Attraction to and repulsion from a subset of the unit sphere for isotropic stable Levy processes

joint work with A.E.Kyprianou and S. Palau

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Overview

Stable process

A process $X = (X_t, t \geq 0)$ with probabilities $\mathbb{P}_x, x \in \mathbb{R}^d$ is a d-dimensional *stable Lévy process* with stability index α, if it is a

(i) Lévy process: cádlág paths with stationary and independent increments and

(ii) Self-similar Markov process: $\exists$ a self-similarity index α s.t. for $\forall c > 0$ and $\forall x \in \mathbb{R}^d \backslash \{0\}$,

$$\text{under } \mathbb{P}_x, \text{ the law of } (cX_{c^{-\alpha}t}, t \geq 0) \text{ is equal to } \mathbb{P}_{cx}$$

Furthermore, let X_t be isotropic, that is

- for all orthogonal $U\colon \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the law of $(UX_t, t \geq 0)$ under $\mathbb{P}_x$ is equal to $(X_t, t \geq 0)$ under $\mathbb{P}_{Ux}$

Remarks

- Lévy-Khintchine + Scaling $\Rightarrow \alpha \in (0,2]$
- When $\alpha = 2$, BM with continuous path
- For $\alpha \in (0,2)$, pure jump process with Lévy triplet $(0,0,\Pi)$ where jump measure is given as

$$\Pi(B) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|} \int_B \frac{1}{|y|^{\alpha+d}} dy, \quad B \subset \mathcal{B}(\mathbb{R}^d)$$

Aim of the work

Want to characterize

- X_t conditioned to hit $\Omega \subseteq \mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ continuously either from outside or from inside of $\mathbb{S}^{d-1}$
- its time reversal from the hitting point on Ω

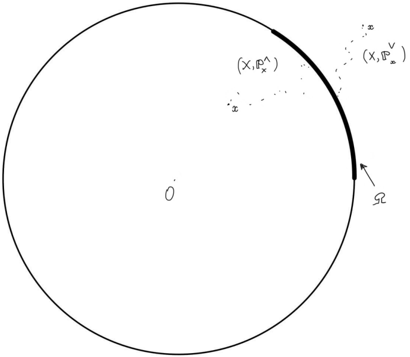
Remark:

- The case $d = 1$ has characterized by Döring, L. and Weissman, P. (2018)
- hence, we will only consider $d \geq 2$

Attraction towards Ω

Let $\Omega \subseteq \mathbb{S}_d$ be a measurable subset with $\sigma_1(\Omega) > 0$ or $\Omega = \{\vartheta\}$

- Attraction towards Ω
  $(X, \mathbb{P}_x^{\vee})$ - conditioned to hit Ω continuously from outside
  $(X, \mathbb{P}_x^{\wedge})$ - conditioned to hit Ω continuously from inside



Conditioning to hit Ω continuously from outside

Let $\mathbb{B}_d := \{x \in \mathbb{R}^d : |x| < 1\}$ and $\overline{\mathbb{B}}_d^c := \mathbb{R}^d \setminus \overline{\mathbb{B}}_d$

- Define $\underline{G}(t) := \sup\{s \le t : |X_s| = \inf_{u \le s} |X_u|\}, \quad t \ge 0$
- When $d \ge 2, (X, \mathbb{P})$ is transient, e.g. $\lim_{t\to\infty} |X_t| = \infty$ a.s.
- Due to monotonicity and transience, $\exists \underline{G}(\infty) := \lim_{t\to\infty} \underline{G}(t)$
- $X_{\underline{G}(\infty)}$ - the point of closest reach to the origin
  - characterized in [Kyprianou, Rivero, Satitkhanitkul '18]

Definition of conditioning

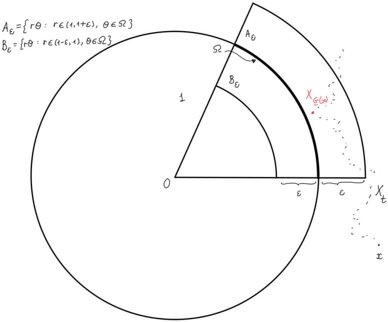
Conditioning event to hit Ω continuously from outside:

- $C_\varepsilon^\vee := \{X_{\underline{G}(\infty)} \in A_\varepsilon\}$, for $\forall \varepsilon > 0$
- $\tau_1^\oplus = \inf\{t > 0 \colon |X_t| < 1\}$

For $x \in \overline{\mathbb{B}}_d^c$, define asymptotic conditioning as

$$\mathbb{P}_x^\vee(A, t < \tau_1^\oplus) := \lim_{\varepsilon \to 0} \mathbb{P}_x(A, t < \tau_1^\oplus | C_\varepsilon^\vee)$$

for all $A \in \mathcal{F}_t := \sigma(X_s, s \le t)$



Conditioning to hit Ω continuously from inside

Similarly, define

- $\overline{G}(t) := \sup\{s \leq t \colon |X_s| = \sup_{u \leq s} |X_u|\}, \quad t \geq 0$
- $\tau_1^{\ominus} := \inf\{t > 0 \colon |X_t| > 1\}$ then
- $X_{\overline{G}(\tau_1^{\ominus}-)}$ - the point of furthest reach from the origin before exiting $\overline{\mathbb{B}}_d$
  - characterized in [Kyprianou, Rivero, Satitkhanitkul '18]

Definition of Conditioning

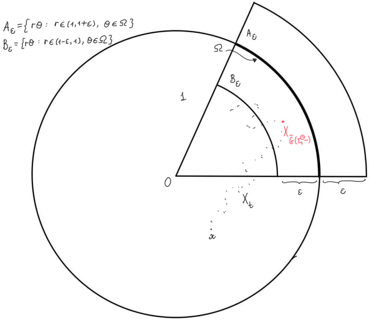
Conditioning event to hit Ω continuously from inside:

- for $\forall \varepsilon > 0$, define $C_\varepsilon^\wedge := \{X_{\overline{G}(\tau_1^\ominus -)} \in B_\varepsilon\}$

When $x \in \mathbb{B}_d$, define asymptotic conditioning as

$$\mathbb{P}_x^\wedge(A, t < k) := \lim_{\varepsilon \to 0} \mathbb{P}_x(A, t < \tau_1^\ominus | C_\varepsilon^\wedge)$$

for all $A \in \mathcal{F}_t$.



Results: Distribution of $(X, \mathbb{P}^{\vee})$ and $(X, \mathbb{P}^{\wedge})$

Denote

$$H_{\Omega}(x) = \begin{cases} \int_{\Omega} |\theta - x|^{-d} ||x|^2 - 1|^{\alpha/2-1} \sigma_1(d\theta), & \sigma_1(\Omega) > 0 \\ |\vartheta - x|^{-d} ||x|^2 - 1|^{\alpha/2-1}, & \Omega = \{\vartheta\} \end{cases}$$

Theorem (Distribution of the conditioned process)

Let $\Omega \subseteq \mathbb{S}_d$ be an open set with $\sigma_1(\Omega) > 0$ or $\Omega = \{\vartheta\}$ for a fixed point $\vartheta \in \mathbb{S}_d$, then for all point of issue $x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}$, we have

(i) *if* $|x| > 1$,

$$\left.\frac{d\mathbb{P}_x^{\vee}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{\{t < \tau_1^{\oplus}\}} \frac{H_{\Omega}(X_t)}{H_{\Omega}(x)}$$

(ii) *if* $|x| < 1$,

$$\left.\frac{d\mathbb{P}_x^{\wedge}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{\{t < \tau_1^{\ominus}\}} \frac{H_{\Omega}(X_t)}{H_{\Omega}(x)}$$

Results: Distribution of the hitting location

- Let $\Omega' \subset \Omega$ and
  - $A'_\varepsilon :=$ the restriction of A_ε on Ω'
  - $B'_\varepsilon :=$ the restriction of B_ε on Ω'
- Want $\mathbb{P}^\vee_x(X_k \in \Omega')$ and $\mathbb{P}^\wedge_x(X_k \in \Omega')$, where k is their lifetime

Define

$$\mathbb{P}^\vee_x(X_k \in \Omega') := \lim_{\varepsilon \to 0} \mathbb{P}_x(X_{\underline{G}(\infty)} \in A'_\varepsilon | C^\vee_\varepsilon), \quad \text{for} \quad |x| > 1,$$

$$\mathbb{P}^\wedge_x(X_k \in \Omega') := \lim_{\varepsilon \to 0} \mathbb{P}_x(X_{\underline{G}(\tau_1^\ominus)} \in B'_\varepsilon | C^\wedge_\varepsilon), \quad \text{for} \quad |x| < 1.$$

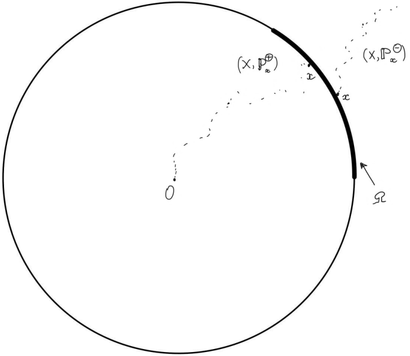
Theorem (Distribution of the hitting location)

Let $\Omega' \subset \Omega$. If $\sigma_1(\Omega') > 0$ then the conditioned process has a hitting distribution given as

$$\mathbb{P}^\vee_x(X_k \in \Omega') = \mathbb{P}^\wedge_x(X_k \in \Omega') = \frac{\int_{\Omega'} |\theta - x|^{-d} d\theta}{\int_{\Omega} |\theta - x|^{-d} d\theta}$$

Repulsion from Ω

- Repulsion from Ω
  $(X, \mathbb{P}^{\ominus})$ - conditioned to avoid the unit ball, issued from Ω
  $(X, \mathbb{P}^{\oplus})$ - conditioned to hit the origin continuously and stay inside the unit ball, issued from Ω



Definitions of $(X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\oplus})$

Denoting $Kx = x/|x|^2, x \in \mathbb{R}^d$, let us introduce

$$H^{\ominus}(x) = \mathbb{P}_x(\tau_1^{\oplus} = \infty), \quad \textit{for} \quad |x| > 1,$$

$$H^{\oplus}(x) = |x|^{\alpha-d} H^{\ominus}(Kx), \quad \textit{for} \quad |x| < 1,$$

Define two families of probabilities

$$\mathbb{P}^{\ominus} = (\mathbb{P}_x^{\ominus}, |x| > 1) \quad \textit{and} \quad \mathbb{P}^{\oplus} = (\mathbb{P}_x^{\oplus}, |x| < 1)$$

via the change of measures

$$\left.\frac{d\mathbb{P}_x^{\ominus}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{\{t<\tau_1^{\oplus}\}} \frac{H^{\ominus}(X_t)}{H^{\ominus}(x)} \quad |x| > 1$$

$$\left.\frac{d\mathbb{P}_x^{\oplus}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \mathbf{1}_{\{t<\tau_1^{\ominus}\}} \frac{H^{\oplus}(X_t)}{H^{\oplus}(x)} \quad |x| < 1,$$

Characterization of $(X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\oplus})$

From the definition, one can observe that

- $(X, \mathbb{P}^{\ominus})$ is a stable process conditioned to avoid $\mathbb{S}^{d-1}$
- $(X, \mathbb{P}^{\oplus})$ is a stable process conditioned to stay inside $\mathbb{S}^{d-1}$ and absorb to the origin
  - thanks to Reisz-Bogdan-Żak transform in [Kyprianou '18]

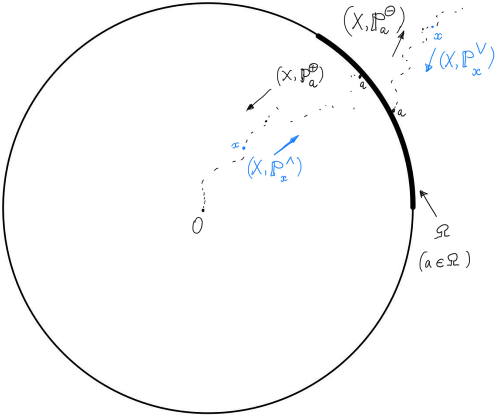
Theorem (Relationship between $(X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\oplus})$)

For $|x| > 1, (KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^{\ominus}$ is equal in law to $(X, \mathbb{P}^{\oplus}_{Kx})$, where $\eta(t) = \inf\{s > 0\colon \int_0^s |X_u|^{-2\alpha} du > t.\}$
For $|x| < 1, (KX_{\eta(t)}, t \geq 0)$ under $\mathbb{P}^{\oplus}$ is equal in law to $(X, \mathbb{P}^{\ominus}_{Kx})$.

Duality

Pairing up $(X, \mathbb{P}^{\vee}), (X, \mathbb{P}^{\ominus})$ and $(X, \mathbb{P}^{\wedge}), (X, \mathbb{P}^{\oplus})$

- via Nagasawa's duality theorem for time reversal [Nagasawa '64]



Idea behind Duality by Nagasawa

Markov processes $(X, \mathbb{P})$ and $(\hat{X}, \hat{\mathbb{P}})$ are dual in the sense that "time reversed" process of $(X, \mathbb{P})$ "is" $(\hat{X}, \hat{\mathbb{P}})$

if

$\exists$ a special measure η (with some properties) s.t.

$$p_t(x, dy)\eta(dx) = \hat{p}_t(y, dx)\eta(dy)$$

To construct "time reversed" process of $(X, \mathbb{P})$, need to

- guess $(\hat{X}, \hat{\mathbb{P}})$ and η
- then verify

Duality result

We guessed and verified that

- Time reversed process of $(X, \mathbb{P}^{\ominus})$ "is" $(X, \mathbb{P}^{\vee})$ with $\eta(dx) = H_{\Omega}(x) H^{\ominus}(x) dx$
- Time reversed process of $(X, \mathbb{P}^{\oplus})$ "is" $(X, \mathbb{P}^{\wedge})$ with $\eta(dx) = H_{\Omega}(x) H^{\oplus}(x) dx$

A bit more (but not full) precisely

Theorem

- *For every $\mathcal{L}$-time* $\mathbf{k}$ *of* $(X, \mathbb{P}^{\ominus})$*, the process* $(X_{(\mathbf{k}-t)-}, t < \mathbf{k})$ *under* $\mathbb{P}^{\ominus}_{\nu}$ *has Markov increments which agrees with those of* $(X, \mathbb{P}^{\vee})$,
- *For every $\mathcal{L}$-time* $\mathbf{k}$ *of* $(X, \mathbb{P}^{\oplus})$ *the process* $(X_{(\mathbf{k}-t)-}, t < \mathbf{k})$ *under* $\mathbb{P}^{\ominus}_{\nu}$ *has Markov increments which agrees with those of* $(X, \mathbb{P}^{\wedge})$

where

$$\nu(da) := \begin{cases} \frac{\sigma_1(da)}{\sigma_1(\Omega)}, & \sigma_1(\Omega) > 0 \\ \delta_{\{\vartheta\}}(da), & \Omega = \{\vartheta\} \end{cases}$$

Remark: Examples of $\mathcal{L}$-time includes killing time and last exit time.

Summary

RBZ stands for Reisz-Bogdan-Żak transform

$$
\begin{tikzcd}
\text{Inside } \mathbb{B}_d & (X,\mathbb{P}^{\wedge}) \arrow[r, leftrightarrow, "\text{Nagasawa}", "\text{duality}"'] \arrow[d, leftrightarrow, "\text{RBZ}"] & (X,\mathbb{P}^{\oplus}) \arrow[d, leftrightarrow, "\text{RBZ}"] \\
\text{Outside } \mathbb{B}_d & (X,\mathbb{P}^{\vee}) \arrow[r, leftrightarrow, "\text{Nagasawa}", "\text{duality}"'] & (X,\mathbb{P}^{\ominus}) \\
 & \text{Attraction towards } \Omega & \text{Repulsion from } \Omega
\end{tikzcd}
$$

References

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THANK YOU!