Yaglom limits for general non-local Branching Markov processes

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(Manchester edition)

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Classical Bienaymè-Galton-Watson Yaglom limit

- $(Z_n, n \geq 0)$ is a BGW process i.e.

$$Z_{n+1} = \sum_{i=1}^{Z_n} A_i, \qquad A_i \sim^{\text{iid}} A \quad \text{(copies of family offspring numbers)}$$

- Assume $\mathbb{E}[A^2] < \infty$ and define $\sigma^2 = \mathbb{E}[A^2] - \mathbb{E}[A]^2$.
- Assume criticality $\mathbb{E}[A] = 1$, recalling that $\zeta = \inf\{n > 0 : Z_n = 0\}$ is almost surely finite.
- Kolmogorov limit:

$$\lim_{n\to\infty} n\mathbb{P}(\zeta > n) = \frac{2}{\sigma^2}.$$

- Yaglom limit:

$$\mathbb{E}\left[\exp\left(-\theta\frac{Z_n}{n}\right)\middle|\zeta > n\right] = \frac{1}{1+\theta\sigma^2/2},$$

  i.e. the QSD limit of Z_n/n conditional on survival is exponential with parameter $2/\sigma^2$.

$(\mathtt{P}, \mathtt{G})$-Branching Markov Process

- Particles will live in E a Lusin space (e.g. a Polish space would be enough)
- Let $\mathtt{P} = (\mathtt{P}_t, t \geq 0)$ be a semigroup on E.
- Write $B^+(E)$ for non-negative bounded measurable functions on E

- Particles evolve independently according to a $\mathtt{P}$-Markov process.
- In an event which we refer to as 'branching', particles positioned at x die at rate $\beta \in B^+(E)$ and instantaneously, new particles are created in E according to a point process.
- The configurations of these offspring are described by the random counting measure

$$\mathcal{Z}(A) = \sum_{i=1}^{N} \delta_{x_i}(A),$$

  with probabilities $\mathcal{P}_x$, where $x \in E$ is the position of death of the parent.
- Without loss of generality we can assume that $\mathcal{P}_x(N = 1) = 0$. On the other hand, we do allow for the possibility that $\mathcal{P}_x(N = 0) > 0$ for some or all $x \in E$.
- Henceforth we refer to this spatial branching process as a $(\mathtt{P}, \mathtt{G})$-branching Markov process.

$(\mathtt{P}, \mathtt{G})$-Branching Markov Process

- Define the so-called branching mechanism

$$\mathtt{G}[f](x) := \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N} f(x_i) - f(x)\right], \qquad x \in E,$$

where we recall $f \in B_1^+(E) := \{f \in B^+(E) : \sup_{x\in E} f(x) \le 1\}$.
- Configuration of particles at time t is denoted by $\{x_1(t), \ldots, x_{N_t}(t)\}$ and, on the event that the process has not become extinct or exploded,

$$X_t(\cdot) = \sum_{i=1}^{N_t} \delta_{x_i(t)}(\cdot), \qquad t \ge 0.$$

is Markovian in $N(E)$, the space of integer atomic measures.
- Its probabilities will be denoted $\mathbb{P} := (\mathbb{P}_\mu, \mu \in N(E))$.
- Define,

$$\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \ge 0.$$

- Non-linear evolution semigroup

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \ge 0.$$

k-TH MOMENT

- Our main results concern understanding the growth of the k-th moment functional in time

$$\mathtt{T}_t^{(k)}[f](x) := \mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k], \qquad x \in E, f \in B^+(E), k \geq 1, t \geq 0.$$

- **Notational convenience**: Write $\mathtt{T}_t$ in place of $\mathtt{T}_t^{(1)}$
- **Related historical work**: A number of papers have opened the topic of moments for branching particle systems and superprocesses, including e.g. :

  ◦ E. Dumonteil and A. Mazzolo. Residence times of branching diffusion processes. *Phys. Rev. E, 94:012131, 2016.*

  ◦ J. Fleischman. Limiting distributions for branching random fields. *Trans. Amer. Math. Soc., 239:353–389, 1978.*

  ◦ I. Iscoe. On the supports of measure-valued critical branching Brownian motion. *Ann. Probab., 16(1):200–221, 1988.*

  ◦ A. Klenke. Multiple scale analysis of clusters in spatial branching models. *Ann. Probab., 25(4):1670–1711, 1997.*

- Our objective: to show that for $k \geq 2$ and any positive bounded measurable function f on E,

$$\lim_{t\to\infty} g_k(t)\mathbb{E}_{\delta_x}[\langle f, X_t\rangle^k] = C_k(x, f)$$

  where the constant $C_k(x, f)$ can be identified explicitly.
- We need **two** fundamental assumptions.

Assumption (H1): Asmussen-Hering class

There exists an eigenvalue $\lambda \in \mathbb{R}$ and a corresponding right eigenfunction $\varphi \in B^+(E)$ and finite left eigenmeasure $\tilde{\varphi}$ such that, for $f \in B^+(E)$,

$$\langle \mathtt{T}_t[\varphi], \mu \rangle = \mathrm{e}^{\lambda t} \langle \varphi, \mu \rangle \text{ and } \langle \mathtt{T}_t[f], \tilde{\varphi} \rangle = \mathrm{e}^{\lambda t} \langle f, \tilde{\varphi} \rangle,$$

for all $\mu \in N(E)$ if $(X, \mathbb{P})$ is a branching Markov process (resp. a superprocess). Further let us define

$$\Delta_t = \sup_{x \in E, ||f|| \leq 1} |\varphi(x)^{-1} \mathrm{e}^{-\lambda t} \mathtt{T}_t[f](x) - \langle \tilde{\varphi}, f \rangle|, \qquad t \geq 0.$$

We suppose that

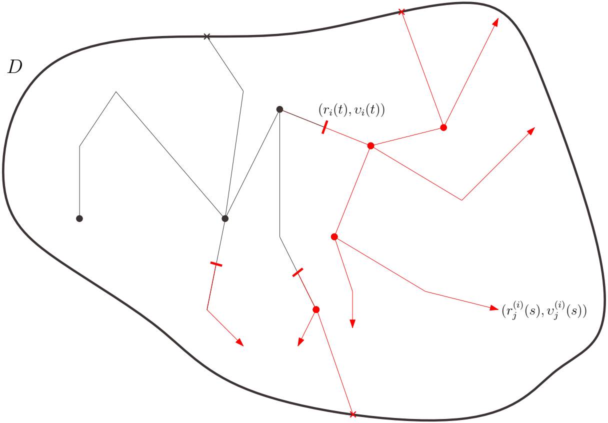
$$\sup_{t \geq 0} \Delta_t < \infty \text{ and } \lim_{t \to \infty} \Delta_t = 0.$$

NOTE: This assumption allows us to talk about criticality of the $(\mathtt{P}, \mathtt{G})$-BMP:

$$\lambda = 0 \text{ (critical)} \mid \lambda > 0 \text{ (supercritical)} \mid \lambda < 0 \text{ (subcritical)}$$

Who lives in the Asmussen-Hering class?

- Branching Brownian Motion in a bounded domain
- Neutron Branching process in a Bounded domain
- Multi-type (cts-time) Bienaymé~~-Galton-~~Watson process



Assumption $(\mathrm{H2})_k$

$$\sup_{x \in E} \mathcal{E}_x(\langle 1, \mathcal{Z} \rangle^k) < \infty.$$

Theorem: The critical case ($\lambda = 0$)

Suppose that (H1) holds along with $(\mathrm{H2})_k$ for some $k \geq 2$ and $\lambda = 0$. Define

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| t^{-(\ell-1)} \varphi(x)^{-1} \mathrm{T}_t^{(\ell)}[f](x) - 2^{-(\ell-1)} \ell! \langle f, \tilde{\varphi} \rangle^{\ell} \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{\ell-1} \right|,$$

where

$$\mathbb{V}[\varphi](x) = \beta(x) \mathcal{E}_x \left(\langle \varphi, \mathcal{Z} \rangle^2 - \langle \varphi^2, \mathcal{Z} \rangle \right).$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

In short, subject to (H1) at criticality and $(\mathrm{H2})_k$, we have, for $f \in B_1^+(E)$,

$$\lim_{t \to \infty} t^{-(k-1)} \mathbb{E}_{\delta_x} \left[\langle f, X_t \rangle^k \right] = 2^{-(k-1)} k! \langle f, \tilde{\varphi} \rangle^k \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{k-1} \varphi(x)$$

"At criticality the k-th moment scales like t^{k-1}"

Ideas from the proof

- The obvious starting point:

$$\mathtt{T}_t^{(k)}[f](x) = (-1)^k \frac{\partial^k}{\partial\theta^k} \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f, X_t\rangle}]\Big|_{\theta=0}$$

- Recall that

$$\mathtt{v}_t[f](x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} f(x_i(t))\right], \qquad f \in B_1^+(E), t \geq 0.$$

- Non-linear evolution semigroup

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s, \qquad t \geq 0.$$

- Hence

$$\mathtt{v}_t[\mathrm{e}^{-\theta f}](x) = \mathbb{E}_{\delta_x}[\mathrm{e}^{-\theta\langle f, X_t\rangle}]$$

- We need a new representation of the non-linear semigroup $(\mathtt{v}_t, t \geq 0)$ which connects us to the assumption (H1).

Linear to non-linear semigroup

- Recall

$$\mathtt{T}_t[f](x) = \mathtt{T}_t^{(1)}[f](x) = \mathbb{E}_{\delta_x}[\langle f, X_t\rangle], \qquad t \geq 0, f \in B_1^+(E), x \in E.$$

- For $f \in B^+(E)$, it is well known that the mean semigroup evolution satisfies

$$\mathtt{T}_t[f](x) = \mathtt{P}_t[f] + \int_0^t \mathtt{P}_s\left[\mathtt{F}\,\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s \qquad t \geq 0, x \in E, \tag{1}$$

where

$$\mathtt{F}[f](x) = \beta(x)\mathcal{E}_x\left[\sum_{i=1}^{N} f(x_i) - f(x)\right], \qquad x \in E.$$

Linear to non-linear semigroup

We now define a variant of the non-linear evolution semigroup equation

$$\mathtt{u}_t[f](x) = \mathbb{E}_{\delta_x}\left[1 - \prod_{i=1}^{N_t} f(x_i(t))\right], \qquad t \geq 0,\ x \in E,\ f \in B_1^+(E).$$

For $f \in B_1^+(E)$, define

$$\mathtt{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^{N}(1 - f(x_i)) - 1 + \sum_{i=1}^{N} f(x_i)\right], \qquad x \in E.$$

$$\mathtt{v}_t[f](x) = \hat{\mathtt{P}}_t[f](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f]]\right](x)\mathrm{d}s \quad \text{and} \quad \mathtt{T}_t[f](x) = \mathtt{P}_t[f] + \int_0^t \mathtt{P}_s\left[\mathtt{F}\mathtt{T}_{t-s}[f]\right](x)\mathrm{d}s$$

gives us.....

Lemma

For all $g \in B_1^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{u}_t[g](x)$ satisfies

$$\mathtt{u}_t[g](x) = \mathtt{T}_t[1 - g](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[g]]\right](x)\mathrm{d}s.$$

Nonlinear to k-th moment evolution equation

In terms of our new semigroup equation:

$$\mathtt{T}_t^{(k)}[f](x) = (-1)^{k+1}\frac{\partial^k}{\partial\theta^k}\mathtt{u}_t[\mathrm{e}^{-\theta f}](x)\Big|_{\theta=0}.$$

Theorem

Fix $k \geq 2$. Assuming (H1) and (H2)$_k$, with the additional assumption that

$$\sup_{x\in E, s\leq t} \mathtt{T}_s^{(\ell)}[f](x) < \infty, \qquad \ell \leq k-1, f \in B^+(E), t \geq 0, \tag{2}$$

it holds that

$$\mathtt{T}_t^{(k)}[f](x) = \mathtt{T}_t[f^k](x) + \int_0^t \mathtt{T}_s\left[\beta\eta_{t-s}^{(k-1)}[f]\right](x)\,\mathrm{d}s, \qquad t \geq 0, \tag{3}$$

where

$$\eta_{t-s}^{(k-1)}[f](x) = \mathcal{E}_x\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j)\right],$$

and $[k_1,\dots,k_N]_k^2$ is the set of all non-negative N-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the k_i are strictly positive.

Induction: $k \mapsto k+1$

- Suppose the result is true for the first k moments.
- Recall $\mathtt{T}_t[f](x) \to \langle f, \tilde{\varphi} \rangle \varphi(x)$ so that, for $k \geq 2$,

$$\lim_{t\to\infty} t^{-k} \mathtt{T}_t[f^{k+1}](x) \to 0$$

- Hence:

$$\lim_{t\to\infty} t^{-k} \mathtt{T}_t^{(k+1)}[f](x)$$

$$= \lim_{t\to\infty} t^{-k} \int_0^t \mathtt{T}_s \left[\mathcal{E}_\cdot \left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \prod_{j=1}^N \mathtt{T}_{t-s}^{(k_j)}[f](x_j) \right]\right](x)\mathrm{d}s$$

$$= \lim_{t\to\infty} t^{-(k-1)} \int_0^1 \mathtt{T}_{ut} \left[\mathcal{E}_\cdot \left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \prod_{j=1}^N \mathtt{T}_{t(1-u)}^{(k_j)}[f](x_j) \right]\right](x)\mathrm{d}u$$

$$= \lim_{t\to\infty} \int_0^1 \mathtt{T}_{ut} \left[\mathcal{E}_\cdot \left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}} \prod_{j=1}^N \frac{\mathtt{T}_{t(1-u)}^{(k_j)}[f](x_j)}{(t(1-u))^{k_j-1}} \right]\right](x)\mathrm{d}u$$

Rough outline of the induction: $k \mapsto k+1$

- From the last slide:

$$\lim_{t\to\infty} t^{-k}\mathtt{T}_t^{(k+1)}[f](x)$$

$$= \lim_{t\to\infty}\int_0^1 \mathtt{T}_{ut}\left[\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \frac{(t(1-u))^{k+1-\#\{j:k_j>0\}}}{t^{k-1}} \prod_{j=1}^N \frac{\mathtt{T}^{(k_j)}_{t(1-u)}[f](x_j)}{(t(1-u))^{k_j-1}}\right]\right](x)\mathrm{d}u$$

- Largest terms in blue correspond to those summands for which $\#\{j : k_j > 0\} = 2$
- The induction hypothesis plus $\sum_{i=1}^N k_j = k+1$ ensures that the product term is asymptotically a constant
- The simple identity

$$\sum_{[k_1,\dots,k_N]^2_{k+1}} \binom{k+1}{k_1,\dots,k_N} \le N^{k+1}$$

  shows us where the need for the hypothesis (H2) comes in.
- We need an ergodic limit theorem that reads (roughly): If

$$F(x,u) := \lim_{t\to\infty} F(x,u,t), \qquad x\in E, u\in[0,1],$$

  "uniformly" for $(u,x)\in[0,1]\times E$, then

$$\lim_{t\to\infty}\int_0^1 \mathtt{T}_{ut}[F(\cdot,u,t)](x)\mathrm{d}u = \int_0^1 \langle\tilde{\varphi}, F(\cdot,u)\rangle\mathrm{d}u$$

  "uniformly" for $x\in E$.

What about the occupation measure?

- Let us define the running occupation of the branching particle system via

$$\int_0^t X_s(\cdot)\mathrm{d}s, \qquad t \geq 0.$$

- What can we say about its moments?

$$\mathtt{M}_t^{(k)}[g](x) := \mathbb{E}_{\delta_x}\left[\left(\int_0^t \langle g, X_s\rangle \mathrm{d}s\right)^k\right], \qquad x \in E, g \in B^+(E), k \geq 1, t \geq 0.$$

- We know that the pair

$$\left(X_t, \int_0^t X_s \mathrm{d}s\right)$$

is Markovian and that its semigroup

$$\mathtt{v}_t[f,g] = \mathbb{E}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle\,\mathrm{d}\,s}\right], \qquad t \geq 0,\ x \in E,\ f, g \in B^+(E),$$

solves

$$\mathtt{v}_t[f,g](x) = \hat{\mathtt{P}}_t[\mathrm{e}^{-f}](x) + \int_0^t \mathtt{P}_s\left[\mathtt{G}[\mathtt{v}_{t-s}[f,g]) - g\mathtt{v}_{t-s}[f,g]\right](x)\mathrm{d}s.$$

Playing the same game as before

Define a variant of the non-linear evolution equation associated with $(X_t, \int_0^s X_s \mathrm{d}s)$ via

$$\mathtt{u}_t[f,g](x) = \mathbb{E}_{\delta_x}\left[1 - \mathrm{e}^{-\langle f, X_t\rangle - \int_0^t \langle g, X_s\rangle \,\mathrm{d}\,s}\right], \qquad t \geq 0,\ x \in E,\ ||f|| < \infty, ||g|| < \infty.$$

For $f \in B_1^+(E)$, define

$$\mathtt{A}[f](x) = \beta(x)\mathcal{E}_x\left[\prod_{i=1}^N (1 - f(x_i)) - 1 + \sum_{i=1}^N f(x_i)\right], \qquad x \in E.$$

A re-arrangement of the joint semigroup of $(X_t, \int_0^t X_s \mathrm{d}s)$ is captured by:

Lemma

For all $f, g \in B^+(E)$, $x \in E$ and $t \geq 0$, the non-linear semigroup $\mathtt{u}_t[f,g](x)$ satisfies

$$\mathtt{u}_t[f,g](x) = \mathtt{T}_t[1 - \mathrm{e}^{-f}](x) - \int_0^t \mathtt{T}_s\left[\mathtt{A}[\mathtt{u}_{t-s}[f,g]] - g(1 - \mathtt{u}_{t-s}[f,g])\right](x)\mathrm{d}s.$$

Theorem: Critical case ($\lambda = 0$)

Suppose that (H1) holds along with (H2) for $k \geq 2$ and $\lambda = 0$. Define

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| t^{-(2\ell-1)} \varphi(x)^{-1} \mathrm{M}_t^{(\ell)}[g](x) - 2^{-(\ell-1)} \ell! \, \langle g, \tilde{\varphi} \rangle^{\ell} \langle \mathbb{V}[\varphi], \tilde{\varphi} \rangle^{\ell-1} L_\ell \right|,$$

where $L_1 = 1$ and L_k is defined through the recursion $L_k = (\sum_{i=1}^{k-1} L_i L_{k-i})/(2k-1)$. Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

Theorem

Suppose that

- *(H1) holds (mean-semigroup ergodicity),*
- *the number of offspring is uniformly bounded by a constant N_{max},*
- *for all t sufficiently large*
$$\sup_{x\in E} \mathbb{P}_{\delta_x}(t < \zeta) < 1,$$
- *there exists a constant $C > 0$ such that for all $g \in B^+(E)$,*

$$\langle \tilde{\varphi}, \beta\mathbb{V}[g]\rangle \geq C\langle \tilde{\varphi}, g\rangle^2, \qquad \textit{where} \qquad \mathbb{V}[g](x) = \beta(x)\mathcal{E}_x\left[\langle g, \mathcal{Z}\rangle^2 - \langle g^2, \mathcal{Z}\rangle\right]$$

Then

$$\lim_{t\to\infty} t\mathbb{P}_{\delta_x}(\zeta > t) = \frac{2\varphi(x)}{\langle \tilde{\varphi}, \beta\mathbb{V}[g]\rangle},$$

$$\lim_{t\to\infty} \mathbb{E}_{\delta_x}\left[\left(\frac{\langle f, X_t\rangle}{t}\right)^k \middle| \zeta > t\right] = k!\,\langle f, \tilde{\varphi}\rangle^k \left(\frac{\langle \tilde{\varphi}, \beta\mathbb{V}[g]\rangle}{2}\right)^k$$

and hence

$$\mathtt{Law}\left(\frac{\langle f, X_t\rangle}{t} \middle| \zeta > t\right) \to \exp\left(\frac{2}{\langle \tilde{\varphi}, \beta\mathbb{V}[g]\rangle\langle f, \tilde{\varphi}\rangle}\right)$$

Thank you!

In case you asked the question about non-criticality

Theorem: Supercritical ($\lambda > 0$)

Suppose that (H1) holds along with $(\mathrm{H2})_k$ for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} \mathrm{e}^{-\ell\lambda t} \mathrm{T}_t^{(\ell)}[f](x) - \ell! \langle f, \tilde{\varphi} \rangle^{\ell} L_\ell(x) \right|,$$

where $L_1(x) = 1$ and we define iteratively for $k \geq 2$,

$$L_k(x) = \int_0^\infty \mathrm{e}^{-\lambda_* ks} \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E}_\cdot \left[\sum_{[k_1, \ldots, k_N]_k^2} \prod_{\substack{j=1 \\ j : k_j > 0}}^{N} \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \mathrm{d}s,$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $\mathrm{e}^{\lambda kt}$ (i.e. the first moment to the power k)"

THEOREM: SUBCRITICAL ($\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||f|| \leq 1} \left| \varphi(x)^{-1} \mathrm{e}^{-\lambda t} \mathrm{T}_t^{(\ell)}[f](x) - L_\ell \right|,$$

where we define iteratively $L_1 = \langle f, \tilde{\varphi} \rangle$ and for $k \geq 2$,

$$L_k = \tilde{\varphi}[f^k] + \int_0^\infty \mathrm{e}^{-\lambda_* s} \tilde{\varphi}\left[\gamma \mathcal{E}_\cdot \left[\sum_{[k_1,\ldots,k_N]_k^2} \binom{k}{k_1, \ldots, k_N} \prod_{\substack{j=1 \\ j: k_j > 0}}^N \psi_s^{(k_j)}[f](x_j) \right] \right] \mathrm{d}s.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

"At subcriticality the k-th moment scales like $\mathrm{e}^{\lambda t}$ (i.e. like the first moment)"

Theorem: Supercritical case ($\lambda > 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda > 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1} \mathrm{e}^{-\ell\lambda t} \mathrm{M}_t^{(\ell)}[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $L_1 = 1/\lambda$ and for $k \geq 2$ we define iteratively,

$$L_k(x) = \int_0^\infty \mathrm{e}^{-\lambda_* ks} \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E}. \left[\sum_{[k_1, \ldots, k_N]_k^2} \prod_{\substack{j=1 \\ j:k_j > 0}}^N \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \mathrm{d}s,$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$

Theorem: Subcritical case ($\lambda < 0$)

Suppose that (H1) holds along with (H2) for some $k \geq 2$ and $\lambda < 0$. Redefine

$$\Delta_t^{(\ell)} = \sup_{x \in E, ||g|| \leq 1} \left| \varphi(x)^{-1} \mathrm{M}_t^{(\ell)}[g](x) - \ell! \langle g, \tilde{\varphi} \rangle^\ell L_\ell(x) \right|,$$

where $||g|| < \infty$, $L_1 = 1/|\lambda|$ and for $k \geq 2$, the constants L_k are defined recursively via

$$L_k(x) = \int_0^\infty \varphi(x)^{-1} \psi_s \left[\gamma \mathcal{E}_\cdot \left[\sum_{[k_1, \ldots, k_N]_k^2} \binom{k}{k_1, \ldots, k_N} \prod_{\substack{j=1 \\ j : k_j > 0}}^N \varphi(x_j) L_{k_j}(x_j) \right] \right](x) \, \mathrm{d}s$$

$$- k \int_0^\infty \varphi(x)^{-1} \psi_s \left[g \varphi L_{k-1} \right](x) \, \mathrm{d}s.$$

Then, for all $\ell \leq k$

$$\sup_{t \geq 0} \Delta_t^{(\ell)} < \infty \text{ and } \lim_{t \to \infty} \Delta_t^{(\ell)} = 0.$$