De Finetti’s control problem and spectrally negative Lévy processes

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Cramér-Lundberg processes

- A classic actuarial problem concerns the ruin problem centred around the surplus process defined by:

\[ X_t = x + ct - \sum_{i=1}^{N_t} \xi_i \]

where \( x, c > 0 \), \( \{N_t: t \geq 0\} \) is a Poisson process with rate \( \lambda > 0 \) and \( \{\xi_i: i \geq 1\} \) is a sequence of i.i.d. random variables.

The ruin problem looks at the behaviour of the surplus process up to and on the event \( \{\tau + 0 < \infty\} \) where \( \tau + 0 = \inf\{t > 0: X_t < 0\} \).

Under the assumption that \( c - \lambda E(\xi_1) > 0 \), i.e. \( \lim_{t \to \infty} X_t = \infty \).
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- The stochastic process under $\mathbb{P}$

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where $x, c > 0$, $\{N_t : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and $\{\xi_i : i \geq 1\}$ is a sequence of i.i.d. random variables.
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- In this talk, you have the option to think of \(X = \{X_t : t \geq 0\}\) as a spectrally negative Lévy process.
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- In this talk, you have the option to think of \(X = \{X_t : t \geq 0\}\) as a spectrally negative Lévy process.
- In either case, for \(\theta \geq 0\) we may work with the Laplace exponent

\[
\psi(\theta) := \log \mathbb{E}_0(e^{\theta X_1}),
\]

which is strictly convex, respects the condition \(\psi'(0+) > 0\), passes through the origin and so tends to \(+\infty\) at \(\infty\).
de Finetti’s view of the ruin problem


- Consider $L = \{L_t : t \geq 0\}$ is a stream of dividend payments or a ‘dividend strategy’: left continuous, non-negative, non-decreasing process adapted to the filtration generated by $X$. 
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- $U_t = X_t - L_t$ is the residual surplus after dividends are paid,

$$\sigma^L = \inf\{t > 0 : X_t - L_t < 0\}$$

is the ruin time. (Also impose that $L$ is such that ruin cannot be caused by a jump of $L$).
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- de Finetti’s control problem: find the value function and matching dividend strategy $L^*$ such that

$$v(x) = \sup_L \mathbb{E}_x \left( \int_0^{\sigma^L} e^{-qt} dL_t \right) = \mathbb{E}_x \left( \int_0^{\sigma^{L^*}} e^{-qt} dL^*_t \right)$$

where $q > 0$ and the supremum is taken over all admissible dividend strategies.
Reflection strategies

• It has been shown that the optimal strategy is of a ‘barrier type with reflection’:

\[ L^a_t = (a \vee \sup_{s \leq t} X_s) - a \]

for some optimal level \( a \). Below a realisation of \( X_t - L^a_t \)

These cases are:

1. (Gerber 1969) Cramér-Lundberg process with exponentially distributed jumps
   \[ X_t = c t - \sum_{i=1}^{N_t} e_i \]

2. (Jeanblanc & Shiryaev 1995 and many others) Linear Brownian motion:
   \[ X_t = \mu t + \sigma B_t \]

• However, it has also been shown that the above strategy is not optimal, even by straying not too far from the above models!

3. (Ascue & Muler 2005) Cramér-Lundberg process with gamma distributed jumps having density proportional to \( xe^{-x} \).
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Scale functions are a natural tool

- It turns out there is a very natural tool for analysing path functionals of spectrally negative Lévy processes (and in particular Cramér-Lundberg processes).
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- For each \( q \geq 0 \) there exists a function \( W(q) : [0, \infty) \to [0, \infty) \) defined by its Laplace transform

\[
\int_0^\infty e^{-\beta x} W(q)(x) \, dx = \frac{1}{\psi(\beta) - q}
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for \( \beta \) sufficiently large.
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- For all \( a > 0 \),

\[
v^a(x) := \mathbb{E}_x \left( \int_0^L e^{-qx} \, dL_t^a \right) = \begin{cases} 
\frac{W(q)(x)}{W(q)'(a)} & \text{when } x \leq a \\
(x - a) + \frac{W(q)(a)}{W(q)'(a)} & \text{when } x > a
\end{cases}
\]
Loeffen (2008)

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1. The refraction strategy at level

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a^* := \sup\{a \geq 0 : W^{(q)'}(a) \leq W^{(q)'}(x) \text{ for all } x \geq 0\}
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is optimal as soon as one assumes that \( W^{(q)} \) is a convex function on \((a^*, \infty)\).
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2 The above condition is satisfied if the distribution of the i.i.d. claims \{\xi_i : i \geq 1\} has a density $f$ which is completely monotone.\(^1\) i.e.

$$(-1)^n \frac{d^n f}{dx^n} \geq 0 \text{ for all } n \geq 1.$$  

\(^1\)For Lévy-friendly readers: the Lévy measure when projected onto $(0, \infty)$ has a completely monotone density.
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• The latter condition expands vastly the claim distributions in the Cramér-Lundberg model for which the reflection barrier strategy is optimal.

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• The latter condition expands vastly the claim distributions in the Cramér-Lundberg model for which the reflection barrier strategy is optimal.

• Moreover, it gives some hint as to why the Azcue & Muler example fails: In that case the claim distribution has a density which is not completely monotone!

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Restricted class of control strategies

- Many variations on this theme have been examined for the case of diffusions (Jeanblanc & Shiryaev 1995, Elena Boguslavskaya’s Ph.D. thesis) as well as the Cramér-Lundberg case with exponential jumps (Gerber & Shiu 2006) including the following:

\[ \mathcal{L}_t = \int_0^t \phi(s) \, ds \]

where \( \phi \) is measurable and uniformly bounded by, say, \( \delta > 0 \). In the Cramér-Lundberg setting we need that \( \delta < c \). We should now think of \( \phi \) as the control.

What was the optimal strategy appeared in the aforementioned articles?
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- The class of admissible strategies is further restricted to the case that

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Refraction strategies

- A refraction strategy refers to the control $\phi(x) = \delta \mathbf{1}_{(x>b)}$ for some threshold level $b \geq 0$. Thus the controlled process would need to solve the stochastic differential equation

$$U_t = X_t - \delta \int_0^t \mathbf{1}_{(U_s>b)} \, ds.$$
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- Note in the case that \( X \) is a general spectrally negative Lévy process the above SDE is highly non-trivial if there is no Gaussian component.
K., and Loeffen (2010)

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- Write $W(q)$ for the scale function associated with $X_t - \delta t$. 

$$v_b(x) := E_x\left( \int_{\kappa - 0}^{\infty} e^{-qt} \mathbb{1}_{\{U_t > b\}} \, ds \right) = -\delta \int_{0}^{\infty} (x-b) \vee 0 \, W(q)(z) \, dz + W(q)(x) + \delta \mathbb{1}_{\{x \geq b\}} \int_{x}^{b} W(q)(x-y) W(q)'(y) \, dy,$$

where $\phi(q)$ is the unique solution in $(0, \infty)$ to

$$\psi(\theta) - \delta \theta = 0.$$
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- Existence and uniqueness of a strong solution to SDE established in the general Lévy case.
- Write $\mathbb{W}(q)$ for the scale function associated with $X_t - \delta t$.
- Suppose that
  \[ \kappa_0^- := \inf\{t > 0 : U_t < 0\}. \]

For $q \geq 0$ and $x \geq 0$

\[
v^b(x) := \mathbb{E}_x \left( \int_0^{\kappa_0^-} e^{-qt} \delta \mathbf{1}_{\{U_t > b\}} \, ds \right)
\]

\[
= -\delta \int_0^{(x-b)\vee 0} \mathbb{W}(q)(z) \, dz
\]

\[
+ \frac{W(q)(x) + \delta \mathbf{1}_{\{x \geq b\}} \int_b^x \mathbb{W}(q)(x-y) W(q)'(y) \, dy}{\varphi(q) \int_0^\infty e^{-\varphi(q)y} W(q)'(y+b) \, dy},
\]

where $\varphi(q)$ is the unique solution in $(0, \infty)$ to $\psi(\theta) - \delta \theta = 0$. 

• Define the function

\[ h(x) = \varphi(q) \int_{0}^{\infty} e^{-\varphi(q)y} W^{(q)'}(y + b) \, dy \]
K., Loeffen and Pérez (2010)

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\[ b^* = \sup\{b \geq 0 : h(b) \leq h(x) \text{ for all } x \geq 0\} \]
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b^* = \sup\{b \geq 0 : h(b) \leq h(x) \text{ for all } x \geq 0\}
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- The refraction strategy at level \(b^*\) is optimal amongst the absolutely continuous \(\delta\)-bounded strategies as soon as we assume that the common distribution of the claims is absolutely continuous with completely monotone density.\(^2\)

\(^2\)For Lévy-friendly readers: the Lévy measure when projected onto \((0, \infty)\) has a completely monotone density.
How explicit is explicit? How natural is natural?

- The results presented here have been possible thanks to a deep understanding of analytical properties of scale functions.
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- Whilst the conditions on the claim distribution (resp. Lévy measure) are very straightforward to check, the expressions for the optimal value can only be written in terms of a mysterious "scale function".
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- Whilst the conditions on the claim distribution (resp. Lévy measure) are very straightforward to check, the expressions for the optimal value can only be written in terms of a mysterious "scale function".
- There has been significant work recently in pushing forward methodology which allows one to develop either closed form or semi-explicit expressions for $W(q)$. See the forthcoming review of the theory of scale functions in the Springer Lecture Notes in Mathematics series "Lévy Matters": K., Rivero and Kuznetsov (2011).
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- Two more papers, K., Rivero, Song (2010) and Loeffen and Renaud (2010) explore this idea further and the final word from the latter papers shows that insisting that the tail of the jump distribution is a log-convex function is still sufficient.
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