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Attraction to and repulsion from a subset of the unit sphere for isotropic stable Lévy processes

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Abstract

Taking account of recent developments in the representation of d -dimensional isotropic stable Lévy processes as self-similar Markov processes, we consider a number of new ways to condition its path. Suppose that S is a region of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. We construct the aforesaid stable Lévy process conditioned to approach S continuously from either inside or outside of the sphere. Additionally, we show that these processes are in duality with the stable process conditioned to remain inside the sphere and absorb continuously at the origin and to remain outside of the sphere, respectively. Our results extend the recent contributions of Döring and Weissman (2020), where similar conditioning is considered, albeit in one dimension as well as providing analogues of the same classical results for Brownian motion, cf. Doob (1957). As in Döring and Weissman (2020), we appeal to recent fluctuation identities related to the deep factorisation of stable processes, cf. Kyprianou (2016), Kyprianou et al. (2020) and Kyprianou et al. (2017).

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1. Introduction

Let $X = (X_t, t \geq 0)$ be a d -dimensional stable Lévy process with probabilities $(\mathbb{P}_x, x \in \mathbb{R}^d)$. This means that X has càdlàg paths with stationary and independent increments as well as

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respecting a property of self-similarity: There is an $\alpha > 0$ such that, for $c > 0$, and $x \in \mathbb{R}^d$, under \mathbb{P}_x , the law of $(cX_{c^{-\alpha}t}, t \geq 0)$ is equal to \mathbb{P}_{cx} . It turns out that stable Lévy processes necessarily have the scaling index $\alpha \in (0, 2]$. The case $\alpha = 2$ pertains to a standard d -dimensional Brownian motion, thus has a continuous path. The processes we construct are arguably less interesting in the diffusive setting and thus we restrict ourselves to the isotropic pure jump setting of $\alpha \in (0, 2)$ in dimension $d \geq 2$.

To be more precise, this means, for all orthogonal transformations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\text{the law of } (UX_t, t \geq 0) \text{ under } \mathbb{P}_x \text{ is equal to } (X_t, t \geq 0) \text{ under } \mathbb{P}_{Ux}.$$

For convenience, we will henceforth refer to X just as a *stable process*.

The stable Lévy process has the jump measure Π that satisfies

$$\Pi(B) = \frac{2^\alpha \Gamma((d + \alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} dy, \quad B \subseteq \mathcal{B}(\mathbb{R}^d).$$

The constant in the definition of $\Pi(B)$ can be arbitrary, however, our choice corresponds to the one that allows us to identify the characteristic exponent Lévy process as

$$\Psi(\theta) = -\frac{1}{t} \log \mathbb{E}(e^{i\theta \cdot X_t}) = |\theta|^\alpha, \quad \theta \in \mathbb{R}^d,$$

where we write \mathbb{P} in preference to \mathbb{P}_0 ; more precisely, the coefficient of $|\theta|^\alpha$ is one.

In this article, we characterise the law of a stable process conditioned to hit continuously a part of the surface, say Borel $S \subseteq \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, either from the inside or from the outside of the unit sphere. We develop an expression for the law of the limiting point of contact on S . Moreover, we show that, when time reversed from the strike point on S , the resulting process can also be seen as a conditioned stable process. The extreme cases that $S = \mathbb{S}^{d-1}$ (the whole unit sphere) and $S = \{\vartheta\} \in \mathbb{S}^{d-1}$ (a single point on the unit sphere) are included in our analysis, however, we will otherwise insist that the Lebesgue surface measure of S is strictly positive.

Our results relate to the recent work of [12], who considered a real valued Lévy process conditioned to continuously approach the boundary of the interval $[-1, 1]$ from the outside. *In order to avoid repetition, we always remain in two or more dimensions.* As in [12], we rely heavily on recent fluctuation identities that are connected to the deep factorisation of the stable process; cf. [16,20,21]. The results here are also related the classical results of Doob [11], who deals with similar conclusions for Brownian motion and as well as echoing the general theory of conditioned stochastic processes in the potential-analytic sense (via a Doob h -transform), see e.g. Chapter 14 of [10].

2. Attraction towards S

For convenience, we will work with the definition $\mathbb{B}_d = \{x \in \mathbb{R}^d : |x| < 1\}$. Let $\mathbb{D}(\mathbb{R}^d)$ denote the space of càdlàg paths $\omega : [0, \infty) \rightarrow \mathbb{R}^d \cup \partial$ with lifetime $\kappa(\omega) = \inf\{s > 0 : \omega(s) = \partial\}$, where ∂ is a cemetery point. The space $\mathbb{D}(\mathbb{R}^d)$ will be equipped with the Skorokhod topology, with its Borel σ -algebra \mathcal{F} and natural filtration $(\mathcal{F}_t, t \geq 0)$. The reader will note that we will also use a similar notion for $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ later on in this text in the obvious way. We will always work with $X = (X_t, t \geq 0)$ to mean the coordinate process defined on the space $\mathbb{D}(\mathbb{R}^d)$. Hence, the notation of the introduction indicates that $\mathbb{P} = (\mathbb{P}_x, x \in \mathbb{R}^d)$ is such that (X, \mathbb{P}) is our stable process.

Consider a subset $S \subseteq \mathbb{S}^{d-1}$ such that it has strictly positive Lebesgue surface measure or it is a point. We want to construct the law of X conditioned to approach S continuously from within $\bar{\mathbb{B}}_d^c := \mathbb{R}^d \setminus \bar{\mathbb{B}}_d$. From a potential-theoretic perspective, this law can be obtained as a Doob h -transform of the killed stable process in $\bar{\mathbb{B}}_d^c$, provided that h is a positive harmonic function in $\bar{\mathbb{B}}_d^c$ which is equal to zero in $\bar{\mathbb{B}}_d$ and which goes to zero at infinity and at $\mathbb{S}^{d-1} \setminus S$; cf. [10, Chapter 14]. In this paper, we want to give a probabilistic construction, which identifies a more physical meaning to the conditioning in terms of the paths of the stable process; see e.g. the classical work of [3,9]. Similarly, we want the law of X conditioned to approach S continuously from within \mathbb{B}_d . More precisely, via an appropriate limiting procedure, we want to build a new family of probabilities $\mathbb{P}^\vee = (\mathbb{P}_x^\vee, x \in \bar{\mathbb{B}}_d^c)$ such that

$$\mathbb{P}_x^\vee(X_s \in \bar{\mathbb{B}}_d^c, s < k \text{ and } X_{k-} \in S) = 1, \quad x \in \bar{\mathbb{B}}_d^c,$$

with a similar statement holding when the conditioning is undertaken from within \mathbb{B}_d .

As we are considering two or higher dimensions, the process (X, \mathbb{P}) is transient in the sense that $\lim_{t \rightarrow \infty} |X_t| = \infty$ almost surely. Defining

$$\underline{G}(t) := \sup\{s \leq t : |X_s| = \inf_{u \leq s} |X_u|\}, \quad t \geq 0,$$

we thus have by monotonicity and the transience of (X, \mathbb{P}) that $\underline{G}(\infty) := \lim_{t \rightarrow \infty} \underline{G}(t)$ exists and, moreover, $X_{\underline{G}(\infty)}$ describes the point of closest reach to the origin in the range of X .

We can similarly define $\bar{G}(t) = \sup\{s \leq t : |X_s| = \sup_{u \leq s} |X_u|\}$, $t \geq 0$, so that $\bar{G}(\tau_1^\ominus -)$ is the point of furthest reach from the origin prior to exiting $\bar{\mathbb{B}}_d$, where

$$\tau_1^\ominus = \inf\{t > 0 : |X_t| > 1\}.$$

Let us turn to what we mean by conditioning to attract to the set S from either the interior or the exterior of the sphere. If S is not a point, we define $A_\varepsilon = \{r\theta : r \in (1, 1 + \varepsilon), \theta \in S\}$ and $B_\varepsilon = \{r\theta : r \in (1 - \varepsilon, 1), \theta \in S\}$, for $0 < \varepsilon < 1$ and define the corresponding events $C_\varepsilon^\vee := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$, and $C_\varepsilon^\wedge := \{X_{\bar{G}(\tau_1^\ominus -)} \in B_\varepsilon\}$. Let

$$\tau_1^\oplus = \inf\{t > 0 : |X_t| < 1\}.$$

We are interested in the asymptotic conditioning

$$\mathbb{P}_x^\vee(A, t < k) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(A, t < \tau_1^\oplus | C_\varepsilon^\vee), \tag{1}$$

when $x \in \bar{\mathbb{B}}_d^c$ and

$$\mathbb{P}_x^\wedge(A, t < k) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(A, t < \tau_1^\ominus | C_\varepsilon^\wedge), \tag{2}$$

when $x \in \mathbb{B}_d$, for all $A \in \mathcal{F}_t$.

When $S = \{\vartheta\} \in \mathbb{S}^{d-1}$, we need to adapt slightly the sets A_ε and B_ε so that $A_\varepsilon = \{r\phi : r \in (1, 1 + \varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$ and $B_\varepsilon = \{r\phi : r \in (1 - \varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$.

We will go a little further in due course and give a fuller description of these two conditioned processes by including the cases that X is issued from the unit sphere itself but not within S , i.e. $\mathbb{S}^{d-1} \setminus S$. For now, we have our first main result, given immediately below, for which we define the function

$$H_S(x) = \begin{cases} |x|^2 - 1|^{\alpha/2} \int_S |\theta - x|^{-d} \sigma_1(d\theta) & \text{if } \sigma_1(S) > 0, \\ |x|^2 - 1|^{\alpha/2} |\vartheta - x|^{-d} & \text{if } S = \{\vartheta\}, \end{cases} \tag{3}$$

for $|x| \neq 1$, where $\sigma_1(d\theta)$ is the Lebesgue surface measure on \mathbb{S}^{d-1} normalised to have unit mass. It is worthy of note that, when $\mathbf{S} = \mathbb{S}^{d-1}$, the integral in (3) can be computed precisely. Indeed, up to an unimportant (for our purposes) multiplicative constant, $C > 0$, which may change from line to line, we note that, for $|x| > 1$,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} |x - \theta|^{-d} \sigma_1(d\theta) &= C \int_0^\pi \frac{(\sin \phi)^{d-2}}{(|x|^2 - 2|x| \cos \phi + 1)^{d/2}} d\phi \\ &= C|x|^{-d} {}_2F_1(d/2, 1; d/2, |x|^{-2}) \\ &= C|x|^{-d} \left(1 - \frac{1}{|x|^2}\right)^{-1}, \end{aligned}$$

where we have used the hypergeometric identity in (A.1) of the Appendix. We can perform a similar calculation when $|x| < 1$ and, obtain, up to a multiplicative constant, $C > 0$, that

$$\int_{\mathbb{S}^{d-1}} |x - \theta|^{-d} \sigma_1(d\theta) = C(1 - |x|^2)^{-1}. \tag{4}$$

All together, noting that we may ignore multiplicative constants, we have

$$H_{\mathbb{S}^{d-1}}(x) = \begin{cases} |x|^{\alpha-d} (1 - |x|^{-2})^{\frac{\alpha}{2}-1} & \text{if } |x| > 1 \\ (1 - |x|^2)^{\frac{\alpha}{2}-1} & \text{if } |x| < 1. \end{cases} \tag{5}$$

As the next result will make clear, $H_{\mathbf{S}}$ is a positive harmonic function for both $(X_t, t < \tau_1^\ominus)$ and $(X_t, t < \tau_1^\oplus)$. From the potential-theoretic perspective, it can be described as an integral of the Martin kernel over \mathbf{S} . Then, by the Martin boundary theory, the h -conditioned process will approach \mathbf{S} with probability one, see [10, Chapter 14] as well as the classical results of Doob for Brownian motion, cf. Theorem 7.1 [11].

Theorem 1 (Stable Process Conditioned to Attract to \mathbf{S} Continuously from One Side). *Let $\mathbf{S} \subseteq \mathbb{S}^{d-1}$ be an closed set with $\sigma_1(\mathbf{S}) > 0$ or $\mathbf{S} = \{\vartheta\}$ for a fixed point $\vartheta \in \mathbb{S}^{d-1}$. Then for all points of issue $x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}$ we have*

$$\frac{d\mathbb{P}_x^\vee}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \mathbf{1}_{(t < \tau_1^\oplus)} \frac{H_{\mathbf{S}}(X_t)}{H_{\mathbf{S}}(x)}, \quad \text{if } x \in \bar{\mathbb{B}}_d^c \tag{6}$$

and otherwise

$$\frac{d\mathbb{P}_x^\wedge}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \mathbf{1}_{(t < \tau_1^\ominus)} \frac{H_{\mathbf{S}}(X_t)}{H_{\mathbf{S}}(x)}, \quad \text{if } x \in \mathbb{B}_d. \tag{7}$$

In particular, $(\mathbb{P}_x^\vee, x \in \bar{\mathbb{B}}_d^c)$ and $(\mathbb{P}_x^\wedge, x \in \bar{\mathbb{B}}_d^c)$ are Markovian families.

Remark 1. The choice of limiting conditioning procedure that we have used reflects a similar approach taken in [12] in one dimension. It is worth noting at this point that the choice of C_ε^\vee and C_ε^\wedge are by no means the only possibilities as far as performing a limiting conditioning that results in (6) and (7). For example, once the reader is familiar with the proof of Theorem 1, it will quickly become clear that, when \mathbf{S} is not a singleton, by defining e.g. $C_\varepsilon^\vee = \{X_{\tau_1^\oplus} \in B_\varepsilon\}$, or indeed $C_\varepsilon^\vee = \{X_{\tau_1^\oplus} \in A_\varepsilon\}$, the limit (1) will still produce the change of measure (6). Once the reader is familiar with the proof of Theorem 1, it is a worthwhile exercise to verify the two proposed alternative definitions of C_ε^\vee for the limiting process by appealing to the fluctuation identities in e.g. [20]. Other definitions of C_ε^\vee giving a consistent limit may indeed also be possible.

Whilst the above theorem deals with the construction of the conditioned process up to but not including its terminal position, we characterise the latter in the next result, which resonates with Theorem 14.8 of [10].

Proposition 1 (Distribution of the Hitting Location). *Suppose that $S \subseteq \mathbb{S}^{d-1}$ be a closed set with $\sigma_1(S) > 0$. Let S' be a closed subset of S . Then for any $x \in \mathbb{R}^d \setminus \mathbb{B}_d$, we have*

$$\mathbb{P}_x^\vee(X_{k-} \in S') = \frac{\int_{S'} |\theta - x|^{-d} \sigma_1(d\theta)}{\int_S |\theta - x|^{-d} \sigma_1(d\theta)}, \tag{8}$$

with an identical result holding for X_{k-} under \mathbb{P}_x^\wedge , with $x \in \mathbb{B}_d$.

3. Lamperti–Kiu representation and radial excursions

The basic definition of the stable process conditioned to attract continuously to S from one side is not quite complete. Strictly speaking, we could think about defining the process to include the points of issue in $\mathbb{S}^{d-1} \setminus S$. It turns out that this is possible. However, we first need to remind the reader of the recently described radial excursion theory, see [20,19]. The starting point for the aforementioned is the Lamperti–Kiu transform which identifies the stable process as a self-similar Markov process.

To describe it, we need to introduce the notion of a Markov Additive Process, henceforth written MAP for short. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$. We say that $(\Xi, \Upsilon) = ((\Xi_t, \Upsilon_t), t \geq 0)$ is a MAP if it is Feller process on $\mathbb{R}^n \times \mathbb{S}^{d-1}$, with probabilities $P_{x,\theta}, x \in \mathbb{R}^n, \theta \in \mathbb{S}^{d-1}$, such that, for any $t \geq 0$, the conditional law of the process $((\Xi_{s+t} - \Xi_t, \Upsilon_{s+t}) : s \geq 0)$, given $((\Xi_u, \Upsilon_u), u \leq t)$, is that of (Ξ, Υ) under $P_{0,\theta}$, with $\theta = \Upsilon_t$. For a MAP pair $((\Xi_t, \Upsilon_t), t \geq 0)$, we call Ξ the *ordinate* and Υ the *modulator*.

According to one of the main results in [1], there exists a MAP on $\mathbb{R} \times \mathbb{S}^{d-1}$, which we will henceforth write as (ξ, Θ) , with probabilities $\mathbf{P} = (P_{x,\theta}, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$ such that the d -dimensional stable process can be written

$$X_t = \exp\{\xi_{\varphi(t)}\} \Theta_{\varphi(t)} \quad t \geq 0, \tag{9}$$

where

$$\varphi(t) = \inf \left\{ s > 0 : \int_0^s e^{\alpha \xi_u} du > t \right\}. \tag{10}$$

Whilst Θ alone is a Feller process, it is not necessarily true that ξ alone is. However, it is a consequence of isotropy that this is the case here. Moreover, ξ alone is a Lévy process whose characteristic exponent is known (but not important in the current context); see for example [8]. What is important for our purposes is to note for now that it has paths of unbounded variation, and therefore is regular for the upper and lower half line (in the sense of Definition 6.4 of [15]).

It is not difficult to show that the pair $((\xi_t - \underline{\xi}_t, \Theta_t), t \geq 0)$, forms a strong Markov process, where $\underline{\xi}_t := \inf_{s \leq t} \xi_s, t \geq 0$ is the running minimum of ξ . On account of the fact that ξ , alone, is a Lévy process, $(\xi_t - \underline{\xi}_t, t \geq 0)$ is also a strong Markov process. Suppose we denote by $\ell = (\ell_t, t \geq 0)$ the local time at zero of $\xi - \underline{\xi}$, then we can introduce the following processes

$$H_t^- = -\xi_{\ell_t^-} \text{ and } \Theta_t^- = \Theta_{\ell_t^-}, \quad t \geq 0,$$

and define $(H_{\ell_t^-}^-, \Theta_{\ell_t^-}^-) = (\partial, \dagger)$, a cemetery state, if $\ell_t^- = \infty$. Then, the pair (ℓ_t^-, H_t^-) , without reference to the associated modulation Θ^- , are Markovian and play the role of the

descending ladder time and height subordinators of ξ . Moreover, the strong Markov property tells us that $(\ell_t^-, H_t^-, \Theta_t^-), t \geq 0$, defines a Markov Additive Process on $\mathbb{R}^2 \times \mathbb{S}^{d-1}$, whose first two elements are ordinates that are non-decreasing. In this sense, ℓ also serves as an adequate choice for the local time of the Markov process $(\xi - \underline{\xi}, \Theta)$ on the set $\{0\} \times \mathbb{S}^{d-1}$. (See [20]).

Suppose we define $g_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$, and recall that the regularity of ξ for $(-\infty, 0)$ and $(0, \infty)$ ensures that it is well defined, as is $g_\infty = \lim_{t \rightarrow \infty} g_t$. Set

$$d_t = \inf\{s > t : \xi_s = \underline{\xi}_s\}.$$

For all $t > 0$ such that $d_t > g_t$ the process

$$(\epsilon_{g_t}(s), \Theta_{g_t}^\epsilon(s)) := (\xi_{g_t+s} - \xi_{g_t}, \Theta_{g_t+s}), \quad s \leq \zeta_{g_t} := d_t - g_t,$$

codes the excursion of $(\xi - \underline{\xi}, \Theta)$ from the set $(0, \mathbb{S}^{d-1})$ which straddles time t . Such excursions live in the space $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$, the space of càdlàg paths in $\mathbb{R} \times \mathbb{S}^{d-1}$, written in canonical form

$$(\epsilon, \Theta^\epsilon) = ((\epsilon(t), \Theta^\epsilon(t)) : t \leq \zeta) \text{ with lifetime } \zeta = \inf\{s > 0 : \epsilon(s) < 0\},$$

such that $(\epsilon(0), \Theta^\epsilon(0)) \in \{0\} \times \mathbb{S}^{d-1}$, $(\epsilon(s), \Theta^\epsilon(s)) \in (0, \infty) \times \mathbb{S}^{d-1}$, for $0 < s < \zeta$, and $\epsilon(\zeta) \in (-\infty, 0]$.

Taking account of the Lamperti–Kiu transform (9), it is natural to consider how the excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ translates into a radial excursion theory for the process

$$Y_t := e^{\xi_t} \Theta_t, \quad t \geq 0.$$

Ignoring the time change in (9), we see that the radial minima of the process Y agree with the radial minima of the stable process X . Indeed, each excursion of $(\xi - \underline{\xi}, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ is uniquely associated to exactly one excursion of $(Y_t / \inf_{s \leq t} |Y_s|, t \geq 0)$, from \mathbb{S}^{d-1} , or equivalently an excursion of Y from its running radial infimum. Moreover, we see that, for all $t > 0$ such that $d_t > g_t$,

$$Y_{g_t+s} = e^{\xi_{g_t}} e^{\epsilon_{g_t}(s)} \Theta_{g_t}^\epsilon(s) = |Y_{g_t}| e^{\epsilon_{g_t}(s)} \Theta_{g_t}^\epsilon(s), \quad s \leq \zeta_{g_t}.$$

This will be useful to keep in mind for the forthcoming excursion computations.

For $t > 0$, let $R_t = d_t - t$, and define the set $\mathbb{G} = \{t > 0 : R_{t-} = 0, R_t > 0\} = \{g_s : s \geq 0\}$. The classical theory of exit systems in [22] (see Theorems (4.1) and (6.3) therein) now implies that there exists an additive functional $(A_t, t \geq 0)$ and a family of *excursion measures*, $(\mathbb{N}_\theta, \theta \in \mathbb{S}^{d-1})$ such that:

- (i) A is an additive functional of (ξ, Θ) , has a bounded 1-potential and is carried by the set of times $\{t \geq 0 : (\xi_t - \underline{\xi}_t, \Theta_t) \in \{0\} \times \mathbb{S}^{d-1}\}$,
- (ii) the map $\theta \mapsto \mathbb{N}_\theta$ is an \mathbb{S}^{d-1} -indexed kernel on $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$ such that $\mathbb{N}_\theta(1 - e^{-\zeta}) < \infty$;
- (iii) we have the *exit formula*

$$\begin{aligned} \mathbf{E}_{x,\theta} \left[\sum_{g \in \mathbb{G}} F((\xi_s, \Theta_s) : s < g) H((\epsilon_g, \Theta_g^\epsilon)) \right] \\ = \mathbf{E}_{x,\theta} \left[\int_0^\infty F((\xi_s, \Theta_s) : s < t) \mathbb{N}_{\Theta_t} (H(\epsilon, \Theta^\epsilon)) dA_t \right], \end{aligned} \tag{11}$$

for $x \neq 0$, where F is continuous on the space of càdlàg paths $\mathbb{D}(\mathbb{R} \times \mathbb{S}^{d-1})$ and H is measurable on the space of càdlàg paths $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$;

- (iv) under any measure \mathbb{N}_θ the process $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ is a strong Markov process with the same semigroup as (ξ, Θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$.

The couple $(\Lambda, \underline{\mathbb{N}})$ is called an exit system. Note that in Maisonneuve’s original formulation, the pair $\Lambda, \underline{\mathbb{N}} := (\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$ is not unique, but once Λ is chosen the measures $(\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$ exist however, are only unique up to Λ -neglectable sets, i.e. sets \mathcal{A} such that $\mathbf{E}_{x,\theta}(\int_0^\infty \mathbf{1}_{\{\theta_s \in \mathcal{A}\}} d\Lambda_s) = 0$. Another example of where this theory has been used is in the construction of excursions from a set is that of Brownian motion away from a hyperplane; see [7].

Now referring back to ℓ , the local time of $\xi - \underline{\xi}$ at 0, since it is an additive functional with a bounded 1-potential, there is an exit system which corresponds to $(\ell, \underline{\mathbb{N}})$. With this choice of ℓ we assume that the choice of $\underline{\mathbb{N}}$ is fixed despite the fact that we can induce subtle variations in $\underline{\mathbb{N}}$ on a Λ -negligible set of $\theta \in \mathbb{S}^{d-1}$ e.g. by setting $\underline{\mathbb{N}}_\theta \equiv 0$ there. The reader is referred to Chapter VII of [4] for further discussion on this matter. Note that $\underline{\mathbb{N}}_\theta$ is not isotropic in θ . For example, excursions that begin at the ‘North Pole’, say 1, are, with high frequency, arbitrarily small and hence will end near to 1. That said, depending on the event A , it is possible that $\underline{\mathbb{N}}_\theta(A)$ does not depend on $\theta \in \mathbb{S}^{d-1}$; for example, $\underline{\mathbb{N}}_\theta(\zeta = \infty)$. The reason for this is that it must agree with the rate at which the infinite excursion of $\xi - \underline{\xi}$ occurs, according to the local time ℓ . More generally, we have that, for all orthogonal transformations $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ and f such that $\underline{\mathbb{N}}_\theta(f(\epsilon, \Theta^\epsilon)) < \infty, \theta \in \mathbb{S}^{d-1}$, isotropy implies that $\underline{\mathbb{N}}_\theta(f(\epsilon, U\Theta^\epsilon)) = \underline{\mathbb{N}}_{U\theta}(f(\epsilon, \Theta^\epsilon))$. On account of the fact that ℓ is only defined up to a multiplicative constant, we can use the common value of $\underline{\mathbb{N}}_\theta(\zeta = \infty)$ to fix a normalisation the local time, or equivalently, of the excursion measures $(\underline{\mathbb{N}}_\theta, \theta \in \mathbb{S}^{d-1})$. We thus fix it to take the value of unity. The place at which this choice of normalisation becomes relevant is when we cite certain identities from (cf. (39) below) from [20], in which this assumption was also made. Henceforth, this is the exit system we will work with and the system of excursion associated to it is what we call our *radial excursion theory*.

Later in our proofs we will use a variant of the above excursion theory based on the MAP $(\bar{\xi} - \xi, \Theta)$, where $\bar{\xi}$ is the process $\bar{\xi}_t = \sup_{s \leq t} \xi_s, t \geq 0$. We leave the details until that point in the text. With our excursion theory in hand, we can now proceed to identify the completion of Theorem 1.

Theorem 2. Let $\mathbf{S} \subseteq \mathbb{S}^{d-1}$ be an closed set with $\sigma_1(\mathbf{S}) > 0$ or $\mathbf{S} = \{\vartheta\}$ for a fixed point $\vartheta \in \mathbb{S}^{d-1}$. The processes (X, \mathbb{P}^\vee) and (X, \mathbb{P}^\wedge) can be extended in a consistent way to include points of issue $x \in \mathbb{S}^{d-1} \setminus \mathbf{S}$ with pathwise continuous entry via

$$\mathbb{P}_x^\vee(X_t \in dy, t < k) := \frac{\Gamma(d/2)}{\Gamma(\alpha/2 + 1)\Gamma((d - \alpha)/2)} \frac{H_{\mathbf{S}}(y)}{h(x)} \underline{\mathbb{N}}_x(X^\epsilon(t) \in dy, t < \zeta), \tag{12}$$

for $|y| > 1$ and

$$\mathbb{P}_x^\wedge(X_t \in dy, t < k) := \frac{\Gamma(d/2)}{\Gamma(\alpha/2 + 1)\Gamma((d - \alpha)/2)} \frac{H_{\mathbf{S}}(y)}{h(x)} \bar{\mathbb{N}}_x(X^\epsilon(t) \in dy, t < \zeta), \tag{13}$$

for $|y| > 1$, where,

$$h(x) = \int_{\mathbf{S}} |x - \theta|^{-d} \sigma_1(d\theta)$$

and, for $(\epsilon, \Theta^\epsilon)$ selected from $\underline{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$ or $\bar{\mathbb{U}}(\mathbb{R} \times \mathbb{S}^{d-1})$, respectively,

$$X^\epsilon(t) = e^{\epsilon(\varphi(t))} \Theta^\epsilon(\varphi(t)) \text{ and } \zeta = \varphi^{-1}(\zeta) = \int_0^\zeta |X^\epsilon(u)|^\alpha du. \tag{14}$$

Here, pathwise continuous entry means that

$$\mathbb{P}_x^\vee(\lim_{t \rightarrow 0} X_t = x) = \mathbb{P}_x^\wedge(\lim_{t \rightarrow 0} X_t = x) = 1 \tag{15}$$

for all $x \in \mathbb{S}^{d-1} \setminus \mathbb{S}$.

Note, referring to the discussion preceding [Theorem 2](#), the choice of local time ℓ leaves a free choice of multiplicative constant in the definition of local time, which may depend on $x \in \mathbb{S}^{d-1} \setminus \mathbb{S}$. In the proof of [Theorem 2](#), we use a method of continuity of resolvents to pin down the aforesaid constants. We also note that extending the notion of a Doob h -transformed process to include certain ‘boundary points’ in the way we have seen in [Theorem 2](#) can be seen in e.g. [\[24,9\]](#) as well as the classical work of Doob [\[11\]](#).

4. Repulsion and duality

In this section, we want to introduce two new processes, which will turn out to be dual to (X, \mathbb{P}^\vee) and (X, \mathbb{P}^\wedge) in the sense of time reversal. The two processes we are interested give meaning to the stable process conditioned to remain in $\bar{\mathbb{B}}_d^c$ and \mathbb{B}_d , respectively, in an appropriate sense.

An important tool that we will make use of in analysing the aforesaid time reversed processes comes through the so-called Riesz–Bogdan–Žak transform, cf. [\[6,17\]](#), which relates path behaviour of the stable process outside of the unit sphere to its behaviour inside the unit sphere. In order to state it, we need to introduce the process (X, \mathbb{P}°) , where the probabilities $\mathbb{P}^\circ = (\mathbb{P}_x^\circ, x \neq 0)$ are given by

$$\frac{d\mathbb{P}_x^\circ}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{|X_t|^{\alpha-d}}{|x|^{\alpha-d}}, \quad \text{on } t < \tau_\varepsilon := \inf\{t > 0 : |X_t| < \varepsilon\} \tag{16}$$

for all $\varepsilon > 0$. Since $\alpha < 2 \leq d$, we note that the change of measure rewards paths that approach the origin and punishes paths that wander far from the origin. Intuitively, it is clear that (X, \mathbb{P}°) describes the stable process conditioned to continuously approach the origin. Nonetheless, this heuristic can be made into a rigorous statement, see for example [\[17,20,19,21\]](#). The reader will also note from these references (and it is easy to prove that) that (X, \mathbb{P}°) is also a self-similar Markov process with the same index of self-similarity as (X, \mathbb{P}) .

Theorem 3 (*Riesz–Bogdan–Žak Transform, [6]*). *Suppose we write $Kx = x/|x|^2$, $x \in \mathbb{R}^d$ for the classical inversion of space through the sphere \mathbb{S}^{d-1} . Then, in dimension $d \geq 2$, for $x \neq 0$, $(KX_{\eta(t)}, t \geq 0)$ under \mathbb{P}_x is equal in law to $(X, \mathbb{P}_{Kx}^\circ)$, where $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}$.*

Let us return to our duality concerns. To this end, let us introduce the probabilities

$$H^\ominus(x) = \mathbb{P}_x(\tau_1^\oplus = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{|x|^2-1} (u+1)^{-d/2} u^{\alpha/2-1} du, \tag{17}$$

for $|x| > 1$, where the second inequality is lifted from [\[5\]](#), and,

$$H^\oplus(x) = |x|^{\alpha-d} H^\ominus(Kx),$$

for $|x| < 1$.

These two functions are positive harmonic for X and can be used to define the two families of probabilities $\mathbb{P}^\ominus = (\mathbb{P}_x^\ominus, |x| > 1)$ and $\mathbb{P}^\oplus = (\mathbb{P}_x^\oplus, |x| < 1)$ via the Doob h -transforms,

$$\frac{d\mathbb{P}_x^\ominus}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{H^\ominus(X_t)}{H^\ominus(x)} \mathbf{1}_{(t < \tau^\ominus)}, \quad t \geq 0, |x| > 1 \tag{18}$$

and,

$$\frac{d\mathbb{P}_x^\oplus}{d\mathbb{P}_x} \Big|_{\mathcal{F}_t} = \frac{H^\oplus(X_t)}{H^\oplus(x)} \mathbf{1}_{(t < \tau^\oplus)}, \quad t \geq 0, |x| < 1. \tag{19}$$

The first of these two changes of measure corresponds to the stable process conditioned to avoid entering \mathbb{B}_d by a simple restriction on the probability space (remembering that $\lim_{t \rightarrow \infty} |X_t| = \infty$). Note from [Theorem 3](#) that

$$H^\ominus(Kx) = \mathbb{P}_{Kx}(\tau_1^\oplus = \infty) = \mathbb{P}_x^\circ(\tau^{(0)} < \tau_1^\ominus),$$

where $\tau^{(0)} = \inf\{t > 0 : |X_{t-}| = 0\}$. The second change of measure, [\(19\)](#), is a composition of conditioning the stable process to be absorbed continuously at the origin, followed by conditioning it not to exit \mathbb{B}_d via a simple restriction on the probability space (noting that $\lim_{t \rightarrow \infty} |X_t| = 0$ under \mathbb{P}°).

The reader will also note that the Riesz–Bogdan–Żak transform also implies a similar spatial inversion and time change must hold for the pair (X, \mathbb{P}^\ominus) and (X, \mathbb{P}^\oplus) .

Corollary 1. *For $|x| > 1$, $(KX_{\eta(t)}, t \geq 0)$ under \mathbb{P}_x^\ominus is equal in law to $(X, \mathbb{P}_{Kx}^\oplus)$, where $\eta(t) = \inf\{s > 0 : \int_0^s |X_u|^{-2\alpha} du > t\}$. Similarly, for $|x| < 1$, $(KX_{\eta(t)}, t \geq 0)$ under \mathbb{P}_x^\oplus is equal in law to $(X, \mathbb{P}_{Kx}^\ominus)$.*

Proof. Suppose that $F(X_s, s \leq t)$ is a bounded \mathcal{F}_t -measurable function for each $t \geq 0$. Then, for $|x| > 1$, appealing to [Theorem 3](#), we have

$$\begin{aligned} \mathbb{E}_x^\ominus [F(KX_{\eta(s)}, s \leq t)] &= \mathbb{E}_x \left[F(KX_{\eta(s)}, s \leq t) \frac{H^\ominus(K(KX_{\eta(t)}))}{H^\ominus(x)} \mathbf{1}_{(\eta(t) < \tau^\ominus)} \right] \\ &= \mathbb{E}_{Kx}^\circ \left[F(X_s, s \leq t) \frac{H^\ominus(KX_t)}{H^\ominus(x)} \mathbf{1}_{(t < \tau^\ominus)} \right] \\ &= \mathbb{E}_{Kx} \left[F(X_s, s \leq t) \frac{|X_t|^{\alpha-d}}{|Kx|^{\alpha-d}} \frac{H^\ominus(KX_t)}{H^\ominus(K(Kx))} \mathbf{1}_{(t < \tau^\ominus)} \right] \\ &= \mathbb{E}_{Kx}^\oplus [F(X_s, s \leq t)]. \end{aligned}$$

This shows the first half of the claim. The second part of the claim is proved using the same technique and the details are omitted for brevity given how straightforward they are. \square

In the spirit of other cases of conditionings from an extreme boundary point (e.g. conditioning a Lévy process to avoid the origin, cf. [\[24\]](#), or to stay positive, cf. [\[9\]](#)), we can extend the definitions given in [\(18\)](#) and [\(19\)](#) by appealing to the Markov property of the excursion measures $\underline{\mathbb{N}}_x$ and $\overline{\mathbb{N}}_x, x \in \mathbb{S}^{d-1}$.

Theorem 4. *The processes (X, \mathbb{P}^\ominus) and (X, \mathbb{P}^\oplus) can be extended in a consistent way to include points of issue on \mathbb{S}^{d-1} . Specifically,*

$$\mathbb{P}_x^\ominus(X_t \in dy) = H^\ominus(y) \underline{\mathbb{N}}_x(X^\epsilon(t) \in dy, t < \varsigma), \quad x \in \mathbb{S}^{d-1}, |y| > 1 \tag{20}$$

and similarly

$$\mathbb{P}_x^\oplus(X_t \in dy) = H^\oplus(y)\overline{N}_x(X^\epsilon(t) \in dy, t < \zeta), \quad x \in \mathbb{S}^{d-1}, |y| < 1, \tag{21}$$

(specifically, the normalisation of the excursion measure is unity in both cases) where we have used the notation given in (14). As in Theorem 2, there is pathwise continuous entry.

Our objective is to pair up (X, \mathbb{P}^\vee) , (X, \mathbb{P}^\ominus) and (X, \mathbb{P}^\wedge) , (X, \mathbb{P}^\oplus) via Nagasawa’s duality theorem for time reversal; cf [23]. To this end we need to introduce the notion of L -times.

Suppose that $Y = (Y_t, t \leq \zeta)$ with probabilities \mathbb{P}_x , $x \in E$, is a regular Markov process on an open domain $E \subseteq \mathbb{R}^d$ (or more generally, a locally compact Hausdorff space with countable base), with cemetery state Δ and killing time $\zeta = \inf\{t > 0 : Y_t = \Delta\}$. Let us additionally write $\mathbb{P}_\nu = \int_E \nu(da)\mathbb{P}_a$, for any probability measure ν on the state space of Y .

Suppose that \mathcal{G} is the σ -algebra generated by Y and write $\overline{\mathcal{G}}(\mathbb{P}_\nu)$ for its completion by the null sets of \mathbb{P}_ν . Moreover, write $\overline{\mathcal{G}} = \bigcap_\nu \overline{\mathcal{G}}(\mathbb{P}_\nu)$, where the intersection is taken over all probability measures on the state space of Y , excluding the cemetery state. A finite random time \mathfrak{k} is called an L -time (generalised last exit time) if

- (i) \mathfrak{k} is measurable in $\overline{\mathcal{G}}$, and $\mathfrak{k} \leq \zeta$ almost surely with respect to \mathbb{P}_ν , for all ν ,
- (ii) $\{s < \mathfrak{k}(\omega) - t\} = \{s < \mathfrak{k}(\omega_t)\}$ for all $t, s \geq 0$,

where ω_t is the Markov shift of ω to time t . The most important examples of L -times are killing times and last exit times.

Theorem 5. *In what follows, we work with the probability distribution*

$$\nu(da) := \frac{\sigma_1(da)|_{\mathbb{S}}}{\sigma_1(\mathbb{S})}, \quad a \in \mathbb{R}^d, \tag{22}$$

if \mathbb{S} is closed and $\sigma_1(\mathbb{S}) > 0$ and, otherwise, if $\mathbb{S} = \{\vartheta\}$, $\vartheta \in \mathbb{S}^{d-1}$, we understand

$$\nu(da) = \delta_{\{\vartheta\}}(da), \quad a \in \mathbb{R}^d. \tag{23}$$

- (i) For every L -time \mathfrak{k} of (X, \mathbb{P}^\ominus) , the time reversed process $(X_{(\mathfrak{k}-t)-}, t < \mathfrak{k})$ under \mathbb{P}_ν^\ominus is a time-homogeneous Markov process whose transition probabilities agree with those of (X, \mathbb{P}^\vee) .
- (ii) Similarly, for every L -time \mathfrak{k} of (X, \mathbb{P}^\oplus) , the time reversed process $(X_{(\mathfrak{k}-t)-}, t < \mathfrak{k})$ under \mathbb{P}_ν^\oplus is a time-homogeneous Markov process whose transition probabilities agree with those of (X, \mathbb{P}^\wedge) .

Nagasawa’s result, [23, Theorem 3.5], allows the definition of the time reversed process only for $t > 0$, however we can extend it for $t = 0$. Indeed, in (i), $\mathfrak{k} < \zeta = \infty$ with probability \mathbb{P}^\ominus one, and the time-reversal can include $t = 0$; in (ii), we may have $\mathfrak{k} = \zeta < \infty$ with positive \mathbb{P}^\oplus -probability, but in this case $X_{\zeta-} = 0$ with \mathbb{P}^\oplus -probability one, and therefore again $t = 0$ can be included in the time reversed process. That means, if the duality is true for $t > 0$, it must be true for all $t \geq 0$.

5. Proof of Theorem 1

We start by recalling two useful identities. In Theorem 1.1 in [20], the law of $X_{\underline{G}(\infty)}$ is given by

$$\mathbb{P}_x(X_{\underline{G}(\infty)} \in dz) = c_{\alpha,d} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^\alpha} |x - z|^{-d} dz, \quad |x| > |z| > 0, \tag{24}$$

where

$$c_{\alpha,d} = \pi^{-d/2} \frac{\Gamma(d/2)^2}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}.$$

Similarly, from Corollary 1.1 of [20], it was also shown that

$$\mathbb{P}_x(X_{\bar{G}(\tau_1^\ominus)} \in dz, X_{\tau_1^\ominus} \in dv) = C_{\alpha,d} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{(|v|^2 - |z|^2)^{\alpha/2} |z - v|^d |z - x|^d} dz dv, \tag{25}$$

for $|x| < |z| < 1$ and $|v| > 1$, where

$$C_{\alpha,d} = \frac{\Gamma(d/2)^2}{\pi^d |\Gamma(-\alpha/2)| \Gamma(\alpha/2)}.$$

First take $x \in \bar{\mathbb{B}}_d^c$. Let $\tau_\beta^\oplus := \inf\{t > 0: |X_t| < \beta\}$ for any $\beta > 1$. For any $A \in \mathcal{F}_t$, define

$$\mathbb{P}_x^\vee(A, t < \tau_\beta^\oplus) = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(A, t < \tau_\beta^\oplus | C_\varepsilon^\vee). \tag{26}$$

The Markov property gives us

$$\mathbb{P}_x(A, t < \tau_\beta^\oplus | C_\varepsilon^\vee) = \mathbb{E}_x \left[\mathbf{1}_{\{A, t < \tau_\beta^\oplus\}} \frac{\mathbb{P}_{X_t}(C_\varepsilon^\vee)}{\mathbb{P}_x(C_\varepsilon^\vee)} \right]. \tag{27}$$

In order to prove Theorem 1, it is enough to prove that, for all $\beta > 1$, (6) is true for sets of the form $A \cap \{t < \tau_\beta^\oplus\} \in \mathcal{F}_t$, in which case the full statement (6) follows by the Monotone Convergence Theorem as we take $\beta \downarrow 1$. Next note from (24) that

$$\begin{aligned} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon) &= c_{\alpha,d} \int_{z \in A_\varepsilon} \frac{(|x|^2 - |z|^2)^{\alpha/2}}{|z|^\alpha} |x - z|^{-d} dz \\ &= c'_{\alpha,d} \int_1^{1+\varepsilon} \int_{\mathbb{S}} \frac{(|x|^2 - r^2)^{\alpha/2}}{r^\alpha} |x - r\theta|^{-d} r^{d-1} dr \sigma_1(d\theta), \end{aligned}$$

where $c'_{\alpha,d}$ is an unimportant constant.

Since $(|x|^2 - r^2)^{\alpha/2} |x - r\theta|^{-d}$ is continuous at $r = 1$ with fixed $|x| > 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1 < r < 1 + \varepsilon$,

$$(1 - \delta)(|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} < (|x|^2 - r^2)^{\alpha/2} |x - r\theta|^{-d} < (1 + \delta)(|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d}$$

and

$$\int_1^{1+\varepsilon} r^{d-\alpha+1} dr = c\varepsilon^{d-\alpha} + o(\varepsilon^{d-\alpha}),$$

where c is an unimportant constant. Hence, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\alpha-d} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon) = c'_{\alpha,d} \int_{\mathbb{S}} (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(d\theta),$$

where $c'_{\alpha,d}$ does not depend on x and may change from the previous one. Note, moreover, that for all fixed $\beta > 1$

$$\sup_{|x| > \beta} \frac{\int_{\mathbb{S}} (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(d\theta)}{|x|^{\alpha-d}} < \infty. \tag{28}$$

We can both make use of the limit

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_{X_t}(X_{\underline{G}(\infty)} \in A_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} = \frac{\int_{\mathbb{S}} |\theta - X_t|^{-d} (|X_t|^2 - 1)^{\alpha/2} \sigma_1(d\theta)}{\int_{\mathbb{S}} |\theta - x|^{-d} (|x|^2 - 1)^{\alpha/2} \sigma_1(d\theta)}, \quad t < \tau_\beta^\oplus. \tag{29}$$

as well as (28) and the Dominated Convergence Theorem to ensure the limit may be passed through the expectation in (27) to give (6) on $\{t < \tau_\beta^\oplus\}$, thus giving the desired result.

Next we look at the proof of (7). In a similar way, it is enough to work with sets of the form $A \cap \{t < \tau_\beta^\ominus\} \in \mathcal{F}_t$, with $\beta < 1$. From (25), recalling $C_\varepsilon^\wedge := \{X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon\}$, we have

$$\begin{aligned} \mathbb{P}_x(C_\varepsilon^\wedge) &= \mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon) \\ &= C_{\alpha,d} \int_{z \in B_\varepsilon} \int_{v \in \mathbb{B}_d^c} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{(|v|^2 - |z|^2)^{\alpha/2} |z - v|^d |z - x|^d} dz dv \\ &= C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} dz \int_1^\infty \frac{r^{d-1} dr}{(r^2 - |z|^2)^{\alpha/2}} \int_{\mathbb{S}^{d-1}(0,r)} \frac{1}{|z - \theta|^d} \sigma_r(d\theta), \end{aligned} \tag{30}$$

where $\sigma_r(d\theta)$ is the surface measure on $\mathbb{S}^{d-1}(0, r)$, the sphere centred at 0 of radius r , normalised to have unit mass and $C'_{\alpha,d}$ is henceforth a constant whose value may change from line to line, which depends only on α and d . The Poisson formula (giving the probability that a d -dimensional Brownian motion issued from z (with $|z| < 1$) will hit the sphere $\mathbb{S}^{d-1}(0, r)$) tells us that

$$\int_{\mathbb{S}^{d-1}(0,r)} \frac{r^{d-2}(r^2 - |z|^2)}{|z - \theta|^d} \sigma_r(d\theta) = 1, \quad |z| < 1 < r, \tag{31}$$

see for example Remark III.2.5 in [17]. Putting (31) in (30) gives us

$$\begin{aligned} \mathbb{P}_x(C_\varepsilon^\wedge) &= C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} dz \int_1^\infty \frac{r^{d-1}}{(r^2 - |z|^2)^{\alpha/2}} \frac{1}{r^{d-2}(r^2 - |z|^2)} dr \\ &= C'_{\alpha,d} \int_{z \in B_\varepsilon} \frac{(|z|^2 - |x|^2)^{\alpha/2}}{|z - x|^d} \frac{1}{(1 - |z|^2)^{\alpha/2}} dz \\ &= C'_{\alpha,d} \int_{1-\varepsilon}^1 \int_{\mathbb{S}} \frac{(u^2 - |x|^2)^{\alpha/2}}{(1 - u^2)^{\alpha/2} |u\theta - x|^d} u^{d-1} du \sigma_1(d\theta). \end{aligned}$$

Since $(u^2 - |x|^2)^{\alpha/2} |x - u\theta|^{-d}$ is continuous at $u = 1$ with fixed $0 < |x| < 1$, for any $\delta > 0$, there exists $\varepsilon > 0$ such that for all $1 - \varepsilon < u < 1$,

$$(1 - \delta)(1 - |x|^2)^{\alpha/2} |x - \theta|^{-d} < (u^2 - |x|^2)^{\alpha/2} |x - u\theta|^{-d} < (1 + \delta)(1 - |x|^2)^{\alpha/2} |x - \theta|^{-d}$$

and

$$\int_{1-\varepsilon}^1 \frac{u^{d-1}}{(1 - u^2)^{\alpha/2}} du = \int_0^\varepsilon \frac{(1 - r)^{d-1}}{r^{\alpha/2}(2 - r)^{\alpha/2}} dr = c\varepsilon^{1-\alpha/2} + o(\varepsilon^{1-\alpha/2}),$$

for an unimportant constant $c > 0$.

It is now clear that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\alpha/2-1} \mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon) = C'_{\alpha,d} \int_{\mathbb{S}} (1 - |x|^2)^{\alpha/2} |x - \theta|^{-d} \sigma_1(d\theta).$$

Finally, we get again

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_{X_t}(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon)}{\mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon)} = \frac{\int_{\mathbb{S}} |\theta - X_t|^{-d} (1 - |X_t|^2)^{\alpha/2} \sigma_1(d\theta)}{\int_{\mathbb{S}} |\theta - x|^{-d} (1 - |x|^2)^{\alpha/2} \sigma_1(d\theta)}, \quad t < \tau_\beta^\ominus. \tag{32}$$

and we can proceed as in (26), noting the application of dominated convergence and that for every fixed $\beta < 1$,

$$\sup_{|x|<\beta} \frac{\int_{\mathbb{S}} (|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d} \sigma_1(d\theta)}{|x|^{\alpha-d}} < \infty.$$

In a similar manner, when $\mathbf{S} = \{\vartheta\}$, we work with sets of the form $A \cap \{t < \tau_\beta^\ominus\} \in \mathcal{F}_t$ or $A \cap \{t < \tau_{\beta'}^\ominus\} \in \mathcal{F}_t$, with $\beta' < 1 < \beta$, respectively. In this case, $A_\varepsilon = \{r\phi : r \in (1, 1 + \varepsilon), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$ and $B_\varepsilon = \{r\phi : r \in (1 - \varepsilon, 1), \phi \in \mathbb{S}^{d-1}, |\phi - \vartheta| < \varepsilon\}$, thus it is clear by similar analysis that

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_{X_t}(X_{\underline{G}(\infty)} \in A_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} = \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_{X_t}(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon)}{\mathbb{P}_x(X_{\overline{G}(\tau_1^\ominus)} \in B_\varepsilon)} = \frac{|\theta - X_t|^{-d} ||X_t|^2 - 1|^{\alpha/2}}{|\theta - x|^{-d} ||x|^2 - 1|^{\alpha/2}}. \tag{33}$$

The rest of the proof is otherwise a minor adjustment of what we have seen previously, now taking account of the continuity of $(u, \theta) \mapsto |u^2 - |x|^2|^{\alpha/2} |x - u\theta|^{-d}$ as well as the fact that $\sup_{|x|>\beta} ((|x|^2 - 1)^{\alpha/2} |x - \theta|^{-d}) / |x|^{\alpha-d} < \infty$ and $\sup_{|x|<\beta'} ((1 - |x|^2)^{\alpha/2} |x - \theta|^{-d}) / |x|^{\alpha-d} < \infty$, in order to use dominated convergence. \square

5.1. Proof of Proposition 1

To calculate the hitting distribution, recall that \mathbb{P}^\vee is the law of a stable process conditioned to attract to \mathbf{S} continuously from the outside, and $A'_\varepsilon = \{r\theta : r \in (1, 1 + \varepsilon), \theta \in \mathbf{S}'\}$, that is the restriction of A_ε from the set \mathbf{S} to its subset $\mathbf{S}' \subset \mathbf{S}$. Then, due to Theorem 1.3 in [20], we have $\mathbb{P}_x^\vee(X_{\mathbb{k}^-} \in \mathbf{S}') = \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | C_\varepsilon^\vee)$. Then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | C_\varepsilon^\vee) &= \lim_{\varepsilon \rightarrow 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | X_{\underline{G}(\infty)} \in A_\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon)}{\mathbb{P}_x(X_{\underline{G}(\infty)} \in A_\varepsilon)} \\ &= \frac{\int_{\mathbf{S}'} |\theta - x|^{-d} \sigma_1(d\theta)}{\int_{\mathbf{S}} |\theta - x|^{-d} \sigma_1(d\theta)}, \end{aligned} \tag{34}$$

which concludes the statement in Proposition 1 for the case when X is issued from outside. Similar computations give the result when X is issued from inside \mathbb{B}_d . \square

6. Proof of Theorems 2 and 4

Proof of Theorem 2. Let us restrict our attention to the extension of (X, \mathbb{P}^\vee) to include $\mathbb{S}^{d-1} \setminus \mathbf{S}$. We need to prove that the proposed definition of \mathbb{P}_θ^\vee , for any $\theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$, is well defined as a finite entity, conforms to the correct normalisation to represent a probability measure and is consistent with the definition of (X, \mathbb{P}^\vee) given in Theorem 1 on \mathbb{B}_d^c , as well as offering continuous entry from the boundary $\mathbb{S}^{d-1} \setminus \mathbf{S}$.

We start with finiteness. To this end, we must show that, for $\theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$

$$\mathbb{N}_\theta(H_{\mathbf{S}}(X^\varepsilon(t)); t < \varsigma) < \infty, \quad t > 0. \tag{35}$$

Noting from (5) that $H_{\mathbf{S}}(x) \leq H_{\mathbb{S}^{d-1}}(x) = |x|^{\alpha-d} (1 - |x|^{-2})^{\frac{\alpha}{2}-1}$, which tends to 0 as $|x| \rightarrow \infty$, it suffices to prove that, for any $R > 1$,

$$\mathbb{N}_\theta(|X^\varepsilon(t)| \geq R, t < \varsigma) + \mathbb{N}_\theta(H_{\mathbf{S}}(X^\varepsilon(t)); |X^\varepsilon(t)| < R, t < \varsigma) < \infty, \quad t > 0. \tag{36}$$

Abusing notation and using $\tau_R^\ominus = \inf\{t > 0 : |X_t^\epsilon| > R\}$ in the canonical sense,

$$\mathbb{N}_\theta(|X^\epsilon(t)| \geq R, t < \zeta) \leq \mathbb{N}_\theta(t < \zeta \wedge \tau_R^\ominus) \leq \underline{n}(t < \kappa_{\log R} \wedge \zeta) < \infty \tag{37}$$

where $\kappa_{\log R} = \inf\{t > 0 : \epsilon(t) > \log R\}$ and \underline{n} is the excursion measure of $\xi - \underline{\xi}$. (The fact that the second expression in (37) is finite is a well known fact from the theory of Lévy processes; otherwise there would be an infinite rate of having arbitrary large excursions, which occurs with probability zero.)

Our objective now will be to show that

$$I_S(\theta) := \int_0^\infty \mathbb{N}_\theta(H_S(X^\epsilon(t)); |X^\epsilon(t)| < R, t < \zeta) dt < \infty, \tag{38}$$

which ensures that (35) is finite for Lebesgue almost all $t > 0$ and hence, thanks to stochastic continuity of the excursion measure, for all $t > 0$.

To prove (38) consider $|y| \geq 1$ and $\theta \in \mathbb{S}^{d-1} \setminus \mathbb{S}$, we can appeal to Proposition 5.2 of [20], which identifies, for $x \in \mathbb{R}^d \setminus \{0\}$, and continuous $g : \mathbb{R}^d \mapsto \mathbb{R}$ whose support is compactly embedded in the exterior of the ball of radius $|x|$,

$$\mathbb{N}_{\arg(x)} \left(\int_0^\zeta g(|x|e^{\epsilon(u)} \Theta^\epsilon(u)) du \right) = \frac{\Gamma((d - \alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)} \int_{|x| < |z|} g(z) \frac{(|z|^2 - |x|^2)^{\alpha/2}}{|z|^\alpha |x - z|^d} dz. \tag{39}$$

This gives us

$$\begin{aligned} \rho^\vee(\theta, dy) &:= \int_0^\infty \mathbb{P}_\theta^\vee(X_t \in dy, t < \kappa) dt \\ &= \frac{\Gamma(d/2)}{\Gamma((d - \alpha)/2)\Gamma(\alpha/2 + 1)} \frac{H_S(y)}{h(\theta)} \int_0^\infty \mathbb{N}_\theta(X^\epsilon(t) \in dy, t < \zeta) dt \\ &= \frac{\Gamma(d/2)}{\Gamma((d - \alpha)/2)\Gamma(\alpha/2 + 1)} \frac{H_S(y)}{h(\theta)} |y|^\alpha \int_0^\infty \mathbb{N}_\theta(e^{\epsilon(u)} \Theta^\epsilon(u) \in dy, u < \zeta) du \\ &= \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)\Gamma(\alpha/2 + 1)} \frac{||y|^2 - 1|^{\alpha/2}}{|\theta - y|^d} \frac{H_S(y)}{h(\theta)} dy \\ &= \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)\Gamma(\alpha/2 + 1)} \frac{H_{\{\theta\}}(y)H_S(y)}{h(\theta)} dy \end{aligned}$$

where we recall $h(\theta) = \int_S |\theta - \vartheta|^{-d} \sigma_1(d\vartheta)$, the representation of X^ϵ is given in (14) and the fact that $e^{\alpha\epsilon(\varphi(t))} d\varphi(t) = |X_t^\epsilon|^\alpha d\varphi(t) = dt$ on $t < \zeta$.

It now follows that, up to a multiplicative constant C (which, in the following calculations, will play the role of different constants that may change from line to line)

$$\begin{aligned} I_S(\theta) &= \int_{1 < |y| < R} \rho^\vee(\theta, dy) \\ &= \frac{C}{h(\theta)} \int_{1 < |y| < R} H_{\{\theta\}}(y)H_S(y) dy \\ &= \frac{C}{h(\theta)} \int_{1 < |y| < R} \int_{\phi \in \mathbb{S}} \frac{||y|^2 - 1|^\alpha}{|\theta - y|^d |\phi - y|^d} dy \sigma_1(d\phi). \end{aligned} \tag{40}$$

Since $\mathbb{S}^{d-1} \setminus \mathbf{S}$ is open, it is easy to see that we can choose ε small enough such that, for $\theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$,

$$\begin{aligned} \{y \in \mathbb{R}^d : |y| > 1\} &= \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y - \theta| > \varepsilon\} \\ &\cup \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y - \phi| > \varepsilon, \text{ for all } \phi \in \mathbf{S}\}, \end{aligned} \tag{41}$$

such that

$$\begin{aligned} \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y - \theta| \leq \varepsilon\} \\ \cap \{y \in \mathbb{R}^d : |y| > 1 \text{ and } |y - \phi| \leq \varepsilon, \text{ for some } \phi \in \mathbf{S}\} = \emptyset. \end{aligned}$$

Making use of (4), (5) and (A.1), and that, for $r > 1$, ${}_2F_1(d/2, 1; d/2, r^{-2}) = (1 - r^{-2})^{-1}$, allowing C to again play the role of a strictly positive constant that may change from line to line, we have, for $\theta \notin \mathbf{S}$,

$$\begin{aligned} I_{\mathbf{S}}(\theta) &= \frac{C}{h(\theta)} \int_{1 < |y| < R} \int_{\phi \in \mathbf{S}} \frac{\|y\|^2 - 1\|^\alpha}{|\theta - y|^d |\phi - y|^d} dy \sigma_1(d\phi) \\ &\leq \frac{C}{h(\theta)} \int_{1 < |y| < R, |\theta - y| \geq \varepsilon} \int_{\phi \in \mathbf{S}} \frac{\|y\|^2 - 1\|^\alpha}{|\theta - y|^d |\phi - y|^d} dy \sigma_1(d\phi) \\ &\quad + \frac{C}{h(\theta)} \int_{1 < |y| < R, |\phi - y| \geq \varepsilon} \int_{\phi \in \mathbf{S}} \frac{\|y\|^2 - 1\|^\alpha}{|\theta - y|^d |\phi - y|^d} dy \sigma_1(d\phi) \\ &\leq \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_{1 < |y| < R} \int_{\phi \in \mathbf{S}} \frac{\|y\|^2 - 1\|^\alpha}{|\phi - y|^d} dy \sigma_1(d\phi) + \int_{1 < |y| < R} \int_{\phi \in \mathbf{S}} \frac{\|y\|^2 - 1\|^\alpha}{|\theta - y|^d} dy \sigma_1(d\phi) \right) \\ &\leq \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_{1 < |y| < R} \int_{\phi \in \mathbb{S}^{d-1}} \frac{\|y\|^2 - 1\|^\alpha}{|\phi - y|^d} dy \sigma_1(d\phi) + \sigma_1(\mathbf{S}) \int_{1 < |y| < R} \frac{\|y\|^2 - 1\|^\alpha}{|\theta - y|^d} dy \right) \\ &= \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_{1 < |y| < R} \|y\|^2 - 1\|^\alpha |y|^{\alpha-d} (1 - |y|^{-2})^{\frac{\alpha}{2}-1} dy \right. \\ &\quad \left. + \int_1^R \int_0^\pi \frac{r^{d-1} (r^2 - 1)^\alpha (\sin \vartheta)^{d-2}}{(r^2 - 2r \cos(\vartheta) + 1)^{d/2}} dr d\vartheta \right) \\ &= \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_1^R (r^2 - 1)^{\frac{3\alpha}{2}-1} r dr + \int_1^R (r^2 - 1)^{\alpha-1} r dr \right) \\ &= \varepsilon^{-d} \frac{C}{h(\theta)} \left(\int_1^{R^2} (u - 1)^{\frac{3\alpha}{2}-1} du + \int_1^{R^2} (u - 1)^{\alpha-1} du \right) < \infty. \end{aligned}$$

Now let us turn to the issue of consistency. Recall that (A, \mathbb{N}_θ) is an exit system for the process (ξ, θ) . In particular, under any measure \mathbb{N}_θ the process $((\varepsilon(s), \theta^\varepsilon(s)), s < \zeta)$ is a strong Markov process with the same semigroup as (ξ, θ) killed at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$, see [22, Theorem 6.3]. As a consequence, for $\theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$,

$$\begin{aligned} \mathbb{E}_\theta^\vee [g(X_{t+s})] &= \frac{C}{h(\theta)} \mathbb{N}_\theta (H_{\mathbf{S}}(X_{t+s}^\varepsilon) g(X_{t+s}^\varepsilon) \mathbf{1}_{(s+t < \zeta)}) \\ &= \frac{C}{h(\theta)} \mathbb{N}_\theta \left(H_{\mathbf{S}}(X_t^\varepsilon) \mathbf{1}_{(t < \zeta)} \mathbb{E}_{X_t^\varepsilon} \left[\frac{H_{\mathbf{S}}(X_s)}{H_{\mathbf{S}}(X_t^\varepsilon)} g(X_s) \mathbf{1}_{(s < \tau_t^\ominus)} \right] \right) \\ &= \frac{C}{h(\theta)} \mathbb{N}_\theta \left(H_{\mathbf{S}}(X_t^\varepsilon) \mathbf{1}_{(t < \zeta)} \mathbb{E}_{X_t^\varepsilon}^\vee [g(X_s)] \right) \\ &= \mathbb{E}_\theta^\vee \left[H_{\mathbf{S}}(X_t^\varepsilon) \mathbf{1}_{(t < \zeta)} \mathbb{E}_{X_t^\varepsilon}^\vee [g(X_s)] \right], \end{aligned} \tag{42}$$

where $C = \Gamma(d/2)/\Gamma(\alpha/2 + 1)\Gamma((d - \alpha)/2)$. Hence, using the notation $\mathcal{P}_t^\vee[g](x) := \mathbb{E}_x^\vee[g(X_t)]$, we have $\mathcal{P}_{t+s}^\vee[g](x) = \mathcal{P}_t^\vee[\mathcal{P}_s^\vee[g]](x)$ for any $x \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathbf{S})$, and the required consistency follows.

To demonstrate the consistent choice of normalisation in our definition of \mathbb{P}^\vee , we will reconsider a different derivation of the resolvent ρ^\vee . To this end, suppose that $x \in \bar{\mathbb{B}}^c$ and we can similarly consider the resolvent of (X, \mathbb{P}_x^\vee) . This calculation can be developed using the nature of the Doob h -transform (6) and Theorem III.3.4 in [17] and takes the form

$$\rho^\vee(x, dy) = \frac{H_S(y)}{H_S(x)} \rho^\oplus(x, dy), \quad |x|, |y| > 1, \tag{43}$$

where

$$\rho^\oplus(x, dy) = \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2} |x - y|^{\alpha-d} \int_0^{\zeta^\oplus(x,y)} (u + 1)^{-d/2} u^{\alpha/2-1} du dy \tag{44}$$

and $\zeta^\oplus(x, y) = (|x|^2 - 1)(|y|^2 - 1)/|x - y|^2$. To show continuity as $x \rightarrow \theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$, and hence that the choice of normalisation in (12) is correct, we note that, as $r \rightarrow 1$,

$$\begin{aligned} \frac{H_S(y)}{H_S(x)} \rho^\oplus(r\theta, dy) &\sim \frac{\Gamma(d/2)|r\theta - y|^{\alpha-d} H_S(y)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)^2 h(\theta)} \\ &\quad \times \frac{2r(|y|^2 - 1)|r\theta - y|^{-2} \zeta(r\theta, y)^{\alpha/2-1} (1 + \zeta(r\theta, y))^{-d/2}}{2r(\alpha/2)(r^2 - 1)^{\alpha/2-1}} dy \\ &\sim \frac{\Gamma(d/2)(|y|^2 - 1)^{\alpha/2} |\theta - y|^{-d} H_S(y)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2) \Gamma(\alpha/2 + 1) h(\theta)} dy \\ &= \frac{\Gamma(d/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2) \Gamma(\alpha/2 + 1)} \frac{H_{\{\theta\}}(y) H_S(y)}{h(\theta)} dy \end{aligned} \tag{45}$$

Now, we need to show that $\mathbb{P}_\theta^\vee(X_{0+} = \theta) = 1$ for any $\theta \in \mathbb{S}^{d-1} \setminus \mathbf{S}$. Since $\lim_{t \downarrow 0} \varphi(t) = 0$, it suffices to show that

$$\mathbb{P}_\theta^\vee(X_0 \neq \theta) = \underline{\mathbb{N}}_\theta (\{\lim_{t \downarrow 0} \epsilon(t) = 0, \lim_{t \downarrow 0} \Theta^\epsilon(t) = \theta\}^c) = 0. \tag{46}$$

Let us first observe ϵ is an excursion of ξ from its running minimum and ξ is a hypergeometric Lévy process with unbounded variation, hence 0 is regular for $(0, \infty)$, that is

$$\mathbf{P}_{0,\theta}(\tau_0^+ = 0) = 1, \quad \theta \in \mathbb{S}^{d-1},$$

where $\tau_0^+ = \inf\{t > 0 : \xi_t > 0\}$. Classical excursion theory for Lévy processes implies that the excursions of ξ from its infimum begin continuously. Thanks to isotropy, this is equivalent to saying

$$\underline{\mathbb{N}}_\theta (\{\lim_{t \downarrow 0} \epsilon(t) = 0\}^c) = 0. \tag{47}$$

Since the jump measure of X in radial form is

$$\Pi(dr, d\theta) = \frac{1}{r^{1+\alpha}} \sigma_1(d\theta) dr, \quad r > 0, \theta \in \mathbb{S}^{d-1},$$

as a consequence, the process (ξ, Θ) has the property that both the modulator and the ordinate must jump simultaneously (the precise jump rate was explored in [17]). If it were the case that $\underline{\mathbb{N}}_\theta (\{\lim_{t \downarrow 0} \Theta^\epsilon(t) = \theta\}^c) > 0$ (and hence for all $\theta \in \mathbb{S}^{d-1}$ by rotational symmetry), this would be tantamount to a discontinuity in Θ but not in ξ , which is a contradiction since $((\epsilon(s), \Theta^\epsilon(s)), s < \zeta)$ under $\underline{\mathbb{N}}_\theta$ has the same semigroup as the isotropic process (ξ, Θ) killed

at its first hitting time of $(-\infty, 0] \times \mathbb{S}^{d-1}$. The requirement (46) now follows. This completes the proof of Theorem 2 as far as \mathbb{P}^\vee is concerned.

The proof of Theorem 2 for (X, \mathbb{P}^\wedge) is essentially the same as soon as we have an analogous identity for (39), but for $\overline{\mathbb{N}}_\theta$. Unfortunately this does not seem to be available in the literature, and so we spend a little time developing it here. However the remaining details of the proof of Theorem 2 we leave to the reader.

The main idea behind the derivation of an analogue to (39) for $\overline{\mathbb{N}}_\theta$ lies with the use of the Riesz–Bogdan–Żak transform in Theorem 3. Let us consider a variant of the radial excursion process which is based on the MAP $(\overline{\xi} - \xi, \Theta)$, that is associated to X but now under the change of measure (16). The reader will recall that the probabilities $\mathbb{P}^\circ = (\mathbb{P}_x^\circ, x \neq 0)$ correspond to conditioning the process X to be continuously absorbed at the origin. It turns out that if (ξ, Θ) , with probabilities $\mathbf{P}^\circ = (\mathbf{P}_{x,\theta}^\circ, x \in \mathbb{R}, \theta \in \mathbb{S}^{d-1})$, is the MAP whose Lamperti transform gives (X, \mathbb{P}°) , then $(-\xi, \Theta)$, is the MAP whose Lamperti transform gives (X, \mathbb{P}) ; see Theorem 1.3.6 and Corollary 1.3.17 of [17].

In the spirit of (11) we can write down the exit system for the radial excursion process of $(\overline{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ under \mathbb{P}° . Suppose that ℓ° , and $(\overline{\mathbb{N}}_\theta^\circ, \theta \in \mathbb{S}^{d-1})$ denote the associated local time and system of excursion measures. As with excursion theory from the radial minimum of X , isotropy allows us to conclude that we may choose ℓ° to be the local time at 0 of $\overline{\xi} - \xi$, and that ξ (without its modulator Θ) is necessarily a Lévy process under \mathbf{P} . Since $\lim_{t \rightarrow \infty} \overline{\xi}_t = -\infty$ under \mathbf{P}° , we can also appeal to isotropy again to normalise ℓ° in such a way that $\overline{\mathbb{N}}^\circ(\zeta = \infty) = 1$.

With this set up we can follow the reasoning in [20] and deduce that, for positive, bounded and measurable g on \mathbb{R}^d ,

$$\overline{\mathbb{N}}_{\arg(x)}^\circ \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) e^{\alpha\epsilon(s)} ds \right) = \lim_{|x| \uparrow 1} \frac{\mathbb{E}_x^\circ \left(\int_0^{\tau_1^\ominus} g(X_s) ds \right)}{\mathbb{P}_x^\circ(\tau_1^\ominus = \infty)}, \tag{48}$$

where we recall that $\tau_1^\ominus = \inf\{t > 0 : |X_t| > 1\}$. Note that the choice of normalisation of ℓ° is implicit in the aforementioned limiting equality. Appealing to numerous calculations involving the Riesz–Bogdan–Żak transformation e.g. in [17] or indeed [18], we can rewrite the limit

$$\lim_{|x| \uparrow 1} \frac{\mathbb{E}_x^\circ \left(\int_0^{\tau_1^\ominus} g(X_s) ds \right)}{\mathbb{P}_x^\circ(\tau_1^\ominus = \infty)} = \lim_{|x| \uparrow 1} \frac{\mathbb{E}_{Kx} \left(\int_0^{\tau_1^\oplus} g(KX_s) |X_s|^{-2\alpha} ds \right)}{\mathbb{P}_{Kx}(\tau_1^\oplus = \infty)},$$

where $Kx = x/|x|^2$. Appealing to the identities provided in (17) and (44), the limiting ratio is computable directly giving us in (48)

$$\overline{\mathbb{N}}_{\arg(x)}^\circ \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) e^{\alpha\epsilon(s)} ds \right) = \int_{|z|>1} g(Kz) |z|^{-2\alpha} \frac{(|z|^2 - 1)^{\alpha/2}}{|\arg(x) - z|^d} dz.$$

An easy change of variables $y = Kz$, noting the classical analytical facts that $dz = |y|^{-2d} dy$ and $|\theta - Ky| = |\theta - y|/|y|$, for $\theta \in \mathbb{S}^{d-1}$,

$$\overline{\mathbb{N}}_\theta^\circ \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) e^{\alpha\epsilon(s)} ds \right) = \int_{|y|<1} g(y) |y|^{\alpha-d} \frac{(1 - |y|^2)^{\alpha/2}}{|\theta - y|^d} dz. \tag{49}$$

As noted in [17], the change of measure (16) when understood as a change of measure affecting (ξ, Θ) , is equivalent to the martingale change of measure,

$$\frac{d\mathbf{P}_{x,\theta}^\circ}{d\mathbf{P}_{x,\theta}} \Big|_{\sigma((\xi_s, \Theta_s), s \leq t)} = e^{(\alpha-d)(\xi_t-x)}. \tag{50}$$

We can use this to compare the left-hand side of (49) with an analogous object albeit for $\bar{\mathbb{N}}_\theta$, the excursion measure of $(\bar{\xi} - \xi, \Theta)$ from $\{0\} \times \mathbb{S}^{d-1}$ under \mathbf{P} , by studying the effect of (50) on the exit formula (11). It is straightforward to show that, for $\theta \in \mathbb{S}^{d-1}$ and positive, bounded and measurable g ,

$$\bar{\mathbb{N}}_\theta^\circ \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) ds \right) = \bar{\mathbb{N}}_\theta \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) e^{(\alpha-d)\epsilon(s)} ds \right).$$

Note that the normalisation of local time for (ξ, Θ) under \mathbf{P} is, in effect, chosen by the above equality. It follows that

$$\bar{\mathbb{N}}_\theta \left(\int_0^\zeta g(e^{\epsilon(s)} \Theta^\epsilon(s)) ds \right) = \int_{|y|<1} g(y) \frac{(1 - |y|^2)^{\alpha/2}}{|y|^\alpha |\theta - y|^d} dy. \tag{51}$$

The reader will note that, aside from the domain of integration on the right-hand side, this agrees with (39).

With (51) in hand, as alluded to above, we can now leave the reader to verify that the proof of Theorem 2 for (X, \mathbb{P}^\wedge) is essentially verbatim the same as for (X, \mathbb{P}^\vee) . \square

Proof of Theorem 4. Given the proof of Theorem 2 above, we refrain from giving the proof of Theorem 4, noting only that it is a variant of the arguments given there. The details are, once again, left to the reader. We additionally note that e.g. in the case of \mathbb{P}^\ominus , the excursion may begin anywhere on \mathbb{S}^{d-1} and, when proving that e.g. $\bar{\mathbb{N}}_\theta(H^\ominus(X^\epsilon(t)); t < \zeta) < \infty$, it is much easier to show that the analogue of (38) is finite without needing to split space up as in (41). \square

7. Proof of Theorem 5

Recall the notation for a general Markov process (Y, P) on E preceding the statement of Theorem 5. We will additionally write $\mathcal{P} := (\mathcal{P}_t, t \geq 0)$ for the semigroup associated to (Y, P) .

Theorem 3.5 of Nagasawa [23], shows that, under suitable assumptions on the Markov process, L -times form a natural family of random times at which the pathwise time-reversal

$$\overleftarrow{Y}_t := Y_{(k-t)-}, \quad t \in (0, k),$$

is again a Markov process. Let us state Nagasawa’s principle assumptions.

(A) The potential measure $U_Y(a, \cdot)$ associated to \mathcal{P} , defined by the relation

$$\int_E f(x) U_Y(a, dx) = \int_0^\infty \mathcal{P}_t[f](a) dt = \mathbb{E}_a \left[\int_0^\infty f(X_t) dt \right], \quad a \in E, \tag{52}$$

for bounded and measurable f on E , is σ -finite. Assume that there exists a probability measure, ν , such that, if we put

$$\mu(A) = \int U_Y(a, A) \nu(da) \quad \text{for } A \in \mathcal{B}(\mathbb{R}), \tag{53}$$

then there exists a Markov transition semigroup, say $\hat{\mathcal{P}} := (\hat{\mathcal{P}}_t, t \geq 0)$ such that

$$\int_E \mathcal{P}_t[f](x)g(x) \mu(dx) = \int_E f(x)\hat{\mathcal{P}}_t[g](x) \mu(dx), \quad t \geq 0, \tag{54}$$

for bounded, measurable and compactly supported test-functions f, g on E .

(B) For any continuous test-function $f \in C_0(E)$, the space of continuous and compactly supported functions, and $a \in E$, assume that $\mathcal{P}_t[f](a)$ is right-continuous in t for all $a \in E$ and, for $q > 0$, $U_{\hat{Y}}^{(q)}[f](\tilde{Y}_t)$ is right-continuous in t , where, for bounded and measurable f on E ,

$$U_{\hat{Y}}^{(q)}[f](a) = \int_0^\infty e^{-qt} \hat{\mathcal{P}}_t[f](a)dt, \quad a \in E,$$

is the q -potential associated to $\hat{\mathcal{P}}$.

Nagasawa’s duality theorem, Theorem 3.5. of [23], now reads as follows.

Theorem 6 (Nagasawa’s Duality Theorem). *Suppose that assumptions (A) and (B) hold. For the given starting probability distribution ν in (A) and any L -time k , the time-reversed process \tilde{Y} under \mathbb{P}_ν is a time-homogeneous Markov process with transition probabilities*

$$\mathbb{P}_\nu(\tilde{Y}_t \in A \mid \tilde{Y}_r, 0 < r < s) = \mathbb{P}_\nu(\tilde{Y}_t \in A \mid \tilde{Y}_s) = p_{\hat{Y}}(t - s, \tilde{Y}_s, A), \quad \mathbb{P}_\nu\text{-almost surely,} \tag{55}$$

for all $0 < s < t$ and Borel A in \mathbb{R} , where $p_{\hat{Y}}(u, x, A)$, $u \geq 0$, $x \in \mathbb{R}$, is the transition measure associated to the semigroup $\hat{\mathcal{P}}$.

Proof of Theorem 5. We give the proof of (i), the proof of (ii) is almost identical albeit requiring some straightforward adjustments. Once again, we leave the details to the reader. When $t > 0$, we use Nagasawa’s duality theorem. However, since the process is conditioned to hit continuously, its dual processes from the hitting time must leave the sphere continuously. That means, if the duality is true for $t > 0$, it must be true for all $t \geq 0$.

We will make a direct application of Theorem 6, with Y taken to be the process $(X, \mathbb{P}_\nu^\ominus)$ where ν satisfies (22) or (23) according to the nature of \mathbf{S} . Accordingly, we will write U^\ominus in place of U_Y , \mathcal{P}^\ominus in place of \mathcal{P} etc. Moreover, the dual process, formerly \hat{Y} , is taken to be (X, \mathbb{P}^\vee) and we will, in the obvious way, work with the notation U^\vee in place of $U_{\hat{Y}}$, \mathcal{P}^\vee in place of $\hat{\mathcal{P}}$ and so on. In essence we need only to verify the two assumptions (A) and (B). Let us momentarily take the former of these two cases.

In order to verify (A) we will make use of (39). Noting that $e^{\alpha\epsilon_\nu(t)}d\varphi(t) = dt$, we have for $a \in \mathbb{S}^{d-1} \setminus \mathbf{S}$ and bounded measurable $f : \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathbf{S}) \rightarrow [0, \infty)$,

$$\begin{aligned} U^\ominus[f](a) &= \mathbb{E}_a^\ominus \left[\int_0^\infty f(X_t)dt \right] \\ &= \mathbb{N}_a \left(\int_0^5 H^\ominus(X_t^\epsilon) f(X_t^\epsilon) dt \right) \\ &= \mathbb{N}_a \left(\int_0^5 H^\ominus(e^{\epsilon(u)} \Theta^\epsilon(u)) f(e^{\epsilon(u)} \Theta^\epsilon(u)) e^{\alpha\epsilon u} du \right) \\ &= C \int_{\mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathbf{S})} H^\ominus(y) f(y) (|y|^2 - 1)^{\alpha/2} |a - y|^{-d} dy, \end{aligned} \tag{56}$$

where $U^\ominus[f](a) = \int_{\mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathbb{S})} f(y)U^\ominus(a, dy)$, $C > 0$ is an unimportant constant and we have used (20) in the second equality.

Next, we need to develop an expression for the reference measure μ . This only needs to be identified up to a multiplicative constant. As such, in the setting that $\sigma_1(\mathbb{S}) > 0$, recalling (53), (22) and (3), we can take (ignoring multiplicative constants in each line)

$$\begin{aligned} \mu(dy) &= \int_{\mathbb{S}} v(da)U^\ominus(a, dy) \\ &= \int_{\mathbb{S}} \sigma_1(da)H^\ominus(y)(|y|^2 - 1)^{\alpha/2}|a - y|^{-d}dy \\ &= H_{\mathbb{S}}(y)H^\ominus(y)dy, \quad y \in \mathbb{R}^d \setminus (\mathbb{B}_d \cup \mathbb{S}). \end{aligned} \tag{57}$$

When $\mathbb{S} = \{\vartheta\}$, we replace the use of (22) by (23) in the above calculation and the same answer comes out (up to a multiplicative constant).

Next, we need to verify that (54) holds. Indeed, using Hunt’s switching identity (cf. Chapter II.1 of [2]) for the process $(X_t, t < \tau_1^\oplus)$, we have for $x, y \in \mathbb{R}^d \setminus \tilde{\mathbb{B}}_d$

$$\begin{aligned} \mu(dy)\mathcal{P}_t^\ominus(y, dx) &= \mathcal{P}_t^\ominus(y, dx)H_{\mathbb{S}}(y)H^\ominus(y)dy \\ &= \frac{H^\ominus(x)}{H^\ominus(y)}\mathcal{P}_t^{\mathbb{B}_d}(y, dx)H_{\mathbb{S}}(y)H^\ominus(y)dy \\ &= \mathcal{P}_t^{\mathbb{B}_d}(x, dy)H_{\mathbb{S}}(y)H^\ominus(x)dx \\ &= \mathcal{P}_t^\vee(x, dy)\mu(dx), \end{aligned}$$

where $\mathcal{P}_t^{\mathbb{B}_d}(x, dy) = \mathbb{P}_x(X_t \in dy, t < \tau_1^\oplus)$. Note, as the measure μ is absolutely continuous with respect to Lebesgue measure, we do not need to deal with the case that x or y belong to $\mathbb{S}^{d-1} \setminus \mathbb{S}$.

Let us now turn to the verification of assumption **(B)**. This assumption is immediately satisfied on account of the fact that both \mathcal{P}^\ominus and \mathcal{P}^\vee are right-continuous semigroups by virtue of their definition as a Doob h -transform with respect to the Feller semigroup $\mathcal{P}^{\mathbb{B}_d}$ of the stable process killed on entry to \mathbb{B}_d . With both **(A)** and **(B)** in hand, we can invoke Theorem 6 and the desired result follows. \square

8. Concluding remarks

The results in this paper have considered the setting of conditioning a relatively special class of Markov process to continuously hit a subset of the unit sphere with a one-sided approach. Taking a step back, one would ideally like to drop a number of the specialisms specific to our approach e.g. moving to a general Markov process and conditioning it continuously hit a suitably general domain. The current proofs rely on too many particular features of stable Lévy processes for the results to directly generalise in this respect. For example, suppose that we drop the assumption that the stable process continuously approaches \mathbb{S} from just one side, but instead we allow it to continuously approach without radial confinement. This is a topic that has been addressed in follow-on work [14], for which a mixture of features that are specific to stable Lévy processes together with general potential-analytic considerations are used. The classical work of Doob [11] for the setting of Brownian motion also gives insight in how one may go about dealing with greater generality.

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Appendix. Hypergeometric identity

An identity for the hypergeometric function that has been used twice in the main body of the text is taken from formula 3.665(2) in [13]. It states that, for any $0 < |a| < r$ and $\nu > 0$,

$$\int_0^\pi \frac{\sin^{d-2} \phi}{(a^2 + 2ar \cos \phi + r^2)^\nu} d\phi = \frac{1}{r^{2\nu}} B\left(\frac{d-1}{2}, \frac{1}{2}\right) {}_2F_1\left(\nu, \nu - \frac{d}{2} + 1; \frac{d}{2}; \frac{a^2}{r^2}\right). \quad (\text{A.1})$$

References

- [1] L. Alili, L. Chaumont, P. Graczyk, Zak. T., Inversion, duality and doob h -transforms for self-similar Markov processes, *Electron. J. Probab.* 22 (20) (2017) 18.
- [2] J. Bertoin, *Lévy Processes*, Cambridge University Press, 1996.
- [3] J. Bertoin, R.A. Doney, On conditioning a random walk to stay nonnegative, *Ann. Probab.* 22 (1994) 2152–2167.
- [4] R.M. Blumenthal, *Excursions of Markov Processes*, Birkhäuser, Boston, Basel, Berlin, 1992.
- [5] R.M. Blumenthal, R.K. Gettoor, D.B. Ray, On the distribution of first hits for the symmetric stable processes, *Trans. Amer. Math. Soc.* 99 (1961) 540–554.
- [6] K. Bogdan, T. Zak, On Kelvin transformation, *J. Theoret. Probab.* 19 (2006) 89–120.
- [7] K. Burdzy, Brownian excursions from hyperplanes and smooth surfaces, *Trans. Amer. Math. Soc.* 295 (1) (1986) 35–57.
- [8] M.E. Caballero, J.C. Pardo, J.L. Pérez, Explicit identities for Lévy processes associated to symmetric stable processes, *Bernoulli* 17 (1) (2011) 34–59.
- [9] L. Chaumont, R.A. Doney, On Lévy processes conditioned to stay positive, *Electron. J. Probab.* 10 (28) (2005) 948–961.
- [10] K.L. Chung, J.B. Walsh, *Markov Processes, Brownian Motion, and Time Symmetry*, Springer-Verlag New York, 2005.
- [11] J.L. Doob, Conditional Brownian motion and the boundary limits of harmonic functions, *Bull. Société Mathématique Fr.* 85 (1957) 431–458.
- [12] L. Döring, P. Weissman, Stable processes conditioned to hit an interval continuously from the outside, *Bernoulli* 26 (2) (2020) 980–1015.
- [13] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York and London, 1965.
- [14] M. Kwaśniski, A.E. Kyprianou, S. Palau, Saizmaa Ts, Oscillatory attraction and repulsion from a subset of the unit sphere or hyperplane for isotropic stable Lévy processes, Preprint.
- [15] A.E. Kyprianou, *Fluctuations of Lévy Processes with Applications*, Springer, 2014.
- [16] A.E. Kyprianou, Deep factorisation of the stable process, *Electron. J. Probab.* 21 (23) (2016) 1–28.
- [17] A.E. Kyprianou, Stable processes, self-similarity and the unit ball, *ALEA Lat. Am. J. Probab. Math. Stat.* 15 (2018) 617–690.
- [18] A.E. Kyprianou, J.C. Pardo, *Stable Lévy Processes Via Lamperti-Type Representations*, Cambridge University Press, 2021.
- [19] A.E. Kyprianou, V. Rivero, W. Satitkanitkul, Stable Lévy processes in a cone, 2018, arXiv:1804.08393v2 [math.PR].
- [20] A.E. Kyprianou, V. Rivero, W. Satitkanitkul, Deep factorization of the stable process III: Radial excursion theory and the point of closest reach, *Potential Anal.* 53 (2020) 1347–1375.
- [21] A.E. Kyprianou, V. Rivero, B. Sengul, Conditioning subordinators embedded in Markov processes, *Stochastic Process. Appl.* 127 (2017) 1234–1254.

- [22] B. Maisonneuve, Exit systems, *Ann. Probab.* 3 (3) (1975) 399–411.
- [23] M. Nagasawa, Time reversion of Markov processes, *Nagoya. Math. J.* 24 (1964) 177–204.
- [24] H. Panti, On Lévy processes conditioned to avoid zero, *ALEA Lat. Am. J. Probab. Math. Stat.* 14 (2017) 657–690.