Wiener-Hopf Factorisations for Lévy processes

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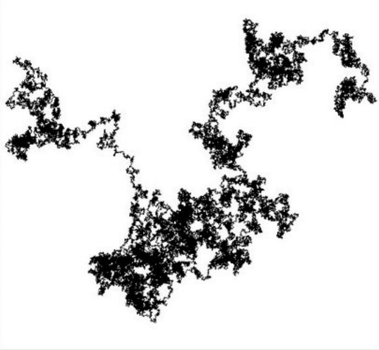
YOUR FAVOURITE MARKOV PROCESS

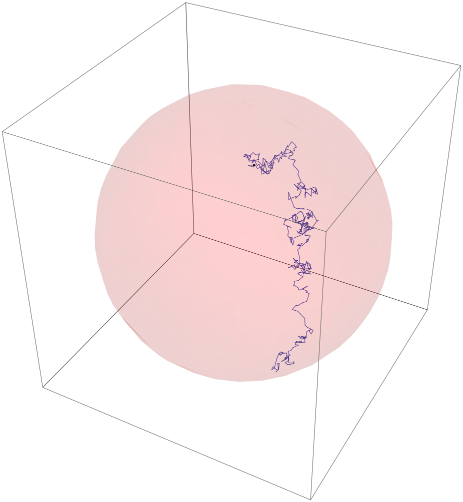
Brownian motion in $\mathbb{R}^d$, $B := \{B_t : t \geq 0\}$, has the defining property that:

- For $t > s > 0$, $B_t - B_s \overset{d}{=} B_{t-s} \sim \mathcal{N}_d(0, \mathbf{I}(t-s))$
- For $t > s > 0$, $B_t - B_s$ is independent of $\{B_u : u \leq s\}$
- B has continuous paths



With Brownian motion, you can.....

Solve a Dirichlet boundary value problem....



WITH BROWNIAN MOTION, YOU CAN.....

Try to model the stock market........

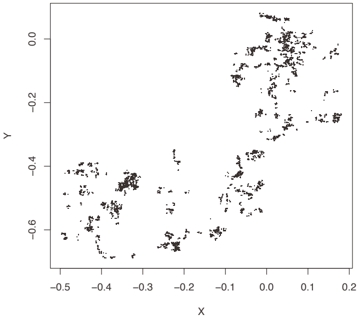
With Brownian motion, you can.....

Try to model the stock market........and fail....

Take it to the next level: Lévy processes

The last 30 years has seen interest in bigger class of Lévy processes (that contains Brownian motion). An $\mathbb{R}^d$ valued Lévy process, $X := \{X_t : t \geq 0\}$ has almost the same properties as Brownian motion:

- For $t > s > 0$, $X_t - X_s \stackrel{d}{=} X_{t-s}$
- For $t > s > 0$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$
- X has paths that are right-continuous with left limits



- **Roughly speaking:** A Lévy process is made up of a linear Brownian motion plus a process of (up to a countable infinity) of jumps (over any finite time horizon), e.g. in one-dimension

$$X_t = at + \sigma B_t + J_t.$$

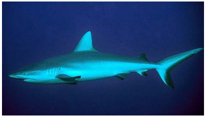
- The process is entirely characterised by: a, σ and Π, the latter is a measure on $\mathbb{R}\backslash\{0\}$.
- The measure Π can be thought as a rate measure:

$$P(\text{Jump of size } x \text{ arrives at time } t) = \Pi(\mathrm{d}x)\mathrm{d}t + o(\mathrm{d}t).$$

With Lévy processes you can.....

Try to model the foraging/flight/feeding patterns of various animals......






WITH LÉVY PROCESSES YOU CAN.....

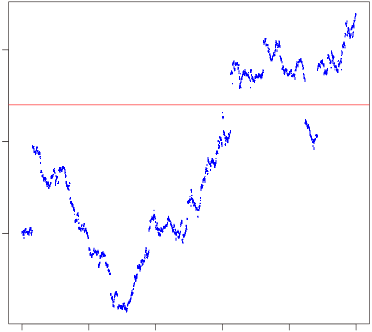
Try to model the foraging/flight/feeding patterns of various animals......

.....as well as humans....

IKEA®

WITH LÉVY PROCESSES YOU CAN.....

Try to model the stock market .....



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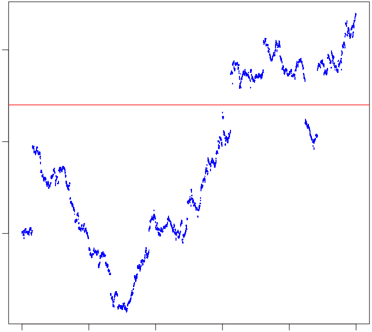
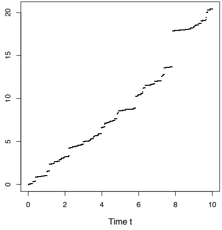


Characteristic vs Laplace exponent

- To characterise a Lévy process, its enough to know its distribution at each fixed time $\mathbb{E}[e^{i\theta X_t}] = e^{-\Psi(\theta)t}, t \geq 0, \theta \in \mathbb{R}$
- Sometimes a Laplace exponent will do $\mathbb{E}[e^{\lambda X_t}] = e^{\psi(\lambda)t}$ where, if it exists, $\psi(\lambda) = -\Psi(-i\lambda)$ in the sense of an analytic extension.
- A good case in point is that of a subordinator: a Lévy process with non-decreasing paths. In which case ψ is the negative of a Bernstein function:

$$\psi(\lambda) = -\left(\delta\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi(dx), \right)$$

where $\delta \geq 0$, and $\int_{(0,\infty)} (1 \wedge x)\Pi(dx) < \infty$.



The WHF: What Alexey should have told you

- Let us extend the definition of a Lévy process and include the possibility of killing, so

$$X_t = \begin{cases} \text{Lévy process } \tilde{X}_t & t < \mathbf{e}_q \\ \text{Cemetery state } \partial & t \geq \mathbf{e}_q \end{cases}$$

  where $\mathbf{e}_q$ is independent and exponentially distributed with parameter $q \geq 0$. We also take as a convention $\mathbf{e}_0 = \infty$.

- In that case, if $\tilde{\Psi}$ is the exponent of $\tilde{X}$ then

$$\Psi = q + \tilde{\Psi}$$

- For a given characteristic exponent of a Lévy process, Ψ, there exist unique Bernstein functions, κ and $\hat{\kappa}$ such that, up to a multiplicative constant,

$$\Psi(\theta) = \kappa(-\mathrm{i}\theta)\hat{\kappa}(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}.$$

- As Bernstein functions, κ and $\hat{\kappa}$ can be seen as the Laplace exponents of (killed) subordinators.
- The probabilistic significance of these subordinators, is that their range corresponds precisely to the range of the running maximum of X and of $-X$ respectively.

The WHF: What Alexey may not have told you

- Until around 5-10 years ago (basically not before Alexey started to think about it), It was very difficult to find non-trivial (jumping in two directions) examples of Lévy processes.
- This talk: expose a recent method that brought a number of examples forward

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α-STABLE PROCESS AS A LÉVY PROCESS

A Lévy process X with probabilities $(\mathbb{P}_x, x \in \mathbb{R})$ is called (strictly) α-stable if it is also a self-similar Markov process, i.e. for all $c > 0$

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

- Necessarily $\alpha \in (0, 2]$. [$\alpha = 2 \to$ BM, exclude this.]
- Lévy measure takes the form

$$\frac{\Gamma(1+\alpha)}{\pi} \frac{1}{|x|^{1+\alpha}} \left(\sin(\pi\alpha\rho)\mathbf{1}_{\{x>0\}} + \sin(\pi\alpha\hat{\rho})\mathbf{1}_{\{x<0\}}\right)$$

- The characteristic exponent $\Psi(\theta) := -t^{-1} \log \mathbb{E}(\mathrm{e}^{\mathrm{i}\theta X_t})$ satisfies

$$\Psi(\theta) = |\theta|^{\alpha}(\mathrm{e}^{\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta>0)} + \mathrm{e}^{-\pi\mathrm{i}\alpha(\frac{1}{2}-\rho)}\mathbf{1}_{(\theta<0)}), \qquad \theta \in \mathbb{R}.$$

  where $\rho = \mathrm{P}_0(X_t \geq 0)$ will frequently appear as will $\hat{\rho} = 1 - \rho$
- Assume jumps in both directions ($0 < \alpha\rho, \alpha\hat{\rho} < 1$), then, up to multiplicative constants

$$\kappa(\lambda) = \lambda^{\alpha\rho} \text{ and } \hat{\kappa}(\lambda) = \lambda^{\alpha\hat{\rho}}, \qquad \lambda \geq 0.$$

α-STABLE PROCESS AND POSITIVE SELF-SIMILAR MARKOV PROCESS

- We are interested in how to examine the stable process, not as a Lévy process, but through the theory of positive self similar Markov processes (pssMp)
- i.e. $[0, \infty)$-valued (regular, strong) Markov process with probabilities $(\mathbb{P}_x, x \in \mathbb{R})$, for which there is an $\alpha > 0$ such that, for all $c > 0$,

$$(cX_{c^{-\alpha}t}, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal in law to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{cx}.$$

- The characterisation of this class can help us see a bit deeper into the class of α-stable processes, out of which we will find some new WHFs.

Notation

- Use $\xi := \{\xi_t : t \geq 0\}$ to denote a Lévy process which is killed and sent to the cemetery state $-\infty$ at an independent and exponentially distributed random time, $\mathbf{e}_q$, with rate in $q \in [0, \infty)$. The characteristic exponent of ξ is thus written

$$-\log E(\mathrm{e}^{\mathrm{i}\theta\xi_1}) = \Psi(\theta) = q + \text{Lévy–Khintchine}$$

- Define the associated integrated exponential Lévy process

$$I_t = \int_0^t \mathrm{e}^{\alpha\xi_s} \mathrm{d}s, \qquad t \geq 0. \tag{1}$$

and its limit, $I_\infty := \lim_{t\uparrow\infty} I_t$.

- Also interested in the inverse process of I:

$$\varphi(t) = \inf\{s > 0 : I_s > t\}, \qquad t \geq 0. \tag{2}$$

As usual, we work with the convention $\inf \emptyset = \infty$.

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Lamperti transform for POSITIVE ssMp: Part (i)

Fix $\alpha > 0$. If $Z^{(x)}$, $x > 0$, is a positive self-similar Markov process with index of self-similarity α, then up to absorption at the origin, it can be represented as follows. For $x > 0$,

$$Z^{(x)}_t \mathbf{1}_{(t<\zeta^{(x)})} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}, \qquad t \geq 0,$$

where $\zeta^{(x)} = \inf\{t > 0 : Z^{(x)}_t = 0\}$ and either

(1) $\zeta^{(x)} = \infty$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\limsup_{t\uparrow\infty} \xi_t = \infty$,

(2) $\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} = 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process satisfying $\lim_{t\uparrow\infty} \xi_t = -\infty$, or

(3) $\zeta^{(x)} < \infty$ and $Z^{(x)}_{\zeta^{(x)}-} > 0$ almost surely for all $x > 0$, in which case ξ is a Lévy process killed at an independent and exponentially distributed random time.

In all cases, we may identify $\zeta^{(x)} = x^\alpha I_\infty$.

Lamperti transform for POSITIVE ssMp: Part (ii)

Conversely, suppose that ξ is a given (killed) Lévy process. For each $x > 0$, define

$$Z_t^{(x)} = x\exp\{\xi_{\varphi(x^{-\alpha}t)}\}\mathbf{1}_{(t<x^\alpha I_\infty)}, \qquad t \geq 0.$$

Then $Z^{(x)}$ defines a positive self-similar Markov process, up to its absorption time $\zeta^{(x)} = x^\alpha I_\infty$, with index α.

LAMPERTI TRANSFORM FOR POSITIVE SSMP

$(Z, \mathrm{P}_x)_{x>0}$ pssMp

$Z_t = \exp(\xi_{S(t)})$,

S a random time-change

$\leftrightarrow$

$(\xi, \mathbb{P}_y)_{y \in \mathbb{R}}$ killed Lévy

$\xi_s = \log(Z_{T(s)})$,

T a random time-change

$$\left.\begin{array}{c} Z \text{ never hits zero} \\ Z \text{ hits zero continuously} \\ Z \text{ hits zero by a jump} \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \\ \xi \text{ is killed} \end{array}\right.$$

Lamperti transform for POSITIVE ssMp

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STABLE PROCESS KILLED ON ENTRY TO $(-\infty, 0)$

- Suppose that X is a stable process with two-sided jumps. Such processes always pass below the origin by a jump.
- This puts $Z^*_t := X_t \mathbf{1}_{(\underline{X}_t > 0)}$, $t \geq 0$, in the class of pssMp for which the underlying Lévy process experiences exponential killing.
- Write $\xi^* = \{\xi^*_t : t \geq 0\}$ for the underlying (killed) Lévy process.
- Its characteristic exponent is given by

$$\Psi^*(z) = \frac{\Gamma(\alpha - \mathrm{i}z)}{\Gamma(\alpha\hat{\rho} - \mathrm{i}z)} \times \frac{\Gamma(1 + \mathrm{i}z)}{\Gamma(1 - \alpha\hat{\rho} + \mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- A rare example where we can see the factorisation as

$$\kappa(\lambda) = \frac{\Gamma(\alpha - \lambda)}{\Gamma(\alpha\hat{\rho} - \lambda)}$$

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Censored stable processes

- Start with X, the stable process.
- Let $A_t = \int_0^t \mathbf{1}_{(X_t>0)}\,\mathrm{d}t$.
- Let γ be the right-inverse of A, and put $\check{Z}_t := X_{\gamma(t)}$.
- Finally, make zero an absorbing state: $Z_t = \check{Z}_t \mathbf{1}_{(t<T_0)}$ where

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

  Note $T_0 < \infty$ a.s. if and only if $\alpha \in (1,2)$ and otherwise $T_0 = \infty$ a.s.
- This is the censored stable process.
- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- Suppose that the underlying Lévy process for the censored stable process is denoted by $\overset{\smile}{\xi}$. Then its characteristic exponent is given by

$$\overset{\smile}{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

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Take care!

- The characteristic exponent of the censored stable:

$$\overrightarrow{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(-\mathrm{i}z)} \times \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(1-\alpha+\mathrm{i}z)}, \qquad z \in \mathbb{R}.$$

- if $\alpha \in (0,1]$, then the factorisation is the one that you see either side of the multiplication sign.
- If $\alpha \in (1,2)$, then we have the term $\Gamma(1-\alpha+\mathrm{i}z)$ which (in Laplace format) can take negative values (remember we are looking for the product of Bernstein functions κ, $\hat{\kappa}$).
- The factorisation for $\alpha \in (1,2)$ turns out to be

$$\overrightarrow{\Psi}(z) = \frac{\Gamma(\alpha\rho - \mathrm{i}z)}{\Gamma(1-\mathrm{i}z)}(\alpha - 1 - \mathrm{i}z) \times \frac{\Gamma(1-\alpha\rho+\mathrm{i}z)}{\Gamma(2-\alpha+\mathrm{i}z)}(\mathrm{i}z), \qquad z \in \mathbb{R}.$$

The radial part of a stable process

- Suppose that X is a symmetric stable process, i.e $\rho = 1/2$.
- We know that $|X|$ is a pssMp.
- Suppose that the underlying Lévy process for $|X|$ is written $\xi^{\odot}$, then it characteristic exponent is given by

$$\Psi^{\odot}(z) = 2^{\alpha}\frac{\Gamma(\frac{1}{2}(-\mathrm{i}z+\alpha))}{\Gamma(-\frac{1}{2}\mathrm{i}z)} \times \frac{\Gamma(\frac{1}{2}(\mathrm{i}z+1))}{\Gamma(\frac{1}{2}(\mathrm{i}z+1-\alpha))}, \qquad z \in \mathbb{R}.$$

Hypergeometric Lévy processes (Alexey and others)

For $(\beta, \gamma, \hat{\beta}, \hat{\gamma})$ in

$$\{\ \beta \leq 2,\ \gamma, \hat{\gamma} \in (0,1)\ \hat{\beta} \geq -1,\ \text{and}\ 1 - \beta + \hat{\beta} + \gamma \wedge \hat{\gamma} \geq 0\ \}$$

there exists a (killed) Lévy process, henceforth referred to as a hypergeometric Lévy process, having the characteristic function

$$\Psi(z) = \frac{\Gamma(1 - \beta + \gamma - \mathrm{i}z)}{\Gamma(1 - \beta - \mathrm{i}z)} \frac{\Gamma(\hat{\beta} + \hat{\gamma} + \mathrm{i}z)}{\Gamma(\hat{\beta} + \mathrm{i}z)} \qquad z \in \mathbb{R}.$$

The Lévy measure of Y has a density with respect to Lebesgue measure is given by

$$\pi(x) = \begin{cases} -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \hat{\gamma})\Gamma(-\gamma)} \mathrm{e}^{-(1-\beta+\gamma)x} {}_2F_1\left(1 + \gamma, \eta; \eta - \hat{\gamma}; \mathrm{e}^{-x}\right), & \text{if } x > 0, \\ -\dfrac{\Gamma(\eta)}{\Gamma(\eta - \gamma)\Gamma(-\hat{\gamma})} \mathrm{e}^{(\hat{\beta}+\hat{\gamma})x} {}_2F_1\left(1 + \hat{\gamma}, \eta; \eta - \gamma; \mathrm{e}^{x}\right), & \text{if } x < 0, \end{cases}$$

where $\eta := 1 - \beta + \gamma + \hat{\beta} + \hat{\gamma}$, for $|z| < 1$, ${}_2F_1(a, b; c; z) := \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} z^k$.

Deep Factorisation

Deep factorisation of the stable process

- All of the previous examples were generated by path functionals of stable processes which were non-negative + identification of them as pssMp.
- Another factorisation also exists, which is more ‘deeply’ embedded in the stable process.
- Based around the representation of the stable process as a real-valued self-similar Markov process (rssMp):

An $\mathbb{R}$-valued regular strong Markov process $(X_t : t \geq 0)$ with probabilities $\mathbb{P}_x$, $x \in \mathbb{R}$, is a rssMp if, there is a stability index $\alpha > 0$ such that, for all $c > 0$ and $x \in \mathbb{R}$,

$$(cX_{tc^{-\alpha}} : t \geq 0) \text{ under } \mathrm{P}_x \text{ is } \mathrm{P}_{cx}.$$

A special Markov additive processes

- Let $(J(t), t \geq 0)$ be a continuous-time, irreducible Markov chain on $\{1, -1\}$.
- process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbf{P}_{x,i}$, $x \in \mathbb{R}, i = \pm 1$, if, for any $i = \pm 1$, $s, t \geq 0$: Given $\{J(t) = i\}$,
- $(\xi(t+s) - \xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
- $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$.

A SPECIAL MARKOV ADDITIVE PROCESSES

- The pair (ξ, J) can be represented as follows:
  - there exist a sequence of iid Lévy processes $(\xi_i^n, n \geq 0)$
  - and a sequence of iid random variables $(U_{ij}^n, n \geq 0)$, independent of the chain J,
  - such that if $T_0 = 0$ and $(T_n, n \geq 1)$ are the jump times of J,

  the process ξ has the representation

$$\xi(t) = \mathbf{1}_{(n>0)}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

  for $t \in [T_n, T_{n+1})$, $n \geq 0$.

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\boldsymbol{Q} = (q_{ij})_{i,j \in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$ $(i \neq j)$ and define $G_{ii}(z) = 1$.
- Let

  $$\boldsymbol{F}(z) = \operatorname{diag}(\psi_1(z), \ldots, \psi_N(z)) + \boldsymbol{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

  $$\mathbb{E}_i(e^{z\xi(t)}; J(t) = j) = \left(e^{\boldsymbol{F}(z)t}\right)_{i,j}, \qquad i, j \in E,$$

  (when it exists).

Lamperti-Kiu transform

- Let
$$X_t = |x|e^{\xi(\tau(t))}J(\tau(t)) \qquad 0 \le t < T_0,$$
where
$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))\mathrm{d}u > t|x|^{-\alpha}\right\}$$
and
$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}\mathrm{d}u.$$
- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \ge 0)$ under P_x is P_{cx}.
- The converse is also true (more or less - some slight adjustment is needed to the definition of the MAP).

AN α-STABLE PROCESS IS A RSSMP

- An α-stable process with two-sided jumps is a rssMp. Remarkably we can compute precisely its matrix exponent explicitly
- Denote the underlying MAP (ξ, J), we prefer to give the matrix exponent of (ξ, J) as follows:

$$\boldsymbol{F}(z) = \begin{bmatrix} -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1-\alpha\hat{\rho}+z)} & \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha\hat{\rho})} \\ \\ \dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1+z)}{\Gamma(\alpha\rho - z)\Gamma(1-\alpha\rho+z)} \end{bmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$.

MAP WHF

- For $\theta \in \mathbb{R}$, up to a multiplicative factor,

$$-\boldsymbol{F}(\mathrm{i}\theta) = \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} \hat{\boldsymbol{\kappa}}(\mathrm{i}\theta)^{\mathrm{T}} \boldsymbol{\Delta}_{\boldsymbol{\pi}} \boldsymbol{\kappa}(-\mathrm{i}\theta),$$

  where $\boldsymbol{\Delta}_{\boldsymbol{\pi}} = \mathrm{diag}(\boldsymbol{\pi})$, $\boldsymbol{\pi}$ is the stationary distribution of $\boldsymbol{Q}$, $\hat{\boldsymbol{\kappa}}$ plays the role of $\boldsymbol{\kappa}$, but for the dual MAP to (ξ, J).
- The dual process, or time-reversed process is equal in law to the MAP with exponent

$$\hat{\boldsymbol{F}}(z) = \boldsymbol{\Delta}_{\pi}^{-1} \boldsymbol{F}(-z)^{\mathrm{T}} \boldsymbol{\Delta}_{\pi},$$

- It turns out to be more natural to consider the factorisation in the form

$$-\boldsymbol{F}(\mathrm{i}\theta)^{-1} = \boldsymbol{\kappa}(-\mathrm{i}\theta)^{-1} \boldsymbol{\Delta}_{\boldsymbol{\pi}}^{-1} [\hat{\boldsymbol{\kappa}}(\mathrm{i}\theta)^{-1}]^{\mathrm{T}} \boldsymbol{\Delta}_{\boldsymbol{\pi}}$$

Suppose that X is an α-stable process then we have that the factors $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{\kappa}}$ are given as follows. For $a, b, c \in \mathbb{R}$ define

$$\Psi(a,b,c) := \int_0^1 u^a (1-u)^b (1+u)^c \underset{\cdot}{u}. \tag{3}$$

Then, up to the multiplicative constant $2^{-\alpha}\Gamma(1-\alpha)^{-1}$,

$$\boldsymbol{\kappa}^{-1}(\lambda) = \begin{pmatrix} \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)} \Psi(\lambda-1, \alpha\rho-1, \alpha\hat{\rho}) & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)} \Psi(\lambda-1, \alpha\rho, \alpha\hat{\rho}-1) \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})} \Psi(\lambda-1, \alpha\hat{\rho}, \alpha\rho-1) & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})} \Psi(\lambda-1, \alpha\hat{\rho}-1, \alpha\rho) \end{pmatrix}$$

and

$$\hat{\boldsymbol{\kappa}}^{-1}(\lambda) = \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})} \Psi(\lambda-\alpha, \alpha\hat{\rho}-1, \alpha\rho) & \frac{\sin(\alpha\pi\hat{\rho})\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat{\rho})} \Psi(\lambda-\alpha, \alpha\hat{\rho}, \alpha\rho-1) \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat{\rho})}{\sin(\alpha\pi\hat{\rho})\Gamma(\alpha\rho)} \Psi(\lambda-\alpha, \alpha\rho, \alpha\hat{\rho}-1) & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)} \Psi(\lambda-\alpha, \alpha\rho-1, \alpha\hat{\rho}) \end{pmatrix}.$$

Deep-WHF stable $\alpha \in (1,2)$

$$
\begin{aligned}
&\boldsymbol{\kappa}^{-1}(\lambda)\\
&\quad= \frac{\alpha-1}{2}\begin{pmatrix}\Psi(\lambda-1,\alpha\rho-1,\alpha\hat{\rho}) & \Psi(\lambda-1,\alpha\rho,\alpha\hat{\rho}-1)\\ \Psi(\lambda-1,\alpha\hat{\rho},\alpha\rho-1) & \Psi(\lambda-1,\alpha\hat{\rho}-1,\alpha\rho)\end{pmatrix}\\
&-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}\Psi(\lambda-1,\alpha\rho-1,\alpha\hat{\rho}-1) & \Psi(\lambda-1,\alpha\rho-1,\alpha\hat{\rho}-1)\\ \Psi(\lambda-1,\alpha\hat{\rho}-1,\alpha\rho-1) & \Psi(\lambda-1,\alpha\hat{\rho}-1,\alpha\rho-1)\end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{\boldsymbol{\kappa}}^{-1}(\lambda)\\
&= \frac{\alpha-1}{2}\begin{pmatrix}\Psi(\lambda-\alpha,\alpha\hat{\rho}-1,\alpha\rho) & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}\Psi(\lambda-\alpha,\alpha\hat{\rho},\alpha\rho-1)\\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}\Psi(\lambda-\alpha,\alpha\rho,\alpha\hat{\rho}-1) & \Psi(\lambda-\alpha,\alpha\rho-1,\alpha\hat{\rho})\end{pmatrix}\\
&-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}\Psi(\lambda-\alpha,\alpha\hat{\rho}-1,\alpha\rho-1) & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}\Psi(\lambda-\alpha,\alpha\hat{\rho}-1,\alpha\rho-1)\\ \frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}\Psi(\lambda-\alpha,\alpha\rho-1,\alpha\hat{\rho}-1) & \Psi(\lambda-\alpha,\alpha\rho-1,\alpha\hat{\rho}-1)\end{pmatrix}
\end{aligned}
$$

Thank you!