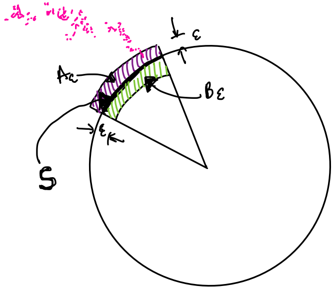
$$\begin{aligned}
\mathbb{P}^S_x(A|X_{\zeta-} = \theta) &= \mathbb{E}^S_x\left[\mathbf{1}_\varepsilon \frac{\mathbb{P}^S_{X_t}(X_{\zeta-} = \theta)}{\mathbb{P}^S_x(X_{\zeta-} = \theta)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau_1^\oplus)} \frac{M_S(X_t)}{M_S(x)} \frac{M_{\{\theta\}}(X_t)}{M_S(X_t)} \frac{M_S(x)}{M_{\{\theta\}}(x)}\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{(A,\, t<\tau_1^\oplus)} \frac{M_{\{\theta\}}(X_t)}{M_{\{\theta\}}(x)}\right] \\
&= \mathbb{P}^{\{\theta\}}_x(A), \qquad A \in \sigma(\xi_u : u \leq t)
\end{aligned}$$

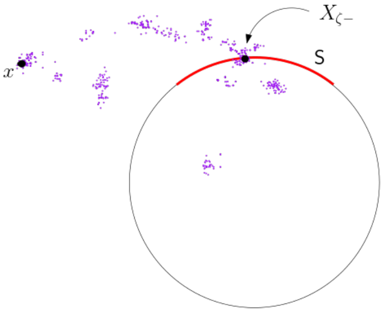
- So

$$\mathbb{P}^S_x(A) = \int_S \mathbb{P}^{\{\theta\}}_x(A) \frac{|\theta - x|^{-d} \sigma_1(\mathrm{d}\theta)}{\int_S |\vartheta - x|^{-d} \sigma_1(\mathrm{d}\vartheta)}.$$

"pick a target uniformly in S with the terminal strike distribution and condition to hit it."

Conditioning to continuously hit $S \subseteq \mathbb{S}^{d-1}$ from either side





- Now define

$$\mathbb{P}^S_x(A,\, t < \zeta) = \lim_{\varepsilon \to 0} \mathbb{P}_x \left(A \middle| \, \tau_{S_\varepsilon} < \infty \right),$$

  where

$$\tau_{S_\varepsilon} = \inf\{t > 0 : X_t \in S_\varepsilon\} \text{ and } S_\varepsilon := A_\varepsilon \cup B_\varepsilon.$$

- Note: need to insist on $\alpha \in (0, 1]$ because $\mathbb{P}_x(\tau_S < \infty) > 0$ if $\alpha \in (1, 2)$.

Theorem

Suppose that $\alpha \in (0,1]$ and the closed set $S \subseteq \mathbb{S}^{d-1}$ is such that $\sigma_1(S) > 0$. For $\alpha \in (0,1]$, the process $(X, \mathbb{P}^S)$ is well defined such that

$$\left.\frac{\mathrm{d}\,\mathbb{P}^S_x}{\mathrm{d}\,\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{H_S(X_t)}{H_S(x)}, \qquad t \geq 0, x \notin S, \tag{1}$$

where

$$H_S(x) = \int_S |x-\theta|^{\alpha-d}\sigma_1(\mathrm{d}\,\theta), \qquad x \notin S.$$

Note, if $S = \{\theta\}$ then it was previously understood[2] that

$$H_S(x) = |x-\theta|^{\alpha-d}, \qquad x \notin S.$$

So it is still the case for a general S that

$$\mathbb{P}^S_x(A) = \int_S \mathbb{P}^{\{\theta\}}_x(A)\frac{|x-\theta|^{\alpha-d}\sigma_1(\mathrm{d}\,\theta)}{\int_S |x-\vartheta|^{\alpha-d}\sigma_1(\mathrm{d}\,\vartheta)}.$$

"pick a target uniformly in S with the terminal strike distribution and condition to hit it."

[2]K. Rivero, Statitkanitkul 2019

CONDITIONING TO CONTINUOUSLY HIT $S \subseteq \mathbb{S}^{d-1}$ FROM EITHER SIDE

Theorem

Let $S \subseteq \mathbb{S}^{d-1}$ be a closed subset such that $\sigma_1(S) > 0$.

(i) *Suppose $\alpha \in (0,1)$. For $x \notin S$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = 2^{1-2\alpha} \frac{\Gamma((d+\alpha-2)/2)}{\pi^{d/2}\Gamma(1-\alpha)} \frac{\Gamma((2-\alpha)/2)}{\Gamma(2-\alpha)} H_S(x).$$

(ii) *When $\alpha = 1$, we have that, for $x \notin S$,*

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \, \mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = \frac{\Gamma((d-1)/2)}{\pi^{(d-1)/2}} H_S(x).$$

Heuristic for Proof of Theorem 2

- The potential of the isotropic stable process satisfies $\mathbb{E}\left[\int_0^\infty \mathbf{1}_{(X_t\in \mathrm{d}\, y)}\, \mathrm{d}\, t\right] = |y|^{\alpha-d}$.
- Let μ_ε be a finite measure supported on S_ε, which is absolutely continuous with respect to Lebesgue measure ℓ_d with density m_ε and define its potential by

$$U\mu_\varepsilon(x) := \int_{S_\varepsilon} |x-y|^{\alpha-d}\mu_\varepsilon(\mathrm{d}\, y) = \mathbb{E}_x\left[\int_0^\infty m_\varepsilon(X_t)\, \mathrm{d}\, t\right] \qquad x\in\mathbb{R}^d,$$

- As $m_\varepsilon(y)=0$ for all $y\notin S_\varepsilon$. As such, the Strong Markov Property tells us that

$$U\mu_\varepsilon(x) = \mathbb{E}_x\left[\mathbf{1}_{\{\tau_{S_\varepsilon}<\infty\}}\int_{\tau_{S_\varepsilon}}^\infty m_\varepsilon(X_t)\, \mathrm{d}\, t\right] = \mathbb{E}_x\left[U\mu_\varepsilon(X_{\tau_\varepsilon})\mathbf{1}_{\{\tau_{S_\varepsilon}<\infty\}}\right], \qquad x\notin S_\varepsilon. \tag{2}$$

  Note, the above equality is also true when $x\in S_\varepsilon$ as, in that case, $\tau_{S_\varepsilon}=0$.
- Let us now suppose that μ_ε can be chosen in such a away that, for all $x\in S_\varepsilon$, $U\mu(x)=1$. Then

$$\mathbb{P}_x(\tau_\varepsilon<\infty) = U\mu_\varepsilon(x), \qquad x\notin S_\varepsilon.$$

- Strategy: 'guess' the measure, μ_ε, by verifying

$$U\mu_\varepsilon(x) = 1+o(1), \qquad x\in S_\varepsilon \text{ as } \varepsilon\to 0,$$

  so that

$$(1+o(1))\mathbb{P}_x(\tau_{S_\varepsilon}<\infty) = U\mu_\varepsilon(x), \qquad x\notin S_\varepsilon,$$

- Draw out the the leading order decay in ε from $U\mu_\varepsilon(x)$.

Heuristic for Proof of Theorem 2: Flat Earth theory

- Believing in a flat Earth is helpful
- In one dimension, it is known[3] that for a one-dimensional symmetric stable process,

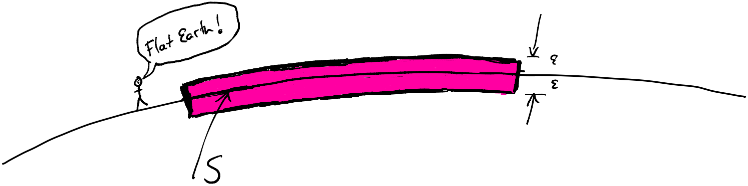
$$\int_{-1}^{1} |x-y|^{\alpha-1}(1-y)^{-\alpha/2}(1+y)^{-\alpha/2}\,\mathrm{d}y = 1, \qquad x \in [-1,1].$$

- Writing $X = |X|\arg(X)$, when X begins in the neighbourhood of S, then $|X|$ begins in the neighbourhood of 1 and $\arg(X)$, essentially, from within S.
- Flat earth theory would imply

$$\mu_\varepsilon(\mathrm{d}y) = m_\varepsilon(y)\ell_d(\mathrm{d}y)\mathbf{1}_{(y\in S_\varepsilon)},$$
$$\text{with} \quad m_\varepsilon(y) = c_{\alpha,d}(|y|-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-|y|)^{-\alpha/2}$$

where ℓ_d is d-dimensional Lebesgue measure and $c_{\alpha,d,\varepsilon}$ is a constant to be determined so that

$$U\mu_\varepsilon(x) = 1 + o(1) \qquad x \in S_\varepsilon$$



[3] Profeta and Simon 2016

The asymptotic does not depend on S

- So far we are guessing:

$$\mu_\varepsilon(\mathrm{d}\,y) = m_\varepsilon(y)\ell_d(\mathrm{d}\,y)\mathbf{1}_{(y\in S_\varepsilon)},$$
$$\text{with}\quad m_\varepsilon(y) = c_{\alpha,d}(|y| - (1-\varepsilon))^{-\alpha/2}(1+\varepsilon - |y|)^{-\alpha/2}$$

where ℓ_d is d-dimensional Lebesgue measure and $c_{\alpha,d,\varepsilon}$ is a constant to be determined so that

$$U\mu_\varepsilon(x) = 1 + o(1) \qquad x \in S_\varepsilon$$

- We don't think that the restriction to S_ε is important so we are going to write

$$\mu_\varepsilon(\mathrm{d}\,y) = \mu_\varepsilon^{(1)}(\mathrm{d}\,y) - \mu_\varepsilon^{(2)}(\mathrm{d}\,y)$$
$$\text{with}\quad \mu^{(1)}(\mathrm{d}\,y) = m_\varepsilon(y)\ell_d(\mathrm{d}\,y) \quad\text{and}\quad \mu_\varepsilon^{(2)}(\mathrm{d}\,y) = \mathbf{1}_{(y\in\mathbb{S}_\varepsilon^{d-1}\setminus S_\varepsilon)} m_\varepsilon(y)\ell_d(\mathrm{d}\,y)$$

where $\mathbb{S}_\varepsilon^{d-1} = \{x \in \mathbb{R}^d : 1-\varepsilon \leq |x| \leq 1+\varepsilon\}$.

NASTY CALCULATIONS: $\alpha \in (0,1)$

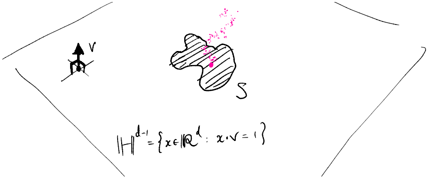
For $x \in \mathbb{S}_\varepsilon^{d-1}$,

$$
\begin{aligned}
&U\mu_\varepsilon^{(1)}(x)\\
&= c_{\alpha,d} \int_{\mathbb{S}_\varepsilon^{d-1}} |x-y|^{\alpha-d} (|y|-(1-\varepsilon))^{-\alpha/2}(1+\varepsilon-|y|)^{-\alpha/2} \ell_d(\mathrm{d}y)\\
&= \frac{2c_{\alpha,d}\pi^{(d-1)/2}}{\Gamma((d-1)/2)} \int_{1-\varepsilon}^{1+\varepsilon} \frac{r^{d-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}} \,\mathrm{d}r \int_0^\pi \frac{\sin^{d-2}\theta \,\mathrm{d}\theta}{(|x|^2-2|x|r\cos\theta+r^2)^{(d-\alpha)/2}}\\
&= \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)} |x|^{\alpha-d} \int_{1-\varepsilon}^{|x|} \frac{{}_2F_1\left(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; (r/|x|)^2\right) r^{d-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}} \,\mathrm{d}r\\
&\qquad + \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)} \int_{|x|}^{1+\varepsilon} \frac{{}_2F_1\left(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; (|x|/r)^2\right) r^{\alpha-1}}{(r-(1-\varepsilon))^{\alpha/2}(1+\varepsilon-r)^{\alpha/2}} \,\mathrm{d}r.\\
&= \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)} \int_{\frac{1-\varepsilon}{|x|}}^{1} \frac{{}_2F_1\left(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; r^2\right) r^{d-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{\alpha/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{\alpha/2}} \,\mathrm{d}r\\
&\qquad + \frac{2c_{\alpha,d}\pi^{d/2}}{\Gamma(d/2)} \int_{1}^{\frac{1+\varepsilon}{|x|}} \frac{{}_2F_1\left(\frac{d-\alpha}{2}, 1-\frac{\alpha}{2}; \frac{d}{2}; r^{-2}\right) r^{\alpha-1}}{\left(r-\frac{1-\varepsilon}{|x|}\right)^{\alpha/2}\left(\frac{1+\varepsilon}{|x|}-r\right)^{\alpha/2}} \,\mathrm{d}r\\
&= \cdots\cdots\cdots = 1+o(1)
\end{aligned}
$$

If we choose

$$
c_{\alpha,d} = \frac{\Gamma((d+\alpha-2)/2)}{2^\alpha \pi^{d/2}\Gamma(1-\alpha)\Gamma((2-\alpha)/2)}
$$

The same concept works with a plane



Theorem

Suppose that $\alpha \in (0,1]$ and the closed and bounded set $S \subseteq \mathbb{H}^{d-1}$ is such that $0 < \ell_{d-1}(S) < \infty$, where we recall that ℓ_{d-1} is $(d-1)$-dimensional Lebesgue measure.

(i) *Suppose $\alpha \in (0,1)$. For $x \notin S$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha-1} \mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = 2^{1-\alpha} \pi^{-(d-2)/2} \frac{\Gamma(\frac{d-2}{2})\Gamma(\frac{d-\alpha}{2})\Gamma(\frac{2-\alpha}{2})^2}{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{d-1}{2})\Gamma(2-\alpha)} K_S(x), \tag{3}$$

where

$$K_S(x) = \int_S |x-y|^{\alpha-d} \ell_{d-1}(\mathrm{d}y), \qquad x \notin S.$$

(ii) *Suppose $\alpha = 1$. For $x \notin S$,*

$$\lim_{\varepsilon \to 0} |\log \varepsilon| \, \mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = \frac{\Gamma(\frac{d-2}{2})}{\pi^{(d-2)/2}} K_S(x), \tag{4}$$

(iii) *The process $(X, \mathbb{P}^S)$ is well defined such that*

$$\left.\frac{\mathrm{d}\,\mathbb{P}^S_x}{\mathrm{d}\,\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{K_S(X_t)}{K_S(x)}, \qquad t \geq 0, x \notin S. \tag{5}$$

Flat earth vs round earth theory

- Consider the case $\alpha \in (0,1)$.
- Recall for conditioning a continuous approach to the patch on the sphere from outside we had a scaling with index $\alpha - d$:

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha - d} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon) = c_{\alpha,d} \int_S (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(\mathrm{d}\,\theta),$$

- Where conditioning a continuous approach to the patch from either side, we had scaling index $\alpha - 1$:

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha - 1} \mathbb{P}_x(\tau_{S_\varepsilon} < \infty) = 2^{1-2\alpha} \frac{\Gamma((d+\alpha-2)/2)}{\pi^{d/2}\Gamma(1-\alpha)} \frac{\Gamma((2-\alpha)/2)}{\Gamma(2-\alpha)} H_S(x).$$

- In the first case, the conditioned path needs to be observant of the entire sphere. In the second case the conditioned path needs only a localised consideration of S, which appears flat in close proximity.

Thank you!